C_0-SEMIGROUPS OF HOLOMORPHIC ENDOMORPHISMS

$\mathbf{E}$ Banach space, $\mathbf{D}$ bounded domain in $\mathbf{E}$

$d_{\mathbf{D}} := \big[\text{Carathéodory distance on } \mathbf{D}\big]$, $\mathrm{Hol}(\mathbf{D}) := \big\{\text{holomorphic maps } \mathbf{D} \to \mathbf{D}\big\}$

Remark. $f \in \mathrm{Hol}(\mathbf{D})$ is a $d_{\mathbf{D}}$-contraction. Taylor series: $f(a+v) = \sum_{n=0}^{\infty} \big[D^n_{z=a} f(z)\big] v^n$.

Cauchy estimates: $\big\| \big[D^n_{z=a} f(z)\big] v^n \big\| \leq \mathrm{diam}(\mathbf{D}) \mathrm{dist}(a, \partial\mathbf{D})^{-(n+1)} \|v\|^n$.

f locally Lipschitzian, $K \subset\subset D$ convex $\Rightarrow \mathrm{Lip}\big(f|K\big) \leq \mathrm{diam}(\mathbf{D}) \mathrm{dist}(K, \partial\mathbf{D})^{-1}$;

$f_j \to f$ pointwise $\Longrightarrow [D^n f_j] v^n\big|_K \rightrightarrows [D^n f] v^n\big|_K$ on compact $K \subset \mathbf{D}$, $\forall\, n\ \forall\, v$.

Definition. $[\Phi^t : t \in \mathbf{R}_+]$ *str.cont.1-prsg* (C_0-semigroup) in $\mathrm{Hol}(\mathbf{D})$ if

$\Phi^0 = \mathrm{Id}$, $\quad \Phi^{t+h} = \Phi^t \circ \Phi^h$ $(t, h \in \mathbf{R}_+)$, $\quad t \mapsto \Phi^t(x)$ continuous $\forall\, x \in \mathbf{D}$.

The *infinitesimal generator* of $[\Phi^t : t \in \mathbf{R}_+]$ is

$\Phi' := \frac{d}{dt}\big|_{t=0+} \Phi^t$, $\quad \mathrm{dom}(\Phi') = \{x : \exists\, v \ \ \Phi^h(x) = x + hv + o(h)\}$

Proposition. $x \in \mathrm{dom}(\Phi') \Longrightarrow\ t \mapsto \Phi^t(x)$ differentiable.

Proof. $\Phi^h(x) = x + hv + o(h) \ \Longrightarrow\ \ \Phi^{t+h}(x) - \Phi^t(x) = \Phi^t\big(x + hv + o(h)\big) - \Phi^t(x) =$

$= h[D_{z=x} \Phi^t(z)] v + o(h)$ In particular $x \in \mathrm{dom}(\Phi') \Rightarrow x \in \mathrm{dom}\big(\frac{d}{ds}\big|_{s=t+0} \Phi^s\big)$ for $h \searrow 0$.

For the left-derivatives:

given $t > 0$ and $x \in \mathrm{dom}(\Phi')$ with $\phi^h(x) = x + hv + w_h$, $w_h = o(h)$ $(h \searrow 0)$ we have

$\big[\Phi^{t-h}(x) - \Phi^t(x)\big]/(-h) = \big[\Phi^{t-h}(x) - \Phi^{t-h}(x + hv + w_h)\big]/(-h) =$

$= \big[D_x \Phi^{t-h}\big] v + \big[D_x \Phi^{t-h}\big](w_h/h) + \sum_{n>1} h^{n-1} \big[D^n_x \Phi^{t-h}\big]\big(v + w_h/h\big)^n$.

Since $\{x\}$ is compact, $\big[D_x \Phi^{t-h}\big] v \to \big[D_x \Phi^t\big] v$ as $h \searrow 0$.

By Cauchy estimates, with $\delta := \mathrm{dist}\big(\{\Phi^s(x) : 0 \le s \le t\}, \partial D\big) > 0$, we have

$\big\| \big[D_x \Phi^{t-h}\big](w_h/h)\big\| \le \mathrm{diam}(D)\delta^{-1}\|w_h/h\| \to 0 \quad (h \searrow 0)$ and

$\big\| \big[D_x^n \Phi^{t-h}\big](v + w_h/h)\big\| \le \mathrm{diam}(D)\delta^{n-1}\|v + w_h/h\|^n$

implying $\left\| \sum_{n>1} h^{n-1}\big[D_x^n \Phi^{t-h}\big](v + w_h/h)\right\| \to 0 \quad (h \searrow 0).$ Q. e. d.

Remark. In course of the proof we have seen

$$\frac{d}{dt}\Phi^t(x) = \Phi'\big(\Phi^t(x)\big) = \big[D_x\Phi^t\big]\Phi'(x) \qquad \big(x \in \mathrm{dom}(\Phi')\big).$$

Corollary. Given $x \in \mathrm{dom}(\Phi')$, the orbit $t \mapsto \Phi^t(x)$ is continuously differentiable. Thus

$$\mathrm{dom}(\Phi') = \big\{x \in \mathbf{D} : t \mapsto \Phi^t(x) \text{ is continuously diff.}\big\}.$$

Proof. Since $\{x\}$ is compact, the function $t \mapsto \big[D_x\Phi^t\big]v$ is continuous for any $v \in \mathbf{E}$.

Proposition. The graph of Φ' is closed.

Let $x_n \in \mathrm{dom}(\Phi')$, $v_n := \Phi'(x_n)$ $(n = 1, 2, \dots)$ and assume $x_n \to x \in \mathbf{D}$, $v_n \to v \in \mathbf{E}$.

$$\frac{\Phi^h(x_n) - x_n}{h} = \int_{s=0}^{h} \Big[\frac{d}{ds}\Phi^s(x_n)\Big]\, ds = \int_{s=0}^{h} \big[D_{x_n}\Phi^s\big]v_n \, ds = \int_{s=0}^{1} \big[D_{x_n}\Phi^{sh}\big]v_n \, ds,$$

$\big[D_{x_n}\Phi^s\big]v_n - v = \big[D_{x_n}\Phi^{sh}\big]v_n - \big[D_{x_n}\Phi^0\big]v = \big[D_{x_n}\Phi^{sh}\big](v_n - v) + \big(\big[D_{x_n}\Phi^{sh}\big] - \big[D_{x_n}\Phi^0\big]\big)v.$

Since $K := \{x\} \cup \{x_n\}_{n=1}^\infty \subset \mathbf{D}$ is compact, $[D\Phi^{sh}]v\big|K \rightrightarrows v = [D\Phi^0]v\big|K$ for $t \searrow 0$. Also $\big\|\big[D_{x_n}\Phi^t\big](v_n - v)\big\| \le M\|v_n - v\|$ with $M := \mathrm{diam}(\mathbf{D})\mathrm{dist}(K, \partial\mathbf{D})^{-1}$. Thus the functions $f_n(t) := \big[D_{x_n}\Phi^t\big]v_n$ satisfy $\|f_n(t) - v\| \le \max_{z\in K}\|v - D_z\Phi^t]v\| + M\|v_{-}v\|$. Hence $h^{-1}\big(\Phi^h(x) - x\big) = \lim_n h^{-1}\big(\Phi^h(x_n) - x_n\big) = \int_{s=0}^1 f_n(sh)\, ds \to v$ as $h \searrow 0$. Q. e. d.

Proposition. Let $[\Phi^t : t \in \mathbf{R}_+], [\Psi^t : t \in \mathbf{R}_+]$ be c_0-semigroups of holomorphic $\mathbf{D} \to \mathbf{D}$ maps with the same generator. Then they coincide on $\mathrm{dom}(\Phi')\big(=\mathrm{dom}(\Phi')\big)$.

Proof. For $t, s, h \geq 0$ with $t \geq s+h$ we have

$$\frac{1}{h}\Big[\Phi^{t-(s+h)}\Big(\Psi^{s+h}(x)\Big) - \Phi^{t-s}\Big(\Psi^s(x)\Big)\Big] =$$

$$= \frac{1}{h}\Big[\Phi^{t-(s+h)}\Big(\Psi^{s+h}(x)\Big) - \Phi^{t-(s+h)}\Big(\Psi^s(x)\Big)\Big] - \frac{1}{h}\Big[\Phi^{t-(s+h)}\Big(\Psi^s(x)\Big) - \Phi^{t-s}\Big(\Psi^s(x)\Big)\Big];$$

$$\frac{1}{h}\Big[\Phi^{t-(s+h)}\Big(\Psi^{s+h}(x)\Big) - \Phi^{t-(s+h)}\Big(\Psi^s(x)\Big)\Big] = \frac{1}{h}\int_{u=0}^1 \Big[\frac{d}{du}\Phi^{t-(s+h)}\Big(\Psi^{s+uh}(x)\Big)\Big] =$$

$$= \int_{u=0}^1 \Big[\mathrm{D}_{\Psi^{s+uh}(x)}\Phi^{t-(s+h)}\Big]\Big[\frac{1}{h}\frac{d}{du}\Psi^{s+uh}(x)\Big]\,du =$$

$$= \int_{u=0}^1 \Big[\mathrm{D}_{\Psi^{s+uh}(x)}\Phi^{t-(s+h)}\Big]\Psi'\big(\Psi^{s+uh}(x)\big)\,du \xrightarrow{h\to 0}$$

$$\xrightarrow{h\to 0} \Big[\mathrm{D}_{\Psi^{s+uh}(x)}\Phi^{t-(s+h)}\Big]\Psi'\big(\Psi^s(x)\big);$$

$$\frac{1}{h}\Big[\Phi^{t-(s+h)}\Big(\Psi^s(x)\Big) - \Phi^{t-(s+h)}\Big(\Psi^s(x)\Big)\Big] = -\frac{1}{h}\int_{u=0}^1 \Big[\frac{d}{du}\Phi^{t-(s+h)}\Big(\Phi^h\big(\Psi^s(x)\big)\Big)\Big] =$$

$$= -\int_{u=0}^1 \Big[\mathrm{D}_{\Psi^s(x)}\Phi^{t-(s+h)}\Big]\Big[\frac{1}{h}\frac{d}{du}\Phi^{uh}\big(\Psi^s(x)\big)\Big]\,du \xrightarrow{h\to 0}$$

$$\xrightarrow{h\to 0} -\Big[\mathrm{D}_{\Psi^s(x)}\Phi^{t-(s+h)}\Big]\Phi'\big(\Psi^s(x)\big)$$

because $(y,\tau,w) \mapsto \Big[\mathrm{D}_y\Phi^\tau\Big]w$ resp. $(y,\tau,w) \mapsto \Big[\mathrm{D}_y\Psi^\tau\Big]w$ are continuous on domains $\mathbf{K} \times [0,t] \times \mathbf{W}$ with compact $\mathbf{K} \subset \mathbf{D}$ (actually $\mathbf{K} := \{\Psi^s(x) : s \in [0,t]\}$) and compact balanced $\mathbf{W} \subset \mathbf{E}$ with $\mathbf{K}+\mathbf{W} \subset \mathbf{D}$. It follows $\frac{d}{ds}\Phi^{t-s}\Big(\Psi^s(x)\Big) = \Psi'\big(\Psi^s(x)\big) - \Phi'\big(\Psi^s(x)\big) = 0$ implying that $[0,t] \ni s \mapsto \Phi^{t-s}\Big(\Psi^s(x)\Big)$ is constant. In particular, by considering $s=0$ resp. $s=t$ we get $\Phi^t(x) = \Psi^t(x)$. Qu. e. d.

Open problem. $\exists?$ $[\Phi^t : t \in \mathbf{R}_+]$ nowhere diff. in t?

HOLOMORPHIC CARATHÉODORY ISOMETRIES OF THE UNIT BALL

Definition. $\mathrm{Iso}_h(\mathbf{D}) := \{\text{holomorphic } d_{\mathbf{D}}\text{-isometries}\}$.

We write $\mathbf{B} := \{x \in \mathbf{E} : \|x\| < 1\}$ and $\partial\mathbf{B} := \{x \in \mathbf{E} : \|x\| = 1\}$ in the sequel.

The *infinitesimal Carathéodory metric* of $\mathbf{D}$ at a point $a \in \mathbf{D}$ is

$\delta_{\mathbf{D}}(a, v) := \frac{d}{dt}\big|_{t=0+} d_{\mathbf{D}}(a + tv, a)$.

Remark. In the case of the unit ball ($\mathbf{D} = \mathbf{B}$) we have

$d_{\mathbf{B}}(0, x) = \operatorname{arth} \|x\|$ $(x \in \mathbf{B})$ and $\delta_{\mathbf{B}}(v) = \|v\|$ $(v \in \mathbf{E})$.

Notation. Throughout this section we consider a holomorphic endomorphism $\Phi \in \mathrm{Iso}(d_{\mathbf{B}})$ leaving the origin fixed: $0 = \Phi(0)$. We write its Taylor series in the form

$\Phi = Ux + \Omega(x) = Ux + \sum_{n=2}^{\infty} \Omega_n(x) \qquad (x \in \mathbf{B})$.

It is well-known [Vesentini-Franzoni] that the Fréchet derivatives $D_a\Psi = D_{z=a}\Psi(z) : v \mapsto \frac{d}{d\zeta}\big|_{\zeta=0} \Psi(a+\zeta v)$ of a holomorphic $d_{\mathbf{D}_1} \to d_{\mathbf{D}_2}$ isometry $\Psi : \mathbf{D}_1 \to \mathbf{D}_2$ between two bounded domains are (linear) $\delta_{\mathbf{D}_1}(a, \cdot) \to \delta_{\mathbf{D}_2}(\Psi(a), \cdot)$ isometries.

In particular U is necessarily an $\mathbf{E}$-isometry: $\|Ux\| = \|x\|$ $(x \in \mathbf{E}$.

Furthermore, since $\Phi \in \mathrm{Iso}_{\mathbf{B}}$, fo any $x \in \mathbf{B}$ we have

$\operatorname{arth} \|x\| = d_{\mathbf{B}}(0, x) = d_{\mathbf{B}}\big(\Phi(0), \Phi(x)\big) = d_{\mathbf{B}}\big(0, \Phi(x)\big) = \operatorname{arth} \|\Phi(x)\|$.

Thus Φ maps the spheres $\rho\partial\mathbf{B} = \{x : \|x\| = \rho\}$ resp. the balls $\rho\mathbf{B} = \{x : \|x\| < \rho\}$ $(0 \le \rho < 1)$ into themselves.

Question. Under which hypothesis is Φ linear (i.e. $\Phi = U$)?

Lemma. If range$(\Phi) \subset$ range(U) then $\Phi = U$.

Proof. By assumption, the map $\widetilde{\Phi} := U^{-1} \circ \Phi$ is a well-defined $\mathbf{B} \to \mathbf{B}$ holomophy with $\widetilde{\Phi}(0) = 0$ and $D_0\widetilde{\Phi} = U^{-1}D_0\Phi = U^{-1}U = \mathrm{id}_{\mathbf{E}}$. From the classical Cartan's Uniqueness Theorem it follows $\widetilde{\Phi} = \mathrm{id}_{\mathbf{B}}$ whence the statement is immediate.

Notation. Given a unit vector $y \in \partial\mathbf{B}$, we write $\mathcal{S}(y) := \big\{L \in \mathcal{L}(\mathbf{E}, \mathbf{C}) : 1 = \langle L, y\rangle = \|L\|\big\}$ for the family of all *supporting* $\mathbf{C}$*-linear functionals* of $\mathbf{B}$ at its boundary point y.

Lemma. Given $x \in \partial\mathbf{B}$ along with a vector $v \in \mathbf{E}$ such that $x + \Delta v \subset \partial\mathbf{B}$, we have* $\big\langle L, \Phi\big(\zeta(x + \eta v)\big)\big\rangle = 1 \ \ (\zeta, \eta \in \Delta)$ for all $L \in \mathcal{S}(Ux)$.

Proof. Let $L \in \mathcal{S}(Ux)$ and consider the holomorphic map $\Phi_{x,v} : \Delta^2 \to \mathbf{C}$ defined as

$\Phi_{x,v}(\zeta, \eta) := U(x + \eta v) + \sum_{n=2}^{\infty} \zeta^{n-1}\eta^n\Omega_n\big(\zeta(x + \eta v)\big) \quad (\zeta, \eta \in \Delta = \{\xi \in \mathbf{C} : |\xi| < 1\}).$

Observe that, for any $0 \neq \zeta, \eta \in \Delta$, we have $\Phi_{x,v}(\zeta, \eta) = \zeta^{-1}\Phi\big(\zeta(x + \eta v)\big)$ implying

$\|\Phi_{x,v}(\zeta, \eta)\| = |\zeta|^{-1}\|\Phi\big(\zeta(x + \eta v)\big)\| = |\zeta|^{-1}\|\zeta(x + \eta v)\| = \|\zeta(x + \eta v)\| = 1.$

Thus $\Phi_{x,v.L} : (\zeta, \eta) \mapsto \langle L, \Phi_{x,v}(\zeta, \eta)\rangle$ is a holomorphic function on Δ^2 with

$|\Phi_{x,v,L}(\zeta, \eta)| \le \|L\| = 1$ and $\Phi_{x,v,L}(0, 0) = \lim_{0 \neq \zeta, \eta \to 0} \Phi_{x,v,L}(\zeta, \eta) = \langle L, \Phi_{x,v}(0, 0)\rangle = \langle L, Ux\rangle = 1.$

By the Maximum Principle, $\Phi_{x,v,L} \equiv 1$ which completes the proof.

Corollary. $\langle L, \Omega_n(Uy)\rangle = 0$ for all $y \in \partial\mathbf{B}$ and $L \in \mathcal{S}(Uy)$.

Proof. Given $L \in \mathcal{S}(Uy)$ where $y \in \partial\mathbf{B}$, for all $\zeta \in \Delta$ (even with $\zeta = 0$) we have

$$1 \equiv \big\langle L, \zeta^{-1}\Phi(\zeta y)\big\rangle = \Phi_{\zeta,0} = \Big\langle L, Uy + \sum_{n=2}^{\infty} \zeta^{n-1}\Omega_n(Uy)\Big\rangle. \quad \text{Qu.e.d.}$$

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* $\Delta := \{\zeta \in \mathbf{C} : |\zeta| < 1\}$ is the unit disc, $\mathbf{T} := \{\zeta \in \mathbf{C} : |\zeta| = 1\} = \partial\Delta$ is the unit circle.

Notation. In terms of the Taylor expansion $\Phi(x) = Ux + \sum_{n=2}^{\infty} \Omega_n(x)$, let

$F(\zeta, x) := \zeta^{-1}\Phi(\zeta x), \;\; F(0,x) := Ux \quad (0 \neq \zeta \in \Delta, \; x \in \mathbf{B})$.

Remark. F is holomorphic around the origin: $F(\zeta,x) = Ux + \sum_{n=1}^{\infty} \zeta^n \Omega_{n+1}(x)$; $\mathrm{ran}(F) \subset \partial\mathbf{B}$.

Lemma. Let $\mathbf{K} \subset \partial\mathbf{B}$ be a convex subset of the unit sphere. Then the convex hull $\mathrm{Conv}\big(F(\Delta, \mathbf{K})\big) \subset \partial\mathbf{B}$.

Proof. Assume $x_1, \ldots, x_k \in \mathbf{K}$, $\zeta_1, \ldots, \zeta_k \in \Delta$ and consider a convex combination $y := \sum_{j=1}^{k} \lambda_j F(\zeta_j, x_j)$ where $\sum_{j=1}^{k} \lambda_j = 1$, $\lambda_1, \ldots, \lambda_k > 0$. We have to see that $y \in \partial\mathbf{B}$.
Consider the points $\quad y_t := \sum_{j=1}^{k} \lambda_j F(e^{2\pi i t}\zeta_j, x_j) \quad (t \in \mathbf{R})$.

We have $\|y_t\| \leq 1$ $(t \in \mathbf{R})$ since F ranges in the unit sphere. On the other hand

$\int_0^1 y_t \, dt = \sum_{j=1}^{k} \lambda_j \int_0^1 \big[Ux_j + \sum_{n=1}^{\infty} e^{2n\pi i t}\Omega_{n+1}(x_j)\big] dt = \sum_{j=1}^{k} \lambda_j U x_j = U \sum_{j=1}^{k} \lambda_j x_j$.
By assumption $x := \sum_{j=1}^{k} \lambda_j x_j \in \mathbf{K}$ implying that $\|Ux\| = 1$ and necessarily $\|y_t\| \equiv 1$.

In particular $y = y_0 \in \partial\mathbf{B}$.

Remark. The map Φ extends holomorphically to some spherical neighborhood of $\overline{\mathbf{B}}$ by a result of Kaup. We denote the extension also by Φ without danger of confusion.

Corollary. If $\mathbf{F}$ is a face of $\mathbf{B}$ then $\Phi(\mathbf{F})$ is contained in some face of $\mathbf{B}$ again.

Proof. We can apply the arguments of the lemma with $\zeta_j = 1$ and the extended Φ.

EXAMPLE OF A NON-LINEAR C0-SEMIGROUP OF $d_{\mathbf{B}}$-ISOMETRIES

$\mathbf{E}$ complex Banach space

$\mathbf{X} := C_0(\mathbf{R}_+, \mathbf{E}) = \Big\{ x : \mathbf{R}_+ \to \mathbf{E} \Big| t \mapsto x(t) \text{ continuous}, \lim_{t\to\infty} x(t) = 0 \Big\}, \quad \big\|x\big\| = \max_{t\geq 0} \|x(t)\|$

Lemma. Let $\big[\varphi^t : t \in \mathbf{R}_+\big]$ be a C0-semigroup of $B(\mathbf{E})$-contractions. Then the maps $\Phi^t : B(\mathbf{X}) \to \mathbf{X}$ $(t \in \mathbf{R}_+)$ defined by

$$\Phi^t(x) : \mathbf{R}_+ \ni \tau \mapsto \Big[\varphi^{t-\tau}\big(x(0)\big) \text{ if } 0 \leq \tau \leq t, \quad x(\tau - t) \text{ if } \tau \geq t \Big]$$

form a C0-semigroup of $B(\mathbf{X})$-isometries.

Proof. Consider any function $x \in B(\mathbf{X})$ and any parameter $t \in \mathbf{R}_+$. The function $\Phi^t(x)$ ranges in $B(\mathbf{X})$ with $\lim_{\tau\to\infty} \Phi^t(x)(\tau) = \lim_{\tau\to\infty} x(\tau - t) = 0$. The continuity of $\Phi^t(x)$ on the intervals $[0, t]$ resp. $[t, \infty]$ is immediate by its definition. Hence $\Phi^t(x) \in \mathbf{X}$ with well-defined $\max_{\tau\geq 0} \|x(\tau)\| < 1$. Given another function $y \in B(\mathbf{X})$, we have

$$\big\|\Phi^t(x) - \Phi^t(y)\big\| = \max \Big\{ \max_{0\leq\tau\leq t} \big\|\varphi^{t-\tau}\big(x(\tau)\big) - \varphi^{t-\tau}\big(y(\tau)\big)\big\|, \max_{\sigma\geq t} \big\|x(\sigma - t) - y(\sigma - t)\big\| \Big\} \leq$$

$$\leq \max \Big\{ \max_{0\leq\tau\leq t} \big\|x(\tau) - y(\tau)\big)\big\|, \max_{\sigma\geq t} \big\|x(\sigma - t) - y(\sigma - t)\big\| \Big\} \leq$$

$$= \max_{\tau\geq 0} \big\|x(\tau) - y(\tau)\big)\big\| = \|x - y\|.$$

Since trivially

$$\big\|\Phi^t(x) - \Phi^t(y)\big\| \geq \max_{\sigma\geq t} \big\|x(\sigma - t) - y(\sigma - t)\big\| \Big\} = \max_{\tau\geq 0} \big\|x(\tau) - y(\tau)\big\| \Big\} = \|x - y\|,$$

we conclude that each map Φ^t is a $B(\mathbf{X})$-isometry.

Next we check the semigroup property of $[\Phi^t : t \in \mathbf{R}_+]$. Let $s, t \geq$. Then we have

$$\Phi^s \circ \Phi^t(x) : \tau \mapsto \Big[\varphi^{s-\tau}\big(\Phi^t(x)(0)\big) \text{ if } \tau \leq s, \quad \varphi^t(x)(\tau - s) \text{ if } \tau \geq s\Big],$$

$$\Phi^{s+t}(x) : \tau \mapsto \Big[\varphi^{(s+t)-\tau}\big(x(0)\big) \text{ if } \tau \leq s+t, \quad x\big(\tau - (s+t)\big) \text{ if } \tau \geq s+t\Big].$$

Thus if $0 \leq \tau \leq s$ then

$$\Phi^s \circ \Phi^t(x)(\tau) = \varphi^{s-\tau}\Big(\Phi^t\big(x(0)\big)\Big) = \varphi^{s-\tau}\Big(\varphi^t\big(x(0)\big)\Big) =$$

$$= \varphi^{s-\tau} \circ \varphi^t\big(x(0)\big) = \varphi^{(s+t)-\tau}\big(x(0)\big) = \Phi^{s+t}(x)(\tau).$$

If $s \leq \tau \leq s+t$ then

$$\Phi^s \circ \Phi^t(x)(\tau) = \Phi^t(x)(\tau - s) =^{\tau-s\leq t}= \varphi^{t-(\tau-s)}\big(x(0)\big) =$$

$$= \varphi^{(s+t)-\tau}\big(x(0)\big) = \Phi^{s+t}(x)(\tau).$$

If $s + t \leq \tau$ then

$$\Phi^s \circ \Phi^t(x)(\tau) = \Phi^t(x)(\tau - s) =^{\tau-s\geq t}= x\big((\tau - s) - t\big) = \Phi^{s+t}(x)(\tau).$$

We complete the proof by checking strong continuity, that is that $\|\Phi^t(x) - \Phi^s(x)\| \to 0$ whenever $s \to t$ in $\mathbf{R}_+$. Recall that the moduli of continuty

$$\Omega(z,\delta) := \max_{|t_1-t_2|\leq\delta} \|z(t_1) - z(t_2)\|, \qquad \omega(e,\delta) := \max_{|t_1-t_2|\leq\delta} \|\varphi^{t_1}(e) - \varphi^{t_2}(e)\|$$

of any function $z \in \mathbf{X}$ resp. any vector $e \in \mathbf{E}$ are well-defined and converge to 0 as $\delta \searrow 0$. Let $0 \leq t_1 \leq t_2$. Then we have

$$\Phi^{t_1}(x) - \Phi^{t_2}(x) = \begin{cases} \varphi^{t_2-\tau}(x(0)) - \varphi^{t_1-\tau}(x(0)) & \text{if } \tau \leq t_1, \\ \varphi^{t_2-\tau}(x(0)) - x(\tau - t_1) & \text{if } t_1 \leq \tau \leq t_2, \\ x(\tau - t_2) - x(\tau - t_1) & \text{if } t_2 \leq \tau. \end{cases}$$

Therefore

$$\|\Phi^{t_1}(x) - \Phi^{t_2}(x)\| \le \begin{cases} \omega\big(x(0), t_2 - t_1\big) & \text{if } \tau \le t_1, \\ \big\|\varphi^{t_2-\tau}\big(x(0)\big) - x(0)\big\| + \big\|x(\tau - t_1) - x(0)\big\| \le & \\ \qquad \le \omega\big(x(0), t_2 - t_1\big) + \Omega(x, t_2 - t_1) & \text{if } t_1 \le \tau \le t_2, \\ \Omega(x, t_2 - t_1) & \text{if } t_2 \le \tau. \end{cases}$$

Hence we see the uniform continuity of the function $t \mapsto \Phi^t(x)$ with modulus of continuity $\delta \mapsto \omega\big(x(0), \delta\big) + \Omega(x, \delta)$.

Remark. The conclusion of the above Lemma holds even if $\mathbf{E}$ is assumed to be a *normed* space and not necessarily a Banach space.

Corollary. If the maps φ^t are holomorphic then each Φ^t is a holomorphic $d_{B(\mathbf{X})}$-isometry because $d_{B(\mathbf{X})}\big(x, y\big) = \max_{\tau \ge 0} d_\Delta\big(x(\tau), y(\tau)\big)$ and the maps φ^t are $d_{B(\mathbf{E})}$-contractions.

Remark. It is well-known [Federer, Geometric measure theory?] that, given a continuously differentiable function $f : \mathbf{R}_+ \to \mathbf{E}$ where $\mathbf{E}$ is a Banach space, we have

$$\frac{d^+}{dt}\big\|f(t)\big\| := \limsup_{h \searrow 0} \big[\|f(t+h)\| - \|f(t)\|\big]/h = \sup_{L \in \mathcal{S}(f(t))} \mathrm{Re}\big\langle L, f'(t)\big\rangle$$

in terms of the family of supporting bounded linear functionals

$$\mathcal{S}(y) := \big\{L \in \mathbf{E}^* : \|L\| = 1,\ \langle L, y\rangle = \|y\|\big\} \qquad (y \in \mathbf{E}).$$

In particular f is non-icreasing whenever $\mathrm{Re}\big\langle L, f'(t)\big\rangle \le 0$ for any $t \in \mathbf{R}_+$ and for any functional $L \in \mathcal{S}(f(t))$.

