1) $e^{it} * e^{-it} = \sin t$

2) $f_1(\mu t) * f_2(\mu t) * \cdots * f_N(\mu t) = \dfrac{1}{\mu^{N-1}} f_1 * \cdots * f_N\big(\mu t\big)$

3) $[f(t)e^{\rho t}] * e^{\rho t} = \left[\int_0^t f\right] e^{\rho t} \quad \Rightarrow \quad \big[e^{\rho t}\big]^{*N} = \dfrac{t^{N-1}}{(N-1)!} e^{\rho t}$

$$\big[\sin t\big]^{*N} = \big[e^{it} * e^{-it}\big]^{*N} = \big[e^{it}\big]^{*N} * \big[e^{-it}\big]^{*N} =$$

$$= \left[\frac{t^{N-1}}{(N-1)!} e^{it}\right] * \left[\frac{t^{N-1}}{(N-1)!} e^{-it}\right] = \int_{s=0}^{t} \frac{s^{N-1}}{(N-1)!} \frac{(t-s)^{N-1}}{(N-1)!} e^{is} e^{-i(t-s)} ds =$$

$$= e^{-it} \int_{s=0}^{t} \frac{s^{N-1}(t-s)^{N-1}}{(N-1)!^2} e^{2is} ds$$

4) $\displaystyle\int_{s=0}^{t} p(s) e^{\rho s} ds = p(s) \frac{e^{\rho} s}{\rho}\Big|_{s=0}^{t} - \int_{s=0}^{t} p'(s) \frac{e^{\rho s}}{\rho} ds = \cdots$

$$\cdots = \sum_{n=0}^{\deg(p)} (-1)^n \frac{1}{\rho^{n+1}} \left[\frac{d^n}{ds^n} p(s)\right] e^{\rho s}\Big|_{s=0}^{t}$$

$$\big[\sin t\big]^{*N} = e^{-it} \sum_{n=0}^{2N-2} e^{2is} \frac{(-1)^n}{(2i)^{n+1}} \frac{d^n}{ds^n} \left[\frac{s^{N-1}}{(N-1)!} \frac{(t-s)^{N-1}}{(N-1)!}\right] \Bigg|_{s=0}^{t}$$

$$\frac{d^n}{ds^n} \left[\frac{s^{N-1}}{(N-1)!} \frac{(t-s)^{N-1}}{(N-1)!}\right] = \sum_{k+\ell=n} \left[\frac{d^k}{ds^k} \frac{s^{N-1}}{(N-1)!}\right] \left[\frac{d^\ell}{ds^\ell} \frac{(t-s)^{N-1}}{(N-1)!}\right] \frac{(k+\ell)!}{k!\ell!} =$$

$$= \sum_{k+\ell=n} \frac{(k+\ell)!}{k!\ell!} \frac{s^{N-1-k}}{(N-1-k)!} (-1)^\ell \frac{(t-s)^{N-1-\ell}}{(N-1-\ell)!} = \begin{cases} 0 & \text{if } s=0,\ k \neq N-1 \\ 0 & \text{if } s=t,\ \ell \neq N-1 \end{cases}$$

$$\big[\sin t\big]^{*N} = e^{-it} \sum_{k,\ell=0}^{N-1} (-1)^{k+\ell+\ell} \frac{e^{2is}}{(2i)^{k+\ell+1}} \frac{(k+\ell)!}{k!\ell!} \frac{s^{N-1-k}}{(N-1-k)!} \frac{(t-s)^{N-1-\ell}}{(N-1-\ell)!}\Bigg|_{s=0}^{t} =$$

$$= e^{-it} \Bigg\{ \sum_{\substack{k=0 \\ [\ell=N-1]}}^{N-1} (-1)^k \frac{e^{2it}}{(2i)^{N+k}} \binom{N-1+k}{N-1} \frac{t^{N-1-k}}{(N-1-k)!} -$$

$$- \sum_{\substack{\ell=0 \\ [k=N-1]}}^{N-1} (-1)^{N-1} \frac{1}{(2i)^{N+\ell}} \binom{N-1+\ell}{N-1} \frac{t^{N-1-\ell}}{(N-1-\ell)!} \Bigg\}$$

