On non-continuous T-norms

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As usually in fuzzy logics, by a T-norm we mean a binary operation $T : [0,1]^2 \to [0,1]$ satisfying the following axioms

(T1) $T(y,x) = T(x,y) \le T(x',y)$ for any $0 \le x \le x'$ and $0 \le y \le 1$,
(T2) $T(x,T(y,z)) = T(T(x,y),z)$ for any $0 \le x,y,z \le 1$,
(T3) $T(0,x) = 0$ and $T(1,x) = x$ for any $0 \le x \le 1$.

Most authors include also the continuity of T into the definition, and actually there is a complete classification for the continuous T-norms. In this paper we focus to the case of non-continuous T-norms. In another terminology, (T1),(T2),(T3) mean that the algebraic structure $\mathbf{T} := \big([0,1], \overset{T}{\bullet}, \ge\big)$ with the binary operation

$$x \overset{T}{\bullet} y := T(x,y) \ , \qquad x,y \in [0,1]$$

is an *ordered Abelian semigroup* on $[0,1]$ with neutral element 1 and sink 0. Also, in accordance with the usual terminology, we say that the T-norm $\overset{T}{\bullet}$ is *strict* if

(T4) $T(x_1,y) < T(x_2,y)$ whenever $0 \le x_1 < x_2 \le 1$ and $0 < y \le 1$.

In the sequel we shall fix an arbitrary T-norm $\overset{T}{\bullet}$, and we shall write simply xy instead of $x \overset{T}{\bullet} y$ without danger of confusion with the notation of the usual numerical product of real numbers (which may appear only as a simple special case of continuous T-norm). Thus, in this terminology axioms (T1),...,(T4) mean simply

(T1) $xy{=}yx$, (T2) $x(yz){=}(xy)z$, (T3) $0x{=}0{\le}1x{=}x$, (T4) $x_1y{<}x_2y$ $(x_1{<}x_2,\ y{\ne}0)$.

We shall also use the customary notation a^n for the n-th $\overset{T}{\bullet}$-power $a^n = \underbrace{a \cdots a}_{n\ terms}$ which is well-defined by the associativity (T2).

1.1. Definition. We introduce the binary relations $\prec$ on the interval $[0,1]$ as follows

$$a \prec b \quad :\Leftrightarrow \quad \inf_n a^n \le \inf_n b^n; \qquad\qquad a \sim b \quad :\Leftrightarrow \quad a \prec b \prec a.$$

1.2. Lemma. (1) *The relation $\prec$ is a linear ordering with $a \prec b$ for $a \le b$. In particular, $\sim$ is an equivalence relation whose equivalence classes are subintervals of* $[0,1]$.
(2) *For any power N we have $a^N \sim a$.*
(3) *We have $ab \sim \min a, b$.*

Proof. As a consequence of axioms (T1)+(T3) the powers

$$T^{(n)}(x) := x^n \qquad (n = 1,2\ldots)$$

are increasing functions $[0,1] \to [0,1]$ with $T^{(1)} \geq T^{(2)} \geq T^{(3)} \geq \cdots$. Therefore their limit $T^{(\infty)}$ is a well-defined with

$$T^{(\infty)}(x) = \inf_n x^n \qquad (0 \leq x \leq 1).$$

By definition, we have $a \prec b$ iff $T^{(\infty)}(a) \leq T^{(\infty)}(b)$. Since the limit of increasing functions is increasing, statement (1) is immediate.

(2) We have $T^{(\infty)}(a^N) = \lim_n a^{Nn} = \lim_n a^n = T^{(\infty)}(a)$.

(3) We may assume $a \leq b$ without loss of generality. Then $a^2 \leq ab \leq 1b = b$. Since $a \sim a^2$ by (2), and since the equivalence classes of $\sim$ are intervals by (1), we conclude $a^2 \sim ab \sim a = \min\{a,b\}$.