Lemma. Let $V : U \to \mathbf{E}$ be a bounded continuously differentiable map (regarded as a vector field) on some open neighborhood U of the closed unit ball $\overline{B(\mathbf{E})}$ with $V(0) = 0$

and let $\mu \geq \sup_{e_1,e_2\in B(\mathbf{E})} \big\|V(e_1) - V(e_2)\big\|$. Then the maximal flow of the vector field $W : B(\mathbf{E}) \ni e \mapsto V(e) - \mu e$ is a well-defined uniformly continuous one-parameter semigroup $[\varphi^t : t \in \mathbf{R}_+]$ consisting of contractive (non-expansive) self maps of $B(\mathbf{E})$.

Proof. By definition, any flow of W is a family $[\varphi^t : t \in I]$ of self maps $\varphi^t : B(\mathbf{E}) \to B(\mathbf{E})$ where I is some (relatively) open subinterval of $\mathbf{R}_+$ and, for any point $e \in B(\mathbf{E})$, the fuction $I \ni t \mapsto \varphi^t(e)$ is the solution of the initial value problem $(*)$ $\frac{d}{dt}z(t) = W\big(z(t)\big)$, $z(0) = e$. By writing I_e for the maximal solution of $(*)$, it is well-known that $\sup I_e > 0$ in any case, furthermore we have $\lim_{t\to\sup I_e} \|z(t)\| = 1$ whenever $\sup I_e < \infty$.

Let $e_1, e_2 \in B(\mathbf{E})$ and consider the function $f(t) := \varphi^t(e_1) - \varphi^t(e_2)$ defined on the interval $I_{e_1} \cap I_{e_2}$. Observe that, given any functional $L \in \mathcal{S}\big(\varphi^t(e_1) - \varphi^t(e_2)\big)$, we have

$$\mathrm{Re}\big\langle L, f'(t)\big\rangle = \mathrm{Re}\big\langle L, W\big(\varphi^t(e_1)\big) - W\big(\varphi^t(e_2)\big)\big\rangle =$$

$$= \mathrm{Re}\big\langle L, V\big(\varphi^t(e_1)\big) - V\big(\varphi^t(e_2)\big)\big\rangle - \mu\mathrm{Re}\big\langle L, \varphi^t(e_1) - \varphi^t(e_2)\big\rangle =$$

$$= \mathrm{Re}\big\langle L, V\big(\varphi^t(e_1)\big) - V\big(\varphi^t(e_2)\big)\big\rangle - \mu\big\|\varphi^t(e_1) - \varphi^t(e_2)\big\| \leq$$

$$\leq \mu\big\|\varphi^t(e_1) - \varphi^t(e_2)\big\| - \mu\big\|\varphi^t(e_1) - \varphi^t(e_2)\big\| = 0.$$

Hence we conclude that the fuction $t \mapsto f(t)$ is decreasing, in particular we have the contraction property $\big\|\varphi^t(e_1) - \varphi^t(e_2)\big\| \leq \big\|\varphi^0(e_1) - \varphi^0(e_2)\big\| = \big\|e_1 - e_2\big\|$ for $t \in I_{e_1} \cap I_{e_2}$. By assumption $W(0) = V(0) = 0$ implying $\varphi^t(0) \equiv 0$ with $I_0 = [0,\infty) = \mathbf{R}_+$. Hence we see also that $\big\|\varphi^t(e)\big\| = \big\|\varphi^t(e) - \varphi^t(0)\big\| \leq \big\|e - 0\big\| = \big\|e\big\| < 1$ for all $e \in B(\mathbf{E})$ and $t \in I_e$. This is possible only if $\sup I_e = \infty$. Therefore the maximal flow of W is defined for all (time) parameters $t \in \mathbf{R}_+$ and consists of $B(\mathbf{E})$-contractions φ^t.

It is well-known that flows parametrized on $\mathbf{R}_+$ are strongly continuous semigroups automatically. The uniform continuity of in our case is a consequence of the fact that $\|\varphi^{t_2}(e) - \varphi^{t_1}(e)\| \leq \int_{t_1}^{t_2} \|\frac{d}{dt}\varphi^t(e)\|dt = \int_{t_1}^{t_2} \|W(\varphi^t(e))\|dt \leq \int_{t_1}^{t_2} 4\mu\ dt \quad (0 \leq t_1 \leq t_2)$, which shows that $\omega(e,\delta) \leq 4\mu\delta \quad (e \in B(\mathbf{E}),\ \delta \in \mathbf{R}_+)$.

Example. Let $\mathbf{E} := \mathbf{C}$ with $B(\mathbf{E}) = \Delta = \{\zeta \in \mathbf{C} : |\zeta| < 1\}$ and let $V(z) \equiv z^2$. Since $|z_1^2 - z_2^2| = |z_1 - z_2| \cdot |z_1 + z_2| \leq 2|z_1 - z_2|$, we can apply the above Lemma with $W(z) := z^2 - 2z$. For the flow $[\varphi^t : t \in \mathbf{R}_+]$ of W we obtain the holomorphic maps

$$\varphi^t(z) = \frac{2z}{\big(1 - e^{2t}\big)z + 2e^{2t}} \qquad (z \in \Delta,\ t \geq 0).$$

Indeed, the solution of the initial value problem $(**)$ $\frac{d}{dt}x(t) = x(t)^2 - 2x(t)$, $x(0) = z$ is $x(t) = 2z/\big[\big(1 - e^{2t}\big)z + 2e^{2t}\big]$ as one can check by direct computation. As for heuristics, we get a real valued solution with real calculus for $(**)$ with initial values $-1 < z < 1$, and the obtained formula extends holomorphically to Δ.

Theorem. Given a complex Banach space $\mathbf{E}$, there is a C0-semigroup of non-linear holomorphic 0-preserving norm and Carathéodory isometries of the open unit ball of the function space $\mathbf{X} := C_0(\mathbf{R}_+, \mathbf{E})$.

Proof. We can apply the construction of the first Lemma with a semigroup $[\varphi^t : t \in \mathbf{R}_+]$ obtained with the construction of the 2nd Lemma with any $\mathbf{E}$-polynomial vector field V.

Example. Let $\mathbf{E} := \mathbf{C}$ and $\mathbf{X} := C_0(\mathbf{R}_+, \mathbf{C})$. Then the maps

$$\Phi^t(x) : \mathbf{R}_+ \ni \tau \mapsto \left[\frac{2x(0)}{\big(1 - e^{2(t-\tau)}\big)x(0) + 2e^{2(t-\tau)}} \text{ if } \tau \le t, \quad x(\tau - t) \text{ if } \tau \ge t \right]$$

form a C0-semigroup of non-linear holomorphic 0-preserving norm and Carathéodory isometries of the unit ball $B(\mathbf{X})$.

Question. Is any holomorphic norm-isometry of the unit ball of a complex Banach space automatically a Carathéodory isometry as well?

Analogous construction in $\mathbf{E} = \mathcal{L}(\mathbf{H})$

$\mathbf{H} := L^2(\mathbf{R}_+), \quad \langle f|g\rangle := \int_0^\infty f(x)\overline{g(x)}\,dx$

$S^t f := [x \mapsto f(x-t) \text{ if } x \ge t,\ 0 \text{ else}] \qquad (t \in \mathbf{R}_+,\ f \in \mathbf{H})$

$(S^t)^* g = [x \mapsto f(x+t)] \quad (t \in \mathbf{R}_+,\ g \in \mathbf{H})$

S^t lin. non surjective $\mathbf{H} \to \mathbf{H}$ isometry:

$(S^t)^* S^t = \mathrm{Id}_{\mathbf{H}}, \quad S^t (S^t)^* g = [x \mapsto g(x) \text{ if } x \ge t,\ 0 \text{ else}].$

$P_t := \mathrm{Pr}_{[0,t]\mathbf{H}} = [f \mapsto 1_{[0,t]} f], \quad \overline{P}_t := 1 - P_t = S^t(S^t)^* = [f \mapsto 1_{(t,\infty)} f]$

Notation. $\mathbf{E} := \mathcal{L}(\mathbf{H}), \quad \mathbf{E}_0 := \bigcup_{t>0} \mathbf{F}_t$ where

$$\mathbf{F}_t := \big\{ A \in \mathbf{E} : P_t A \overline{P}_t = \overline{P}_t A P_t = 0, P_t A P_t = \textstyle\int_0^t \psi(s)\ dP_s \text{ with } \psi \in \mathcal{C}[0,t] \big\},$$

$\Lambda_0 \in \mathbf{E}^*$ lin. functional with norm 1, such that

$$\Lambda_0(A) := \psi(0) \quad \text{whenever} \quad A \in \mathbf{F}_t \text{ with } P_t A P_t = \textstyle\int_0^t \psi(s)\ dP_s,\ \psi \in \mathcal{C}[0,t].$$

Lemma. Λ_0 is well-defined.

Proof. Immediate from the observations that

1) if $0 < t_1 \leq t_2$ and $A \in \mathbf{F}_{t_k}$ with $P_{t_k} A P_{t_k} = \int_0^{t_k} \psi_k(s)\ dP_s$ then

$A \in \mathbf{F}_{t_1}$ with $P_{t_1} A P_{t_1} = \int_0^{t_1} \psi_k(s)\ dP_s$ $(k = 1, 2)$;

2) $A \in \mathbf{F}_t$ with $P_t A P_t = \int_0^t \psi_1(s)\ dP_s = \int_0^t \psi_2(s)\ dP_s$, $\psi_1, \psi_2 \in \mathcal{C}[0, t]$ implies $\psi_1 = \psi_2$

due to continuity of the functions ψ_k.

Definition. $\Lambda := \big[$a Hahn-Banach extension of Λ_0 to $\mathbf{E}$ with norm $1\big]$

CARTAN TYPE LINEARITY THEOREMS WITH NON-SURJECTIVE MAPS

$\mathbf{E}$ Banach space, $\mathbf{B}$ its open unit ball, $\Phi : \mathbf{B} \to \mathbf{B}$ holomorhic

Assumption. $\Phi(0) = 0, \quad \|\Phi(x)\| = \|x\| \quad (x \in \mathbf{B})$.

Remark. If $\Psi \in \mathrm{Iso}(d_{\mathbf{B}})$ and $\Psi(0) = 0$ then necessarily $\|\Psi(x)\| = \tanh d_{\mathbf{B}}\big(\Psi(x), 0\big) = \tanh d_{\mathbf{B}}(x, 0) = \|x\|$ $(x \in \mathbf{B})$. However, it is not known in general whether $\Phi \in \mathrm{Iso}(d_{\mathbf{B}})$. This latter holds if $\mathbf{E}$ is a JB*-triple.

As for the Taylor series of Φ, we can write

$$\Phi(x) = Ux + \sum_{n=2}^{\infty} \Omega_n(x) \qquad (x \in \mathbf{B})$$

where each term Ω_n is a homogeneous polynomial $\mathbf{E} \to \mathbf{E}$ of n-th degree and

U is a linear isometry of $\mathbf{E}$ since

$$\|Ux\| = \lim_{t \to 0+} \big\|\Phi(tx)\big\| = \lim_{t \to 0+} d_{\mathbf{B}} \tanh d_{\mathbf{B}}\big(\Phi(tx), \Phi(0)\big) =$$

$$= \lim_{t \to 0+} \tanh d_{\mathbf{B}}(tx, 0) = \lim_{t \to 0+} d_{\mathbf{B}}\|tx\| = \|x\|.$$

As an easy consequence of Cartan's Uniqueness Theorem, if $\mathrm{range}(\Phi) \subset U\mathbf{B}$ then necessarily $\Phi = U|_{\mathbf{B}}$. Indeed, the mapping $\Psi(x) := U^{-1}\Phi(x)$ $(x \in \mathbf{B})$ is a well-defined holomorphic self-map of $\mathbf{B}$ with $\Psi'(0) = \mathrm{Id}_{\mathbf{E}}$ and hence $\Psi = \mathrm{Id}_{\mathbf{B}}$ with $\Phi = U\Psi = U|_{\mathbf{B}}$.

On the other hand, there is a rather simple example for a non-linear map Φ satisfying our assumptions: If we take the classical sequence space $\mathbf{E} = c_0 = \big\{(\zeta_0, \zeta_1, \ldots) : \lim_n \zeta_n = 0\big\}$ with $\big\|(\zeta_n)_{n=0}^{\infty}\big\| := \max_n |\zeta_n|$ then the mapping $\Phi(\zeta)_{n=0}^{\infty} := (\zeta_0^2, \zeta_0, \zeta_1, \zeta_2, \ldots)$ is clearly a norm preservig holomorphic self-map of the unit ball.

Conjecture. If the underlying space $\mathbf{B}$ is reflexive then necessarily $\Phi = U|_{\mathbf{B}}$.

We achieved the following result which implies the conjecture for uniformly convex spaces:

Theorem. If we have $\sup \dim\big\{\text{faces of } \mathbf{B}\big\} < \infty$ then $\Phi = U|_{\mathbf{B}}$.

Recall that by a *face* of $\mathbf{B}$ we mean a non-empty convex subset of $\partial\mathbf{B} := \{x \in \mathbf{E} : \|x\| = 1\}$. A *norm exposed face* of $\mathbf{B}$ is a non empty intersection of a real affine subspace passing outside the open unit ball with the closed unit ball, i.e. any non-empty set of the form $\bigcap_{\mu\in\mathcal{M}} \big\{x \in \mathbf{E} : \|x\| = 1 = \langle\mu, x\rangle\big\}$ with a family $\mathcal{M}$ of norm-one real-linear functionals $\mathbf{E} \to \mathbf{R}$. By a *norm exposed complex face* of $\mathbf{B}$ we mean a non empty intersection of the form $\bigcap_{L\in\mathcal{L}} \big\{x \in \mathbf{E} : \|x\| = 1 = \langle L, x\rangle\big\}$ with a family $\mathcal{L}$ norm-one complex-linear functionals $\mathbf{E} \to \mathbf{C}$. Notice that norm exposed (complex-)faces are automatically convex subsets of the unit sphere $\partial\mathbf{B}$ ab being the intersection of the closed unit ball witt a real (complex) affine subspace of $\mathbf{E}$.

Given any unit vector $x \in \partial\mathbf{B}$, we shall write $\mathcal{S}_x(\mathbf{B} := \big\{L \in \mathbf{E}^* : \|x\| = \langle L, x\rangle = 1\big\}$ for the family of all *supporting linear functionals* of the unit ball at the point x. By the aid of these terms we introduce the notations

$$\mathrm{Face}_x(\mathbf{B}) := \bigcap_{L\in\mathcal{S}_x(\mathbf{B})} \big\{y \in \partial\mathbf{B} : \mathrm{Re}\langle L, y\rangle = 1\big\}, \quad \mathrm{Face}_x^{\mathbf{C}}(\mathbf{B}) := \bigcap_{L\in\mathcal{S}_x(\mathbf{B})} \big\{y \in \partial\mathbf{B} : \langle L, y\rangle = 1\big\}$$

for the minimal real resp. complex norm exposed face at the point x.

Lemma. Suppose $\Psi : \mathbf{D} \to \mathbf{E}$ is a holomorphic map from a domain (open connected set)

$\mathbf{D}$ in some Banach space into $\mathbf{E}$ such that range$(\Psi)\big(=\Psi(\mathbf{D})\big)\subset\partial\mathbf{B}$. Then range$(\Psi)$ is contained in some norm exposed complex face of $\mathbf{B}$.

Proof. Let $z_0\in\mathbf{D}$ be any point and define $x_0:=\Psi(z)$. Given any support linear functional $L\in\mathcal{S}_{x_0}(\mathbf{B})$, we have

$$|\langle L,\Psi(z)\rangle|\le\|L\|\|\Psi(z)\|=1=|\langle L,\Psi(z_0)\rangle|\qquad(z\in\mathbf{D}).$$

That is the modulus of the holomorphic scalar valued function $L\Psi: z\mapsto\langle L,\Psi(z)\rangle$ assumes its maximum value $(=1)$ at the inner point z_0 of the (open) domain $\mathbf{D}$. Hence, by the Maximum Priciple, necessarily $L\Psi\equiv L\Psi(z_0)=1$ and therefore range$(\Psi)\subset\{y\in\partial\mathbf{B}:\langle L,y\rangle=1\}$. By the arbitrariness of the choice for $z_0\in\mathbf{D}$, we conclude that range$(\Psi)\subset\bigcap_{z_0\in\mathbf{D}}\bigcap_{L\in\mathcal{S}_{\Psi(z_0}}\{y\in\partial\mathbf{B}:\langle L,y\rangle=1\}=\mathrm{Face}^{\mathbf{C}}_{\Psi(z_0)}(\mathbf{B})$.

Corollary. We have $\mathrm{Face}^{\mathbf{C}}_{\Psi(z_0)}(\mathbf{B})=\mathrm{Face}^{\mathbf{C}}_{\Psi(z_1)}(\mathbf{B})\supset\mathrm{range}(\Psi)\quad(z_0,z_1\in\mathbf{D})$.

Proof. It suffices to see that $\mathcal{S}_{\Psi(z_0)}(\mathbf{B})=\mathcal{S}_{\Psi(z_0)}(\mathbf{B})\ \ (z_0,z_1\in\mathbf{D})$.

Let $z_0,z_1\in\mathbf{D}$ and $L\in\mathcal{S}_{\Psi(z_0)}(\mathbf{B})$. Since $L\Psi\equiv1$, we have $1=\langle L,\Psi(z_1)\rangle=\|\Psi(z_1)\|$ that is also $L\in\mathcal{S}_{\Psi(z_1)}(\mathbf{B})$. By the arbitrariness of L in $\mathcal{S}_{\Psi(z_0)}(\mathbf{B})$ we see $\mathcal{S}_{\Psi(z_0)}(\mathbf{B})\subset\mathcal{S}_{\Psi(z_1)}(\mathbf{B})$. With the change $z_0\leftrightarrow z_1$ in the argument, we get the converse inclusion as well.

Proposition. All the polynomial maps

$$\Psi_{N,\delta}: x\mapsto Ux+\frac{\delta}{2}\Omega_N(x)\qquad(|\delta|\le1;\ N=2,3,\ldots)$$

are norm-preserving on the closed unit ball $\overline{\mathbf{B}}$.

Proof. Let $x \in \partial\mathbf{B}$ be fixed arbitrarily and consider the holomorphic map

$$\Phi_x(\zeta) := Ux + \sum_{n=2}^{\infty} \zeta^{n-1}\Omega_n(x) \qquad (\zeta \in \Delta).$$

Actually $\Phi_x(\zeta) := \zeta^{-1}\Phi(\zeta x)$ $(0 \neq \zeta \in \Delta)$ while $\Phi_x(0) := Ux$. Let us choose a supporting (continuous complex-)linear) functional $L \in \mathcal{S}(Ux, \mathbf{B}) := \{L \in \mathbf{E}^* : 1 = \|L\| = |\langle L, x\rangle|\}$. Since $\|Ux\| = \|x\| = 1$, this can be done due to the Hahn-Banach Theorem. Since for $\zeta \neq 0$ we have $\|\Phi_x(\zeta)\| = |\zeta|^{-1}\|\Phi(\zeta x)\| = |\zeta|^{-1}\|\zeta x\| = \|x\| = 1$ implying $|\langle L, \Phi_x(\zeta)\rangle| \leq \|L\| \cdot \|\Phi_x(\zeta)\| = 1 = \langle L, \Phi_x(0)\rangle$, the absolute value of the holomorphic function $\Delta \ni \zeta \mapsto \langle L, \Phi_x(\zeta)\rangle$ assumes its maximum at tha origin. Thus, by the Schwarz Lemma, $|\langle L, \Phi_x(\zeta)\rangle| \equiv 1$ that is the set $\Phi_x(\Delta)(= \{\Phi_x(\zeta) : |\zeta| < 1\})$ is contained in the norm exposed face $\mathrm{Face}_{Ux}(\mathbf{B}) := \bigcap_{L \in \mathcal{S}(Ux,\mathbf{B})} \{y \in \overline{\mathbf{B}} : \langle L, y\rangle = 1\}$ at Ux in $\partial\mathbf{B}$. Since $\mathrm{Face}_{Ux}(\mathbf{B})$ is a convex closed subset of $\mathbf{E}$ containing the point Ux, even the closed convex hull of $\Phi_x(\Delta)$ has the same property

$$\overline{\mathrm{Conv}}\big(\Phi_x(\Delta)\big) \subset \mathrm{Face}_{Ux}(\mathbf{B}).$$

In particular, by weighting with any non-negative continuous function $\lambda : \Delta \to \mathbf{R}_+$ we have

$$\left[\int\limits_{\zeta\in\Delta} \lambda(\zeta)\,\mathrm{area}(d\zeta)\right]^{-1} \int\limits_{\zeta\in\Delta} \lambda(\zeta)\Phi_x(\zeta)\,\mathrm{area}(d\zeta) \in \ \mathrm{Face}_{Ux}(\mathbf{B}).$$

Given N and δ as in the statement of the Proposition, consider this relation with the functions

$$\lambda_m\big(\rho e^{i\varphi}\big) := \rho^m\big[1 + \delta\cos\big((N-1)\varphi\big)\big] \qquad (0 \leq \rho < 1;\ 0 \leq \varphi < 2\pi;\ m = 1, 2, \ldots).$$

Since $\int\limits_{\zeta\in\Delta} |\zeta|^k\zeta^n \,\mathrm{area}(d\zeta) \;=\; \int\limits_{\rho=0}^{1}\int\limits_{\varphi=0}^{2\pi} \rho^k\rho^n e^{in\varphi}\, d\varphi\, \rho d\rho \;=\; \big[2\pi/(k{+}n{+}2) \text{ if } n{=}0,\;\; 0 \text{ else}\big]$,

furthermore $\lambda_m(\zeta) = |\zeta|^m\big[1{+}(\delta/2)|\zeta|^{1-N}\big(\zeta^{N-1}{+}\zeta^{1-N}\big)\big]$ and $\Phi_x(\zeta) = Ux{+}\sum\limits_{n>1}\zeta^{n-1}\Omega_n(x)$,

hence we conclude that

$$Ux + \frac{\delta}{2}\,\frac{2\pi/(m+N+1)}{2\pi/(m+2)}\Omega_N(x) \in \mathrm{Face}_{Ux}(\mathbf{B}) \qquad (m=1,2,\ldots;\; 0\le\delta\le 1).$$

By passing to the limit $m\to\infty$, it follows

$$Ux + \frac{\delta}{2}\Omega_N(x) \in \mathrm{Face}_{Ux}(\mathbf{B}) \qquad (\|x\|=1;\; 0\le\delta\le 1).$$

Given any $0\neq y\in\overline{\mathbf{B}}$, consider the boundary point $x := y/\|y\|$ with the constant $\delta' := \|y\|^{N-2}\delta \in [0,1]$. We have $\|y\|^{-1}Uy + (\delta'/2)\|y\|^{1-N}\Omega_N(y) \in \mathrm{Face}_{Ux}(\mathbf{B}) \subset \partial\mathbf{B}$ whence $1 = \Big\|\|y\|^{-1}Uy{+}(\delta'/2)\|y\|^{1-N}\Omega_N(y)\Big\|$ i.e. $\|y\| = \Big\|Uy + (\|y\|^{2-N}\delta'/2)\Omega_N(y)\Big\| = \Big\|Uy + (\delta/2)\Omega_N(y)\Big\|$. Qu.e.d.

Lemma. Assume $v_0,v_1,\ldots,v_n \in \big[\mathbf{E}\setminus\mathrm{range}(U)\big]\cup\{0\}$ and $\sum\limits_{j=0}^{n} U^jv_j \in \mathrm{range}(U^{n+1})$. Then necessarily $v_0=v_1=\cdots=v_n=0$.

Proof. We proceed by contradiction and let k be the least index with $v_k\neq 0$ i.e. $v_k\notin \mathrm{range}(U)$. Then $\sum\limits_{j=k}^{n} U^jv_j = U^{n+1}w$ that is $U^k\big[v^k{+}Uv_{k+1}{+}\cdots{+}U^{n-k}v_v{-}U^{n-k+1}w\big]=0$ for some $w\in\mathbf{E}$. Since U is an isometry, it follows $v^k{+}Uv_{k+1}{+}\cdots{+}U^{n-k}v_n{-}U^{n-k+1}w=0$ which leads to the contradiction $v_k = U\Big[\sum\limits_{\ell:0<\ell\le n-k} U^{\ell-1}v_{k+\ell} - U^{n-k}\Big] \in \mathrm{range}(U)$.

Lemma. Let $P:\Delta^n\to\mathbf{E}$, $P(\delta_1,\ldots,\delta_n) := \sum_{j_1,\ldots,j_n\in\{0,\ldots,K\}}\delta_1^{j_1}\cdots\delta_n^{j_n}p_{[j_1,\ldots,j_n]}$ with vector coefficients $p_{[j_1,\ldots,j_n]}\in\mathbf{E}$ be a bounded holomorphic map . Then for any constant

$\delta \in \overline{\Delta}$ and for any coefficient multiindex $[k_1, \dots, k_n] \neq [0, \dots, 0]$ we have

$$p_0 + \frac{\delta}{2} p_{[k_1,\dots,k_n]} \in \overline{\mathrm{Conv}}\big(P(\Delta^n)\big).$$

Proof. Notice that given any non-vanishing bounded continuous function $\lambda : \Delta^n \to \mathbf{R}_+$,

$$(*) \qquad \frac{\int\limits_{\xi_1+i\eta_1,\dots,\xi_n+i\eta_n \in \Delta} \lambda(\xi + i\eta) P(\xi + i\eta)\, d\xi_1 \dots d\xi_n \; d\eta_1 \dots d\eta_n}{\int\limits_{\xi_1+i\eta_1,\dots,\xi_n+i\eta_n \in \Delta} \lambda(\xi + i\eta)\, d\xi_1 \dots d\xi_n \; d\eta_1 \dots d\eta_n} \in \overline{\mathrm{Conv}}\big(P(\Delta^n)\big).$$

Let us fix any $\delta \in \overline{\Delta}$ and any pair of non-negative multiindices $[m_1, \dots, m_n], [k_1, \dots, k_n] \neq$

0 and consider the above relation with the choice

$$\lambda\big(\rho_1 e^{i\varphi_1}, \dots, \rho_n e^{i\varphi_n}\big) := \Big[\prod_{j=1}^n \rho_j^{m_j}\Big] \cdot \Big[2 + \overline{\delta} \prod_{j=1}^n e^{ik_j\varphi_j} + \delta \prod_{j=1}^n e^{-ik_j\varphi_j}\Big].$$

Observe that $\lambda(\Delta^n) \geq 0$ and

$$\lambda(\delta_1, \dots, \delta_n) = 2 \prod_{j=1}^n |\delta_j|^{m_j} + \overline{\delta} \prod_{j=1}^n |\delta_j|^{m_j - k_j} \delta_j^{k_j} + \delta \prod_{j=1}^n |\delta_j|^{m_j + k_j} \delta_j^{-k_j}.$$

In general, with polar coordinate integration we get

$$\int\limits_{\delta_1=\xi_1+i\eta_1\in\Delta} \cdots \int\limits_{\delta_n=\xi_n+i\eta_n\in\Delta} \prod_{j=1}^n |\delta_j|^{r_j} \delta_j^{s_j} \, d\xi_1 \dots d\xi_n \; d\eta_1 \dots d\eta_n =$$

$$= \int_{\rho_1=0}^1 \cdots \int_{\rho_n=0}^1 \int_{\varphi_1=0}^{2\pi} \cdots \int_{\varphi_n=0}^{2\pi} \prod_{j=1}^n \rho_j^{r_j} \rho_j^{s_j} e^{is_j\varphi_j} \; \rho_j \quad d\varphi_n \cdots d\varphi_1 \; d\rho_n \cdots d\rho_1 =$$

$$= \left[\frac{(2\pi)^n}{\prod_{j=1}^n (r_j + 2)} \text{ if } s_1 = \dots = s_n = 0, \quad 0 \quad \text{else} \right].$$

In particular, for any non-negative multiindex $[t_1, \dots, t_n]$,

$$\int\limits_{\delta_1=\xi_1+i\eta_1\in\Delta} \cdots \int\limits_{\delta_n=\xi_n+i\eta_n\in\Delta} \lambda(\delta_1, \dots, \delta_n) \prod_{j=1}^n \delta_j^{t_j} \, d\xi_1 \dots d\xi_n \; d\eta_1 \dots d\eta_n =$$

$$= \int \cdots \int\limits_{\delta_j=\xi_j+i\eta_j\in\Delta} \Big[2 \prod_{j=1}^n |\delta_j|^{m_j} \delta_j^{t_j} + \overline{\delta} \prod_{j=1}^n |\delta_j|^{m_j-k_j} \delta_j^{t_j+k_j} + \delta \prod_{j=1}^n |\delta_j|^{m_j+k_j} \delta_j^{t_j-k_j}\Big] d\xi_1 \cdots d\eta_n =$$

$$= \Big[\frac{2 \cdot (2\pi)^n}{\prod_{j=1}^n (m_j + 2)} \text{ if } t = 0\Big] + \Big[\frac{(2\pi)^n \delta}{\prod_{j=1}^n (m_j + k_j + 2)} \text{ if } t = k, \; 0 \quad \text{else}\Big]$$

Since the Taylor series of P coverges locally uniformly, it follows that

$$\int\limits_{\xi_1+i\eta_1,\dots,\xi_n+i\eta_n\in\Delta} \lambda(\xi+i\eta)\, d\xi_1\dots d\xi_n\ d\eta_1\dots d\eta_n = \frac{2\cdot(2\pi)^n}{\prod_{j=1}^n(m_j+2)}$$

and

$$\int\limits_{\xi_1+i\eta_1,\dots,\xi_n+i\eta_n\in\Delta} \lambda(\xi+i\eta)P(\xi+i\eta)\, d\xi_1\dots d\xi_n\ d\eta_1\dots d\eta_n =$$
$$= \frac{2\cdot(2\pi)^n}{\prod_{j=1}^n(m_j+2)} p_{[0,\dots,0]} + \frac{(2\pi)^n\delta}{\prod_{j=1}^n(m_j+k_j)} p_{[k_1,\dots,k_n]}.$$

Hence and from $(*)$ the statement of the Lemma is immediate by passing to the limits $m_1,\dots,m_n\to\infty$.