$$\big[\sin t\big]^{*N} = \sum_{k=0}^{N-1} \frac{1}{(2i)^{N+k}} \binom{N-1+k}{N-1} \frac{t^{N-1-k}}{(N-1-k)!} \Big\{ (-1)^k e^{it} - (-1)^{N-1} e^{-it} \Big\} =^{d=N-1-k}$$

$$= \sum_{d=0}^{N-1} \frac{1}{(2i)^{2N-1-d}} \binom{2N-2-d}{N-1} \frac{t^d}{d!} \Big\{ (-1)^{N-1-d} e^{it} - (-1)^{N-1} e^{-it} \Big\} =$$

$$= \sum_{d=0}^{N-1} \frac{(-1)^{N-1}}{(2i)^{2N-1-d}} \binom{2N-2-d}{N-1} \frac{t^d}{d!} \Big\{ (-1)^d e^{it} - e^{-it} \Big\} =$$

$$= \sum_{d=0}^{N-1} \frac{(2i)^{d-1}}{4^{N-1}} \binom{2N-2-d}{N-1} \frac{t^d}{d!} i^d \Big\{ i^d e^{it} - (-i)^d e^{-it} \Big\} =$$

$$= \frac{1}{4^{N-1}} \sum_{d=0}^{N-1} \binom{2N-2-d}{N-1} \frac{(-2t)^d}{d!} \mathrm{Im}\Big(i^d e^{it} \Big)$$

5) $\quad \left[\frac{t^n}{n!}\right] * \left[\frac{t^m}{m!}\right] = \frac{t^{m+n+1}}{(m+n+1)!}$

$$\left[\frac{t^n}{n!}\right] * \left[\frac{t^m}{m!}\right] = \int_{s=0}^{t} \frac{s^n}{n!} \frac{(t-s)^m}{m!}\, ds =^{\text{PARTS}}=$$

$$= \frac{s^{n+1}}{(n+1)!} \frac{(t-s)^m}{m!}\Big|_{s=0}^{t} - \int_{s=0}^{t} \frac{s^{n+1}}{(n+1)!} \frac{(t-s)^{m-1}(-1)}{(m-1)!} ds =$$

$$= \int_{s=0}^{t} \frac{s^{n+1}}{(n+1)!} \frac{(t-s)^{m-1}}{(m-1)!} ds =^{\text{same argument (m−1) times}}=$$

$$= \int_{s=0}^{t} \frac{s^{n+m}}{(n+m)!} \frac{(t-s)^0}{0!} ds = \frac{t^{n+m+1}}{(n+m+1)!}$$

Corollary: $\quad \left[\frac{t^{m_1}}{m_1!}\right] * \cdots * \left[\frac{t^{m_N}}{m_N!}\right] = \frac{t^{m_1+\ldots+m_M+N-1}}{(m_1+\cdots+m_N+N-1)!}$

6) Analyticity of $\left[\frac{\sin \lambda_1 t}{\lambda_1}\right] * \cdots * \left[\frac{\sin \lambda_N t}{\lambda_N}\right]$

With absolute convergence we have

$$\left[\frac{\sin\lambda_1 t}{\lambda_1}\right] * \cdots * \left[\frac{\sin\lambda_N t}{\lambda_N}\right] =$$

$$= \left[\sum_{m_1=0}^{\infty}(-1)^{m_1}\frac{\lambda_1^{2m_1}t^{2m_1+1}}{(2m_1+1)!}\right] * \cdots * \left[\sum_{m_N=0}^{\infty}(-1)^{m_N}\frac{\lambda_1^{2m_N}t^{2m_1+1}}{(2m_1+1)!}\right] =$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty}(-1)^{m_1+\cdots+m_N}\lambda_1^{2m_1}\cdots\lambda_N^{2m_N}\left[\frac{t^{2m_1+1}}{(2m_1+1)!}\right] * \cdots * \left[\frac{t^{2m_N+1}}{(2m_N+1)!}\right] =$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty}(-1)^{m_1+\cdots+m_N}\lambda_1^{2m_1}\cdots\lambda_N^{2m_N}\frac{t^{2m_1+\ldots+2m_M+N+N-1}}{(2m_1+\cdots+2m_N+N+N-1)!} =$$