$\square$

Henceforth we introduce the notations

$$\mathcal{I} := \big\{ I_\alpha : \ \alpha \in A \big\} := \big\{ \{x : \ x \sim a\} : \ a \in [0,1] \big\}$$

for the family of all equivalence classes of the relation $\sim$. We know already that $\mathcal{I}$ is a set of pairwise disjoint intervals forming a partition of $[0,1]$ such that $I_\prec \leq I_\beta$ (i.e. $a \prec b$ for all couples $(a,b) \in I_\alpha \times I_\beta$) whenever $a \leq b$ for some $a \in I_\alpha$ and $b \in I_\beta$. we shall say simply that the point $e \in [0,1]$ is an *idempotent* if it is idempotent with respect to the product $\overset{T}{\bullet}$, that is $e^2 = e \overset{T}{\bullet} e = T(e,e) = e$.

1.3. Corollary. (1) *If the equivalence class I_α is a left-closed interval then its initial point $e := \min I_\alpha$ is an idempotent.*

(2) *If I_α is a non-degenerate right-closed interval then its endpoint $f := \max I_\alpha$ is no idempotent, moreover $f > f^2 \geq f^3 \geq \cdots \to \inf I$.*

(3) *If I_α is a non-degenerate right-open interval then $f := \max I_\alpha$ is an idempotent.*

(4) If $I_{\alpha_1} < I_{\alpha_2} < \cdots$ is an increasing sequence in $\mathcal{I}$ then the point $g := \sup\big(\bigcup_n I_{\alpha_n}\big)$ is an idempotent.

Proof. (1) Assume $I = \{x : \ x \sim e\}$ with $e = \min I (\in I)$. Then $e = T^{(1)}(e) \geq T^{(2)}(e) = e^2$. By Lemma 1.2(3) we have $e^2 \sim e$ and hence $e^2 \in I$ with $e^2 \geq e = \min I$. However, in general $e = T^{(1)}(e) \geq T^{(2)}(e) = e^2$.

(2) Assume $I = \{x : \ x \sim e\}$ with $f = \max I (\in I)$. Given any element $x \in I$, by definition we have $x \sim e$ with $\inf_n x^n = T^{(\infty)}(x) = T^{(\infty)}(e)$. It follows

$$\inf\{x : \ x \sim f\} = \inf I = T^{(\infty)}(f).$$

Hence the case $f = f^2$ is impossible because this would imply $\inf I = T^{(\infty)}(f) = f$ contradicting the non-degeneracy of I. Thus necessarily $f = 1f > f^2 = 1f^2 \geq f^3 \geq \to T^{(\infty)}(f) = \inf I$.

(3) Assume $I = \{x : \ x \sim e\}$ with $\sup I = f \not\in I)$. By Lemma 1.2(3), the contrary $f^2 < f$ would imply the contradiction $f \sim f^2$ with $f \in I$.

(4) Assume the contrary that is let $g > g^2$. Then $f^2 < I_{\alpha_n} < f$ for some index n. However, by Lemma 1.2(1)+(2), then we would have $f^2 \sim x \sim f$ for all $x \in I_{\alpha_n}$ entailing the contradiction $I_{\alpha_n} > f \in I_{\alpha_n}$. $\square$

1.4. Lemma. *Let $P : [0,1] \to [0,1]$ be an inceasing backward projection (that is $P(y) \leq P(x) = P(P(x)) \leq x$ whenever $0 \leq y \leq x \leq 1$) onto the set Ω. Then the complement $[0,1] \setminus \Omega$ is the union of a family of pairwise disjoint left-open intervals and*

$$P(x) = \max\big(\Omega \cap [0,x]\big) \qquad (x \in [0,1]).$$

Proof. It suffices to see only that, given any point $x \in [0,1] \setminus \Omega$ with $P(x) < x$, every point y from the left-open interval $(P(x), x]$ is mapped into $P(x)$ by P. Let $P(x) < y < x$. By assumption, P is an increasing mapping with $P = P \circ P$. Hence the conclusion $P(x) = P^2(x) \leq P(y) \leq P(x)$ entailing $P(y) = P(x)$ is immediate. $\square$

1.5. Lemma. *Given a T-idempotent $e = e^2 < 1$, with its multiplication range $\Omega_e := \{ex : x \in [0,1]\}$ we have*

$$ex = \max\big(\Omega_e \cap [0,x] \qquad (0 \leq x \leq 1).$$

Also $e = \max \Omega_e$ and $[0,1] \setminus \Omega_e$ is the union of a disjoint family of left-open intervals.