Lemma. Given any index $N>1$, for any unit vector $x\in\partial\mathbf{B}$ we have

$$(*)\qquad (\Delta/2)\, U^{n-k}\Omega_N\big(U^k x\big)\subset \mathrm{Face}_{U^{n+1}x}(\mathbf{B}) \qquad (0\le k\le n=0,1,\dots,n).$$

Proof. We proceed by induction on n. The case $n=0$ is immediate by the Proposition.

Assume that $(\Delta/2)\, U^{n-k}\Omega_N\big(U^k x\big)\subset \mathrm{Face}_{U^{n+1}x}(\mathbf{B})$ $(x\in\partial\mathbf{B})$ holds for some (k,n).

Since U is a (complex-)linear $\mathbf{E}$-isometry, it follows

$$(\Delta/2)U^{n+1-k}\Omega_N\big(U^k x\big)=U\big[(\Delta/2)U^{n+1-k}\Omega_N\big(U^k x\big)\big]\subset U\big[\mathrm{Face}_{U^{n+1}x}(\mathbf{B})\big]\subset \mathrm{Face}_{U^{n+2}x}(\mathbf{B}).$$

On the other hand, by replacing x with Ux, we get

$$(\Delta/2)\, U^{n-k}\Omega_N\big(U^{k+1}x\big)=(\Delta/2)\, U^{n-k}\Omega_N\big(U^k(Ux)\big)\subset \mathrm{Face}_{U^{n+1}(Ux)}(\mathbf{B})=\mathrm{Face}_{U^{n+2}x}(\mathbf{B})$$

which completes the induction argument and hence the proof.

Proof of the Theorem. We show that the assumption $\Omega\not\equiv 0$ leads to contradction.

Assume there is a homogeneous polynomial $\Omega_N \not\equiv 0$ (with $N > 1$) in the Taylor expansion of Ω. It is well-known that then the set $\mathcal{N}(\Omega_N) := \{x \in \mathbf{E} : \Omega_N(x) = 0\}$ is nowhere dense in $\mathbf{E}$. Since U is an isometry, also all the sets

CASE OF JB*-TRIPLES WITH FINITE RANK

$(\mathbf{E}, \{\ldots\})$ is a JB*triple with $\quad \mathrm{rank}(\mathbf{E}) = r < \infty \quad$ in this section.

Remark. $\mathbf{E}$ is reflexive and is a finite ℓ^∞-direct sum of finitely many Cartan factors of which only the types $\mathcal{L}(\mathbf{H}_1, \mathbf{H}_2)$ and Spin factors can be infinite dimensional [Kaup, 1981]. By [Edwards-Rüttiman] or [Peralta-Stachó], the norm exposed faces of the unit ball $\mathbf{B}$ are in a natural one-to-one correspondance with the *tripotents* of $\mathbf{E}$ as being of the form

$$\mathrm{Face}(\mathbf{B}, e) = \big\{ y \in \partial \mathbf{B} : \langle L, y \rangle = 1 \ \text{ for all } L \in \mathcal{S}(e) \big\} =$$

$$= \big\{ e + v : v \perp^{\text{Jordan}} e, \ \|v\| \leq 1 \big\} \qquad (e \in \mathrm{Trip}(\mathbf{E})).$$

Lemma. Let $a, b \in \partial \mathbf{B}$ be unit vectors such that $\|\alpha a + \beta b\| = \max\{|\alpha|, |\beta|\}$ $(\alpha, \beta \in \mathbf{C})$. Then

$$a = e + a_0, \quad a_0, b \perp^{\text{Jordan}} e, \qquad b = f + b_0, \quad b_0, a \perp^{\text{Jordan}} f, \qquad e \perp^{\text{Jordan}} f$$

with suitable tripotents $e, f \in \mathrm{Trip}(\mathbf{E})$ and vectors $a_0, b_0 \in \overline{\mathbf{B}}$.

Proof. Since $a, b \in \partial \mathbf{B}$, we have

$$a \in \mathrm{Face}(\mathbf{B}, e), \ a = a_0 + e, a_0 \perp^{\text{Jordan}} e \quad \text{resp.} \quad b \in \mathrm{Face}(\mathbf{B}, f), \ b = b_0 + e, b_0 \perp^{\text{Jordan}} f$$

with suitable tripontents e, f and vectors $a_0, b_0 \in \overline{\mathbf{B}}$. By assumption $\|a + \beta b\| = 1$ whenever $|\beta| \leq 1$. That is the disc $a + \Delta b = a + a_0 + \Delta b$ is also contained in the face $\mathrm{Face}(\mathbf{B}, e)$ of the point a. Similarly (with the chages $a \leftrightarrow b, e \leftrightarrow f, a_0 \leftrightarrow b_0$), $b + \Delta a \subset \mathrm{Face}(\mathbf{B}, f)$. It follows

$$e \perp^{\text{Jordan}} b = f + b_0, \quad f \perp^{\text{Jordan}} a = e + a_0$$

implying (with the standard notation $L(x,y): z \mapsto \{xy^*z\}$)

$$L(e, f + b_0) = L(f + b_0, e) = 0 \text{ i.e. } L(e,f) = -L(e,b_0),\ L(f,e) = -L(b_0,e);$$

$$L(f, e + a_0) = L(e + a_0, f) = 0 \text{ i.e. } L(f,e) = -L(f,a_0),\ L(e,f) = -L(a_0,f);$$

$$L(e,f) = -L(e,b_0) = -L(a_0,f), \quad L(f,e) = -L(f,a_0) = -L(b_0,e).$$

Since $a_0 \perp^{\text{Jordan}} e$, hence we get

$$-L(f,e)e = -L(f,a_0)e = \{fa_0e\} = \{ea_0f\} = L(e,a_0)f = 0$$

which means the Jordan-orthogonality $\{fee\} = 0$ of the tripotents e, f. Qu.e.d.

Corollary. If $a_1, \ldots, a_r \in \mathbf{E}$ have the property $\left\| \sum_{k=1}^r \alpha_k a_k \right\| = \max_{k=1}^r |\alpha_k|$ $(\alpha_1, \ldots, \alpha_m \in \mathbf{C})$, then necessarily $a_1, \ldots, a_r$ are pairwise Jordan-orthogonal tripotents.

Proof. Recall that $r = \text{rank}(\mathbf{E})$ is the maximal number of pairwise Jordan-ortogonal non-zero vectors in $\mathbf{E}$. By the previous lemma, we can write

$$a_k = e_k + a_{k0}, \quad a_k \perp^{\text{Jordan}} e_j \ (j \neq k)$$

with a maximal Jordan-orthogonal family of tripotents $\{e_1, \ldots, e_r\}$ and suitable vectors $a_{10}, \ldots, a_{r0} \in \overline{\mathbf{B}}$ such that $a_{k0} \perp^{\text{Jordan}} e_k$ $(k = 1, \ldots, r)$. The property $a_k \perp^{\text{Jordan}} e_j$ $(j \neq k)$ along with the maximality of $\{e_1, \ldots, e_r\}$ implies that, for any index k, necessarily $a_k \in \mathbf{C}e_k$ and hence even $a_k = \varepsilon_k e_k \in \text{Trip}(\mathbf{E})$ with $|\varepsilon_k| = 1$ (because $\|a_k\| = 1$). Qu.e.d.

Theorem. The 0-preserving holomorphic Carathéodory isometries of the unit ball of a JB*-triple of finite rank are linear triple product homomorphisms.

Proof. Let $(\mathbf{E}, \{\ldots\})$ be a JB*-triple with rank $r < \infty$ and let $\Phi = U + \Omega \in \mathrm{Iso}(d_{\mathbf{B}})$ with $U := D_0\Phi$ and $\Omega(0) = 0$. According to the results of the previous section, the linear term U is a $\mathbf{E}$-isometry. Consider a maximal family $x_1, \ldots, x_r \in \mathrm{Trip}(\mathbf{E})$ of pairwise orthogonal tripotents. It is well-known that $\|\sum_{k=1}^r \alpha_k x_k\| = \max_{k=1}^r |\alpha_k|$ $(\alpha_1, \ldots, \alpha_r \in \mathbf{C})$ in this case. Thus the vectors $a_k := Ux_k$ satisfy the hypothesis of the Lemma and its Corollary, giving rise to the conclusion that $Ux_1, \ldots, Ux_r$ form also a maximal family of (minimal) tripotents in $\mathbf{E}$. Therefore (by Kaup's description of the extreme points of $\mathbf{B}$), all the vectors $u_{\zeta_1,\ldots,\zeta_r} := \sum_{k=1}^r \zeta_k U x_k$ with $|\zeta_k| = 1$ are extreme points of $\mathbf{B}$ with $\mathrm{Face}(\mathbf{B}, u_{\zeta_1,\ldots,\zeta_r}) = \big\{u_{\zeta_1,\ldots,\zeta_r}\big\}$. According to the last corollary of the previous section, $\Omega(u_{\zeta_1,\ldots,\zeta_r}) = \sum_{n=0}^{\infty} \Omega_n(u_{\zeta_1,\ldots,\zeta_r}) \in \bigcap_{L\in\mathcal{S}(u_{\zeta_1,\ldots,\zeta_r})} \ker(L) = \{0\}$ implying even $\Omega\left(\sum_{k=1}^r \zeta_k U x_k\right) = 0$ for $|\zeta_1|, \ldots, |\zeta_r| \leq 1$. Since every point of the ball $\mathbf{B}$ is a *finite* linear combination of extreme points (because $\mathbf{E}$ is of finite rank), necessarily $\Phi = U|\mathbf{B}$ is a linear isometry. Observe that range(U) is a subtriple of $\mathbf{E}$: if $y = Ux$ then $x = \sum_{k=1}^r \zeta_k e_k$ with suitable orthogonal min tripotens e_k; by the lemma, also $f_k := Ue_k$ are orthogonal tripotens and hence $\{yy^*y\} = \big\{(\sum_k \zeta_k f_k)(\sum_k \zeta_k f_k)^*(\sum_k \zeta_k f_k)\big\} = \sum_k |\zeta_k|^2 \zeta_k f_k \in U\mathbf{E}$.

It is well-known [Kaup, Horn] that linear isometries between JB*-triples are triple product homomorphisms.

Lemma. An endomorphism $U \in \mathcal{L}(\mathbf{E})$ of the triple product maps Cartan factors of $\mathbf{E}$ into Cartan factors.

Proof. First observe that any minimal tripotent (atom) e of $\mathbf{E}$ is mapped into a minimal tripotent by U and Ue belongs to some Cartan factor of $\mathbf{E}$. Indeed, we can find a maximal Jordan-orthogonal system $e_1, \ldots, e_r$ (where $r = \mathrm{rank}(\mathbf{E})$) of minial tripotents with $e = e_1$. The vectors Ue_k form again a maximal Jordan-orthogonal system of (necessarily minimal) tripotents by the definition of rank($\mathbf{E}$). The stetement follows hence because the factor components of any tripotent form a Jordan-orthogonal system of tripotents.

Let $\mathbf{F}$ be a Cartan factor of $\mathbf{E}$ and consider two minimal tripotents in $e_1, e_2 \in \mathbf{F}$. It suffices to see that Ue_1 and Ue_2 belong to the same Cartan factor of $\mathbf{E}$. Suppose the contrary. Then we wotld have $Ue_1 \in \mathbf{F}_1 \perp \mathrm{Jordan}\mathbf{F}_2 \ni Ue_2$ with some Cartan factors $\mathbf{F}_1 \neq \mathbf{F}_2$. However, even if $e_1 \perp^{\mathrm{Jordan}} e_2$, there exists a minimal tripotent $f \in \mathbf{F}$ with $f \not\perp^{\mathrm{Jordan}} e_1, e_2$. (this can be seen elementarily, knowing the structures of Cartan factors) and the relations lead to the contradiction $Ue_k \not\perp^{\mathrm{Jordan}} Uf$ implying $Ue_k, f \in \mathbf{F}_k$ $(k = 1, 2)$.

Corollary. Given a strongly continuous one-parameter family (not necessarily semigroup) $[U_t : t \in \mathbf{R}_+]$ of linear maps in $\mathrm{Iso}(d_{\mathbf{B}})$ (thus necessarily $\{\ldots\}$-homomorphisms), there exists $\varepsilon > 0$ such that $U_t\mathbf{F}$ $t \in [0, \varepsilon]$ for every Cartan factor of $\mathbf{E}$.

Proof. $\mathbf{E}$ is a finite Jordan-orthogonal direct sum of its Cartan factors. Let $\mathbf{F}$ be any of them and consider any minimal tripotent $(0 \neq)e \in \mathbf{F}$. Since each U_t is a $\{\ldots\}$-homomorphism, the vectors U_te are minimal tripotents. By assumption $U_te \to e = U_0e$ $(t \searrow 0)$. Therefore there exists $\varepsilon_{\mathbf{F},e} > 0$ with $U_te \not\perp^{\mathrm{Jordan}} e$ $(t \in [0, \varepsilon_{\mathbf{F},e}])$. Proof:

$\{[U_t e][U_t e]e\} \to \{eee\} = e \neq 0$ as $t \searrow 0$. As we have noticed, non-orthogonal minimal tripotents belong to the same Cartan factor. In particular $U_t e \in \mathbf{F} \quad (t \in [0, \varepsilon_{\mathbf{F},e}])$. Since each U_t maps Cartan factors into Cartan factors, hence also $U_t \mathbf{F} \subset \mathbf{F} \quad (t \in [0, \varepsilon_{\mathbf{F},e}])$. Qu.e.d.

Question. Can we extend the arguments to ℓ^∞-sums of finite rank Cartan factors?

Counter-example. $\mathbf{E} := c_0 \Big(= \big\{ (\zeta_0, \zeta_1, \ldots) : \mathbf{C} \ni \zeta_n \to 0 \big\} \Big), \quad \big\| (\zeta_0, \zeta_1, \ldots) \big\| := \max_n |\zeta_n|$

with $d_{\mathbf{B}}\big((\zeta_0, \zeta_1, \ldots), (\eta_0, \eta_1, \ldots) \big) = \max_n d_\Delta(\zeta_n, \eta_n)$.

Let $\Phi(\zeta_0, \zeta_1, \ldots) := (\zeta_0^2, \zeta_0, \zeta_1, \ldots)$.

Clearly $\Phi : \mathbf{B} \to \mathbf{B}$ holomorphically, with $\Phi(0) = 0$. Since $\zeta \mapsto \zeta^2$ is d_Δ-contractive,

$$d_{\mathbf{B}}\big(\Phi(\zeta_0, \zeta_1, \ldots), \Phi(\eta_0, \eta_1, \ldots) \big) = \max \big\{ d_\Delta(\zeta_0^2, \eta_0^2), \max_n d_\Delta(\zeta_n, \eta_n) \big\} =$$

$$= \max_n d_\Delta(\zeta_n, \eta_n) = d_{\mathbf{B}}\big((\zeta_0, \zeta_1, \ldots), (\eta_0, \eta_1, \ldots) \big).$$

Non-commutative version. $\mathbf{E} := \mathcal{L}(\mathbf{H})$, $\{e_0, e_1, \ldots\}$ orthn.basis in $\mathbf{H}$,

$\Phi(x) := (pxp)^2 + uxu^*$ where $u : e_0 \mapsto e_1 \mapsto \cdots$ unilateral shift, $p := \mathrm{Proj}_{\mathbf{C}e_0}$.

$\Phi(x)$ is reduced by the subspace $\mathbf{K} := \mathrm{Span}_{n>0} e_n$

i.e. $pxp : \mathbf{C}e_0 = \mathbf{K}^\perp \to \mathbf{K}^\perp$, $\mathbf{K} \to 0$ and $uxu^* : \mathbf{K} \to \mathbf{K}$, $\mathbf{K}^\perp \to 0$.

It follows $\|\Phi(x)\| = \max\{ \|(pxp)^2\|, \|uxu^*\| \} = \|x\|$.

Matrix form (wrt. $[e_k]_{k=0}^\infty$): for $x := \big[\xi_{k,\ell} \big]_{k,\ell=0}^\infty$, $\Phi(x) = \begin{bmatrix} \xi_{00}^2 & 0 & 0 & \cdots \\ 0 & \xi_{00} & \xi_{01} & \cdots \\ 0 & \xi_{10} & \xi_{11} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$.

MÖBIUS TRANSFORMATIONS

Definition. The *Möbius transformations* are maximal holomorphic continuations

of holomorphic automorphisms of the unit ball $\mathbf{B}$ of a JB*-triple $(\mathbf{E}, \{\ldots\})$

$\Phi \in \mathrm{Aut}(\mathbf{B})$ extends holomorphically to a neighborhood of $\overline{\mathbf{B}}$.

Canonical form [Kaup MathZ. 1983]: $\quad \Phi = M_a \circ U$

$M_a(x) = a + \mathrm{Bergman}(a)^{1/2}[1 + L(x,a)]^{-1}x, \;\; U$ surj.lin $\mathbf{E}$-isom.

Faces: If $\mathbf{E}$ JBW*-triple and $\mathbf{F}$ is a (norm-exposed) face of $\partial\mathbf{B}$ then

$\exists\, e$ TRIP in $\mathbf{E} \quad \mathbf{F} = \big\{ x \in \partial\mathbf{B} : \; x - e \perp e \big\} = \big\{ M_c(e) : \; c \perp e, \; \|c\| \leq 1 \big\}.$

Tripotents: $e = \{eee\} \in \partial\mathbf{B}$

Möbius equivalence: $\quad \Phi \sim \Psi \quad$ if $\exists\, \Theta$ Möbius trf. with $\Psi = \Theta \circ \Phi \circ \Theta$

Definition. In general, $\quad \mathrm{Iso}_h(\mathbf{D}) := \big\{$holomorphic $d_{\mathbf{D}}$-isometries$\big\}$.

Remark. [Vesentini, 1980] $\Rightarrow \quad \big\{ \Theta|_{\mathbf{B}} : \; \Theta$ Möbius trf.$\big\} = \big\{ \Phi \in \mathrm{Iso}_h(\mathbf{B}) : \; \phi(\mathbf{B}) = \mathbf{B} \big\}$

Proposition. The 0-preserving holomorphic Carathéodory isometries Θ of $\mathbf{B}$ are linear

provided $\quad \mathrm{range}(\Theta) \subset \mathrm{range}\big(D_{z=0}\Theta(z)\big)$.

Proof. Let $\Theta := U + \Omega \in \mathrm{Isom}(d_{\mathbf{B}})$ where U is linear and Ω is holomorphic with Taylor series $\Omega(x) = \sum_{n=2}^{\infty} \Omega_n(\underbrace{x, \ldots, x}_{n})$ around 0. For any vector $v \in \mathbf{B}$ we have $d_{\mathbf{B}}(0,v) = \mathrm{artanh}\|v\|$ and $d_{\mathbf{B}}(0,v) = d_{\mathbf{B}}(0, \Theta(v))$ implying $\|v\| = \|\Theta(v)\|$. Hence, for any $v \in \mathbf{E}$ with $t \searrow 0$ we get

$$\|v\| = t^{-1}\|tv\| = t^{-1}\|U(tv) + \Omega(tv)\| = \|t^{-1}U(tv) + t^{-1}\Omega(tv)\| = \|Uv + t^{-1}o(t^2)\| = \|Uv\|.$$

Since $\mathrm{range}(\Theta) \subset \mathrm{range}(U)$, the mapping $\Psi := U^{-1}\Theta$ is a well-defined holomorphic 0-preserving Carathéodory isometry of $\mathbf{B}$ with $D_{z=0}\Psi(z) = U^{-1}U = 1(= \mathrm{id}_{\mathbf{E}})$. According to Cartan's Uniqueness Theorem, $\Psi = \mathrm{id}_{\mathbf{B}}$.

Remark. $\mathrm{Iso}_h(\mathbf{B}) \supset \big\{M_a \circ U : a \in \mathbf{B},\ U \text{ lin. } \mathbf{E}\text{-isom.}\big\}$ since both Möbius transformations and linear isometries are $d_{\mathbf{B}}$-preserving.

Remark. If V is a linear $\mathbf{E}$-isometry and $a \in \mathbf{B}$ then

$V \circ M_a = M_{V_a} \circ \underbrace{M_{Va}^{-1} \circ V \circ M_a}_{0 \mapsto 0} = M_{V_a} \circ U$ with the linear $\mathbf{E}$-isometry

$$U := D_{z=0}\big[M_{Va}^{-1} \circ V \circ M_a\big] = \big[D_{z=0}M_{Va}(z)\big]^{-1} V \big[D_{z=0}M_a(z)\big] =$$

$$= \mathrm{Bergman}(Va)^{-1/2} V \mathrm{Bergman}(a)^{1/2}.$$

C_0-SEMIGROUPS IN $\mathrm{Iso}_h(d_{\mathbf{B}})$ FOR REFLEXIVE JB*-TRIPLES

Assumption 0:

We consider strongly cont. 1-pr.*semigroups*

$[\Phi^t : t \in \mathbf{R}_+]$, $\quad \Phi^t = M_{a(t)} \circ U_t$, $\ U_t : \mathbf{E} \to \mathbf{E}$ lin. isometry, $\quad$ such that

(1) $\mathrm{dom}(\Phi') \cap \mathbf{B} \neq \emptyset$ or (up to Möbius equ.) $\underline{0 \in \mathrm{dom}(\Phi'),\ t \mapsto a(t) \text{ diff.}}$

Lemma. $x \in \mathrm{dom}(\Phi') \iff t \mapsto U_t x$ diff. $\qquad \big(U_h x \in \mathrm{dom}(\Phi')\big)$.

Proof. $U_t x = M_{-a(t)} \underbrace{\Phi^t}_{M_{a(t)} \circ U_t} (x)$. $\qquad (a,z) \mapsto M_a(z)$ real-anal.

$M_{c+hv+o(h)}(u + hw + o(h)) =$

$= (c+hv+o(h)) + B(c+hv+o(h))^{1/2}\big(1+L(u+hw+o(h), c+hv+o(h))\big)^{-1}(u+hw+o(h)) =$

$= M_c(u) - h(L(w,c) + L(u,v))u + h\big(1 + L(u,c)\big)^{-1} w + o(h)$.

Assumption 1: Hencforth $(\mathbf{E}, \{\ldots\})$ is a *reflexive* JB*-triple.

Remark. Reflexive JB*-triples are finite direct sums of copies of spin factors, $\mathcal{L}(\mathbf{H}_1, \mathbf{H}_2)$ spaces with $\dim(\mathbf{H}_2) < \infty$ and some finite dimensional Cartan factors.

(?) A str.cont. family $[V_t : t \in \mathbf{R}_+]$ with $V_0 = \mathrm{id}$ of lin. isometries $\mathbf{E} \to \mathbf{E}$ maps each factor into itself.

Lemma. The linear isometries of a *spin factor* $\mathbf{E}$ are necessarily JB*-endomorphisms.

Proof. This is contained implicitly in [Apazoglou-Peralta, Quart. J. Math. 65 (2014), 485–503] (even for real setting). Actually there is a simple geometric argument based on the well-known facts [Neher, Edwards] that

1) any $v \in \mathbf{E}$ is a real-linear combination of an orthogonal couple of minimal tripotens, and the JB*-subtriple $\mathcal{C}_0(v)$ generated by v is their ($\mathbf{C}$)-linear span.

2) $e \in \mathbf{E}$ is a minimal tripotent iff $e = a + ib$ with $a, b \in \mathrm{Re}(\mathbf{E})$, $\langle a|b\rangle = 0$, $\langle a\rangle^2 = \langle b\rangle^2 = 1/2$,

2′) e, f is an orthogonal couple of minimal tripotens iff

$e = a + ib, f = a - ib \quad$ with $\quad a, b \in \mathrm{Re}(\mathbf{E})$, $\langle a|b\rangle = 0$, $\langle a\rangle^2 = \langle b\rangle^2 = 1/2$,

3) the (norm exposed) faces of $\mathbf{B}$ are either extreme points or 1-dimensional closed discs of the form $\mathbf{F} = \{e + \zeta f : |\zeta| \le 1\}$ with an orthogonal couple of minimal tripotens.

Thus, given an isometry $U \in \mathcal{L}(\mathbf{E})$, by 1), it suffices to see that the U preserves the linear spans of orthogonal couples of minimal tripotents. Let e, f be an orthogonal couple of minimal tripotents and consider the face $\mathbf{F} := \{e + \zeta f : |\zeta| \le 1\}$. Since U is a linear isometry, $U\mathbf{F}$ is a 1-dimensional disc with radius 1 in the unit sphere $\partial\mathbf{B}$. Thus, according to 3), $U\mathbf{F}$ is also a face of $\mathbf{B}$ and therefore $U\mathbf{F} = \{\widetilde{e} + \zeta\widetilde{f} : |\zeta| \le 1\}$ for some orthogonal couple of minimal tripotents $\widetilde{e}, \widetilde{f}$. The middle point e of $\mathbf{F}$ is mapped into the middle point of $U\mathbf{F}$ whence necessarily $\widetilde{e} = Ue$. On the other hand, $\widetilde{f} = (\widetilde{e} + \widetilde{f}) - \widetilde{e} \in \mathbf{F} - \mathbf{F} \subset \mathrm{range}(U)$. Hence the statement is immediate. Qu.e.d.

Proposition. The the factor preserving linear isometries $\mathbf{E} \to \mathbf{E}$ of any reflexive JB*-triple $\mathbf{E}$ are JB*-homomorhisms.

Proof. 1) The linear isometries of finite dimensional factors are surjective and hencwe necessarily automorphisms of the triple product.

2) [Vesentini 1994] established that, for $\mathbf{E} = \mathcal{L}(\mathbf{H}_1, \mathbf{H_2})$ with $\dim(\mathbf{H}_2) < \infty$ we have

$\mathrm{Iso}(d_{\mathbf{B}}) \cap \{L|\mathbf{B} : L \in \mathbf{E}\} = \big\{[X \mapsto uXv] : u, v \text{ linear isometries}\big\}$.

3) The case of spin factors is setteled by the previous Lemma. Qu.e.d.

Corollary. $\mathrm{dom}(\Phi')$ is closed with respect to the Jordan-prod. $\{\ldots\}$

Proof. $x, y, z \in \mathrm{dom}(\Phi') \Rightarrow t \mapsto U_t\{xyz\} = \{(U_t x)(U_t y)(U_t z)\}$ diff.

Remark: In particular $\mathrm{dom}(\Phi') = \big[\text{Jordan subtriple}\big] \cap \overline{\mathbf{B}}$ and $\{\Phi^t(0) : t \in \mathbf{R}\} \subset \mathrm{dom}(\Phi')$.

Lemma. $x \in \mathrm{dom}(\Phi') \Longrightarrow U_h x \in \mathrm{dom}(\Phi'\ (h \in \mathbf{R})$.

Proof. $U_h x \in \mathrm{dom}(\Phi' \iff t \mapsto U_h U_t x$ diff.

$\Phi^{t+h}(x) = \Phi^t \circ \Phi^h(x) = M_{a(t)} \circ U_t \circ M_{a(h)} \circ U_h x =^{U \circ M_a \circ U^{-1} = M_{Ua}}$

$= M_{a(t)} \circ M_{U_t a(h)} \circ U_t U_h x.$

$U_t U_h x = M_{-U_t a(h)} \circ M_{-a(t)} \circ \Phi^{t+h}(x), \qquad a(h) \in \mathrm{dom}(\Phi') \ \Rightarrow\ t \mapsto U_t a(h)$ diff.

$t \mapsto \Phi^t$ diff., $t \mapsto a(t)$ diff., $(a, b) \mapsto M_a \circ M_b$ real-anal. ; $\Longrightarrow t \mapsto U_t U_h x$ diff.