$$= \sum_{n\in\left\{2N-1,2N+1,2N+3,\ldots\right\}}(-1)^{(n-2N+1)/2}\frac{t^n}{n!}\sum_{m_1+\cdots+m_N=(n-2N+1)/2}\lambda_1^{2m_1}\cdots\lambda_N^{2m_N} =^{m=(n-2N+1)/2}=$$

$$= \sum_{m=0}^{\infty}\Big[\sum_{m_1+\ldots+m_N=m}\lambda_1^{2m_1}\cdots\lambda_N^{2m_N}\Big](-1)^m\frac{t^{2m-1+2N}}{(2m-1+2N)!}$$

Proposition. *With the restricted measures* $p_\Lambda : Z \mapsto p\big([0,\lambda)\cap Z\big)$ $(\Lambda\in\mathbf{R}_+)$, *and the corresponding moments* $M_{m,\Lambda} := \int_{\mathbf{R}_+}\lambda^m\ p_\Lambda(d\lambda)$ $(m=0,1,2,\ldots)$, *with absolute convergence we have*

$$\sum_{N=1}^{\infty}\int_{\mathbf{R}_+^N}\frac{\sin(\lambda_1 t)}{\lambda_1} * \cdots * \frac{\sin(\lambda_1 t)}{\lambda_1}\ p_\Lambda(d\lambda_1)\cdots p_\Lambda(d\lambda_N) =$$

$$= \sum_{m=0}^{\infty}\sum_{N=1}^{\infty}\Big[\sum_{m_1+\ldots+m_N=m}M_{2m_1,\Lambda}\cdots M_{2m_N,\Lambda}\Big](-1)^m\frac{t^{2m-1+2N}}{(2m-1+2N)!}.$$

Proof. Observation: since the total weight $C := p(\mathbf{R}_+) = \|y\|^2 < \infty$, we have $M_{m,\Lambda} \le C\Lambda^m$ $(m=0,2,\ldots)$. Even for complex values $t\subset\mathbf{C}$ it follows

$$\int_{\mathbf{R}_+^N}\Big|\frac{\sin(\lambda_1 t)}{\lambda_1} * \cdots * \frac{\sin(\lambda_1 t)}{\lambda_1}\Big|\ p_\Lambda(d\lambda_1)\cdots p_\lambda(d\lambda_N) =$$

$$= \int_{\lambda\in\mathbf{R}_+^N}\Big|\sum_{m=0}^{\infty}\sum_{m_1+\ldots+m_N=m}\lambda_1^{2m_1}\cdots\lambda_N^{2m_N}(-1)^m\frac{t^{2m-1+2N}}{(2m-1+2N)!}\Big|\ p_\Lambda(d\lambda_1)\cdots p_\Lambda(d\lambda_N) \le$$

$$= \sum_{m=0}^{\infty}\left|\Big[\sum_{m_1+\ldots+m_N=m}M_{2m_1,\Lambda}\cdots M_{2m_N,\Lambda}\Big](-1)^m\frac{t^{2m-1+2N}}{(2m-1+2N)!}\right| \le$$

$$\le \sum_{m=0}^{\infty}\Big[C^N\sum_{m_1+\ldots+m_N=m}\Lambda^{2m_1}\cdots\Lambda^{2m_N}\Big]\frac{|t|^{2m-1+2N}}{(2m-1+2N)!}.$$

Since $\#\Big\{(m_1,\ldots,m_N)\in \mathbf{Z}_+ :\ m_1+\cdots+m_N=m\Big\} =,$

$= \binom{m+N-1}{N-1} = \frac{(m+N-1)!}{m!(N-1)!} = \frac{(m+1)\cdots(m+N-1)}{(N-1)!},$ we have

$$\int_{\mathbf{R}_+^N} \Big|\frac{\sin(\lambda_1 t)}{\lambda_1} * \cdots * \frac{\sin(\lambda_1 t)}{\lambda_1}\Big|\, p_\Lambda(d\lambda_1)\cdots p_\lambda(d\lambda_N) \le$$

$$\le \sum_{m=0}^{\infty} \frac{(m+1)\cdots(m+N-1)}{(N-1)!} C^N \Lambda^{2m} \frac{|t|^{2m-1+2N}}{(2m-1+2N)!}.$$