Proof. According to (T1)+(T2), the mapping $P_e(x) := ex$ is an increasing backward projection of $[0,1]$ onto Ω_e. Indeed, $ey \leq ex = (ee)x = e(ex)$ whenever $0 \leq x \leq y \leq 1$. Since $\omega = P_e(\omega) \leq P_e(1) = e \in \Omega_e$, necessarily $e = \max \Omega_e$. The remaining statements are immediate from Lemma 1.4. $\square$

1.6. Proposition. *Let T be a strict T-norm. Then*
(1) *the only idempotents are* 0 *and* 1,
(2) *we have $\{1\} = \{x : x \sim 1\}$, the interval $\{x : x \sim 0\}$ is closed, and each interval $I_\alpha \in \mathcal{I}$ with $0, 1 \notin I_\alpha$ is non-degenerate, open from left and closed from right,*
(3) *there is no infinite strictly increasing sequence $I_{\alpha_1} < I_{\alpha_2} < \cdots$ in $\mathcal{I}$.*

Proof. (1) Assume $e \in (0,1)$ would be an idempotent. Then, by Lemma 1.5, we would have $ex = e$ for all $e < x \leq 1$ contradicting the strictness of T.

(2) is immediate from statement (1) and Corollary 1.3(1)+(3).

(3) is immediate from from statement (1) and Corollary 1.3(4). $\square$

Recall that a function $\varphi : [0,1]^N \to [0,1]$ is said to be *right* [*left*] semicontinuous if $\phi(x_n^{(1)}, \ldots, x_n^{(1)}) \to \phi(x^{(1)}, \ldots, x^{(1)})$ whenever $x_n^{(1)} \searrow x^{(1)}, \ldots, x_n^{(1)} \searrow x^{(1)}$ [resp. $x_n^{(1)} \nearrow x^{(1)}, \ldots, x_n^{(1)} \nearrow x^{(1)}$]. It is folklore that if ϕ is increasing then the right [left] semicontinuity of all the sections $x \mapsto \phi(a_1, \ldots, a_{k-1}, x, a_{k+1}, \ldots, a_N)$ implies the right [left] semicontinuity of ϕ.

1.7 Lemma. *If $N > 1$ and $T^{(N)}$ is right semicontinuous (in particular if T is right semicontinuous) then all the intervals $I_\alpha \in \mathcal{I}$ are closed from left.*

proof. Assume $T(N)$ to be right semicontinuous and let $x \in I \in \mathcal{I}$. Define $e := \inf I$ and consider the sequence $x^N, x^{2N}, x^{3N}, \ldots$. By definition $x_n \searrow T^{(\infty)}(x) = e$. The right semicontinuity of $T^{(N)}$ entails $x^{nN} = T^N(x) \searrow T^{(N)}(e) = e^N$. However, since $\big(x_{nN}\big)_{n=1}^\infty$ is a subsequence of $\big(x_n\big)_{n=1}^\infty$, we have $e = \lim_n x = \lim_n x^{Nn} = e^N$. Since $e \geq e^2 \geq \cdots \geq e^N$ it follows $e^2 = e$ and hence $e \in I$ by Corollary 1.4. □

1.8 Corollary. *If T is a right semicontinuous strict T-norm then $\mathcal{I} = \big\{[0,1), \{1\}\big\}$.*

Proof. Immediate from Proposition 1.6 and Lemma 1.7. □

1.9 Remark. Assuming the operation T to be *continuous*, we can conclude the following.

(1) The powers $T^{(n)}$ $(n = 1, 2, \ldots)$ are continuous increasing functions and hence their infimum $T^{(\infty)}$ is *left semicontinuous* and increasing.

(2) From (1) it readily follows that the intervals I_α are *closed from left* with idempotent initial point.