Notation: $\mathbf{D} := \overline{\mathrm{dom}(\Phi')}$ closure in $\mathbf{E}$, $\quad \mathbf{F} := \mathrm{Span}(\mathbf{D})$

Proposition. We have seen: $\mathbf{F}$ closed JB*-subtriple in $\mathbf{E}$, $\quad \mathbf{D} = \overline{\mathrm{Ball}(\mathbf{F})}$,

$$\big\{U_t|\mathbf{F} : t \in \mathbf{R}\big\} \subset \mathrm{Aut}(\mathbf{F}, \{\ldots\}), \quad \big\{M_{a(t)}|\mathbf{D} : t \in \mathbf{R}\big\} \subset \mathrm{Aut}_{\mathrm{hol}}(\mathbf{D}).$$

Remark. In case of groups $[\Phi^t : t \in \mathbf{R}]$,

$$\begin{aligned} [\Phi^t]^{-1} = \Phi^{-t} &\iff U_t^{-1} M_{-a(t)} = M_{a(-t)} \circ U_{-t} \\ &\iff M_{-U_t^{-1} a(t)} \circ U_t^{-1} = M_{a(-t)} \circ U_{-t} \\ &\iff U_t^{-1} = U_{-t} \text{ and } -U_t^{-1} a(t) = a(-t). \end{aligned}$$

Lemma. $\mathbf{F}^{\perp\ \mathrm{Jordan}} = 0$.

Proof. Given $\Phi^t = M_{a(t)} \circ U_t$, we have $M_{a(t)}|\mathbf{F} \cap \mathbf{B} = \mathrm{id}$ and $U_t : \mathbf{F} \to \mathbf{F}$ for every $t \in \mathbf{R}_+$. Hence $U_{t+h}|\mathbf{F} = [U_t|\mathbf{F}] \circ [U_h|\mathbf{F}]$ $(t, h \in \mathbf{R}_+)$. Thus $[U_t|\mathbf{F} : t \in \mathbf{R}_+]$ is a str.conr. 1-pr. semigroup and, by the Hille-Yosida theorem, the generator $\Phi'|\mathbf{F} = U'|\mathbf{F}$ is dense in $\mathbf{F}$. By definition, $\Phi'|\mathbf{F} = \{0\}$, which is possible only if $\mathbf{F} = \{0\}$.

Assumption 2 (without loss of generality for reflexive $\mathbf{E}$):

(2) $e = \Phi^t(e) \ \ \forall\, t \in \mathbf{R}_+$ common fixed point

$\Lambda^t := \mathrm{D}_e\Phi^t \ \Big(: z \mapsto \frac{d}{dt}\big|_{t=0}\Phi^t(e+tz) \Big)$ Fréchet derivative

$\Lambda_t z = (2\pi i)^{-1}\int_{|\zeta|=1}\zeta^{-1}\Phi^t(e+\zeta z)d\zeta$ with $z \in \mathbf{B}$, $\mathrm{dom}\big(M_{a(t)}\big) \supset 2\overline{\mathbf{B}}$.

$[\Lambda^t : t \in \mathbf{R}]$ str.cont.1prg LIN $\mathbf{Z} := \mathrm{dom}(\Lambda')$ dense lin. in $\mathbf{E}$

$\Phi = M_a U \ (= M_a \circ U)$ t FIX, $w := w(z) = \Phi(z) - e$

$$w + e = \Phi(e+z) = M_a(Uz + Ue)$$

$$w + e = a + B(a)^{1/2}[1 + L(Ue+Ue,a)]^{-1}(Uz+Ue)$$

$$[1 + L(Uz+Ue,a)]B(a)^{-1/2}(w + (e-a)) = Uz + Ue$$

$$\Phi(e) = e \iff [1 + L(Ue,a)]B(a)^{-1/2}(e-a) = Ue$$

$$[1 + L(Uz+Ue,a)]B(a)^{-1/2}(w+(e-a)) - [1+L(Ue,a)]B(a)^{-1/2}(e-a) = Uz$$

$$[1 + L(Uz+Ue,a)]B(a)^{-1/2}w + L(Uz,a)B(a)^{-1/2}(e-a) = Uz$$

$$w = B(a)^{1/2}[1 + L(Uz+Ue,a)]^{-1}[Uz - L(Uz,a)B(a)^{-1/2}(e-a)]$$

$$\Phi(z+e) - e = w = (A_z + B)^{-1}Cz$$

$$A_z = L(Uz,a)B(a)^{-1/2}, \quad B = [1 + L(Ue,a)]B(a)^{-1/2}, \quad C = U + L(U\bullet,a)B(a)^{-1/2}(a-e)$$

$$\Lambda z = \mathrm{D}_e\Phi = \frac{d}{dz}\Big|_{z=0}(A_z+B)^{-1}Cz = B^{-1}Cz$$

Proposition. As a consequence, under hypothesis (0)+(3) we have

$$\Phi^t(z+e) - e = B(a_t)^{1/2}[1 + L(U_tz + U_te, a_t)]^{-1}[U_tz + L(U_tz,a_t)B(a_t)^{-1/2}(a_t - e)],$$

$\Lambda^t z = B(a_t)^{1/2}[1 + L(U_t e, a_t)]^{-1}[U_t z + L(U_t z, a_t)B(a_t)^{-1/2}(a_t - e)]$.

$\Lambda^t e = B(a_t)^{1/2}[1 + L(U_t e, a_t)]^{-1}[U_t e + L(U_t e, a_t)B(a_t)^{-1/2}(a_t - e)]$

Proposition $\Longrightarrow$ 1) $t \mapsto U_t z$ diff. $\Rightarrow t \mapsto \Lambda^t z$ diff.

2) $t \mapsto \Lambda^t z$ diff. $\Rightarrow t \mapsto U_t z$ diff. at 0

Proof:

$[1 + L(U_t e, a_t)]B(a_t)^{1/2}\Lambda^t z = U_t z + L(U_t z, a_t)B(a_t)^{-1/2}(a_t - e)$

$U_t z = [1 + L(U_t e, a_t)]B(a_t)^{1/2}\Lambda^t z - L(U_t z, a_t)B(a_t)^{-1/2}(a_t - e)$

Suppose $z \in \mathrm{dom}(\Lambda')$ i.e. $\frac{d}{dt}\big|_{t=0+}\Lambda^t$ exits and

$\Lambda^t z = z + tz' + o(t)$ $(t \searrow 0)$ for some $z' \in \mathbf{E}$

We know also: $U_t e = e + te' + o(t),\;\; a_t = ta' + o(t),\;\; U_t z = z + o(1)$

Thus

$$U_t z = \Big[1 + L\big(e + te' + o(t), ta' + o(t)\big)\Big]\big[1 + o(t)\big]\big(z + tz' + o(t)\big) -$$

$$-L\big(z + o(1), ta' + o(t)\big)\big[1 + o(t)\big]\big(ta' + o(t) - e\big) =$$

$$= z + tL\big(z, a')z + tL\big(z, a'\big)e + o(t)$$

$$\Big[\mathrm{Id} + L\big(e + te' + o(t), ta' + o(t)\big)\Big]\big[\mathrm{Id} + o(t)\big]\big(z + tz' + o(t)\big) =$$

$$= U_t z + L\big(z + o(1), a + ta' + o(t)\big)\big[\mathrm{Id} + o(t)\big](ta' - e)$$

$$\Big[1 + L\big(e + te', ta'\big)\Big]\big(z + tz'\big) + o(t) = U_t z + L(z + o(1), ta')(ta' - e) + o(t)$$

$$\Big[1 + tL\big(e, a'\big) + t^2 L\big(e', a'\big)\Big]\big(z + tz'\big) + o(t) =$$

$$= U_t z + t^2 L(z, a')a' + tL(o(t), a') - tL(U_t z, a')e + o(t)$$

$$z + tz' + tL\big(e, a'\big)z + o(t) = U_t z - tL(U_t z, a')e + o(t)$$

Assumption 3:

(3) $\quad e \in \mathbf{Z} = \mathrm{dom}(\Lambda'), \qquad t \mapsto \Lambda^t e \;$ diff.

Remark. We intend to see: $\;(0) + (2) \Rightarrow (3)\;$ up to Möbius equiv.

$$\Lambda^t e = B(a_t)^{1/2}[1 + L(U_t e, a_t)]^{-1}[U_t e + L(U_t e, a_t)B(a_t)^{-1/2}(a_t - e)]$$

e FIXP (2): $\quad e = \Phi^t(e) = M_{a_t}(U_t e) = a_t + B(a_t)^{1/2}[1 + L(U_t e, a_t)]^{-1}U_t e$

$$\Lambda^t e = e - a_t + B(a_t)^{1/2}[1 + L(U_t e, a_t)]^{-1}L(U_t z, a_t)B(a_t)^{-1/2}(a_t - e) =$$

$$= B(a_t)^{1/2}\big\{ - 1 + [1 + L(U_t e, a_t)]^{-1}L(U_t z, a_t)\big\}B(a_t)^{-1/2}(a_t - e) =$$

$$= B(a_t)^{1/2}[1 + L(U_t e, a_t)]^{-1}\big\{ - 1 - L(U_t z, a_t) + L(U_t z, a_t)\big\}B(a_t)^{-1/2}(a_t - e) =$$

$$= B(a_t)^{1/2}[1 + L(U_t e, a_t)]^{-1}B(a_t)^{-1/2}(e - a_t)$$

Another formula for $\Lambda^t e$:

$$\Phi^t(e) = e \implies \quad e = a_t + B(a_t)^{1/2}[1 + L(U_t e, a_t)]^{-1}U_t e$$

$$a_t - e = -B(a_t)^{1/2}[1 + L(U_t e, a_t)]^{-1}U_t e$$

$$\Lambda^t e =$$

$$= B(a_t)^{1/2}[1+L(U_t e, a_t)]^{-1}[U_t e+L(U_t e, a_t)B(a_t)^{-1/2}(-B(a_t)^{1/2})[1+L(U_t e, a_t)]^{-1}U_t e] =$$

$$= B(a_t)^{1/2}[1 + L(U_t e, a_t)]^{-1}[1 - L(U_t e, a_t)[1 + L(U_t e, a_t)^{-1}]U_t e =$$

$$= \underline{B(a_t)^{1/2}[1 + L(U_t e, a_t)]^{-2}U_t e}$$

since $1 - L(1 + L)^{-1} = (1 + L)^{-1}[(1 + L) - L] = (1 + L)^{-1}$.

Question: $\;(3) \;\Rightarrow^{?}\; (2) \qquad t \mapsto \Lambda^t e \;$ diff. $\Rightarrow^{?}\; t \mapsto a_t \;$ diff.

$U_t e = M_{a_t}^{-1}(e) = M_{-a_t}(e) \ \Big(= -a_t + B(a_t)^{1/2}[1 - L(e, a_t)]^{-1} e \ \Big)$

Define: $\quad F(a) := B(a)^{1/2}[1 + L(M_{-a}(e), a)]^{-1} B(a)^{-1/2}(e - a)$

Proposition. (2)+(3) $\Rightarrow \mathrm{dom}(\Phi') = \big[$dense Jordan subtriple $\cap \overline{\mathbf{B}}\big]$.

Proof. $\quad F$ real-analytic, $\quad \Lambda^t e = F(a_t)$.

Lemma 1. $(2) + (3) \Rightarrow 0 \in \mathrm{dom}(\Phi')$.

Proof 1: $\quad$ For $\ a \to 0\ $ we have

$$B(a) = 1 - 2L(a, a) + Q_a^2 = 1 + O(\|a\|^2) = 1 + o(\|a\|) \ \text{ wrt. norm in } \mathcal{L}(\mathbf{E})$$

$$B(a)^{\pm 1/2} = 1 + o(\|a\|)$$

$$M_{-a}(e) = -a + B(a)^{1/2}[1 - L(e, a)]^{-1} e = -a + [1 - L(e, a)]^{-1} e + o(\|a\|) =$$

$$= -a + [1 + L(e, a)]e + o(\|a\|) = -a + \{eae\} + o(\|a\|)$$

$$F(a) = [1 + L(M_{-a}(e), a)](e - a) + o(\|a\|) =$$

$$= [1 + L(-a + Q_e a, a)](e - a) + o(\|a\|) = e - a + o(\|a\|)$$

Implicit Funct. Thm. $\Longrightarrow \quad F$ is invertible real-analytically in a nbh. of $a = 0$

$t \mapsto a_t = \Phi^t(0)$ diff. at $t = 0 \Longrightarrow t \mapsto a_t$ diff. $\qquad$ Q.e.d.

Strategy. Assume $c \in \mathbf{B}$, $V \in \mathcal{L}(\mathbf{E})$ unitary. Let

$$\Theta := M_c \circ V, \quad \widetilde{\Phi}^t := \Theta^{-1} \circ \Phi^t \circ \Theta,$$

$$\widetilde{a}_t := \widetilde{\Phi}^t(0), \quad \widetilde{e} := \Theta^{-1}(e), \quad \widetilde{\Lambda}^t := \mathrm{D}_{\widetilde{e}} \widetilde{\Phi}^t : v \mapsto \tfrac{d}{ds}\big|_{s=0} \widetilde{\Phi}^t(\widetilde{e} + sv).$$

We know: $\ t \mapsto \widetilde{a}_t$ diff. $\iff \ t \mapsto \widetilde{\Lambda}^t \widetilde{e}$ diff. $\quad$ Try to find a suitable Θ with

$t \mapsto \widetilde{\Lambda}^t \widetilde{e}$ diff. so that we have properties (2),(3) for $[\widetilde{\Phi}^t : t \in \mathbf{R}_+]$, on the basis of the fact that $\mathrm{dom}(\Lambda')$ is a dense linear submanifold in $\mathbf{E}$.

Lemma 2: $t \mapsto \widetilde{\Lambda}^t \widetilde{e}$ diff. $\iff$ $[\mathrm{D}_{M_c(e)}] M_{-c}(e) \in \mathrm{dom}(\Lambda')$.

Thus, if $[\mathrm{D}_{M_c(e)}] M_{-c}(e) \in \mathrm{dom}(\Lambda')$ for some $c \in \mathbf{B}$ then $[\Phi^t : t \in \mathbf{R}_+]$ is Möbius equivalent to str.cont.1pr. semigroup $[\widetilde{\Phi}^t : t \in \mathbf{R}_+]$ with $\mathrm{Fix}[\widetilde{\Phi}^t : t \in \mathbf{R}_+] \neq \emptyset$ and $t \mapsto \widetilde{\Phi}^t(0)$ diff. and, in particular, $\mathrm{dom}(\Phi')$ dense in the ball $\mathbf{B}$, which completes the proof of the Proposition.

Proof 2: $\widetilde{\Phi}^t(\widetilde{e}) = \Theta^{-1}\Phi^t\Theta(\Theta^{-1}(e)) = \Theta^{-1}\Phi^t(e) = \widetilde{e}$

$\widetilde{\Lambda}^t = \mathrm{D}_{\widetilde{e}}\widetilde{\Phi}^t = \mathrm{D}_{\Theta^{-1}(e)}\big[\Theta^{-1}\Phi^t\Theta\big] =^{\text{chain rule}}$

$= \big[\mathrm{D}_{\Phi^t\Theta(\widetilde{e})}\Theta^{-1}\big]\big[\mathrm{D}_{\Theta(\widetilde{e})}\Phi^t\big]\big[\mathrm{D}_{\widetilde{e}}\Theta\big]$

$\Theta : \widetilde{e} \mapsto e, \quad \Theta^{-1} : e \mapsto \widetilde{e}, \qquad \mathrm{D}_{\widetilde{e}}\Theta = \big[\mathrm{D}_e\Theta^{-1}\big]^{-1}$

$\mathrm{D}_{\Theta(\widetilde{e})}\Phi^t = \mathrm{D}_e\Phi^t = \Lambda^t, \qquad \mathrm{D}_{\Phi^t\Theta(\widetilde{e})}\Theta^{-1} = \mathrm{D}_{\Phi^t(e)}\Theta^{-1} = \mathrm{D}_e\Theta^{-1}$

$\widetilde{\Lambda}^t = \big[\mathrm{D}_e\Theta^{-1}\big]\Lambda^t\big[\mathrm{D}_e\Theta^{-1}\big]^{-1} = \big[V^{-1}\mathrm{D}_e M_{-c}\big]\Lambda^t\big[V^{-1}\mathrm{D}_e M_{-c}\big]^{-1}$

$\widetilde{\Lambda}^t\widetilde{e} = V^{-1}[\mathrm{D}_e M_{-c}]\Lambda^t[\mathrm{D}_e M_{-c}]^{-1}VV^{-1}M_{-c}(e) = V^{-1}[\mathrm{D}_e M_{-c}]\Lambda^t[\mathrm{D}_e M_{-c}]^{-1}M_{-c}(e)$

$[\mathrm{D}_e M_{-c}]^{-1} =^{[\mathrm{D}_p F]^{-1} = \mathrm{D}_{F(p)}F^{-1}} = \mathrm{D}_{M_{-c}(e)}M_c$

Hence $\widetilde{\Lambda}^t\widetilde{e} = [\mathrm{LINOP}]\Lambda^t[\mathrm{D}_{M_{-c}(e)}M_c]M_{-c}(e) \Rightarrow$ statement Qu.e.d.

Remark. Analogously as the underlined formula for $\Lambda^t e$ was obtained, we get

$[\mathrm{D}_{M_{-c}(e)}M_c]M_{-c}(e) = [\mathrm{D}_f M_c]f = \frac{d}{ds}\big|_{s=0}M_c(f+sf) = \frac{d}{ds}\big|_{s=1}M_c(sf) =$

$= \frac{d}{ds}\big|_{s=1}\big\{c + B(c)^{1/2}[1+L(sf,c)]^{-1}sf\big\} = B(c)^{1/2}[1+L(f,c)]^{-2}f =$

$= \underline{B(c)^{1/2}[1+L(M_{-c}(e),c)]^{-2}M_{-c}(e)}$

Since $\mathrm{dom}(\Lambda')$ is dense in $\mathbf{E}$, if the Fréchet derivative $\mathrm{D}_c G(c) = \big[v \mapsto \frac{d}{ds}\big|_{s=0}G(c+sv)\big]$

with $G(c) := B(c)^{1/2}[1+L(M_{-c}(e),c)]^{-2}M_{-c}(e)$ is an invertible operator for some $c \in \mathbf{B}$ then $\mathrm{ran}(G) \cap \mathrm{dom}(\Lambda') \neq \emptyset$ implying that $[\Phi^t : t \in \mathbf{R}_+]$ is Möbius equivalent to some str.cont. 1pr.sg. with properties (2)+(3)

Corollary. We have

$0 \in \mathrm{dom}(\Phi') \iff \exists\, e \in \mathrm{Fix}(\Phi) \quad e \in \mathrm{dom}\underbrace{[D_e\Phi]}_{\Lambda}{}' \iff \forall\, e \in \mathrm{Fix}(\Phi) \quad e \in \mathrm{dom}\underbrace{[D_e\Phi]}_{\Lambda}{}'$.

Therefore $\quad c = M_c(0) \in \mathrm{dom}(\Phi') \iff 0 \in \mathrm{dom}[M_{-c}\circ\Phi\circ M_c]'$

because, with $M_{-c}(e) \in \mathrm{Fix}(M_{-c}\circ\Phi\circ M_c)$ we have

$c = M_c(0) \in \mathrm{dom}(\Phi') \iff [t \mapsto \Phi^t M_c(0)]$ diff. $\iff [t \mapsto M_{-c}\Phi^t M_c(0)]$ diff. $\quad$ and

$0 \in \mathrm{dom}[M_{-c}\circ\Phi\circ M_c]' \iff M_c(e) \in \mathrm{dom}\big([D_{M_{-c}(e)}M_-c\circ\Phi\circ M_c]'\big)$.

Notation. Henceforth

$$G(c) := B(c)^{1/2}[1+L(M_{-c}(e),c)]^{-2}M_{-c}(e).$$

Lemma. $\mathrm{D}_{c=0}G(c) = -[1+Q(e)]$

Proof. We have to see $\big($with real differentiation $\frac{d^+}{d\tau}\big|_0 = \frac{d}{d\tau}\big|_{\tau=0+}\big)$ that

$$\tfrac{d^+}{d\tau}\big|_0 G(\tau c) = \tfrac{d^+}{d\tau}\big|_0\big\{B(\tau c)^{1/2}[1+L(M_{-\tau c}(e),\tau c)]^{-2}M_{-\tau c}(e)\big\} = -c - \{ece\}.$$

$B(\tau c)^{1/2} = \big(1+\tau^2[-2L(c)+\tau^2 Q(c)^2]\big)^{1/2} = 1 - \frac{\tau^2}{2}[-2L(c)+\tau^2Q(c)^2] + o(\tau^2) = 1 + o(\tau)$,

$M_{-\tau c}(e) = -\tau c + B(\tau c)^{1/2}[1-\tau L(e,c)]^{-1}e =$

$$= -\tau c + [1+o(\tau)][1+\tau L(e,c)+o(\tau)]e = e + \tau[-c+L(e,c)e] + o(\tau) = e - \tau[1-Q(e)]c + o(\tau)$$

$G(\tau c) = \big\{1+o(\tau)\big\}\big\{1+\tau L\big(e-\tau[1-Q(e)]c, c\big)+o(\tau)\big\}^{-2}\big\{e-\tau[1-Q(e)]c+o(\tau)\big\} =$

$$= \big\{1-2\tau L(e,c)+o(\tau)\big\}\{e-\tau[1-Q(e)]c+o(\tau)\} = e - \tau[1-Q(e)]c - 2\tau L(e,c)e + o(\tau) =$$

$$= e - \tau\big[1 + Q(e)\big]c + o(\tau). \qquad \text{Qu.e.d.}$$

Lemma. e TRIP $\Longrightarrow \quad G(\lambda e) = \dfrac{|1-\lambda|^2}{1-|\lambda|^2}e,$

$$\big[\mathrm{D}_{\lambda e}G\big]e = -\frac{2\mathrm{Re}[(1-\lambda)^2]}{(1-|\lambda|^2)^2}e, \quad \big[\mathrm{D}_{\lambda e}G\big](ie) = \frac{4\mathrm{Re}(1-\lambda)\mathrm{Im}\lambda}{(1-|\lambda|^2)^2}e$$

Proof. Let e TRIP. With the Peirce proj. $P_k(e) : \mathbf{E} \to \mathbf{E}_k(e) := \big\{x : \{eex\} = kx/2\big\}$

$L(\lambda e) = \frac{|\lambda|^2}{2}P_1(e) + |\lambda|^2 P_2(e), \quad Q(\lambda e)^2 = |\lambda|^4 P_2(e) \quad$ whence

$B(\lambda(e)|\mathbf{E}_2(e) = [1 - 2|\lambda|^2 + |\lambda|^4]\,\mathrm{id}, \quad [1 - L(e,\lambda e)]|\mathbf{E}_2(e) = [1-\overline{\lambda}]\,\mathrm{id};$

$$M_{-\lambda e}(e) = -\lambda e + B(-\lambda e)^{1/2}[1 + L(-\lambda e)]^{-1}e = \left[-\lambda + \frac{1-|\lambda|^2}{1-\overline{\lambda}}\right]e = \frac{1-\lambda}{1-\overline{\lambda}}e,$$

$$G(\lambda e) = B(\lambda e)^{1/2}[1 + L(M_{-\lambda e}(e), e)]^{-2}M_{-\lambda e}(e) =$$

$$= (1-|\lambda|^2)\frac{(1-\lambda)/(1-\overline{\lambda})}{\big[1+\overline{\lambda}(1-\lambda)/(1-\overline{\lambda})\big]^2}\,e = \frac{(1-|\lambda|^2)(1-\lambda)(1-\overline{\lambda})}{(1-|\lambda|^2)^2}\,e$$

Thus $G(\lambda e) = g(\lambda)e \quad$ with $\quad g(\lambda) := \dfrac{(1-\lambda)(1-\overline{\lambda})}{1-\lambda\overline{\lambda}} = \dfrac{|1-\lambda|^2}{1-|\lambda|^2}.$

With straightforward calculation, $\quad \dfrac{\partial g}{\partial\lambda} = -\dfrac{(1-\overline{\lambda})^2}{(1-|\lambda|^2)^2}, \quad \dfrac{\partial g}{\partial\overline{\lambda}} = -\dfrac{(1-\lambda)^2}{(1-|\lambda|^2)^2}. \quad$ Hence

$$\big[\mathrm{D}_{\lambda e}G\big]e = \frac{d^+}{d\tau}\Big|_0 G(\lambda+\tau) = \frac{d^+}{d\tau}\Big|_0 g(\lambda+\tau)e = \frac{\partial g}{\partial x}e = \frac{\partial g}{\partial\lambda} + \frac{\partial g}{\partial\overline{\lambda}} = -2\mathrm{Re}\left(\frac{(1-\lambda)^2}{(1-|\lambda|^2)^2}\right),$$

$$\big[\mathrm{D}_{\lambda e}G\big](ie) = \frac{d^+}{d\tau}\Big|_0 G(\lambda+i\tau) = \frac{d^+}{d\tau}\Big|_0 g(\lambda+i\tau)e = \frac{\partial g}{\partial y}e = i\frac{\partial g}{\partial\lambda} - i\frac{\partial g}{\partial\overline{\lambda}} = -2\mathrm{Re}\left(i\frac{(1-\lambda)^2}{(1-|\lambda|^2)^2}\right).$$

Lemma. e TRIP, $L(e)v = \kappa v$, $Q(e)v = \varepsilon v$, $|\lambda| < 1 \quad \Longrightarrow$

for $\quad w := [\mathrm{D}_{c=\lambda e}G(c)]v \quad$ we also have $\quad L(e)w = \kappa w$, $Q(e)w = \varepsilon w$.

Proof. Let us write $\mathcal{J}_{k,\ell}$ for the family of all possible Jordan triple product expressions with k terms v and ℓ terms e. E.g.

$\mathcal{J}_{1,4} = \Big\{\{\{vee\}ee\}, \{\{eve\}ee\}, \{\{eev\}ee\}, \{e\{vee\}e\}, \ldots, \{ee\{eev\}\}\Big\}$ has 9 elements.

By definition, $\big[\mathrm{D}_{c=\lambda e}G(c)\big]v = \dfrac{d^+}{d\tau}\Big|_0 G(\lambda e + \tau v) \ =$

$$= \frac{d^+}{d\tau}\Big|_0 \Big\{ B(\lambda e+\tau v)^{1/2}\big[1+L(M_{-\lambda e-\tau v}(e),\lambda e+\tau v)\big]^{-2} M_{-\lambda e-\tau v}(e)\Big\}.$$

Observe that $B(\lambda e+\tau v)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} \big[-2L(\lambda e+\tau v)+Q(\lambda e+\tau v)^2\big]^n$ is a series of Jordan multiplications of the form $\{ee\cdot\},\{e\cdot e\}$ i.e. a power series of the commuting real linear operators $L(e),Q(e)$ acting as muliples of the identity on the Peirce spaces $\mathbf{E}_\kappa^{(\varepsilon)}(e)$.

Also in general we can write $\big[1+L(x,y)\big]^{-r} z = \sum_{n=0}^{\infty} \binom{-r}{n} L(x,y)^n z =$

$= \sum_{k,\ell=0}^{\infty} \mu_{k,\ell}^{(r)}\big[$Jordan expression with k terms x, ℓ terms y and one term $z\big]$

such that $\exists\ \delta^{(r)}>0$ with $\sum_{k,\ell=0}^{\infty} |\mu_{k,\ell}^{(r)}|\|x\|^k\|y\|^\ell<\infty$ whenever $\|x\|,\|y\|<\delta^{(r)}$.

Hence we see that $\big[\mathrm{D}_{c=\lambda e}G(c)\big]v$ admits an expansion of the form

$\big[\mathrm{D}_{c=\lambda e}G(c)\big]v = \sum_{j,k=0}^{\infty} \sum_{J\in\mathcal{J}_{k,\ell}} \gamma_J\tau^k J$

such that $\exists\ \delta>0$ with $\sum_{k,\ell=0}^{\infty} \sum_{J\in\mathcal{J}_{k,\ell}} |\gamma_J\tau^k\|v\|^k<\infty$ whenever $0\le\tau\|v\|<\delta$.

In terms of this expansion we have

$$\big[\mathrm{D}_{c=\lambda e}G(c)\big]v = \frac{d^+}{d\tau}\Big|_0 \sum_{j,k=0}^{\infty} \sum_{J\in\mathcal{J}_{k,\ell}} \gamma_J\tau^k J = \sum_{\ell=0}^{\infty} J\in\mathcal{J}_{1,\ell}\gamma_J J.$$

Our closing observation is that the value of any product $J\in\mathcal{J}_{1,\ell}$ containg only one term v must be a real multiple of v if $\{eev\}=\kappa v$ and $\{eve\}=\varepsilon v$.

Corollary. e TRIP $\implies$ $\exists\ \rho_0,\rho_1,\rho_2^{(1)},\rho_2^{(-1)}:\{\lambda:|\lambda|<1\}\to\mathbf{R}$ real-analytic

$\mathrm{D}_{\lambda e}G=\sum_{k,\varepsilon}\rho_k^{(\varepsilon)}(\lambda)P_k^{(\varepsilon)}$ with Peirce proj. $P_k^{(\varepsilon)}\colon\mathbf{E}\to\mathbf{E}_k^{(\varepsilon)}(e):=\big\{x\colon L(e)x=\frac{k}{2}x,\ Q(e)x=\varepsilon x\big\}$.