On the other hand

$$\binom{m+N-1}{N-1} C^N\Lambda^{2m}\frac{|t|^{2m-1+2N}}{(2m-1+2N)!} = \frac{(m-1+N)!C^N\Lambda^{2m}|t|^{2m-1+2N}}{(2m-1+2N)!m!(N-1)!} \le$$

$$\le \frac{C^N\Lambda^{2m}|t|^{2m-1+2N}}{m!(N-1)!} \le \frac{[C|t|^2]^N\big[\Lambda^2\max\{1,|t|^2\}\big]^m}{m!(N-1)!}.$$

Hence we complete the proof with the observation

$$\sum_{N=1}^{\infty}\sum_{m=0}^{\infty}\binom{m+N-1}{N-1} C^N\Lambda^{2m}\frac{|t|^{2m-1+2N}}{(2m-1+2N)!} \le \sum_{N=1}^{\infty}\sum_{m=0}^{\infty}\frac{[C|t|^2]^N\big[\Lambda^2\max\{1,|t|^2\}\big]^m}{m!(N-1)!} \le$$

$$\le C|t|^2\exp\big(C|t|^2\big)\exp\big(\Lambda^2\max\{1,|t|^2\}\big) < \infty.$$

Corollary. $\displaystyle\sum_{N=1}^{\infty}\int_{\mathbf{R}_+^N}\frac{\sin(\lambda_1 t)}{\lambda_1} * \cdots * \frac{\sin(\lambda_1 t)}{\lambda_1}\ p_\Lambda(d\lambda_1)\cdots p_\Lambda(d\lambda_N) =$

$$= \sum_{\ell=0}^{\infty}(-1)^\ell\frac{t^{2\ell-1}}{(2\ell-1)!}\sum_{N=1}^{\ell}(-1)^N\sum_{m_1+\cdots+m_N=\ell-N} M_{2m_1,\Lambda}\cdots M_{2m_N,\Lambda}.$$

Proof. We make the grouping with respect to the powers of t by means of the index transform $\ell := m+N$ i.e. $m=\ell-N$.

Remark. In the sum $\displaystyle\sum_{m=0}^{\infty}\sum_{N=1}^{\infty}\Big[\sum_{m_1+\ldots+m_N=m} M_{2m_1,\Lambda}\cdots M_{2m_N,\Lambda}\Big](-1)^m\frac{t^{2m-1+2N}}{(2m-1+2N)!}$

the terms $M_{2m_1,\Lambda}\cdots M_{2m_N,\Lambda}$ can be written in the form

$M_{2m_1,\Lambda}\cdots M_{2m_N,\Lambda} = M_{0,\Lambda}^{\#S_0} M_{2,\Lambda}^{\#S_1}\cdots M_{2m,\Lambda}^{\#S_m}$

in terms of the sets $S_j := \{r \in \{1,\dots,N\} : m_r = j\}$. Therefore, for given m, N we have

$$\sum_{m_1+\cdots+m_N=m} M_{2m_1,\Lambda}\cdots M_{2m_N,\Lambda} = \sum_{\sum_{n=0}^{m} j\cdot k_j=m} P\big(m,N\big|k_0,\dots,k_m\big) M_{0,\Lambda}^{k_0} M_{2,\Lambda}^{k_1}\cdots M_{2m,\Lambda}^{k_m}$$

where (for $k_0+k_1+\cdots+k_m=N$)

$$P\big(m,N\big|k_0,\dots,k_m\big) :=$$

$$:= \#\Big\{\big[S_0,\dots,S_m\big] : \bigcup_j S_j = \{1,\dots,N\},\ S_i\cap S_j=\emptyset\ (i\neq j),\ \#S_j=k_j\ (j=0,\dots,m)\Big\} =$$

$$= \Big[\text{Polynomial distr. } \binom{N}{[k_j : k_j\neq 0]}\Big] = \frac{N}{\prod_{k_j\neq 0} k_j!} = \frac{N!}{k_0!k_1!\cdots k_m!} \overset{k_0=N-(k_1+\cdots+k_m)}{=\!=\!=}$$

$$= \binom{N}{k_1+\cdots k_m}\frac{(k_1+\cdots+k_m)!}{k_1!\cdots k_m!}.$$

Since $\binom{N}{k}=0$ if $k>N$ (as N being integer) and since $k_1+\cdots k_m \le \sum_{j=0}^{m} k_j = m$ we get the following.