(3) It is well-known that the idempotents of a continuous T-norm form a closed subset of $[0,1]$ whose complement is the union of a countable family of pairwise disjoint open intervals. Hence one can deduce that the intervals I_α are either closed from the left and open from right or consist of a single point which is necessarily an idempotent. The points of continuity of $T^{(\infty)}$ are exactly the idempotents of a continuous T-norm.

2. The structure of a $\sim$-equivalence interval

Henceforth let $\mathbf{S} := \big([\omega, a], \cdot, \geq\ \big)$ be an ordered Abelian semigroup on the real interval $[a, \omega]$ such that

(S1) $xy_1 \leq xy_2$ whenever $y_1 \leq y_2$,

(S2) $a > a^2 > a^3 > \cdots$ and $a^n \searrow \omega$ $(n \to \infty)$.

Since, by (S2), $(\omega, a]$ is the disjoint union of the intervals $(a^{n+1}, a^n]$ $(n = 1, 2, \ldots)$, for any element $b \in (\omega, a]$ and for any index $k = 1, 2, \ldots$ we can define

$$n_k(b) := \big[n : \ a^{n+1} < b^k \leq a^n\big] \ .$$

2.1 Lemma. *Given any $b \in (\omega, a]$, the intervals $\big[n_k(b)/k, (n_k(b)+1)/k\big]$, $k = 1, 2, \ldots$ have a unique common point.*

Proof. Since the for the lengths we have $\big|\big[n_k(b)/k, (n_k(b)+1)/k\big]\big| = 1/k \to 0$ $(k \to \infty)$, at most one common point may exists. To establish its existence, according to

Helly's theorem, it suffices to see that each pair of them admits a non-empty intersection, that is

$$(2.2) \qquad n_k(b)/k \le (n_\ell(b)+1)/\ell \qquad \text{for all } k,\ell = 1,2,\ldots.$$

Consider any couple of indices $k \neq \ell$. By definition, $a^{n_k(b)+1} < b^k \le a^{n_k(b)}$ and hence, by (S1), also $a^{\ell(n_k(b)+1)} \le b^{k\ell} \le a^{\ell n_k(b)}$. Similarly $a^{k(n_\ell(b)+1)} \le b^{k\ell} \le a^{k n_\ell(b)}$. It follows $a^{k(n_\ell(b)+1)} \le b^{k\ell} \le a^{\ell n_k(b)}$ and hence by (S2) we conclude $k(n_\ell(b)+1) \ge \ell n_k(b)$ which is equivalent to (1.2). $\square$

2.3 Definition. Henceforth we write

$$L(b) := \Big[t:\ \{t\} = \bigcap_{k=1}^{\infty} \big[n_k(b)/k, (n_k(b)+1)/k\big]\Big] \qquad \text{for any } \ b \in (\omega, a].$$

Furthermore $\Lambda := L\big((\omega,a]\big)$ shall denote the rage of the function L.

2.4 Remarks. (1) $n_k(b) \in \big[\lceil kL(b)\rceil - 1, \lceil kL(b)\rceil + 1\big]$ for all $k = 1,2,\ldots$ and $b \in (\omega,a]$.

(2) If $b \in (a^{n+1}, a^n]$ then $L(b) \in [n, n+1]$. In particular $L(a^n) = n$ $(n = 1,2,\ldots)$.

(3) The mapping L is decreasing trivially, but not necessarily strictly decreasing. Example: $\mathbf{S} := \big((-\infty,1],\cdot,\ge\big)$ with $xy := \lceil x\rceil + \lceil y\rceil$ and $L(b) = \lfloor -b\rfloor$.