Proof. We know that the linear operators $L(e),Q(e)$ commute. [Indeed, with $\mathcal{K}:=\{(1,1),(1,-1),(1/2,0),(0,0)\}$ and the Peirce spaces $\mathbf{E}_{(\kappa,\varepsilon)}:=\{x:L(e)x=\kappa x,\ Q(e)x=\varepsilon x\}$ we have $\mathbf{E}=\oplus_{(\kappa,\varepsilon)\in\mathcal{K}}\mathbf{E}_{(\kappa,\varepsilon)}$. Given $x\in\mathbf{E}_{(\kappa,\varepsilon}$, $L(e)Q(e)x=Q(e)L(e)x=\kappa\varepsilon x$.] Hence given any $J \in \mathcal{J}_{1,\ell}$ we can write $J = Q(e)^m L(e)^{\ell/2-m} v = \varepsilon^m \kappa^{\ell/2-m} v$ independently of the choice of $v \in \mathcal{E}_{\kappa,\varepsilon)}$.

Proposition. Assume e TRIP and $[\Phi^t : t \in \mathbf{R}_+]$ str.cont.1-prg. in Aut($\mathbf{B}$) with $e \in \bigcap_{t\in\mathbf{R}_+} \mathrm{Fix}(\Phi^t)$. Then dom($\Phi'$) is dense in $\mathbf{B}$.

Proof. With the previous notations, it suffices to see only that range(G) contains an inner point. By the Inverse Mapping Theorem, to this it is enough that the Fréchet derivative $\mathrm{D}_{c=\lambda e} G(c)$ is an invertible operator for some λ with $|\lambda| < 1$.

By the previous corollary, with real-analytic coefficient functions, we have

$\mathrm{D}_{c=\lambda e} G(c) = \rho_0(\lambda P_0 + \rho_1(\lambda)_1 + \rho_2^{(+)}(\lambda) P_2^{(+)} + \rho_2^{(+)}(\lambda) P_2^{(+)}$.

By the first lemma, $\mathrm{D}_{c=0} G(c) = -[1 + Q(e)] = -P_0 - \frac{1}{2} P_1 - 2P_2^{+)}$ that is $\rho_0(0) = -1$, $\rho_1(0) = -1/2$, $\rho_2^{(+)}(0) = -2$, $\rho_2^{(-)}(0) = 0$.

Observation: $\rho_2^{(-)}(\lambda) e = P_2^{(-)} \big[\mathrm{D}_{c=0} G(c) \big](ie)$.

By the second Lemma, $\big[\mathrm{D}_{c=0} G(c) \big](ie) = \dfrac{4\mathrm{Re}(1-\lambda)\mathrm{Im}\lambda}{(1-|\lambda|^2)^2}$

that is $\rho_2^{(-)}(\lambda) = \dfrac{4\mathrm{Re}(1-\lambda)\mathrm{Im}\lambda}{(1-|\lambda|^2)^2} \neq 0$ for $0 \neq |\lambda| < 1$.

By the continuity of the functions $\rho_k^{(\pm)}$, for some $\delta \in (0,1)$ (in particular around $\lambda = 0$), we have $\rho_0(\lambda), \rho_1(\lambda), \rho_2^{(+)}(\lambda), \rho_2^{(-)}(\lambda) \neq 0$,

implying the invertibility of $\mathrm{D}_{c=\lambda e} G(c) = \sum_{(k,\varepsilon)} \rho_k^{(\varepsilon)}(\lambda) P_k^{(\varepsilon)}$ whenever $0 \neq |\lambda| < \delta$. Qu.e.d.

Remark. We can calculate the precise form of the functions $\rho_k^{\pm}$ as follows.

Recall that $\mathbf{E} = \oplus_{(\kappa,\varepsilon)\in\mathcal{K}} \mathbf{E}_{2\kappa}^{(\varepsilon)}(e)$ for the Peirce spaces

$$\mathbf{E}^{\varepsilon}_{2\kappa}(e):=\big\{x\in\mathbf{E}: L(e)x=\kappa e,\ Q(e)x=\varepsilon x\big\},\qquad \mathcal{K}:=\big\{(0,0),(\tfrac{1}{2},0),(1,1),(1,-1)\big\}.$$

Fix $(\kappa,\varepsilon)\in\mathcal{K}$ and $v\in\mathbf{E}^{(\varepsilon)}_{2\kappa}(e)$ arbitrarily. Define

$$B_{\lambda,\tau}:=B(\lambda e+\tau v)^{1/2},\ B'_\lambda:=\frac{d^+}{d\tau}\Big|_0 B_{\lambda,\tau},\ b'_\lambda:=B'_{\lambda,\tau}e,$$

$$M_{\lambda,\tau}:=M_{-(\lambda e+\tau v)},\quad m_{\lambda,\tau}:=M_{\lambda,\tau}(e),\quad m'_\lambda:=\frac{d^+}{d\tau}\Big|_0 m_{\lambda,\tau},\ \ ,$$

$$R_{\lambda,\tau}:=L(m_{\lambda,\tau},\lambda e+\tau v),\quad R'_\lambda:=\frac{d^+}{d\tau}\Big|_0 R_{\lambda,\tau}$$

Notice that by Peirce arithmetics, for some scalars,

$$B_{\lambda,0}e=\beta_\lambda e,\quad B_{\lambda,0}v=\widetilde{\beta}_\lambda v,\quad B'_\lambda e=\frac{d^+}{d\tau}\Big|_0 B(\lambda e+\tau v)e=\beta'_\lambda v,$$

$$m_{\lambda,0}=\mu_\lambda e,\quad m'_\lambda=\mu'_\lambda v,\quad R_{\lambda,0}e=\rho_\lambda e,\quad R_{\lambda,0}v=\widetilde{\rho}_\lambda v,\quad R'_\lambda e=\rho'_\lambda v.$$

With the rule of product differentiation we get

$$\mathrm{D}_{\lambda e}G=B'_\lambda\big[1+R_{\lambda,0}\big]^{-2}m_{\lambda,0}+B_{\lambda,0}\Big\{\frac{d}{d\tau}\Big|_{\tau=0}\big[1+R_{\lambda,\tau}\big]^{-2}\Big\}m_{\lambda,0}+B_{\lambda,0}\big[1+R_{\lambda,0}\big]^{-2}m'_\lambda$$

where $\displaystyle\frac{d^+}{d\tau}\Big|_0\big[1+R_{\lambda,\tau}\big]^{-2}=-\big[1+R_{\lambda,0}\big]^{-2}R'_\lambda\big[1+R_{\lambda,0}\big]^{-1}-\big[1+R_{\lambda,0}\big]^{-1}R'_\lambda\big[1+R_{\lambda,0}\big]^{-2}.$

It follows

$$\begin{aligned}\big[\mathrm{D}_{\lambda e}G\big]v&=B'_\lambda\big[1+R_{\lambda,0}\big]^{-2}\mu_\lambda e-B_{\lambda,0}\big[1+R_{\lambda,0}\big]^{-2}R'_\lambda\big[1+R_{\lambda,0}\big]^{-1}\mu_\lambda e-\\&\qquad-\big[1+R_{\lambda,0}\big]^{-1}R'_\lambda\big[1+R_{\lambda,0}\big]^{-2}\mu_\lambda e+B_{\lambda,0}\big[1+R_{\lambda,0}\big]^{-2}\mu'_\lambda v=\\&=B'_\lambda\frac{\mu_\lambda}{(1+\rho_\lambda)^2}e-B_{\lambda,0}\big[1+R_{\lambda,0}\big]^{-1}R'_\lambda\frac{\mu_\lambda}{(1+\rho_\lambda)^2}e-B_{\lambda,0}\big[1+R_{\lambda,0}\big]^{-2}R'_\lambda\frac{\mu_\lambda}{1+\rho_\lambda}e+\\&\qquad+B_{\lambda,0}\frac{\mu'_\lambda}{(1+\widetilde{\rho}_\lambda)^2}v\qquad\text{and continuing similarly,}\end{aligned}$$

$$\big[\mathrm{D}_{\lambda e}G\big]v=\frac{\beta'_\lambda\mu_\lambda}{(1+\rho_\lambda)^2}v-\frac{\widetilde{\beta}_\lambda\rho'_\lambda\mu_\lambda}{(1+\widetilde{\rho}_\lambda)(1+\rho_\lambda)^2}v-\frac{\widetilde{\beta}_\lambda\rho'_\lambda\mu_\lambda}{(1+\widetilde{\rho}_\lambda)^2(1+\rho_\lambda)}v+\frac{\widetilde{\beta}_\lambda\mu'_\lambda}{(1+\widetilde{\rho}_\lambda)^2}v.$$

Here we caculate the constants as follows.

$$\mu_\lambda=\frac{1-\lambda}{1-\overline{\lambda}}\qquad\text{because}\qquad m_{\lambda,0}=M_{-\lambda e}(e)=-\lambda e+[1-2L(e)+Q(e)^2]^{1/2}[1+L(e,-\lambda e)]^{-1}e=$$

$$= -\lambda e + [1 - 2L(\lambda e) + Q(\lambda e)^2]^{1/2} \frac{1}{1-\overline{\lambda}} e = \Big[-\lambda + \frac{1}{1-\overline{\lambda}} [1-2|\lambda|^2+|\lambda|^4]^{1/2} \Big] e = \frac{1-\lambda}{1-\overline{\lambda}} e$$

Next we determine β' along with β_λ and $\widetilde{\beta}_\lambda$:

$$B(\lambda e + \tau v)x = 1 - 2\big\{(\lambda e+\tau v)(\lambda e+\tau v)x\big\} + \big\{(\lambda e+\tau v)\big\{(\lambda e+\tau v)x(\lambda e+\tau v)\big\}(\lambda e+\tau v)\big\},$$

In particular $\quad B(\lambda e)e = (1 - 2|\lambda|^2 + |\lambda|^4)e \quad B(\lambda e)v = (1 - 2|\lambda|^2\kappa + |\lambda|^4\varepsilon^2)v, \quad$ whence

$$\beta_\lambda e = B(\lambda e)^{1/2} e = (1 - |\lambda|^2)e \ , \quad \widetilde{\beta}_\lambda v = B(\lambda e)^{1/2} v = \big[1 - 2|\lambda|^2\kappa + |\lambda|^4\varepsilon^2\big]^{1/2} v.$$

$$\frac{d^+}{d\tau}\Big|_0 B(\lambda e + \tau v)x =$$

$$= -2\overline{\lambda}\{vex\} - 2\lambda\{evx\} + \overline{\lambda}^2\lambda\{v\{exe\}e\} + \lambda\overline{\lambda}\lambda\{e\{vxe\}e\} + \lambda\overline{\lambda}\lambda\{e\{exv\}e\} + \lambda\overline{\lambda}^2\{e\{exe\}v\} =$$

$$= 2\Big[-\overline{\lambda}L(v,e) - \lambda L(e,v) + \overline{\lambda}^2\lambda Q(v,e)Q(e) + \lambda^2\overline{\lambda}Q(e)Q(v,e)\Big]x$$

$$\frac{d^+}{d\tau}\Big|_0 B(\lambda e + \tau v)e = \frac{d^+}{d\tau}\Big|_0 \Big[B(\lambda e + \tau v)^{1/2}\Big]^2 e = \big[B'_\lambda B_{\lambda,0} + B_{\lambda,0}B'_\lambda\big]e =$$

$$= \beta_\lambda B'_\lambda e + \beta'_\lambda B_{\lambda,0} v = \beta'_\lambda \beta_\lambda v + \beta'_\lambda \widetilde{\beta}_\lambda v \qquad \text{that is}$$

$$\beta'_\lambda(\beta_\lambda + \widetilde{\beta}_\lambda)v = \frac{d^+}{d\tau}\Big|_0 B(\lambda e + \tau v)e =$$

$$= -2\overline{\lambda}\{vee\} - 2\lambda\{eve\} + \overline{\lambda}^2\lambda\{v\{eee\}e\} + \lambda\overline{\lambda}\lambda\{e\{vee\}e\} + \lambda\overline{\lambda}\lambda\{e\{eev\}e\} + \lambda\overline{\lambda}^2\{e\{eee\}v\} =$$

$$= 2\big[-\overline{\lambda}\kappa - \lambda\varepsilon + |\lambda|^2\overline{\lambda}\kappa + |\lambda|^2\lambda\kappa\varepsilon\big]v,$$

$$\beta'_\lambda = 2\frac{-\lambda\varepsilon - \overline{\lambda}\kappa + |\lambda|^2\kappa(\overline{\lambda} + \lambda\varepsilon)}{(1 - |\lambda|^2) + (1 - 2|\lambda|^2\kappa + |\lambda|^4\varepsilon^2)^{1/2}}$$

In terms of β'_λ, we get $\qquad \mu'_\lambda = -1 + \dfrac{\beta'_\lambda}{1-\overline{\lambda}} + \dfrac{\widetilde{\beta}_\lambda\varepsilon}{(1-\overline{\lambda}\kappa)(1-\overline{\lambda})}$

since $\quad m'_\lambda = \dfrac{d^+}{d\tau}\Big|_0 \Big\{ -(\lambda e+\tau v) + B_{\lambda,\tau}\big[1 - L(e, \lambda e + \tau v)\big]^{-1} e\Big\} = -v+$

$+B'_\lambda[1-L(e,\lambda e)]^{-1}e + B_{\lambda,0}\dfrac{d^+}{d\tau}\Big|_0 [1-L(e,\lambda e+\tau v)]^{-1}e$ where $B'_\lambda[1-L(e,\lambda e)]^{-1}e = \dfrac{\beta'_\lambda}{1-\overline{\lambda}}v,$

$$\frac{d^+}{d\tau}\Big|_0 [1-L(e,\lambda e+\tau v)]^{-1}e = -[1-L(e,\lambda e)]^{-1}\Big\{\frac{d^+}{d\tau}\Big|_0 [1-L(e,\lambda e+\tau v)]\Big\}[1-L(e,\lambda e)]^{-1}e =$$

$$= [1-\overline{\lambda}L(e)]^{-1}L(e,v)\frac{1}{1-\overline{\lambda}}e = [1-\overline{\lambda}L(e)]^{-1}\frac{\varepsilon}{1-\overline{\lambda}}v = \frac{\varepsilon}{(1-\overline{\lambda}\kappa)(1-\overline{\lambda})}v.$$

Finally, for the constants $\rho_\lambda, \widetilde{\rho}_\lambda, \rho'_\lambda$, in terms of $\mu_\lambda, \mu'_\lambda$ we obtain

$\rho_\lambda = \overline{\lambda}\mu_\lambda, \quad \widetilde{\rho}_\lambda = \overline{\lambda}\mu_\lambda\kappa, \quad \rho'_\lambda = \overline{\lambda}\mu'_\lambda\kappa + \mu_\lambda\varepsilon \qquad$ because

$$R_{\lambda,0}e = L(m_\lambda, \lambda e)e = \mu_\lambda\overline{\lambda}L(e)e = \overline{\lambda}\mu_\lambda e, \quad R_{\lambda,0}v = \mu_\lambda\overline{\lambda}L(e)v = \overline{\lambda}\mu_\lambda\kappa v,$$

$$\frac{d^+}{d\tau}\Big|_0 R_{\lambda,\tau}e = \frac{d^+}{d\tau}\Big|_0 L(m_\lambda, \lambda e + \tau v)e = L(m'_\lambda, \lambda e)e + L(m_\lambda, v)e = \mu'_\lambda\overline{\lambda}L(v,e)e + \mu_\lambda L(e,v)e.$$

In particular, hence we can get reasonably simple formulas for the following cases:

(1) if $\mu(=\lambda) \in \mathbf{R}$ and $v \in \mathbf{E}_\kappa^{(\varepsilon)}(e)$ then

(1a) $\big[\mathrm{D}_{\mu e}G\big]v = -v$ for $(\kappa,\varepsilon) = (0,0)$, (1b) $\big[\mathrm{D}_{\mu e}G\big]v = -\dfrac{1}{1+\mu}$ for $(\kappa,\varepsilon) = (1/2,0)$,

(1c) $\big[\mathrm{D}_{\mu e}G\big]v = -\dfrac{2}{(1+\mu)^2}$ for $(\kappa,\varepsilon) = (1,1)$, (1d) $\big[\mathrm{D}_{\mu e}G\big]v = 0$ for $(\kappa,\varepsilon) = (1,-1)$;

(2) if $i\nu(=\lambda) \in i\mathbf{R}$ and $v \in \mathbf{E}_\kappa^{(\varepsilon)}(e)$ then

(2a) $\big[\mathrm{D}_{i\nu e}G\big]v = -v$ for $(\kappa,\varepsilon) = (0,0)$, (2b) $\big[\mathrm{D}_{i\nu e}G\big]v = -\dfrac{1+i\nu}{1-\nu^2}$ for $(\kappa,\varepsilon) = (1/2,0)$,

(2c) $\big[\mathrm{D}_{i\nu e}G\big]v = -\dfrac{2}{1-\nu^2}$ for $(\kappa,\varepsilon) = (1,1)$, (2d) $\big[\mathrm{D}_{i\nu e}G\big]v = -\dfrac{4i\nu}{(1-\nu^2)^2}$ for $(\kappa,\varepsilon) = (1,-1)$.

Theorem. If $0 \in \mathrm{dom}(\Phi')$ and $\bigcap_{t\in\mathbf{R}_+} \mathrm{Fix}(\Phi^t) \neq \emptyset$ then the generator Φ' is of Kaup's type:

$\mathrm{dom}(\Phi')$ is a subtriple in $\mathbf{E}$, $\Phi'(z) = a - \{zaz\} + iAz$ closed.

Proof. $\mathrm{dom}(\Phi') = \{x : t \mapsto U_t x \text{ diff.}\} = \mathrm{dom}(\Lambda')$ dense in $\mathbf{E}$, Λ' closed lin. op.

$$\Phi^t(z+e) - e = (A_{t,z} + B_t)^{-1}C_t z$$

$$\Psi'(z+e) = -(A_{t,z}+B_t)^{-1}\big[\tfrac{d}{dt}(A_{t,z}+B_t)\big](A_{t,z}+B_t)^{-1}C_t\big|_{t=0} + (A_{t,z}+B_t)^{-1}\tfrac{d}{dt}C_t z\big|_{t=0}$$

$$\Lambda'(z) = -B_t^{-1}\big[\tfrac{d}{dt}B_t\big]B_t^{-1}C_t\big|_{t=0} + B_t^{-1}\big[\tfrac{d}{dt}B_t\big]\big|_{t=0}$$

Let $x_n \to x$, $\Psi'(x_n) \to y$.

$z_n := x_n - e$,

$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$

Let $x \in \mathrm{dom}(\Psi')$, $\|x\| = 1$, $\varphi \in \mathbf{E}^*$, $\langle \varphi, x \rangle = \|\varphi\| = 1$

Φ' is a TANGENT vector field to $\partial \mathbf{B}$

$0 = \mathrm{Re}\langle \varphi \circ \overline{\kappa}, \Phi'(\kappa x)\rangle \qquad \Leftarrow |\kappa| = 1$

$\zeta \mapsto \langle \varphi, \Phi'(\zeta x)\rangle = \sum_{n=0}^\infty \alpha_n \zeta^n$ holomorphic

$\mathrm{Re}\big(\overline{\kappa} \sum_{n=0}^\infty \alpha_n \kappa^n\big) = 0$

$\sum_{n=0}^\infty (\alpha_n \kappa^{n-1} + \overline{\alpha_n} \kappa^{1-n}) = 0 \quad (|\kappa| = 1)$

$\sum_{n=-\infty}^\infty \beta_n \kappa^n = 0 \qquad \beta_n = \alpha_{n+1} \ (n \geq 2), \quad \beta_n = \overline{\alpha_{1-n}} \ (n \leq -2),$

$$\beta_1 = \alpha_2 + \overline{\alpha_0}, \quad \beta_{-1} = \alpha_0 + \overline{\alpha_2}, \quad \beta_0 = \alpha_1 + \overline{\alpha_1}$$

$\alpha_n = 0 \ (|n| \geq 2), \quad \alpha_1 + \overline{\alpha_1} = 0, \quad \alpha_2 = -\overline{\alpha_0}$

CONSIDER $\Omega(x) := \Phi'(x) - \{xbx\}$ INSTEAD OF Φ', $\quad b := \Psi'(0) = \frac{d}{dt} a(t)\big|_{t=0}$

This is also tangent to $\partial bf B$ with $\Omega(0) = 0$

$\Omega(\zeta x) = \zeta \Omega(x)$ HOMOGENITY

SPIN FACTORS

$\big(\mathbf{H},\langle\cdot|\cdot\rangle\big)$ Hilbert space, $\quad x\mapsto\overline{x}$ conjugation, $\quad \langle x|y\rangle^- = \langle\overline{x}|\overline{y}\rangle$

$\mathcal{S}:=\mathcal{S}(\mathbf{H},\bar{\cdot})$ is the JB*-triple with the triple product

$$\{xay\} = \langle x|a\rangle y + \langle y|a\rangle x - \underbrace{\langle x|\overline{y}\rangle}_{\langle y|\overline{x}\rangle}\,\overline{a}$$

$$\big[\ \text{TRIPOTENTS}\,\big] = \Big\{\lambda e:\ e\in\mathrm{Re}(\mathbf{H}),\ \lambda\in\mathbf{T},\ \langle e|e\rangle=1\Big\}\cup$$
$$\cup\Big\{u+iv:\ u,v\in\mathrm{Re}(\mathbf{H}),\ \langle u|u\rangle=\langle v|v\rangle=1/2,\ \langle u|v\rangle=0\Big\}$$

$U_t=\kappa_t V_t$: $\quad V_t \quad$ real $\langle\cdot|\cdot\rangle$-unitary, $\mathrm{Re}(\mathbf{E})\to\mathrm{Re}(\mathbf{H})$, $\kappa_t\in\mathbf{T}$.

Norm formula. Given $a=x+iy\in\mathbf{H}$ with $x=\overline{x}, y=\overline{y}$, by writing $\langle z\rangle^2:=\langle z|z\rangle$,

$$\|a\| = \big\|x+iy\big\| = \Big[\big[\langle x\rangle^2+\langle y\rangle^2\big]+2\big[\langle x\rangle^2\langle y\rangle^2-\langle x|y\rangle^2\big]^{1/2}\Big]^{1/2}$$

Direct proof: By [Kaup, 1983], since $\mathrm{Span}\{L(a)^n a: n=1,2,\ldots\}=\mathbf{C}a+\mathbf{C}\overline{a}$,

$$\|a\|^2 = \mathrm{radSp}\big(L(a)\big) = \mathrm{radSp}\big(L(a)|\mathbf{C}a+\mathbf{C}\overline{a}\big) = \mathrm{radSp}\big(L(x+iy)|\mathbf{C}x+\mathbf{C}y\big).$$

Here we have $L(a)z=\langle a|a\rangle z+\langle z|a\rangle a-\langle z|\overline{a}\rangle\overline{a}$, that is

$$L(a)=\big[\langle x\rangle^2+\langle y\rangle^2\big]\mathrm{id}+a\otimes a^*-\overline{a}\otimes\overline{a}^* = \big[\langle x\rangle^2+\langle y\rangle^2\big]\mathrm{id}+2i\big[y\otimes x^*-x\otimes y^*\big]\quad\text{and}$$

$$L(a)x=\big[\langle x\rangle^2+\langle y\rangle^2\big]x+2i\big[\langle x\rangle^2 y-\langle x|y\rangle x\big],\quad L(a)y=\big[\langle x\rangle^2+\langle y\rangle^2\big]y+2i\big[\langle x|y\rangle y-\langle y\rangle^2 x\big];$$

$$\mathrm{Sp}\big(L(a)|\mathbf{C}x+\mathbf{C}y\big) = \big[\langle x\rangle^2+\langle y\rangle^2\big]+2i\,\mathrm{Sp}\begin{bmatrix}-\langle x|y\rangle & -\langle y\rangle\\ \langle x\rangle^2 & \langle x|y\rangle\end{bmatrix} =$$

$$= \big[\langle x\rangle^2+\langle y\rangle^2\big]+2i\,\mathrm{roots}\big(\lambda^2-\langle x|y\rangle^2+\langle x\rangle^2\langle y\rangle^2\big) = \big[\langle x\rangle^2+\langle y\rangle^2\big]\pm 2\big[\langle x\rangle^2\langle y\rangle^2-\langle x|y\rangle^2\big]^{1/2}.$$

Unit ball: $\quad\Big\{z\in\mathbf{H}:\ \langle z\rangle^2<\dfrac{1}{2}\big(1+\big|\langle z|\overline{z}\rangle\big|^2\big)<1\Big\}.$

Str.cont one-parameter semigroups in $\mathrm{Iso}(d_{\mathbf{B}(\mathcal{S})})$

$[\Phi^t : t \in \mathbf{R}_+]$ str.cont.1-prsg in $\mathrm{Iso}(d_{\mathbf{B}(\mathcal{S})})$

Vesentini (1992)*: $\exists\, M_t \in \mathrm{Re}\big(\mathcal{L}(\mathbf{H})\big)\ \exists\, b_1^t, b_2^t, c_1^t, c_2^t \in \mathrm{Re}(\mathbf{H})\ \exists\, E^t \in \mathrm{Mat}(2,2,\mathbf{R})$

$\Phi^t(x) = F^t(x)/\varphi^t(x)$ where (with transposition $X^{\mathrm{T}} := \overline{X^*}$)

$F^t(x) = (b_1^t - ib_2^t) + 2M_t x + (x^{\mathrm{T}}x)(b_1^t + ib_2^t)$

$\varphi^t(x) = (E_{11}^t + E_{22}^t - iE_{12}^t + iE_{21}^t) + 2(c_1^t + ic_2^t)^{\mathrm{T}}x + (E_{11}^t - E_{22}^t + iE_{12}^t + iE_{21}^t)x^{\mathrm{T}}x$

such that, with $B_t := [b_1^t, b_2^t], C_t := [c_1^t, c_2^t]$, the matrices

$$G^t = \begin{bmatrix} M_t & B_t \\ C_t^{\mathrm{T}} & E^t \end{bmatrix} \qquad (t \in \mathbf{R}_+)$$

form a str.cont.1prsg. such that

$[G^t]^*\mathrm{diag}(I, -I_2)G^t = \mathrm{diag}(I, -I_2),\ \det(E^t) > 0 \quad (t \in \mathbf{R}_+),$ that is

$C_t E^t = M_t^{\mathrm{T}} B_t,\ \ M_t^{\mathrm{T}} = I + C_t C_t^{\mathrm{T}},\ \ [E^t]^{\mathrm{T}} E^t = I_2 + B_t^{\mathrm{T}} B_t.$

Remark. In Rend.Sem.Mat.Univ Pol.Torino, there is a misprint on p.438 line 11: it should be "$\delta G(X) = 2(X|C_1 - iC_2) + \cdots$" instead of "$\delta G(X) = 2(X|C_1 - C_2) + \cdots$".

It also seems that Vesentini's results rely upon the tacitly used hypothesis that the origin belongs to the domain of the holomorphic infinitesimal generator Φ' of $[\Phi^t : t \in \mathbf{R}_+]$.

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* Note di Mat. 9-Suppl.(1989)123-144; Ann.Mat.Pura Appl., 161/4(1992)281-297, Rend. Mat.Acc.Lincei, 3/9(1992)287-294. Rend.Sem.Mat.Univ Pol.Torino, 50/4(1992)427-455. Forerunners: U. Hierzbruch, Math Ann., 152 (1964) 395-417; L.A. Harris, Lecture Notes in Math. (Springer , 1974), Proc. London Math. Soc., 42/3 (1981) 331-361.

With the convention $Z' := \frac{d}{dt}\big|_{t=0+} Z^t$(or Z_t), we calculate the infinitesimal generator Φ' in terms of G' that is of M', B', C', E', respectively (provided $0 \in \mathrm{dom}(\Phi')$).

$\Phi' := \frac{d}{dt}\big|_{t=0+} \frac{F^t}{\varphi^t} = -\frac{\varphi'}{(\varphi^0)^2} F^0 + \frac{1}{\varphi^0} F' \quad$ where, for $x \in \mathrm{dom}(G')$.

Since $G^0 = \mathrm{Id}_{\mathbf{H}\oplus\mathbf{C}^2} = \mathrm{diag}(I, I_2)$, we have

$M_0 = I, \quad b_k^0 = c_k^0 = E_{12}^0 = E_{21}^0 = 0, \quad E_{11}^0 = E_{22}^0 = 1,$

$0 = (C_t E^t - M_t^{\mathrm{T}} B_t)' = C' - B', \quad 0 = I' = (M_t^{\mathrm{T}} M_t - C_t^{\mathrm{T}})' = [M']^{\mathrm{T}} + M',$

$0 = I_2' = ([E^t]^{\mathrm{T}} E^t) - B_t^{\mathrm{T}} B_t)' = [E']^{\mathrm{T}} + E' \quad$ i.e. $E_{11}' = E_{22}' = 0$, $E_{12}' = -E_{21}'$.