Remark. In the subexpression $\displaystyle\sum_{m_1+\ldots+m_N=N-\ell} M_{2m_1,\Lambda}\cdots M_{2m_N,\Lambda}$ above the products $M_{2m_1,\Lambda}\cdots M_{2m_N,\Lambda}$ can be written in the form

$$M_{2m_1,\Lambda}\cdots M_{2m_N,\Lambda} = M_{0,\Lambda}^{\#S_0} M_{2,\Lambda}^{\#S_1}\cdots M_{2m,\Lambda}^{\#S_m}$$

in terms of the sets $S_j := \{r \in \{1,\dots,N\} : m_r = j\}$. Notice that the sets S_j are pairwise disjoint (some of them may be empty) and $\bigcup_{j=0}^{\ell} S_j = \{1,\dots,N\}$. Therefore, for given ℓ, N (with $\ell \ge N \ge 1$) we have

$$\sum_{m_1+\cdots+m_N=N-\ell} M_{2m_1,\Lambda}\cdots M_{2m_N,\Lambda} = \sum_{\Sigma_{j=0}^{\ell} j\cdot k_j=N-\ell} P\big(N\big|k_0,\dots,k_\ell\big) M_{0,\Lambda}^{k_0} M_{2,\Lambda}^{k_1}\cdots M_{2m,\Lambda}^{k_m}$$

where (for $k_0+k_1+\cdots+k_\ell=N$)

$$P\big(N\big|k_0,\dots,k_\ell\big):=$$

$$:=\#\Big\{\big[S_0,\dots,S_m\big]:\bigcup_j S_j=\{1,\dots,N\},\ S_i\cap S_j=\emptyset\ (i\neq j),\ \#S_j=k_j\ (j=0,\dots,m)\Big\}=$$

$$=\Big[\text{Polynomial distr. }\binom{N}{[k_j:k_j\neq 0]}\Big]=\frac{N!}{\prod_{k_j\neq 0}k_j!}=\frac{N!}{k_0!k_1!\cdots k_\ell!}\overset{k_0=N-(k_1+\cdots+k_m)}{=\!=}$$

$$=\binom{N}{k_1+\cdots k_\ell}\frac{(k_1+\cdots+k_\ell)!}{k_1!\cdots k_\ell!}.$$

Since $\binom{N}{k}=0$ if $k>N$ (as N being integer) and since $k_1+\cdots k_m\le\sum_{j=0}^m k_j=\ell-N$ we get the following.

Corollary. $\displaystyle\sum_{N=1}^\infty\int_{\mathbf{R}_+^N}\frac{\sin(\lambda_1 t)}{\lambda_1}*\cdots*\frac{\sin(\lambda_1 t)}{\lambda_1}\ p_\Lambda(d\lambda_1)\cdots p_\Lambda(d\lambda_N)=$

$$=\sum_{\ell=0}^\infty(-1)^\ell\frac{t^{2\ell-1}}{(2\ell-1)!}\ \sum_{N=1}^\ell(-1)^N\sum_{\substack{k_0,k_1,\dots,k_\ell\in\mathbf{Z}_+\\ \Sigma_j k_j=N,\ \Sigma_j jk_j=\ell-N}}\frac{N!}{k_0!\cdots k_\ell!}\ M_{0,\Lambda}^{k_0}M_{2,\Lambda}^{k_1}\cdots M_{2\ell,\Lambda}^{k_\ell}=$$

$$=\sum_{\ell=0}^\infty(-1)^\ell\frac{t^{2\ell-1}}{(2\ell-1)!}\sum_{\substack{k_0,k_1,\dots,k_\ell\in\mathbf{Z}_+\\ \Sigma_j k_j(j+1)=\ell}}(-1)^{\Sigma_j k_j}\frac{\big(\Sigma_j k_j\big)!}{k_0!\cdots k_\ell!}\ M_{0,\Lambda}^{k_0}M_{2,\Lambda}^{k_1}\cdots M_{2\ell,\Lambda}^{k_\ell}\ .$$

Remark. We developed a MAPLE5 code [`vazlat_kieg3.mws`] for enumerating the tuples $\Big\{(k_0,k_1,\dots,k_\ell)\in\mathbf{Z}_+:\sum_{j=0}^\ell k_j(j+1)=\ell\Big\}$.