2.5 Lemma. *We have* $L(bc) = L(b) + L(c)$ *for all* $b,c \in (\omega,a]$.

Proof. According to Remark 2.5(1), $L(bc) = \lim_{k\to\infty} n_k(bc)/k$. By definition, $a^{n_k(b)} \ge b^k > a^{n_k(b)+1}$ and $a^{n_k(c)} \ge c^k > a^{n_k(c)+1}$. Hence $a^{n_k(b)+n_k(c)} \ge (bc)^k \ge a^{n_k(b)+n_k(c)+2}$. By the definition of the value $n_k(bc)$ and axiom (T2) it follows $n_k(b) + n_k(c) - 1 \le n_k(bc) \le n_k(b) + n_k(c) + 3$. Therefore $L(b) + L(c) = \lim_{k\to\infty} \big(n_k(b)+n_k(c)\big)/k = \lim_{k\to\infty} n_k(bc)/k = L(bc)$. $\square$

2.6 Corollary. (1) *The range* Λ *of* L *is a subsemigroup of* $\big([1,\infty),+\big)$.

(2) *In particular* Λ *is countable under the hypotheses that* L *is not strictly increasing and*
(T1*) $xy_1 < xy_2$ whenever $y_1 \le y_2$.

(3) Λ *is Lebesgue-measurable. If it has positive Lebesgue measure, for some* n *we have* $[n,\infty) \subset \Lambda$.

Proof. (1) is immediate from Lemma 2.5.

(2) The inverse images $L^{-1}\{\xi\} := \{b:\ L(b) = \xi\}$, $\xi \in \Lambda$ are pairwise disjoint intervals since the function L is decreasing. If L is not strictly increasing, some interval $L^{-1}\{\xi_0\}$ has positive length. By 1.5 we have $L^{-1}\{\xi_0 + \eta\} \supset L^{-1}\{\xi_0\} + L^{-1}\{\eta\}$ and $L^{-1}\{\xi_0+\eta\}$ is also a non-degenerate interval for any $\eta \in \Lambda$ if (T1*) holds. Since there may only be countably many pairwise disjoint non-degenerate real intervals, we conclude (2).

(3) It is well-known that the range of a decreasing real function is a Borel set (actually a sequence of points added to an interval minus a countable union of intervals). In particular

$\Lambda = \mathrm{range}(L)$ is Borel measurable. Suppose $\mathrm{mes}(\Lambda) > 0$ (mes denoting Lebesgue measure). Then almost every point of Λ is a Lebesgue point. In particular, $\mathrm{mes}\big(\Lambda \cap [\alpha,\beta] > (\beta-\alpha)/2$ for some $1 \le \alpha < \beta$. Recall that given any set Ω of real numbers with density$> 1/2$, the sum $\Omega + \Omega := \{\omega_1 + \omega_2 : \ \omega_1, \omega_2 \in \Omega\}$ contains an interval with positive length.* Hence we conclude that $\Lambda \supset \Lambda + \Lambda \supset \big(\Lambda \cap [\alpha,\beta]\big) + \big(\Lambda \cap [\alpha,\beta]\big)$ contains some interval I of length $\delta > 0$. It is immediate that $\Lambda \supset \Lambda + \cdots + \Lambda$ with $\lceil 1/\delta \rceil$ terms contains the interval $J := I + \cdots + I$ with length > 1. According to Remark 2.4(2), we have $\{1,2,\ldots\} \subset \Lambda$. It follows $\Lambda \supset \bigcup_{k=0}^{\infty} k + J \supset \big[\lceil \inf J \rceil, \infty\big)$. $\square$

2.7 Lemma. (1) *If the underlying product is left semicontinuous* [*i.e.* $x_i y \nearrow xy$ *whenever* $x_i \nearrow x$] *then its logarithm* L *is also left semicontinuous.*