It follows $\quad \varphi^0(x) = (E_{11}^0 + E_{22}^0 - iE_{12}^0 + iE_{21}^0) = 2, \quad F^0(x) = 2M_0 x = 2x,$

$F'(x) = (b_1' - ib_2') + 2M'x + x^{\mathrm{T}}x(b_1' + ib_2'),$

$$\varphi'(x) = (E_{11}' + E_{22}' - iE_{12}' + iE_{21}') + 2(c_1' + ic_2')^{\mathrm{T}}x + (E_{11}' - E_{22}' + iE_{12}' + iE_{21}')x^{\mathrm{T}}x =$$

$$= 2iE_{21}' + 2(b_1' + ib_2')^{\mathrm{T}}x,$$

$$\Phi'(x) = -\tfrac{1}{4}[2iE_{21}' + 2(b_1' + ib_2')^{\mathrm{T}}x]2x + \tfrac{1}{2}[(b_1' - ib_2') + 2M'x + x^{\mathrm{T}}x(b_1' + ib_2')] =$$

$$= -iE_{21}'x - [(b_1' + ib_2')^{\mathrm{T}}x]x + \tfrac{1}{2}(b_1' - ib_2') + M'x + \tfrac{1}{2}x^{\mathrm{T}}x(b_1' + ib_2') =$$

$$= \big[\tfrac{1}{2}(b_1' - ib_2')\big] + \big[M' - iE_{21}'\big]x - \big[x(b_1' + ib_2')^{\mathrm{T}}x - \tfrac{1}{2}(b_1' - ib_2')x^{\mathrm{T}}x\big].$$

Proposition. If $0 \in \mathrm{dom}(\Phi')$ i.e. Φ' is of Kaup's type as $\Phi'(x) = a + iAx - \{xa^*x\}$ with $a := \Phi'(0)$ and some $\mathcal{S}$-Hermitian $A \in \mathcal{L}(\mathbf{H})$ then

$$G' = \begin{bmatrix} iA - i\varepsilon I & 2\,\mathrm{Re}(a) & -2\,\mathrm{Im}(a) \\ 2\,\mathrm{Re}(a)^{\mathrm{T}} & 0 & -\varepsilon \\ -2\,\mathrm{Im}(a)^{\mathrm{T}} & \varepsilon & 0 \end{bmatrix} \quad \text{where} \quad \varepsilon := E_{21}'$$

and $\quad iA = M + i\varepsilon I \;$ with $M = -M^{\mathrm{T}} : \mathrm{Re}(\mathbf{H}) \to \mathrm{Re}(\mathbf{H})$.

Coordinatization, Möbius transformations

Recall that, by means of SVD-decomposition, we can write

$$B=[b_1',b_2']=Q_1\begin{bmatrix}0&0\\ \lambda_1&0\\ 0&\lambda_2\end{bmatrix}Q_2^{\mathrm T}\quad\text{where}\quad Q_1\in\mathrm{QRT}(\mathrm{Re}(\mathbf{H}),\ Q_2\in\mathrm{ORT}(\mathbf{R}^2),\ \lambda_1\geq\lambda_2\geq 0.$$

Hence with the real orthogonal operator matrix $Q:=Q_1\oplus Q_2=\left[\begin{smallmatrix}Q_1&0\\0&Q_2\end{smallmatrix}\right]$,

$$G^t=Q_1\widetilde{G}^tQ_2^{\mathrm T}\quad(t\in\mathbf{R}_+)\quad\text{where}\quad\widetilde{G}':=\mathrm{gen}[\widetilde{G}^t:t\in\mathbf{R}_+]\ \text{ has the form}$$

$$\widetilde{G}'=\begin{bmatrix}\widetilde{M}_{11}'&\widetilde{M}_{12}'&0\\ \widetilde{M}_{21}'&\left[\begin{smallmatrix}0&-\nu\\ \nu&0\end{smallmatrix}\right]&\left[\begin{smallmatrix}\lambda_1&0\\0&\lambda_2\end{smallmatrix}\right]\\ 0&\left[\begin{smallmatrix}\lambda_1&0\\0&\lambda_2\end{smallmatrix}\right]&\left[\begin{smallmatrix}0&-\varepsilon\\ \varepsilon&0\end{smallmatrix}\right]\end{bmatrix}.$$

Continuing with a similar transformation $\widehat{G}':=\widehat{Q}\widetilde{G}'\widehat{G}^{\mathrm T}$ where $\widehat{Q}=I_1\oplus I_2\oplus I_2$ with suitable real orthogonal $\widehat{Q}_1$, with QR-decomposition we can achieve the form

$$\widehat{G}'=\begin{bmatrix}\widehat{M}_{11}&\widehat{M}_{12}&0&0\\ -\widehat{M}_{12}^{\mathrm T}&\widehat{M}_{22}&L&0\\ 0&-L&\widetilde{M}_{22}&\Lambda\\ 0&0&\Lambda^{\mathrm T}&E\end{bmatrix},\qquad\begin{matrix}\widetilde{M}_{22},\widehat{M}_{22},E\ \text{antisymm.}\\ \Lambda\ \text{pos.diag.},\ L\ \text{lower triangular}\ 2\times 2\ \text{real matr.}\end{matrix}$$

Question. Can we further eliminate Λ in entry $(2,3)$ with a transform $X\mapsto SXS^{-1}$?

In particular the *Möbius transformations* in a spin factor are the maps arising from integrating the vector fields corresponding to generators of the form with $M'=0$. Thus they are contructed as follows. Take an operator matrix of the form

$$G'=\begin{bmatrix}0&B'\\ [B']^{\mathrm T}&0\end{bmatrix}=\begin{bmatrix}0&b_1'&b_2'\\ [b_1']^{\mathrm T}&0&0\\ [b_2']^{\mathrm T}&0&0\end{bmatrix}=Q_1\begin{bmatrix}0&0&0\\ 0&0&\left[\begin{smallmatrix}\lambda_1&0\\0&\lambda_2\end{smallmatrix}\right]\\ 0&\left[\begin{smallmatrix}\lambda_1&0\\0&\lambda_2\end{smallmatrix}\right]&0\end{bmatrix}Q_2^{\mathrm T}.$$

Since G' is a bounded operator in this cases, its integration is simply

$$G^t=\exp(tG')=\sum_{n=0}^{\infty}n!^{-1}t^n[G']^n=$$

$$=\sum_{k=0}^{\infty}\tfrac{t^{2k}}{(2k)!}\begin{bmatrix}(B'[B']^{\mathrm T})^k&0\\0&([B']^{\mathrm T}B')^k\end{bmatrix}+\sum_{k=0}^{\infty}\tfrac{t^{2k+1}}{(2k+1)!}\begin{bmatrix}0&(B'[B']^{\mathrm T})^kB'\\ [B']^{\mathrm T}(B'([B']^{\mathrm T})^k&0\end{bmatrix}=$$

$$= (Q_1 \oplus Q_2) \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cosh\left(\left[\begin{smallmatrix} \lambda_1 t & 0 \\ 0 & \lambda_2 t \end{smallmatrix}\right]\right) & \sinh\left(\left[\begin{smallmatrix} \lambda_1 t & 0 \\ 0 & \lambda_2 t \end{smallmatrix}\right]\right) \\ 0 & \sinh\left(\left[\begin{smallmatrix} \lambda_1 t & 0 \\ 0 & \lambda_2 t \end{smallmatrix}\right]\right) & \cosh\left(\left[\begin{smallmatrix} \lambda_1 t & 0 \\ 0 & \lambda_2 t \end{smallmatrix}\right]\right) \end{bmatrix} (Q_1^{\mathrm{T}} \oplus Q_2^{\mathrm{T}})$$

giving rise to

$$M_{a(t)}(x) = \Phi^t(x) = F^t(x)/\varphi^t(x) \qquad \text{where}$$

$$F^t(x) = (b_1^t - ib_2^t) + 2M_t x + (x^{\mathrm{T}}x)(b_1^t + ib_2^t)$$

$$\varphi^t(x) = (E_{11}^t + E_{22}^t - iE_{12}^t + iE_{21}^t) + 2(c_1^t + ic_2^t)^{\mathrm{T}}x + (E_{11}^t - E_{22}^t + iE_{12}^t + iE_{21}^t)x^{\mathrm{T}}x$$

with $\quad M_t = Q_1 \begin{bmatrix} 0 & 0 \\ 0 & \cosh\left(\left[\begin{smallmatrix} \lambda_1 t & 0 \\ 0 & \lambda_2 t \end{smallmatrix}\right]\right) \end{bmatrix} Q_1^{\mathrm{T}}, \quad B_t = C_t = Q_1 \begin{bmatrix} 0 \\ \sinh\left(\left[\begin{smallmatrix} \lambda_1 t & 0 \\ 0 & \lambda_2 t \end{smallmatrix}\right]\right) \end{bmatrix} Q_2^{\mathrm{T}},$

$E^t = Q_2 \left[\cosh\left(\left[\begin{smallmatrix} \lambda_1 t & 0 \\ 0 & \lambda_2 t \end{smallmatrix}\right]\right)\right] Q_2^{\mathrm{T}}.$

Remark. The maximal faces of the unit ball of a spin factor are discs of the form

$$\mathbf{B}_e := e + \big\{\zeta\overline{e} : |\zeta| \leq 1\big\} \qquad \text{where } e = \tfrac{1}{2}u + \tfrac{i}{2}v \text{ with } u \perp v \in \mathrm{Re}(\mathbf{H}, \langle u\rangle^2 = \langle v\rangle^2 = 1.$$

Lemma. Given a tripotent e as above, for the Möbius group $[M_{a(t)} : t \in \mathbf{R}]$ integrating the vector field $M' : z \mapsto 2\overline{e} - \{z(2\overline{e})^* z\}$ corresponding to the generator $G' := \begin{bmatrix} 0 & u & -v \\ u^{\mathrm{T}} & 0 & 0 \\ -v^{\mathrm{T}} & 0 & 0 \end{bmatrix}$ we have

$$M'(e + \zeta\overline{e}) = 2(1 - \zeta^2)\overline{e}, \quad M_{a(t)}(e + \zeta\overline{e}) = e + \frac{\zeta + \tanh(t)}{1 + \tanh(t)\zeta}\overline{e} \quad (|\zeta| \leq 1).$$

Proof. Since $e \perp \overline{e}$ and $\langle e\rangle^2 = \langle\overline{e}\rangle^2 = 1/2$, we have

$$M'(e + \zeta\overline{e})/2 = \overline{e} - 2\langle e + \zeta\overline{e}|\overline{e}\rangle(e + \zeta\overline{e}) + \langle e + \zeta\overline{e}|e - \zeta\overline{e}\rangle e = \overline{e} - \zeta\overline{e}.$$

Thus the vector field M' is tangent to the complex line $\mathbf{L}_e := e + \mathbf{C}\overline{e}$ and, in terms of the trivial coordinatization $Z(e+\zeta\overline{e}) := \zeta$ it has the form $Z_\# M' : \zeta \mapsto 1 - \zeta^2$ whose integration gives the classical Möbius group $\big[(\zeta + \tanh(t))/(1 + \zeta\tanh(t)) : t \in \mathbf{R}\big]$

Triangularization with fixed points

Assume $e \in \partial\mathbf{B}$ is a common fixed point of $[\Phi^t : t \in \mathbf{R}_+]$ represented with the c_0-sgr. of operator matrices $[G^t : t \in \mathbf{R}_+]$ (in Vesentini's sense). Consider the corresponding generators

$$\Phi'(x) = a + iAx - \{xa^*x\} = \big(\tfrac{1}{2}b_1 - \tfrac{i}{2}b_2\big) + Mx + i\varepsilon x - \langle x|b_1 - ib_2\rangle x + \langle x|\overline{x}\rangle\big(\tfrac{1}{2}b_1 + \tfrac{i}{2}b_2\big),$$

$$G' = \begin{bmatrix} M & b_1 & b_2 \\ b_1^{\mathrm{T}} & 0 & -\varepsilon \\ b_2^{\mathrm{T}} & \varepsilon & 0 \end{bmatrix} \quad \text{where} \quad b_1 := 2\mathrm{Re}(a), b_2 := -2\mathrm{Im}(a), \;\; M = \overline{M} = -M^{\mathrm{T}},\, \varepsilon \in \mathbf{R}.$$

We may assume without loss of generality (by means of Möbius equivalence) that e is a tripotent, that is we have either

1) $e = \overline{e}$, $\langle e|e\rangle = 1$ (real extreme point), or 2) $e \perp \overline{e}$, $\langle e|e\rangle = \frac{1}{2}$ (face middle point).

In any case, $\Phi'(e) = 0$.

Case (1) $0 = \Phi'(e) = a + iAe - \{ea^*e\} =$

$$= \big(\tfrac{1}{2}b_1 - \tfrac{i}{2}b_2\big) + Me + i\varepsilon e - \langle e|b_1 - ib_2\rangle e + \langle e|e\rangle\big(\tfrac{1}{2}b_1 + \tfrac{i}{2}b_2\big).$$

With the orthogonal decompositions $b_j := \rho_j e + x_j$ (i.e. $\rho_j \in \mathbf{R},\; x_j \perp e$), we have

$0 = i(\varepsilon - \rho_2)e + x_1 + Me$ implying $\rho_2 = \varepsilon$ and $Me = -x_1$.

Hence, with the restricted operator $M_0 := \mathrm{P}_{e^\perp} M|e^\perp$,

$$G' = \begin{bmatrix} M & b_1 & b_2 \\ b_1^{\mathrm{T}} & 0 & -\varepsilon \\ b_2^{\mathrm{T}} & \varepsilon & 0 \end{bmatrix} = \begin{bmatrix} 0 & -(Me)^{\mathrm{T}} & \rho_1 & -\varepsilon \\ Me & M_0 & -Me & y \\ \rho_1 & -(Me)^{\mathrm{T}} & 0 & -\varepsilon \\ -\varepsilon & y^{\mathrm{T}} & \varepsilon & 0 \end{bmatrix} = \begin{bmatrix} 0 & x_1^{\mathrm{T}} & \rho_1 & -\varepsilon \\ -x_1 & M_0 & x_1 & x_2 \\ \rho_1 & x_1^{\mathrm{T}} & 0 & -\varepsilon \\ -\varepsilon & x_2^{\mathrm{T}} & \varepsilon & 0 \end{bmatrix}.$$

An almost triagular similar matrix can be obtained with the operator matrices

$$T := \begin{bmatrix} 1/2 & 0 & 0 & 1 \\ 0 & I_0 & 0 & 0 \\ -1/2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & I_0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix}$$

as

$$T^{-1} G' T = \begin{bmatrix} -\rho_1 & 0 & 0 & 0 \\ -x_1 & M_0 & x_2 & 0 \\ -\varepsilon & x_2^{\mathrm{T}} & 0 & 0 \\ 0 & x_1^{\mathrm{T}} & -\varepsilon & \rho_1 \end{bmatrix}.$$

Remark. M_0 is a possibly unbounded skew symmetric closed real-linear operator defined on a dense linear submanifold of $e^{\perp}$. For heuristics see `vazlat6.mws`.

Case (2) $0 = \Phi'(e)$, $e \perp \overline{e}$, $\langle e \rangle^2 = 1/2$ of face middle points. Then

$0 = \Phi'(e) = \left(\frac{1}{2} b_1 - \frac{i}{2} b_2\right) + Me + i\varepsilon e - \langle e | b_1 - i b_2 \rangle e.$

We assume without loss of generality that

$e = \frac{1}{2} u + \frac{i}{2} v$ where $u \perp v$, $u = \overline{u}, v = \overline{v}$ and $\langle u \rangle^2 = \langle v \rangle^2 = 1$.

Since M is real antisymmetric i.e. $M = \overline{M} \subset -M^{\mathrm{T}} = -\overline{M}^* = -\overline{M^*}$ along with $\mathrm{dom}(M) = \overline{\mathrm{dom}(M)}$, we have $u, v \in \mathrm{dom}(M)$ with $\langle Mu | u \rangle = \langle Mv | v \rangle = \langle Mu | v \rangle + \langle Mv | u \rangle = 0$ and $\langle Me | e \rangle = -\frac{i}{2} \langle Mu | v \rangle$ resp. $\langle Me | \overline{e} \rangle = 0$.

Hence, using the identities $\langle b_j | u \rangle = \langle u | b_j \rangle$ resp. $\langle b_j | v \rangle = \langle v | b_j \rangle$, we get

$0 = \langle \Phi'(e) | e \rangle = \left\langle \frac{1}{2} b_1 - \frac{i}{2} b_2 \middle| e \right\rangle + \langle Me | e \rangle + \frac{i}{2} \varepsilon - \left\langle e \middle| \frac{1}{2} b_1 - \frac{i}{2} b_2 \right\rangle = \frac{i}{2} \left[\varepsilon - \langle Mu | v \rangle - \langle b_1 | v \rangle - \langle b_2 | u \rangle \right],$

$0 = \langle \Phi'(e) | \overline{e} \rangle = \left\langle \frac{1}{2} b_1 - \frac{i}{2} b_2 \middle| \overline{e} \right\rangle = \frac{1}{4} \left[\langle b_1 | u \rangle + \langle b_2 | v \rangle + i \langle b_1 | v \rangle - i \langle b_2 | u \rangle \right].$

Considering the real and imaginary parts, therefore

$\langle b_1 | u \rangle = -\langle b_2 | v \rangle, \quad \langle b_1 | v \rangle = \langle b_2 | u \rangle, \quad \langle Mu | v \rangle = \varepsilon - \langle b_1 | v \rangle - \langle b_2 | u \rangle = \varepsilon - 2 \langle b_2 | u \rangle.$

Thus in terms of the orthogonal decompositions

$b_j = \rho_j u + \sigma_j v + x_j$, (where $x_1, x_2 \perp \{u,v\}$)

and with $\mu := \langle Mu|v\rangle$ we have

$\sigma_2 = -\rho_1,\ \ \sigma_1 = \rho_2,\ \ \mu = \varepsilon - 2\rho_2.$

Hence, with the notations $P := \mathrm{P}_{\{u,v\}^\perp},\ \ M_0 := PM|\{u,v\}^\perp,\ \ q_1 := PMu,\ \ q_2 := PMv,$

we can write

$$G' = \begin{bmatrix} M & b_1 & b_2 \\ b_1^{\mathrm{T}} & 0 & -\varepsilon \\ b_2^{\mathrm{T}} & \varepsilon & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\mu & -q_1^{\mathrm{T}} & \rho_1 & \rho_2 \\ \mu & 0 & -q_2^{\mathrm{T}} & \rho_2 & -\rho_1 \\ q_1 & q_2 & M_0 & x_1 & x_2 \\ \rho_1 & \rho_2 & x_1^{\mathrm{T}} & 0 & -\varepsilon \\ \rho_2 & -\rho_1 & x_2^{\mathrm{T}} & \varepsilon & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2\rho_2-\varepsilon & x_1^{\mathrm{T}} & \rho_1 & \rho_2 \\ \varepsilon-2\rho_2 & 0 & -x_2^{\mathrm{T}} & \rho_2 & -\rho_1 \\ -x_1 & x_2 & M_0 & x_1 & x_2 \\ \rho_1 & \rho_2 & x_1^{\mathrm{T}} & 0 & -\varepsilon \\ \rho_2 & -\rho_1 & x_2^{\mathrm{T}} & \varepsilon & 0 \end{bmatrix}$$

because from the relation

$$0 = P\Phi'(e) = P\big[\big(\tfrac{1}{2}b_1 - \tfrac{i}{2}b_2\big) + Me + i\varepsilon e - \langle e|b_1 - ib_2\rangle e\big] = \tfrac{1}{2}\big[x_1 - ix_2 + PM(u+iv) + 0\big]$$

we infer also $q_1 = -x_1$ and $q_2 = x_2$.

Intergration of the almost triangular systems

Case (1) For short we write $\rho := \rho_1$, $x := x_1$, $y := x_2$. We determine the c_0-semigroup $[U^t : t \in \mathbf{R}_+]$, $U^t := (TS)^{-1}G^t(TS)$ with the generator $A+B$ where

$$A := \begin{bmatrix} -\rho & 0 & 0 & 0 \\ -x & M_0 & 0 & 0 \\ -\varepsilon & y^{\mathrm{T}} & 0 & 0 \\ 0 & x^{\mathrm{T}} & -\varepsilon & \rho \end{bmatrix}, \qquad B := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is well-known [Engel-Nagel] that, in terms of the c_0-semigroup $[T^t : t \in \mathbf{R}_+]$ with generator $A = S_0'$ which consits of lower triangular operator matrices, we have the convolution equation of Volterra type

$$\text{(V)} \qquad U^t = \int_{s=0}^{t} T^{t-s} B U^s ds + T^t \qquad (t \in \mathbf{R}_+)$$

and also $\quad U^t = \sum_{n=0}^{\infty} S_n(t) \quad$ with the recursion $\quad S_0(t) := T^t,\ S_{n+1}(t) = \int_0^t T^{t-s} B S_n(s)\ ds.$

The so-called Dyson-Phillips series $\sum_{n=0}^{\infty} S_n(t)$ converges locally uniformly in norm.

In terms of the entries, we can write

$$T^{t-s}B = \begin{bmatrix} T_{11}^{t-s} & & & \\ T_{21}^{t-s} & T_{22}^{t-s} & & \\ T_{31}^{t-s} & T_{32}^{t-s} & T_{33}^{t-s} & \\ T_{41}^{t-s} & T_{42}^{t-s} & T_{43}^{t-s} & T_{44}^{t-s} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & T_{22}^{t-s}y & 0 \\ 0 & 0 & T_{32}^{t-s}y & 0 \\ 0 & 0 & T_{42}^{t-s}y & 0 \end{bmatrix}$$

and

$$T^{t-s}BU^s = \begin{bmatrix} 0 \quad 0 \quad 0 \quad 0 \\ \left[T_{k,2}^{t-s} y U_{3,\ell}^s\right]_{\substack{2\le k\le 4 \\ 1\le \ell\le 4}} \end{bmatrix}, \qquad T^{t-s}BS_n(s) = \begin{bmatrix} 0 \quad 0 \quad 0 \quad 0 \\ \left[T_{k,2}^{t-s} y [S_n(s)]_{3,\ell}\right]_{\substack{2\le k\le 4 \\ 1\le \ell\le 4}} \end{bmatrix}.$$

It follows

$$(\mathrm{V}') \qquad U_{1,\ell}^t \equiv T_{1,\ell}^t, \quad U_{k,\ell}^t = \int_{s=0}^t T_{k,2}^{t-s} y U_{3,\ell}^s\ ds + T_{k,\ell}^t \qquad (t \in \mathbf{R}_+;\ k = 1,2,3;\ \ell = 1,2,3,4).$$

At this point, one more reduction is easily available: Since the matrices T^t are lower triangular, we have $T_{34}^t \equiv 0$ with the consequence that the solution U_{34}^t of the homogeneous Volterra equation $U_{34}^t = \int_{s=0}^t T_{3,2}^{t-s} y U_{3,4}^s + T_{34}^t$ is necesarily $U_{34}^t \equiv 0$ and hence also

$U_{k,4}^t = \int_{s=0}^t \left[T_{k,2}^{t-s} y\right] U_{34}^s\ ds + T_{k,2}^t \equiv T_{k,4}^t \qquad (k = 2,3,4),$

$U_{1,4}^t = U_{2,4}^t = U_{3,4}^t \equiv 0,\ \ U_{4,4}^t \equiv T_{4,4}^t = e^{\rho t}$ and also $U_{1,1}^t \equiv T_{1,1}^t = e^{-\rho t},\ \ U_{1,2}^t = U_{1,3}^t \equiv 0.$

For the remaining cases ($k > 1,\ \ell < 4$) we obtain the following crucial Volterra equations which can control the entries $U_{k,\ell}^t$ by the third row via (V$'$) completely:

$$(\mathrm{V}'') \qquad U_{3,\ell}^t = \int_{r=0}^t \left[T_{32}^{t-r} y\right] U_{3,\ell}^r\ dr + T_{3,\ell}^t \qquad (t \in \mathbf{R}_+;\ \ell = 1,2,3).$$

Notice that the matrices $T_{32}^{t-r}y$ are of type 1×1, thus the effect of left multiplication with them is simply a scalar multiplication. Also the submatrices $T_{k,\ell}^t, U_{k,\ell}^r$ with $(k,\ell) = (3,1),(3,3)$ are of type 1×1.

Since $\left[T^{t-s}BS_n(s)\right]_{3,\ell} = T_{32}^{t-s}y[S_n(s)]_{3,\ell}$, in terms of convolutions with the functions

$$w(t) := T_{32}^t y,\ \ V_\ell(t) := T_{3,\ell}^t \qquad (t \in \mathbf{R}_+,\ \ell = 1,2,3),$$

with uniform convergence on bounded intervals ($t \le M$), we have

$$U_{3,\ell}^t = T_{3,\ell}^t + \sum_{n=1}^{\infty} S_n(t)_{3,\ell} = V_\ell(t) + \big\{w * V_\ell\big\}(t) + \sum_{n=2}^{\infty} \big\{\underbrace{w * \cdots * w}_{n\ \text{terms}} * V_\ell\big\}(t) =$$

$$= \big\{W * V_\ell\big\}(t) \quad \text{where} \quad W := 1 + w + \sum_{n=2}^{\infty} \underbrace{w * \cdots * w}_{n\ \text{terms}} = \sum_{n=0}^{\infty} w^{*n}.$$

Remark. We can achieve useful structure formulas for the functions w^{*n} above by means of the *Laplace transform*

$$\mathcal{L}v = \mathcal{L}_t\big\{v(t)\big\} : s \mapsto \int_{t=0}^{\infty} e^{-st}V(t)\ dt, \quad \mathrm{dom}\big(\mathcal{L}v\big) = \Big\{s \in \mathbf{C} : \int_{t=0}^{\infty} \big|e^{-st}v(t)\big|\ dt < \infty\Big\}$$

and its inverse

$$\mathcal{L}^{-1}V : 0 \le t \mapsto \frac{1}{\pi} \int_{\sigma=-\infty}^{\infty} e^{(\Omega+i\sigma)t}V(\Omega+i\sigma)\, d\sigma \text{ with } \Omega > 0 \text{ satisfying } \int_{\sigma=-\infty}^{\infty} e^{\Omega t}\big|V(\Omega+i\sigma)\big|\, d\sigma < \infty.$$

It is well-known [Deddens, Stachó JMAA] that the c_0-semigroup $[U_0^t : t \in \mathbf{R}_+]$ of real-linear isometries $\mathbf{H}_0 \to \mathbf{H}_0$ with generator M_0 embeds into a c_0-group of isometries of some covering real Hilbert space which can be regarded as the real part of the complexified Hilbert space $\widehat{\mathbf{H}} := \mathbf{H}_0 \oplus i\mathbf{H}_0$ with conjugation $\tau : x \oplus iy \mapsto x \oplus (-i)y \quad (x, y \in \mathbf{H}_0)$. Thus

$$U_0^t z = \int_{\lambda\in\mathbf{R}} e^{i\lambda t}\, P(d\lambda)\, z \qquad \big(z \in \mathrm{Re}(\mathbf{H})\big)$$

in terms of a spectral measure

$$P : \Lambda(\subset \mathbf{R} \text{ Borelian}) \to \big\{ \text{ orthogonal projections on } \widehat{\mathbf{H}}\big\}.$$

Since the operators $\widehat{U}_0^t := \int_{\lambda\in\mathbf{R}} e^{i\lambda t}\, P(d\lambda)$ leave the eigenspace $\mathbf{H}_0 = \big\{\widehat{x} : \tau\widehat{x} = \widehat{x}\big\}$ invariant, we have

$$\tau \widehat{U}_0^t \equiv \widehat{U}_0^t \tau \quad \text{i.e.} \quad \widehat{U}_0^t \equiv \tau \widehat{U}_0^t \tau \qquad (t \in \mathbf{R}).$$

Hence necessarily

$$\int_{\lambda\in\mathbf{R}} e^{i\lambda t} P(d\lambda) = \tau \int_{\lambda\in\mathbf{R}} e^{i\lambda t}\, P(d\lambda)\tau = \int_{\lambda\in\mathbf{R}} e^{-i\lambda t}\, \tau P(d\lambda)\tau = \int_{\lambda\in\mathbf{R}} e^{i\lambda t}\, \tau P(-d\lambda)\tau \qquad (t\in\mathbf{R}).$$

This implies the following symmetry of $P(\cdot)$:

$$P(\Lambda) = \tau P(-\Lambda)\tau \quad \text{i.e.} \ P(-\Lambda) = \tau P(\Lambda)\tau \qquad (\Lambda \subset \mathbf{R} \text{ Borelian}).$$

It is immediate that

$$\begin{aligned} w(t) &= T_{32}^t y = y^{\mathrm{T}} \int_{r=0}^{t} U_0^r\, dr\ y = \Big\langle y \Big| \int_{r=0}^{t} U_0^r\, dr\ y \Big\rangle = \int_{r=0}^{t} \Big\langle y \Big| \int_{\lambda\in\mathbf{R}} e^{i\lambda r}\, P(d\lambda) y \Big\rangle\, dr = \\ &= \Big\langle y \Big| \int_{\lambda\in\mathbf{R}} \int_{r=0}^{t} e^{i\lambda r}\, dr\ P(d\lambda) y \Big\rangle = \int_{\lambda\in\mathbf{R}} \Big[\int_{r=0}^{t} e^{-i\lambda r}\, dr \Big] \Big\langle y \Big| P(d\lambda) y \Big\rangle = \\ &= \Big[\int_{\lambda<0} + \int_{\lambda=0} + \int_{\lambda>0} \Big] \frac{1-e^{-i\lambda t}}{i\lambda} \Big\langle y \Big| P(d\lambda) y \Big\rangle = \\ &= t\, P\{0\} + \int_{\lambda\in\mathbf{R}_{++}} \tfrac{1-e^{-i\lambda t}}{i\lambda} \Big\langle y \Big| P(d\lambda) y \Big\rangle + \int_{\lambda\in\mathbf{R}_{++}} \tfrac{1-e^{i\lambda t}}{(-i)\lambda} \Big\langle y \Big| \tau P(-d\lambda) y \Big\rangle. \end{aligned}$$