Proposition. $\displaystyle\left[\frac{\sin\mu_1 t}{\mu_1}\right]^{N_1}*\cdots*\left[\frac{\sin\mu_R t}{\mu_R}\right]^{*N_R}=\sum_{r=1}^R\left[\prod_{s:s\neq r}\frac{1}{(\mu_s^2-\mu_r^2)^{N_s}}\right]\left[\frac{\sin\mu_r t}{\mu_r}\right]^{*N_r}.$

Proof. Recall that, for distinct real values $\lambda_1,\dots,\lambda_N$ we have

$$\frac{\sin\lambda_1 t}{\lambda_1}*\cdots*\frac{\sin\lambda_N t}{\lambda_N}=\sum_{k=1}^N\left[\prod_{j:j\neq k}\frac{1}{\lambda_j^2-\lambda_k^2}\right]\frac{\sin\lambda_k t}{\lambda_k}$$

Taking into account the integral form of the convolution, we see that the function

$$(t, \lambda_1, \dots, \lambda_N) \mapsto \frac{\sin \lambda_1 t}{\lambda_1} * \cdots * \frac{\sin \lambda_N t}{\lambda_N}$$

is the restriction of an entire holomorphic map $\mathbf{C}^{N+1} \to \mathbf{C}$ from $\mathbf{R}^{N+1}$.

Given any distinct values $\mu_1, \dots, \mu_R (\in \mathbf{R})$, and taking the index groups $I_r := \{k : \sum_{s<r} N_s < k \le \sum_{s \le r} N_s\}$ we apply the lemma below with

$$\varphi_k : z \mapsto \frac{\sin(\mu_r + kz)}{\mu_r + kz} \prod_{j:k \neq j \in I_r} \frac{1}{(\mu_r + jz)^2 - (\mu_r + kz)^2} = *_{j \in I_r} \frac{\sin(\mu_r + jz)}{\mu_r + jz}$$

and $\qquad \psi_k : z \mapsto \prod_{s \neq r} \prod_{j \in I_s} \dfrac{1}{(\mu_s + jz)^2 - (\mu_r + kz)^2}$.

Lemma. *Let* $N = N_1 + \cdots + N_R$ *and* $I_1, \dots, I_R$ *denote the index intervals* $I_r := \sum_{j<r} N_j + \{1, \dots, N_r\}$. *Assume* $\varphi_1, \dots, \varphi_N$ *are meromorphic functions respectively* $\psi_1, \dots, \psi_N$ *are holomorphic functions* (*of one complex variable*) *such that*

1) $\sum_{k=1}^N \varphi_k \psi_k$ is holomorphic, 2) $\Phi_r := \sum_{k \in I_r} \varphi_k$ $(r = 1, \dots, R)$ are holomorphic,

3) *the functions* ψ_k *with* $k \in I_r$ *admit a common value* $\beta_r := \psi_k(0)$ $(k \in I_r)$ *at the origin.*

Then we have

$$\sum_{k=1}^N \varphi_k(0) \psi_k(0) = \sum_{r=1}^R \phi_r(0) \beta_r.$$

Case of smooth distribution

Lemma. *If* $p \in \mathcal{C}^2(\mathbf{R}_+)$ *with* $p' \ge 0 = p(0)$, $M := \sup p < \infty$ *and* $\mu \in \mathbf{R}_{++}$ (*i.e.* $\mu > 0$)

then there is a bounded Borelian function $\theta : (0,\mu) \to (0,\mu)$ such that

$$\int_{\lambda>0:|\lambda-\mu|>\delta} \frac{dp(\lambda)}{\lambda^2-\mu^2} \longrightarrow \frac{1}{\mu}\int_{\xi=0}^{\mu} p''\big(\theta(\xi)\big)\,d\xi - \frac{1}{2\mu}\int_{\xi=0}^{\mu}\left[\frac{p'(\mu-\xi)}{2\mu-\xi}+\frac{p'(\mu+\xi)}{2\mu+\xi}\right]d\xi + \\ + \int_{\eta=\mu}^{\infty}\frac{p'(\mu+\eta)d\eta}{\eta(2\mu+\eta)} \qquad (\delta\searrow 0)$$

with finite integrals in each term.