(2) *If the product is right semicontinuous then* L *is right semicontinuous.*

Proof. Assume the product is left semicontinuous. It is well-known that then we have even $x_i y_i \nearrow xy$ whenever $x_i \nearrow x$ and $y_i \nearrow y$. (Indeed, given any $\varepsilon > 0$, there exists j_0 with $xy \ge xy_{j_0} \ge xy - \varepsilon/2$. Also there exists $j_1 \ge j_0$ with $xy_{j_0} \ge x_{j_1} y_{j_0} \ge xy_{j_0} - \varepsilon/2$ and hence $xy \ge x_{j_1} y_{j_0} \ge xy - \varepsilon$. Given any couple $x_i \nearrow x$ resp. $y_i \nearrow y$ of sequences, for any $i \ge j_1$ we have $xy \ge x_i y_i \ge x_{j_1} y_{j_0} \ge xy - \varepsilon$). In particular the powers $b \mapsto b^k$ $(k = 1,2,\ldots)$ are left semicontinuous. It follows that, for any fixed k, the step function $b \mapsto n_k(b)$ is left semicontinuous. Proof: Fix k arbitrarily. Since the power $b \mapsto b^k$ is increasing, the function $n_k(\cdot)$ decreases. Consider a sequence $b_i \nearrow b > \omega$. Since $\omega < \inf_i b_i \le a$, the decreasing sequence $\{n_k(b_i) : \ i = 1,2,\ldots\}$ is bounded. Since $n_k(\cdot)$ assumes integer values, there is i_0 with $n_k(b_i) = N := \lim_i n_k(b_i)$ for $i \ge i_0$. Then $a^{N+1} = a^{n_k(b_i)-1} < b_i^k \le a^{n_k(b_i)} = a^N$ for any $i \ge i_0$. It follows $a^{N+1} > b \ge a^N$ which means that $n_k(b) = N$ i.e. $n_k(b_i) \nearrow N = n_k(b)$. On the other hand the sequence $n_k(\cdot)/k$ $(k = 1,2,\ldots)$ converges uniformly to $L(\cdot)$ (actually $\sup_b |L(b) - n_k(b)/k| \le 1/k$ for all k). Hence we deduce that left semicontinuity of L, because, in general, the uniform limit of τ-continuous functions is τ-continuous for any topology τ. Thus, in particular L is left semicontinuous. The proof of (2) is analogous with the step functions $\widetilde{n}_k(b) := \big[n : \ a^n \le b^k < a^{n-1}\big]$ in place of $n_k(\cdot)$. $\square$

2.8 Lemma. *For any* $c \in (\omega, a]$, *the functions* $n_k^c(b) := \big[n : \ c^{n+1} < b^k \le c^n\big]$ *and* $L^c(b) := \lim_k n_k^c(b)/k$ *are well-defined, moreover we have* $L^c = L(c)^{-1} L$ *in terms of the logarithm function defined in* 1.3.

Proof. $\mathbf{S}^c := \big((\omega, c], \cdot, \ge\big)$ is an ordered subsemigroup of $\mathbf{S} = \big((\omega, a], \cdot, \ge\big)$. Hence we can apply the previous arguments with c in place of a to establish that all the functions n_k^c

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* Proof. We may assume $\Omega \supset [\alpha,\beta] \setminus \bigcup_{k=1}^{\infty} I_k$ where $I_1, I_2, \ldots$ are pairwise disjoint open intervals with $\sum_{k=1}^{\infty} \mathrm{mes}(I_k) = (\beta - \alpha)(1/2 - \varepsilon)$ for some $\varepsilon > 0$. The vertical resp. horizontal stripes $I_k \times [\alpha,\beta]$ and $[\alpha,\beta] \times I_k$, $k = 1,2,\ldots$ cut at most $2(1/2 - \varepsilon)\sqrt{2}(\beta - \alpha)$ length from the diagonal segments $D_\rho := \{(\omega_1,\omega_2) : \ \alpha \le \omega_1, \omega_2 \le \beta, \ \omega_1 + \omega_2 = \rho\}$ which have length $>\sqrt{2}(\beta - \alpha - \varepsilon)$ whenever $\rho \in (\alpha + \beta - \varepsilon, \alpha + \beta + \varepsilon)$. Therefore $\Omega + \Omega \supset (\alpha + \beta - \varepsilon, \alpha + \beta + \varepsilon)$.

along with L^c are well-defined and decreasing. By definition we have $c^{n_k^c(b)+1} < b^k \leq c^{n_k^c(b)}$, whence

$$(n_k^c(b)+1)L(c) = L\big(c^{n_k^c(b)+1}\big) \geq L(b^k) = kL(b) \geq L\big(c^{n_k^c(b)}\big) = n_k^c(b)L(c).$$

Since $L^c(b) = \lim_k n_k^c(b)/k$, we get $L^c(b)L(c) \geq L(b) \geq L^c(b)L(c)$. $\square$