Since $P(-\Lambda) \equiv \tau P(\Lambda)\tau$, $y = \tau y \in \mathbf{H}_0$ and $\langle \tau\widehat{u} | \tau\widehat{v} \rangle = \langle \widehat{u} | \widehat{v} \rangle^- = \langle \widehat{v} | \widehat{u} \rangle$, it follows

$$0 \le \Big\langle y \Big| P(-\Lambda) y \Big\rangle = \Big\langle \tau y \Big| \tau P(\Lambda) y \Big\rangle = \Big\langle y \Big| P(\Lambda) y \Big\rangle^- = \Big\langle y \Big| P(\Lambda) y \Big\rangle.$$

Thus we get even

$$w(t) = t\, P\{0\} + \int_{\lambda\in\mathbf{R}_{++}} \Big(\tfrac{1-e^{-i\lambda t}}{i\lambda} + \tfrac{1-e^{i\lambda t}}{(-i)\lambda} \Big)\, p(d\lambda) = \int_{\lambda\in\mathbf{R}_{+}} \frac{\sin(\lambda t)}{\lambda}\, dp(\lambda)$$

in terms of the *non-negative real valued* measure

$$p(\Lambda) := 2\Big\langle y \Big| P(\Lambda) y \Big\rangle \qquad (\Lambda \subset \mathbf{R}_{++} \text{ Borelian}), \quad p\big(\{0\}\big) := \Big\langle y \Big| P(\{0\}) y \Big\rangle$$

on $\mathbf{R}_+$ with total mass

$$p(\mathbf{R}_+) = p(\{0\}) + 2p(\mathbf{R}_{++}) = p(\{0\}) + p(\mathbf{R}_{++}) + p(-\mathbf{R}_{++}) = \Big\langle y \Big| P(\mathbf{R}) y \Big\rangle = \Big\langle y \Big| y \Big\rangle = \big\|y\big\|^2 < 1.$$

For its *Laplace transform* we have

$$\mathcal{L}w(s)=\int\limits_{t=0}^{\infty} e^{-st}\int\limits_{\lambda\in\mathbf{R}_+}\frac{\sin(\lambda t)}{\lambda}dp(\lambda)\ dt=\int\limits_{\lambda\in\mathbf{R}_+}\int\limits_{t=0}^{\infty} e^{-st}\frac{\sin(\lambda t)}{\lambda}dt\ dp(\lambda)=$$

$$=\int\limits_{\lambda\in\mathbf{R}_+}\mathcal{L}_t\big\{\sin(\lambda t)/\lambda\big\}(s)\ dp(\lambda)=\int\limits_{\lambda\in\mathbf{R}_+}\frac{1}{s^2+\lambda^2}\ dp(\lambda).$$

Hence

$$\mathcal{L}w^{*n}=\big[\mathcal{L}w\big]^n=\Big[\int\limits_{\lambda\in\mathbf{R}_+}\frac{1}{s^2+\lambda^2}\ dp(\lambda)\Big]^n \qquad (n=1,2,\ldots),$$

$$w^{*n}=\frac{1}{\pi}\int_{\sigma=-\infty}^{\infty}e^{(\Omega+i\sigma)t}\left[\int_{\lambda\in\mathbf{R}_+}\frac{dp(\lambda)}{(\Omega+i\sigma)^2+\lambda^2}\right]^n d\sigma \qquad \text{for sufficiently large } \Omega>0.$$

We can calculate w^{*n} in terms of the product measure $dp^{\otimes n}(\lambda):=dp(\lambda_1)\cdots dp(\lambda_n)$ as follows. Since $w(t)=\int\limits_{\lambda\in\mathbf{R}_+}s_\lambda(t)\ dp(\lambda)$, by induction on n we can see that

$$w^{*n}(t)=\int\limits_{\lambda_1\in\mathbf{R}_+^n}s_{\lambda_1}*\cdots*s_{\lambda_n}(t)\ dp(\lambda_n)\cdots dp(\lambda_1)=\int\limits_{\lambda\in\mathbf{R}_+^n}s_{\lambda_1}*\cdots*s_{\lambda_n}(t)\ dp^{\otimes n}(\lambda).$$

For the functions

$$s_\lambda(t):=\frac{\sin\lambda t}{\lambda}\qquad (0\neq\lambda\in\mathbf{R});\quad s_0\equiv t$$

we have (with computer algebra MAPLE `vazlat5.mws`)

$$s_\alpha*s_\beta(t)=\int\limits_{s=0}^{t}s_\alpha(s)s_\beta(t-s)\,ds=-\frac{\sin\alpha t}{\alpha(\alpha^2-\beta^2)}-\frac{\sin\beta t}{\beta(\beta^2-\alpha^2)}\ .$$

Using this identity, by induction on n we obtain that

$$s_{\lambda_1}*\cdots*s_{\lambda_n}(t)=\sum_{k=1}^{n}\alpha_k^{(n)}\sin\lambda_k t\quad\text{where}\quad \alpha_k^{(n)}=\alpha_k^{(n)}(\lambda_1,\ldots,\lambda_n)=\frac{1}{\lambda_k}\prod_{j:k\neq j\leq n}\frac{1}{\lambda_j^2-\lambda_k^2}.$$

Indeed, for every n with this property, also

$$s_{\lambda_1}*\cdots*s_{\lambda_{n+1}}(t)=\sum_{k=1}^{n}\alpha_k^{(n)}\lambda_k s_{\lambda_k}*s_{\lambda_{n+1}}=$$

$$=\sum_{k=1}^{n}\alpha_k^{(n)}\lambda_k\left[\frac{\sin\lambda_k t}{\lambda_k(\lambda_{n+1}^2-\lambda_k^2)}+\frac{\sin\lambda_{n+1}t}{\lambda_{n+1}(\lambda_k^2-\lambda_{n+1}^2)}\right]=$$

$$=\sum_{k=1}^{n}\left[\frac{1}{\lambda_k}\prod_{k\neq j\leq n}\frac{1}{\lambda_j^2-\lambda_k^2}\right]\frac{\sin\lambda_k t}{(\lambda_{n+1}^2-\lambda_k^2)}+\sum_{k=1}^{n}\left[\frac{1}{\lambda_k}\prod_{k\neq j\leq n}\frac{1}{\lambda_j^2-\lambda_k^2}\right]\frac{\sin\lambda_{n+1}t}{\lambda_{n+1}(\lambda_k^2-\lambda_{n+1}^2)}=$$

$$= \sum_{k=1}^{n} \alpha_k^{(n+1)} \sin \lambda_k t + \sum_{k=1}^{n} \beta(\lambda_1, \cdots, \lambda_{n+1}) \sin \lambda_{n+1} t.$$

We need no direct algebraic argument to prove that $\alpha_{n+1}^{(n+1)} = \beta(\lambda_1, \ldots, \lambda_{n+1})$ in the second sum. Namely the commutativity of the convolution implies that for any permutation γ of the indices $\{1, \ldots, n+1\}$ we can write

$$\sum_{k \le n} \alpha_k^{(n+1)}(\lambda_1, \ldots, \lambda_{n+1}) \sin \lambda_k t + \beta(\lambda_1, \ldots, \lambda_{n+1}) \sin \lambda_{n+1} t \equiv$$

$$\equiv \sum_{k \le n} \alpha_k^{(n+1)}(\lambda_{\gamma(1)}, \ldots, \lambda_{\gamma(n+1)}) \sin \lambda_{\gamma(k)} t + \beta(\lambda_{\gamma(1)}, \ldots, \lambda_{\gamma(n+1)}) \sin \lambda_{\gamma(n+1)} t.$$

Comparing the coefficients of $\sin \lambda_1 t, \ldots, \sin \lambda_{n+1} t$, respectively, we conclude that

$$\alpha_k^{(n+1)}(\lambda_1, \ldots, \lambda_{n+1}) = \alpha_m^{(n+1)}(\lambda_{\gamma(1)}, \ldots, \lambda_{\gamma(n+1)}) \quad \text{if } k \le n \text{ and } \gamma(k) = m \le n,$$

$$\beta(\lambda_1, \ldots, \lambda_{n+1}) = \alpha_k^{(n+1)}(\lambda_{\gamma(1)}, \ldots, \lambda_{\gamma(n+1)}) \quad \text{if } k \le n \text{ and } \gamma(k) = n+1.$$

In particular (with γ transposing 1 and $n+1$),

$$\beta(\lambda_1, \ldots, \lambda_{n+1}) = \alpha_1^{(n+1)}(\lambda_{n+1}, \lambda_2, \ldots, \lambda_n, \lambda_1) = \frac{1}{\lambda_{n+1}} \prod_{j: j \ne n+1} \frac{1}{\lambda_{n+1}^2 - \lambda_j^2}.$$

We check from the definitions, that also $\alpha_{n+1}^{(n+1)}(\lambda_1, \ldots, \lambda_n) = \frac{1}{\lambda_{n+1}} \prod_{j: j \ne n+1} \frac{1}{\lambda_{n+1}^2 - \lambda_j^2}$ which completes the induction argument.

Remark. The equations (V$''$) can be solved by means of the *Laplace transform*

$$\mathcal{L}V = \mathcal{L}_t\{V(t)\} : 0 < s \mapsto \int\limits_{t=0}^{\infty} e^{-st} V(t)\ dt$$

well-defined for bounded(?) continuous functions $V : \mathbf{R}_{++}(= \{t \in \mathbf{R} : t > 0\}) \to \mathbf{Z}$ ranging in Banach spaces with finite norm integral ($\int_0^\infty \|V(t)\|\, dt < \infty$).

Namely, for the convulution $w*V : 0 < t \mapsto \int\limits_{s=0}^{t} w(t-s)V(s)\, ds = \int\limits_{s=0}^{t} w(s)V(t-s)\, ds$ of any couple $w \in \mathcal{C}_{\rm bded}(\mathbf{R}_{++}, \mathbf{C})$, $V \in \mathcal{C}_{\rm bded}(\mathbf{R}_{++}, \mathbf{Z})$ we always have $\mathcal{L}\big(w * V\big) = \big(\mathcal{L}w\big)\big(\mathcal{L}V\big)$.

It is well-known that the operator valued functions $[t \mapsto U^t]$, $[t \mapsto T^t]$ satisfy

$$\text{(L)} \qquad \|V(t)\| \leq M e^{\Omega t} \quad (t \in \mathbf{R}_{++}) \qquad \text{for some } M, \Omega > 1.$$

Thus, in view of (V$''$), for the scaled functions

$$\widetilde{w}(t) := e^{-\Omega t} w(t) = e^{-\Omega t}\big[T_{32}^t y\big], \quad \widetilde{U}_\ell(t) := e^{-\Omega t} U_{3,\ell}^t, \quad \widetilde{V}_\ell(t) := e^{-\Omega t} T_{3,\ell}^t$$

we have

$$\begin{aligned}
\widetilde{U}_\ell(t) &= e^{-\Omega t} U_{3,\ell}^t = e^{-\Omega t} \int\limits_{s=0}^{t} \big[T_{32}^{t-s} y\big] U_{3,\ell}^s\ ds + T_{3,\ell}^t = \\
&= \int\limits_{s=0}^{t} \big[e^{-\Omega(t-s)} T_{32}^{t-s} y\big]\big[e^{-\Omega s} U_{3,\ell}^s\big]\ ds + e^{-\Omega t} T_{3,\ell}^t = \\
&= \int\limits_{s=0}^{t} \widetilde{w}(t-s)\widetilde{U}_\ell(s)\ ds + \widetilde{V}_\ell(t) = \big[\widetilde{w} * \widetilde{U}_\ell\big](t) + \widetilde{V}_\ell(t)
\end{aligned}$$

with the consequence that $\mathcal{L}\widetilde{U}_\ell = \big(\mathcal{L}\widetilde{w}\big)\big(\mathcal{L}\widetilde{U}_\ell\big) + \mathcal{L}\widetilde{V}_\ell$, $\mathcal{L}\widetilde{U}_\ell = \big(1 - \mathcal{L}\widetilde{w}\big)^{-1} \mathcal{L}\widetilde{V}_\ell$. That is

$$\mathcal{L}_t\big\{e^{-\Omega t} U_{3,\ell}^t\big\} = \frac{\mathcal{L}_t\big\{e^{-\Omega t} T_{3,\ell}^t\big\}}{1 - \mathcal{L}_t\big\{e^{-\Omega t} T_{32}^t y\big\}} \qquad (\ell = 1, 2, 3).$$

We shall see that actually $w(t) = \int\limits_{r=0}^{t} y^{\rm T} U_0^r y\, dr$ $(t \in \mathbf{R}_+)$ where the operators U_0^r are linear isometries. Thus we can choose the scaling factor $\Omega > 1$ to be so large that $\max_t \|\widetilde{w}(t)\| < 1$ along with $\int\limits_{t=}^{\infty} \|\widetilde{w}(t)\|\, dt < 1$. Then we may apply the inverse of $\mathcal{L}$ with the result

$$\widetilde{U}_\ell(t)=\mathcal{L}^{-1}\left(\frac{\mathcal{L}\widetilde{V}_\ell}{1-\mathcal{L}\widetilde{w}}\right)=\mathcal{L}_s^{-1}\left(\frac{e^{-\omega s}\mathcal{L}\widetilde{V}_\ell(s)}{e^{-\omega s}\left[1-\mathcal{L}\widetilde{w}(s)\right]}\right)=\lim_{\omega\to 0+}\mathcal{L}_s^{-1}\left(\frac{e^{-\omega s}}{1-\mathcal{L}\widetilde{w}(s)}\right)*\mathcal{L}_s^{-1}\left(e^{\omega s}\widetilde{V}_\ell(s)\right).$$

Next we establish finite explicit formulas for T^t. It is convenient to use the block partitions

$$T^t=\begin{bmatrix}\widetilde{T}^t_{11} & 0\\ \widetilde{T}^t_{21} & \widetilde{T}^t_{22}\end{bmatrix}\ \text{ where }\ \widetilde{T}^t_{11}=\begin{bmatrix}T^t_{11} & T^t_{12}\\ T^t_{21} & T^t_{22}\end{bmatrix},\ \widetilde{T}^t_{21}=\begin{bmatrix}T^t_{31} & T^t_{32}\\ T^t_{41} & T^t_{42}\end{bmatrix},\ \widetilde{T}^t_{22}=\begin{bmatrix}T^t_{33} & T^t_{34}\\ T^t_{43} & T^t_{44}\end{bmatrix},$$
$$A=\begin{bmatrix}\widetilde{A}_{11} & 0\\ \widetilde{A}_{21} & \widetilde{A}_{22}\end{bmatrix}\ \text{ where }\ \widetilde{A}_{11}=\begin{bmatrix}-\rho & 0\\ -x & M_0\end{bmatrix},\ \widetilde{A}_{21}=\begin{bmatrix}-\varepsilon & y^{\mathrm{T}}\\ \rho & x^{\mathrm{T}}\end{bmatrix},\ \widetilde{A}_{22}=\begin{bmatrix}0 & 0\\ -\varepsilon & \rho\end{bmatrix}.$$

Notice that $[\widetilde{T}^t_{11}:t\in\mathbf{R}_+]$ and $[\widetilde{T}^t_{22}:t\in\mathbf{R}_+]$ are c_0-semigroups with the lower triangular generators $\widetilde{A}_{11}$ resp. $\widetilde{A}_{22}$. Furthermore $[-\rho]=\mathrm{gen}[e^{-\rho t}:t\in\mathbf{R}_+]$ and $M_0=\mathrm{gen}[U_0^t:t\in\mathbf{R}_+]$.

Therefore, according to [Stachó JMAA, Lemma],

$$\widetilde{T}^t_{11}=\begin{bmatrix}e^{-\rho t} & 0\\ -\int_{s=0}^t\left[e^{-\rho(t-s)}U_0^s x\right]ds & U_0^t\end{bmatrix},$$
$$\widetilde{T}^t_{22}=\begin{bmatrix}1 & 0\\ -\int_{s=0}^t e^{\rho(t-s)}\varepsilon\,ds & e^{\rho t}\end{bmatrix}=\begin{bmatrix}1 & 0\\ \rho^{-1}(1-e^{\rho t})\varepsilon & e^{\rho t}\end{bmatrix},$$
$$\widetilde{T}^t_{21}=\int\limits_{s=0}^{t}\widetilde{T}^{t-s}_{22}\widetilde{A}_{21}\widetilde{T}^s_{11}ds=\int\limits_{s=0}^{t}\widetilde{T}^{t-s}_{22}\begin{bmatrix}-\varepsilon & y^{\mathrm{T}}\\ \rho & x^{\mathrm{T}}\end{bmatrix}\widetilde{T}^s_{11}ds=$$
$$=\int\limits_{s=0}^{t}\begin{bmatrix}1 & 0\\ \rho^{-1}(1-e^{\rho(t-s)})\varepsilon & e^{\rho(t-s)}\end{bmatrix}\begin{bmatrix}-e^{-\rho s}\varepsilon-y^{\mathrm{T}}\left(\int_{r=0}^s e^{-\rho(s-r)}U_0^r dr\right)x & y^{\mathrm{T}}U_0^s\\ e^{-\rho s}\rho\ -x^{\mathrm{T}}\left(\int_{r=0}^s e^{-\rho(s-r)}U_0^r dr\right)x & x^{\mathrm{T}}U_0^s\end{bmatrix}ds.$$

In particular

$$T^t_{31}=\left[\widetilde{T}^t_{21}\right]_{11}=\int\limits_{s=0}^{t}(-\varepsilon)e^{-\rho s}\,ds-y^{\mathrm{T}}\Big(\int\limits_{s=0}^{t}\int\limits_{r=0}^{s}e^{-\rho(s-r)}U_0^r\ dr\ ds\Big)x=$$
$$=\varepsilon\rho^{-1}(e^{-\rho t}-1)-y^{\mathrm{T}}\Big(\int\limits_{r=0}^{t}\int\limits_{s=t-r}^{t}e^{-\rho(s-r)}U_0^r\ ds\ dr\Big)x,$$
$$T^t_{32}=\left[\widetilde{T}^t_{21}\right]_{12}=\int\limits_{s=0}^{t}\left[y^{\mathrm{T}}U_0^s\right]ds=y^{\mathrm{T}}\Big[\int\limits_{s=0}^{t}U_0^s\ ds\Big],$$
$$T^t_{41}=\left[\widetilde{T}^t_{21}\right]_{21}=\int\limits_{s=0}^{t}\left[\rho^{-1}(1-e^{\rho(t-s)})\varepsilon(-e^{-\rho s}\varepsilon)+e^{\rho(t-s)}e^{-\rho s}\rho\right]ds-$$
$$-\int\limits_{s=0}^{t}\Big[\rho^{-1}(1-e^{\rho(t-s)})\varepsilon y^{\mathrm{T}}\Big(\int_{r=0}^s e^{-\rho(s-r)}U_0^r dr\Big)x+e^{\rho(t-s)}x^{\mathrm{T}}\Big(\int_{r=0}^s e^{-\rho(s-r)}U_0^r dr\Big)x\Big]ds=$$
$$=\rho^{-1}t+\rho^{-2}(\varepsilon^2+\rho)(e^{\rho t}-e^{-\rho t})/2-y^{\mathrm{T}}\Big[\int\limits_{r=0}^{t}\int\limits_{s=t-r}^{t}\rho^{-1}(1-e^{\rho(t-s)})\varepsilon e^{-\rho(s-r)}U_0^r\ ds\ dr\Big]x-$$

$$-x^{\mathrm{T}}\Big[\int\limits_{r=0}^{t}\int\limits_{s=t-r}^{t} e^{\rho(t-s)}e^{-\rho(s-r)}U_0^r\ ds\ dr\Big]x =,$$

$$T_{42}^t = \big[\widetilde{T}_{21}^t\big]_{22} = \int\limits_{s=0}^{t}\Big[\rho^{-1}(1-e^{\rho(t-s)})\varepsilon y^{\mathrm{T}} + e^{\rho(t-s)}x^{\mathrm{T}}\Big]U_0^s ds =$$

$$= y^{\mathrm{T}}\Big[\int\limits_{s=0}^{t}\varepsilon\rho^{-1}(1-e^{\rho(t-s)})U_0^s ds\Big] + x^{\mathrm{T}}\Big[\int\limits_{s=0}^{t}e^{\rho(t-s)}U_0^s ds\Big].$$

It is well-known [Deddens, Stachó JMAA] that $[U_0^t : t \in \mathbf{R}_+]$ embeds into a c_0-group of isometries of some covering complex Hilbert space $\widehat{\mathbf{H}} \supset \mathbf{H}$ with conjugation. Thus

$$U_0^t z = \int\limits_{\lambda\in\mathbf{R}} e^{i\lambda t}\, dP(\lambda)\, z \qquad \big(z \in \mathrm{Re}(\mathbf{H})\big)$$

in terms of a spectral measure $P : \Lambda(\subset \mathbf{R}$ Borelian$)\to \big\{$orthogonal projections on $\widehat{\mathbf{H}}\big\}$.

Since the operators $U_0^t \equiv \overline{U_0^t}$ $(t \in \mathbf{R}_+)$ are real and unitary, necessarily

$$\int\limits_{\lambda\in\mathbf{R}} e^{i\lambda t}dP(\lambda) = \int\limits_{\lambda\in\mathbf{R}} e^{-i\lambda t}d\overline{P(\lambda)} = \int\limits_{\lambda\in\mathbf{R}} e^{i\lambda t}d\overline{P(-\lambda)} \qquad \text{for all} \quad t \geq 0.$$

We achieve formulas suitable for treating the entries $T_{k,\ell}^t$ which involve integrations of $[U_0^t : t \in \mathbf{R}_+]$ with the aid of the Laplace transform in terms of the functional calculus [Halmos] $\mathcal{F}_P : \mathcal{C}(\mathbf{R}) \to \mathcal{L}(\mathbf{H})$,

$$\mathcal{F}\varphi := \int\limits_{\lambda\in\mathbf{R}} \varphi(\lambda)\ dP(\lambda), \qquad \mathcal{F}_\lambda\Phi(\lambda,t) := \int\limits_{\lambda\in\mathbf{R}} \phi(\lambda,t)\ dP(\lambda).$$

Carrying out the integrations $\int_s, \int_r$, it is immediate that

$$T^t = \begin{bmatrix} e^{-\rho t} & 0 & 0 & 0 \\ \big[\mathcal{F}\tau_{12}^t\big]x & \big[\mathcal{F}_\lambda e^{i\lambda t}\big] & 0 & 0 \\ \tau_{31}^{t,0} + y^{\mathrm{T}}\big[\mathcal{F}\tau_{31}^{t,1}\big] & y^{\mathrm{T}}\big[\mathcal{F}\tau_{32}^t\big]x & 1 & 0 \\ \tau_{41}^{t,0} + x^{\mathrm{T}}\big[\mathcal{F}\tau_{41}^{t,1}\big] + y^{\mathrm{T}}\big[\mathcal{F}\tau_{41}^{t,2}\big] & x^{\mathrm{T}}\big[\mathcal{F}\tau_{42}^{t,1}\big] + y^{\mathrm{T}}\big[\mathcal{F}\tau_{42}^{t,2}\big] & (1-e^{\rho t})\frac{\varepsilon}{\rho} & e^{\rho t} \end{bmatrix} \quad \text{where}$$

$$\tau_{21}^t = -\int\limits_{s=0}^{t} e^{-\rho(t-s)}e^{i\lambda s}\, ds = -e^{-\rho t}\frac{(e^{i\lambda t}-1)}{i\lambda}, \quad \tau_{31}^{t,0}(\lambda) = \varepsilon\frac{(e^{-\rho t}-1)}{\rho},$$

$$\tau_{31}^{t,1}(\lambda) = -\frac{\big(e^{-\rho t} + e^{(\rho+i\lambda)t} - 2e^{i\lambda}\big) + i\lambda e^{i\lambda t}(e^{\rho t}-1)/\rho}{(2\rho+\lambda i)(\rho+\lambda i)}, \quad \tau_{32}^t(\lambda) = \frac{e^{i\lambda t}-1}{i\lambda},$$

$$\tau_{41}^{t,0}(\lambda) = \frac{t}{\rho} + \frac{(\varepsilon^2+\rho)(e^{\rho t}-e^{-\rho t})}{2\rho^2}, \quad \tau_{41}^{t,1}(\lambda) = \frac{\rho(3e^{i\lambda t} - 2e^{-\rho t} - e^{(2\rho+i\lambda)t}) + i\lambda e^{i\lambda t}(1-e^{2\rho t})}{2\rho(\rho+\lambda i)(3\rho+\lambda i)},$$

$$\tau_{41}^{t,2}(\lambda) = \varepsilon\frac{12i\rho^3(1-e^{i\lambda t}) + \rho^2\lambda\big(4e^{-\rho t} - 6e^{\rho t} + 2 + e^{i\lambda t}(2e^{2\rho t} - 6e^{\rho t} + 4)\big)}{2\rho^2\lambda(i\lambda+2\rho)(i\lambda-\rho)(i\lambda+3\rho)} +$$
$$+\varepsilon\frac{i\rho\lambda^2\big(e^{i\lambda t}(-3-e^{2\rho t}+4e^{\rho t}) - 5e^{\rho t} - 3e^{-\rho t} + 8\big)}{2\rho^2\lambda(i\lambda+2\rho)(i\lambda-\rho)(i\lambda+3\rho)} +$$
$$+\varepsilon\frac{\lambda^3\big(e^{\rho t} - 2 + e^{-\rho t} + e^{i\lambda t}(e^{2\rho t} - 2e^{\rho t} + 1)\big)}{2\rho^2\lambda(i\lambda+2\rho)(i\lambda-\rho)(i\lambda+3\rho)},$$
$$\tau_{42}^{t,1} = \frac{e^{\rho t} - e^{i\lambda t}}{\rho - i\lambda}, \quad \tau_{42}^{t,1} = \varepsilon\frac{-i\rho I - \lambda + \lambda e^{\rho t} + i\rho e^{\lambda t}}{\lambda(i\lambda-\rho)\rho}.$$

Finally we calculate the terms $U_{3,\ell}$ from $(*)$ and substitute them into (V'') to achieve the closing result.

Theorem. Let $[\Psi^t : t \in \mathbf{R}_+]$ be a c_0-semigroup of holomorphic Carathéodory isometries of the unit ball of the spin factor $\mathcal{S} := \mathrm{SPIN}(\mathbf{H}, \bar{\cdot})$ such that $\Phi^t(e) = e$ $(t \in \mathbf{R}_+)$ for some extreme point e of the unit ball. Then there exists a c_0-group $[\widehat{\Psi}^t : t \in \mathbf{R}]$ of holomorphic Carathéodory isometries of the unit ball of a spin factor $\widehat{\mathcal{S}} := \mathrm{SPIN}(\widehat{\mathbf{H}}, \bar{\cdot})$ with $\widehat{\mathbf{H}} \supset \mathbf{H}$ and with conjugation extending that in $\mathcal{S}$ with the dilation property

$$\Psi^t = \widehat{\Psi}^t|\mathbf{H} \qquad (t \in \mathbf{R}_+).$$

Furthermore the dilation group $[\widehat{\Psi}^t : t \in \mathbf{R}_+]$ is Möbius equivalent to a a c_0-group with Vesentini-generator of the form

$$G' = W\begin{bmatrix} -\rho & 0 & 0 & 0 \\ -x & \widehat{M_0} & y & 0 \\ -\varepsilon & y^{\mathrm{T}} & 0 & 0 \\ \rho & x^{\mathrm{T}} & -\varepsilon & \rho \end{bmatrix} W^{-1} \quad \text{with} \quad W := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & \widehat{I}_0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

where $\widehat{M_0} = -[\widehat{M_0}]^{\mathrm{T}}$ is a possibly unbounded skew-selfadjoint extension of the operator M_0 to $\widehat{\mathbf{H}}$ and $\widehat{I}_0 := \mathrm{Id}_{\widehat{\mathbf{H}} \ominus \mathbf{C}e}$. In terms of the spectral decomposition $\widehat{M_0} = \int_{\lambda \in \mathbf{R}} (i\lambda)\, dP(\lambda)$, the maps Φ^t can be written as finite rational expressions of the terms

$z, e^{\varepsilon t}, e^{\rho t}, x, y, x^T, y^{\mathrm{T}}, \int\limits_{\lambda\in\mathbf{R}} \tau_{k,\ell}^{t,j}\, dP(\lambda), \mathrm{Laplace}^{-1}(\mathrm{Laplace}(w_\Omega)/[1-\mathrm{Laplace}(w_\Omega))$

with a function $w_\Omega(t) := e^{-\Omega t} \int\limits_{s=0}^{t} \int\limits_{\lambda\in\mathbf{R}} e^{i\lambda s}\, d\langle y|P(\lambda)y\rangle\; ds$ for suitable large Ω.