Proof. For $\mu > \delta > 0$ we have

$$\int_{\lambda\in\mathbf{R}_{++}:|\lambda-\mu|>\delta} \frac{dp(\lambda)}{\lambda^2-\mu^2} = \int_{\lambda=0}^{\mu-\delta}\frac{p'(\lambda)d\lambda}{\lambda^2-\mu^2} + \int_{\lambda=\mu+\delta}^{\infty}\frac{p'(\lambda)d\lambda}{\lambda^2-\mu^2} \underset{\eta=\lambda-\mu}{\overset{\xi=\mu-\lambda}{=}}$$

$$= \int_{\xi=\mu}^{\delta}\frac{p'(\mu-\xi)(-1)d\xi}{(-\xi)(2\mu-\xi)} + \int_{\eta=\delta}^{\infty}\frac{p'(\mu+\eta)d\eta}{\eta(2\mu+\eta)} = \otimes\otimes$$

$$= -\frac{1}{2\mu}\int_{\xi=\delta}^{\mu}\Big(\frac{1}{\xi}+\frac{1}{2\mu-\xi}\Big)p'(\mu-\xi)\,d\xi + \frac{1}{2\mu}\int_{\eta=\delta}^{\mu}\Big(\frac{1}{\eta}-\frac{1}{2\mu+\eta}\Big)p'(\mu+\eta)\,d\xi + \int_{\eta=\mu}^{\infty}\frac{p'(\mu+\eta)d\eta}{\eta(2\mu+\eta)}.$$

With partial integration we get $\int_\mu^\infty \eta^{-1}p'(\mu+\eta)d\eta = -\mu^{-1}p(2\mu) + \int_\mu^\infty \eta^{-2}p(\mu+\eta)d\eta$ where $\int_\mu^\infty \eta^{-2}p(\mu+\eta)d\eta \in \int_\mu^\infty \eta^{-2}d\eta\cdot\big[p(2\mu),M\big] = \big[\mu^{-1}p(2\mu),\mu^{-1}M\big]$.

On the other hand,

$$-\frac{1}{2\mu}\int_{\xi=\delta}^{\mu}\Big(\frac{1}{\xi}+\frac{1}{2\mu-\xi}\Big)p'(\mu-\xi)\,d\xi + \frac{1}{2\mu}\int_{\eta=\delta}^{\mu}\Big(\frac{1}{\eta}-\frac{1}{2\mu+\eta}\Big)p'(\mu+\eta)\,d\xi =$$

$$= \frac{1}{\mu}\int_{\xi=\delta}^{\mu}\frac{p'(\mu+\xi)-p'(\mu-\xi)}{2\xi}\,d\xi - \frac{1}{2\mu}\int_{\xi=\delta}^{\mu}\left[\frac{p'(\mu-\xi)}{2\mu-\xi}+\frac{p'(\mu+\xi)}{2\mu+\xi}\right]d\xi \longrightarrow$$

$$\longrightarrow \frac{1}{\mu}\int_{\xi=0}^{\mu}\frac{p'(\mu+\xi)-p'(\mu-\xi)}{2\xi}\,d\xi - \frac{1}{2\mu}\int_{\xi=0}^{\mu}\left[\frac{p'(\mu-\xi)}{2\mu-\xi}+\frac{p'(\mu+\xi)}{2\mu+\xi}\right]d\xi \quad (\delta\searrow 0)$$

as a consequnce of Lebesgue bounded convergence theorem, since the Newton difference $\big[p'(\mu+\xi)-p'(\mu-\xi)\big]/(2\xi)$ is bounded on bounded subintervals of $\mathbf{R}_+$ due to the continuity of p'' (even at 0 from the right).

Remark. We may even only assume p' to be *locally Lipschitzian* and the arguments work.