Case (2) As we have seen, up to Möbius equivalence, we may assume that the Vesentini generator G' has the form

$$G' = \begin{bmatrix} 0 & 2\rho_2-\varepsilon & x_1^{\mathrm{T}} & \rho_1 & \rho_2 \\ \varepsilon-2\rho_2 & 0 & -x_2^{\mathrm{T}} & \rho_2 & -\rho_1 \\ -x_1 & x_2 & M_0 & x_1 & x_2 \\ \rho_1 & \rho_2 & x_1^{\mathrm{T}} & 0 & -\varepsilon \\ \rho_2 & -\rho_1 & x_2^{\mathrm{T}} & \varepsilon & 0 \end{bmatrix}.$$

We can take it into a convenient quasi lower triagular form as

$$T^{-1}G'T = \begin{bmatrix} -\rho_1 & \varepsilon-\rho_2 & 0 & 2\varepsilon & 0 \\ \rho_2-\varepsilon & -\rho_1 & 0 & 0 & 2\varepsilon \\ x_2 & -x_1 & M_0 & 0 & 0 \\ \rho_1 & \rho_1 & x_1^{\mathrm{T}} & \rho_1 & -\varepsilon-\rho_2 \\ -\rho_1 & -\rho_2 & x_2^{\mathrm{T}} & \rho_2+\varepsilon & \rho_1 \end{bmatrix} \quad \text{with} \quad T := \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & I_0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

In terms of $\big(\mathbf{C}^2 \oplus \mathbf{H} \ominus [\mathbf{C}u \oplus \mathbf{C}v] \oplus \mathbf{C}^2\big)$-blocking,

$$T^{-1}G'T = \begin{bmatrix} -\rho & 0 & 0 \\ x & M_0 & 0 \\ \mu & x^{\mathrm{T}} & \rho \end{bmatrix} \text{ with } \rho := \begin{bmatrix} \rho_1 & \varepsilon-\rho_2 \\ -(\varepsilon-\rho_2) & \rho_1 \end{bmatrix},\ \mu := \begin{bmatrix} \rho_1 & \rho_2 \\ \rho_2 & -\rho_1 \end{bmatrix}.$$

It follows (from the triangular lemma [Stachó JMAA]) that

$$G^t = \begin{bmatrix} \exp(-t\rho) & 0 & 0 \\ G_{21}^t & U_0^t & 0 \\ G_{31}^t & G_{32}^t & \exp(t\rho) \end{bmatrix} \qquad \text{with}$$

$$G_{21}^t = \int\limits_{s=0}^{t} U_0^s x \exp\big((s-t)\rho\big)\, ds, \qquad G_{32}^t = \int\limits_{s=0}^{t} \exp\big((t-s)\rho\big)x^{\mathrm{T}}U_0^s\, ds,$$

$$\big[G_{31}^t\ G_{32}^t\big] = \int\limits_{r=0}^{t} G_{33}^{t-r}\,\big[\mu\ \ x^{\mathrm{T}}\big] \begin{bmatrix} G_{11}^r & 0 \\ G_{21}^r & G_{22}^r \end{bmatrix} dr \quad \text{i.e.} \quad G_{31}^t = \int\limits_{r=0}^{t} G_{33}^{t-r}\,[\mu G_{11}^r + x^{\mathrm{T}} G_{21}^r]dr,$$

$$G_{31}^t = \int\limits_{r=0}^{t} \exp\big((t-r)\rho\big)\Big[\mu \exp\big(-r\rho\big) + \int\limits_{s=0}^{r} x^{\mathrm{T}}U_0^s x \exp\big((s-r)\rho\big)\, ds\Big]\, dr$$

Since $\exp\big(t\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]\big) = \left[\begin{smallmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{smallmatrix}\right]$, here we have

$$\exp\big(t\rho\big) = e^{\rho_1 t}\begin{bmatrix} \cos\big(t(\varepsilon-\rho_2)\big) & \sin\big(t(\varepsilon-\rho_2)\big) \\ -\sin\big(t(\varepsilon-\rho_2)\big) & \cos\big(t(\varepsilon-\rho_2)\big) \end{bmatrix}.$$

Problem. $z \in \mathbf{D} \Rightarrow^{?} \overline{z} \in \mathbf{D}$ $\quad$ $\big($i.e. $t \mapsto U_t z$ diff. $\Rightarrow^{?}$ $t \mapsto U_t \overline{z}$ diff.$\big)$ $\quad$ [YES]

Lemma. $\exists\, x \quad t \mapsto U_t x, U_t \overline{x}$ diff. $\implies \quad \exists\, t \mapsto \varepsilon_t \in \{\pm 1\} \quad t \mapsto \varepsilon_t \kappa_t$ diff.

Proof. $t \mapsto \overline{U_t \overline{x}} = \overline{\kappa_t V_t \overline{x}} = \overline{\kappa_t} V_t x$ diff.

$t \mapsto \langle \kappa_t V_t x | \overline{\kappa_t} V_t x \rangle = \kappa_t^2$ diff.

$\forall\, h \in \mathbf{R} \quad \exists\, I_h$ open intv. around h, $\qquad \mathrm{Re}(\kappa_t^2/\kappa_h^2) > 0$ $(t \in I_h)$

$\ldots, J_{-2}, J_{-1}, J_0, J_1, J_2, \ldots$ $\;$ chain of intervals $\;$ $J_k \subset I_{h_k}$ $(k = 0, \pm 1, \ldots)$

$\exists\, k \mapsto \nu_k \in \{\pm 1\} \qquad \varepsilon_t := \nu_k \mathrm{sgn}(\kappa_t/\kappa_h)$ $(t \in J_k)$ well-def. and suits

Corollary. $\mathbf{F} \cap \mathrm{conj}(\mathbf{F}) \neq 0 \Rightarrow \mathbf{F} = \mathrm{conj}(\mathbf{F})$

Proof. $0 \neq x \in \mathbf{F} \cap \mathrm{conj}(\mathbf{F}) \Rightarrow t \mapsto U_t x, U_t \overline{x}$ diff. $\Rightarrow t \varepsilon_t \kappa_t, \varepsilon_t \kappa_t^{-1}$ diff.

$z \in \mathbf{F} \Rightarrow t \mapsto \varepsilon_t V_t z = \varepsilon_t \kappa_t^{-1} U_t z$ diff. $\Rightarrow t \mapsto \mathrm{conj}\big(\varepsilon_t \kappa_t^{-1} V_t z\big) = U_t \overline{z}$ diff.

Proposition. $\mathbf{F}$ is closed under conjugation in any case.

Proof. The only case of a JB*-subtriple $\mathbf{H}$ such that $\mathbf{H} \cap \mathrm{conj}(\mathbf{H}) = 0$ is if $\mathbf{H}$ is a Hilbert space spanned by a collinear grid $\{2^{-1/2}(u_k + i v_k) : k \in \mathcal{K}\}$ where $\{a_k, b_k : k \in |\mathcal{K}\}$ is $\langle \cdot | \cdot \rangle$-orthononormed. Also $\mathrm{TRIP}(\mathbf{H}) = \{w + iT(w) : w \in \mathbf{G}, \langle w | w \rangle = 1/2\}$ with some subspace $\mathbf{G} \subset \mathrm{Re}(\mathbf{E})$ and an isometry $T : \mathrm{Sphere}(\mathbf{G}) \to \mathrm{Re}(\mathbf{E})$. The case $\mathbf{F} = \mathbf{H}$ is impossible: then $t \mapsto a_t = w_t + iT(w_t)$ diff. $\Rightarrow t \mapsto \overline{a_t} = w_t - iT(w_t)$ diff. $\Rightarrow \{a_t, \overline{a_t} : t \in \mathbf{R}\} \subset \mathbf{F}$.

Assumption without loss of gen.: $U_t = \kappa_t V_t, \quad t \mapsto \kappa_t$ diff.

Notation: $\mathbf{F}^{\perp} := \{x \in \mathbf{E} : \langle x | \mathbf{F} \rangle = 0\}$. $\quad (\neq \mathbf{F}^{\perp \mathrm{Jordan}})$

Proposition. $\mathbf{E} = \mathbf{F}$ $\;$ (i.e. $\mathbf{F}^{\perp} = 0$).

Proof. $\mathbf{F} = \mathrm{conj}(\mathbf{F}) \Rightarrow \mathbf{F}^\perp = \mathrm{conj}(\mathbf{F}^\perp)$ spin factor. $\dim(\mathbf{F}^\perp) > 0 \Rightarrow \exists\, y \in \mathbf{F}^\perp \quad 0 \neq y = \overline{y}$

Calculate $t \mapsto \Phi^t(y) = M_{a(t)} \circ U_t y$.

$M_a(x) = a + B(a)^{1/2}[1 + L(x,a)]^{-1}x, \quad B(a) = 1 - 2L(a) + Q_a^2 : z \mapsto z - 2\{aaz\} + \{a\{aza\}a\}$

$y \in \mathbf{F}^\perp,\ a \in \mathbf{F} \Rightarrow \qquad \langle y|f\rangle = \langle y|\overline{f}\rangle = 0\ (f \in \mathbf{F})$

$\{fgy\} = \langle f|g\rangle y + \langle y|g\rangle f - \langle y|\overline{f}\rangle \overline{g} = \langle f|g\rangle y,\ \ \{fyg\} = \langle f|y\rangle g + \langle g|y\rangle f - \langle g|\overline{f}\rangle \overline{y} = -\langle g|\overline{f}\rangle \overline{y}$

$x_1 + y_1 = (1 + L(y,a))^{-1} y$

$y = (1 + L(y,a))(x_1 + y_1) = x_1 + y_1 + \{yax_1\} + \{yay_1\}$

$0 = x_1 - \langle y|\overline{y_1}\rangle \overline{a} \quad (\mathbf{F}\text{-component}), \qquad y = y_1 + \langle x_1|a\rangle y \quad (\mathbf{F}^\perp\text{-component})$

$\gamma = \gamma(y, y_1) := \langle y|\overline{y_1}\rangle = \langle y_1|\overline{y}\rangle$

$x_1 = \langle y|\overline{y_1}\rangle \overline{a} = \gamma \overline{a}, \qquad y_1 = (1 - \langle x_1|a\rangle) y = (1 - \gamma\langle \overline{a}|a\rangle) y$

$$\gamma = \langle y_1|y\rangle = (1 - \gamma\langle \overline{a}|a\rangle)\langle y|\overline{y}\rangle, \quad \Longrightarrow \quad \gamma = \frac{\langle y|\overline{y}\rangle}{1 + \langle \overline{a}|a\rangle\langle y|\overline{y}\rangle}$$

$$[1 + L(y,a)]^{-1} y = x_1 + y_1 = \gamma\overline{a} + (1 - \gamma\langle a|\overline{a}\rangle) y = \frac{\langle y|\overline{y}\rangle \overline{a} + y}{1 + \langle \overline{a}|a\rangle\langle y|\overline{y}\rangle}$$

$z \perp \mathbf{F} \Rightarrow \quad B(a)z = z - 2\{aaz\} + \{a\{aza\}a\} = z - 2\langle a|a\rangle z + |\langle a|\overline{a}\rangle|^2 z$

$B(a)^{1/2} z = \beta(a) z \quad \beta(a) := \sqrt{1 - 2\langle a|a\rangle + |\langle a|\overline{a}\rangle|^2}$

$U_t y = \kappa_t V_t y, \quad t \mapsto \langle U_t y|\overline{U_t y}\rangle = \kappa_t^2 \langle y|\overline{y}\rangle$ diff.

$$\begin{aligned} t \mapsto \Phi^t(y) &= M_{a(t)} \circ U_t y = a(t) + B(a(t))^{1/2}[1 + L(U_t y, a(t))]^{-1} U_t y = \\ &= a(t) + \beta(a(t)) \frac{\langle y|\overline{y}\rangle \overline{a(t)} + U_t y}{1 + \langle \overline{a(t)}|a(t)\rangle\langle y|\overline{y}\rangle} \end{aligned}$$

IF $\dim(\mathbf{F}^\perp = 1$ THEN $V_t y = y$ and $T_t y = \kappa_t y \Longrightarrow \dim(\mathbf{F}^\perp) = 1$ impossible

CASE $\dim(\mathbf{F}^\perp) > 1$

We can find $y \in \mathbf{F}^{\perp}$ with $0 \neq y \perp \overline{y}$

Calculate $t \mapsto \Phi^t(x+y) = M_{a(t)} \circ U_t(x+y)$.

$M_a(x+y) = a + B(a)^{1/2}[1 + L(x+y,a)]^{-1}(x+y), \quad B(a) = 1 - 2L(a) + Q_a^2 : z \mapsto z - 2\{aaz\} + \{a\{aza\}a\}$

$y \in \mathbf{F}^{\perp},\ a \in \mathbf{F} \Rightarrow \qquad \langle y|f\rangle = \langle y|\overline{f}\rangle = 0\ (f \in \mathbf{F})$

$\{fgy\} = \langle f|g\rangle y + \langle y|g\rangle f - \langle y|\overline{f}\rangle\overline{g} = \langle f|g\rangle y,\ \ \{fyg\} = \langle f|y\rangle g + \langle g|y\rangle f - \langle g|\overline{f}\rangle\overline{y} = -\langle g|\overline{f}\rangle\overline{y}$

$x_1 + y_1 = (1 + L(x+y,a))^{-1}(x+y)$

$x + y = (1 + L(x+y,a))(x_1+y_1) = x_1 + y_1 + \{xax_1\} + \{xay_1\} + \{yax_1\} + \{yay_1\}$

$x = x_1 + \{xax_1\} - \langle y|\overline{y_1}\rangle\overline{a}$ ($\mathbf{F}$-component), $\qquad y = y_1 + \langle x|a\rangle y_1 + \langle x_1|a\rangle y$ ($\mathbf{F}^{\perp}$-component)

$\gamma_0 = \gamma_0(x_1,a) := \big(1 - \langle x_1|a\rangle\big)/(1 + \langle x|a\rangle\big)$

$y_1 = \gamma_0 y$

Consider vectors y with $0 \neq y \perp \overline{y}$: $\quad x = x_1 + \{xax_1\} - \langle y\big|\overline{\gamma_0\, y}\rangle\overline{a} = x_1 + \{xax_1\}$

$$x_1 = [1 + L(x,a)]^{-1}x, \qquad y_1 = \frac{1 - \langle [1 + L(x,a)]^{-1}x|a\rangle}{1 + \langle x|a\rangle} = \gamma(x,a)y$$

$x_2 + y_2 = B(a)^{1/2}(x_1 + y_1)$

$$\begin{aligned} M_a(x+y) &= a + B(a)^{1/2}(x_1+y_1) = a + B(a)^{1/2}\big([1 + L(x,a)]^{-1}x + \gamma(x,a)y\big] = \\ &= M_a(x) + \gamma(x,a)B(a)^{1/2}y \qquad \text{if } y \perp \overline{y} \in \mathbf{F}^{\perp} \end{aligned}$$

$z \perp \mathbf{F} \Rightarrow \quad B(a)z = z - 2\{aaz\} + \{a\{aza\}a\} = z - 2\langle a|a\rangle z + |\langle a|\overline{a}\rangle|^2 z$

$B(a)^{1/2}z = \beta(a)z \quad \beta(a) := \sqrt{1 - 2\langle a|a\rangle + |\langle a|\overline{a}\rangle|^2}$

If $y \perp \overline{y} \in \mathbf{F}^{\perp}$ then $U_t y \in \mathbf{F}^{\perp}$, $\langle U_t y|\overline{U_t y}\rangle = \langle \kappa_t V_t|\overline{\kappa_t V_t y}\rangle = \kappa_t^2\langle y|\overline{y}\rangle = 0$,
$\Phi^t(x+y) = M_a(U_t x + U_t y) = M_{a(t)}(U_t x) + \beta(a(t))\gamma(U_t x, a(t))U_t y =$

$$= \Phi^t(x) + \beta(a(t))\gamma(U_t x, a(t))U_t y$$

$\gamma(0,a) \equiv 0$, $t \mapsto a(t)$ diff. $\Rightarrow$

$t \mapsto \Phi^t(y) = \underbrace{\Phi^t(0)}_{a(t)} + \beta(a(t))y$ diff. whenever $y \perp \overline{y} \in \mathrm{Ball}(\mathbf{F}^\perp)$

Thus $0 \neq y \in \mathbf{F}^\perp = 0$ contradiction if we assume $\dim(\mathbf{F}^\perp) > 1$

Proof. $\mathbf{F} = \mathrm{conj}(\mathbf{F}) \Rightarrow \mathbf{F}^\perp = \mathrm{conj}(\mathbf{F}^\perp)$ spin factor. $\dim(\mathbf{F}^\perp) > 1 \Rightarrow \exists\, y \in \mathbf{F}^\perp \quad 0 \neq y \perp \overline{y}$

Calculate the effect of $\Phi^t = M_{a(t)} \circ U_t$ on $\mathbf{F}^\perp$.

$M_a(x) = a + B(a)^{1/2}[1 + L(x,a)]^{-1}x, \quad B(a) = 1 - 2L(a) + Q_a^2 : z \mapsto z - 2\{aaz\} + \{a\{aza\}a\}$

$y \in \mathbf{F}^\perp \Rightarrow \qquad \langle y|f\rangle = \langle y|\overline{f}\rangle = 0 \ (f \in \mathbf{F})$

$\{fgy\} = \langle f|g\rangle y + \langle y|g\rangle f - \langle y|\overline{f}\rangle \overline{g} = \langle f|g\rangle y, \ \ \{fyg\} = \langle f|y\rangle g + \langle g|y\rangle f - \langle g|\overline{f}\rangle \overline{y} = -\langle g|\overline{f}\rangle \overline{y}$

$x_1 + y_1 = (1 + L(x+y, a))^{-1}(x+y)$

$x + y = (1 + L(x+y,a))(x_1 + y_1) = x_1 + y_1 + \{xax_1\} + \{xay_1\} + \{yax_1\} + \{yay_1\}$

$x = x_1 + \{xax_1\} - \langle y|\overline{y_1}\rangle \overline{a}, \qquad y = y_1 + \langle x|a\rangle y_1 + \langle x_1|a\rangle y$

$y_1 = \dfrac{1 - \langle x_1|a\rangle}{1 + \langle x|a\rangle} y = \gamma_0(x_1, a) y$

Consider vectors y with $y \perp \overline{y}$: $\quad x = x_1 + \{xax_1\} - \langle y|\overline{\gamma_0 y}\rangle \overline{a} = x_1 + \{xax_1\}$

$x_1 = [1 + L(x,a)]^{-1}x, \quad y_1 = \dfrac{1 - \langle [1 + L(x,a)]^{-1}x|a\rangle}{1 + \langle x|a\rangle} y \ \underline{= \gamma(x,a)y} \quad (y \perp \overline{y})$

$x_2 + y_2 = B(a)^{1/2}(x_1 + y_1)$

$$M_a(x+y) = a + B(a)^{1/2}(x_1 + y_1) = a + B(a)^{1/2}\big([1 + L(x,a)]^{-1}x + \gamma(x,a)y\big] =$$
$$= M_a(x) + \gamma(x,a)B(a)^{1/2}y \qquad \text{if } y \perp \overline{y} \in \mathbf{F}^\perp$$

$z \perp \mathbf{F} \Rightarrow \quad B(a)z = z - 2\{aaz\} + \{a\{aza\}a\} = z - 2\langle a|a\rangle z + |\langle a|\overline{a}\rangle|^2 z$

$B(a)^{1/2}z = \beta(a)z \quad \beta(a) := \sqrt{1 - 2\langle a|a\rangle + |\langle a|\overline{a}\rangle|^2}$

If $y \perp \overline{y} \in \mathbf{F}^\perp$ then $U_t y \in \mathbf{F}^\perp$, $\langle U_t y | \overline{U_t y}\rangle = \langle \kappa_t V_t | \overline{\kappa_t V_t y}\rangle = \kappa_t^2 \langle y|\overline{y}\rangle = 0$,

$$\begin{aligned}\Phi^t(x+y) &= M_a(U_t x + U_t y) = M_{a(t)}(U_t x) + \beta(a(t))\gamma(U_t x, a(t))U_t y = \\ &= \Phi^t(x) + \beta(a(t))\gamma(U_t x, a(t))U_t y\end{aligned}$$

$\gamma(0,a) \equiv 0$, $t \mapsto a(t)$ diff. $\Rightarrow$

$t \mapsto \Phi^t(y) = \underbrace{\Phi^t(0)}_{a(t)} + \beta(a(t))y$ diff. whenever $y \perp \overline{y} \in \mathrm{Ball}(\mathbf{F}^\perp)$

Thus $0 \neq y \in \mathbf{F}^\perp = 0$ contradiction if we assume $\dim(\mathbf{F}^\perp) > 1$

$x_1 := \Phi^t(x)$, $y_1 := \beta(a(t))\gamma(U_t x, a(t))U_t y \qquad \langle y_1 | \overline{y_1}\rangle = 0$

$$\begin{aligned}\Phi^{t+h}(x+y) &= \Phi^h\big(\Phi^t(x+y)\big) = \Phi^h(x_1+y_1) = \Phi^h(x_1) + \beta(a(h))\gamma(U_h x_1, a(h))U_h y_1 = \\ &= \Phi^{t+h}(x) + \beta(a(h))\gamma\big((U_h \Phi^t(x), a(h)\big)\beta(a(t))\gamma\big((U_t x, a(t)\big)U_h U_t y\end{aligned}$$

$$\Phi^{t+h}(x+y) = \Phi^{t+h}(x) + \beta(a(t+h))\gamma\big(U_{t+h}x, a(t+h)\big)U_{t+h}y$$

$$U_h U_t y = \frac{\beta(a(h))\gamma\big(U_h \Phi^t(x), a(h)\big)\beta(a(t))\gamma\big(U_t x, a(t)\big)}{\beta(a(t+h))\gamma\big(U_{t+h}x, a(t+h)\big)} U_{t+h} \quad \big(\mathrm{Span}\{\text{admissible } y\} = \mathbf{F}^\perp\big)$$

$x := 0 \Rightarrow \quad x_1 = \Phi^t(x) = a(t)$, $\Phi^h(x_1) = a(t+h)$, $\gamma(0,a) = 1$

$$U_h U_t = \lambda(h,t)U_{t+h}, \qquad \lambda(h,t) := \frac{\beta(a(h))\gamma\big(U_h a(t), a(h)\big)\beta(a(t))}{\beta(a(t+h))}$$

Formula for Möbius transformations in SPIN factor

$M_a(x) = a + B(a)^{1/2}[1 + L(x,a)]^{-1}x$

Consider the case when $\{a, \overline{a}, z, \overline{z}\}$ ORTN wrt. $\langle\cdot|\cdot\rangle$ and $\langle a|a\rangle = \langle z|z\rangle = 1/2$.

Well-known: $a, \overline{a}, z, \overline{z}$ TRIPs, moreover

$J_{a,z} : \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right] \mapsto \alpha a + \beta z + \gamma \overline{z} + \delta \overline{a}$ JB*-isom. $\mathrm{Mat}(2,2,\mathbf{C}) \leftrightarrow \mathrm{Span}\{a, z, \overline{z}, \overline{a}\}$

Hence, with $A := \left[\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right]$, $X := \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]$,

$$M_{\lambda a + \mu \overline{a}}(\alpha a + \beta z + \gamma \overline{z} + \delta \overline{a}) = J_{a,z} M_{\left[\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right]}\left(\left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]\right) =$$

$$= J_{a,z}\Big([1 - AA^*]^{-1/2}\big(X + A\big)\big(1 + A^*X\big)^{-1}[1 - A^*A]^{1/2}\Big) =^{\text{vazlat2.mws}}=$$

$$= J_{a,z} \begin{bmatrix} \frac{-\alpha - \alpha\overline{\mu}\delta - \lambda - \lambda\overline{\mu}\delta + \beta\overline{\mu}\gamma}{-1 - \overline{\mu}\delta - \overline{\lambda}\alpha - \overline{\lambda}\alpha\overline{\mu}\delta + \overline{\lambda}\beta\overline{\mu}\gamma} & \frac{\beta(\lambda\overline{\lambda} - 1)\sqrt{1 - \mu\overline{\mu}}}{\sqrt{1 - \lambda\overline{\lambda}}(-1 - \overline{\mu}\delta - \overline{\lambda}\alpha - \overline{\lambda}\alpha\overline{\mu}\delta + \overline{\lambda}\beta\overline{\mu}\gamma)} \\ \frac{\gamma(-1 + \mu\overline{\mu})\sqrt{1 - \lambda\overline{\lambda}}}{\sqrt{1 - \mu\overline{\mu}}(-1 - \overline{\mu}\delta - \overline{\lambda}\alpha - \overline{\lambda}\alpha\overline{\mu}\delta + \overline{\lambda}\beta\overline{\mu}\gamma)} & \frac{\overline{\lambda}\beta\gamma - \delta - \overline{\lambda}\alpha\delta - \mu - \mu\overline{\lambda}\alpha}{-1 - \overline{\mu}\delta - \overline{\lambda}\alpha - \overline{\lambda}\alpha\overline{\mu}\delta + \overline{\lambda}\beta\overline{\mu}\gamma} \end{bmatrix} =$$

$$= J_{a,z} \begin{bmatrix} \frac{\alpha + \alpha\overline{\mu}\delta + \lambda + \lambda\overline{\mu}\delta - \beta\overline{\mu}\gamma}{1 + \overline{\mu}\delta + \overline{\lambda}\alpha + \overline{\lambda}\alpha\overline{\mu}\delta - \overline{\lambda}\beta\overline{\mu}\gamma} & \frac{\beta\sqrt{(1 - |\lambda|^2)(1 - |\mu|^2)}}{1 + \overline{\mu}\delta + \overline{\lambda}\alpha + \overline{\lambda}\alpha\overline{\mu}\delta - \overline{\lambda}\beta\overline{\mu}\gamma} \\ \frac{\gamma\sqrt{(1 - |\lambda|^2)(1 - |\mu|^2)}}{1 + \overline{\mu}\delta + \overline{\lambda}\alpha + \overline{\lambda}\alpha\overline{\mu}\delta - \overline{\lambda}\beta\overline{\mu}\gamma} & \frac{\delta + \overline{\lambda}\alpha\delta + \mu + \mu\overline{\lambda}\alpha - \overline{\lambda}\beta\gamma}{1 + \overline{\mu}\delta + \overline{\lambda}\alpha + \overline{\lambda}\alpha\overline{\mu}\delta - \overline{\lambda}\beta\overline{\mu}\gamma} \end{bmatrix}$$

$$= \frac{1}{1 + \overline{\mu}\delta + \overline{\lambda}\alpha + \overline{\lambda}\alpha\overline{\mu}\delta - \overline{\lambda}\beta\overline{\mu}\gamma} J_{a,z} \begin{bmatrix} \alpha + \alpha\overline{\mu}\delta + \lambda + \lambda\overline{\mu}\delta - \beta\overline{\mu}\gamma & \beta\sqrt{(1 - |\lambda|^2)(1 - |\mu|^2)} \\ \gamma\sqrt{(1 - |\lambda|^2)(1 - |\mu|^2)} & \delta + \overline{\lambda}\alpha\delta + \mu + \mu\overline{\lambda}\alpha - \overline{\lambda}\beta\gamma \end{bmatrix}$$

$$= \frac{1}{(1 + \overline{\mu}\delta)(1 + \overline{\lambda}\alpha) - \overline{\lambda}\beta\overline{\mu}\gamma} J_{a,z} \begin{bmatrix} \alpha + \alpha\overline{\mu}\delta + \lambda + \lambda\overline{\mu}\delta - \beta\overline{\mu}\gamma & \beta\sqrt{(1 - |\lambda|^2)(1 - |\mu|^2)} \\ \gamma\sqrt{(1 - |\lambda|^2)(1 - |\mu|^2)} & \delta + \overline{\lambda}\alpha\delta + \mu + \mu\overline{\lambda}\alpha - \overline{\lambda}\beta\gamma \end{bmatrix}$$

$(X + A)(1 + A^*X)^{-1} = \frac{1}{\det(A)}(X + A)[1 + (A^*X)^\sim]$ where $\left[\begin{smallmatrix} \xi & \eta \\ \zeta & \omega \end{smallmatrix}\right]^\sim := \left[\begin{smallmatrix} \omega & -\eta \\ -\zeta & \xi \end{smallmatrix}\right]$

Fractional linear approach to spin factors

$\mathbf{H}$ Hilbert space (no conjugation is fixed)

Remark. The general form for spin factors is the following:

A subtriple $\mathbf{S}$ of $\mathcal{L}(\mathbf{H})$ is a spin factor if (and only if) $S^2 \in \mathbf{C}\,\mathrm{id}_{\mathbf{H}},\ S^* \in \mathbf{S}$ whenever $S \in \mathbf{S}$.

In the case the conjugation on $\mathbf{S}$ is simply taking adjoints,

the scalar product on $\mathbf{S}$ is given by $\quad \langle A \big| B\rangle \mathrm{id}_{\mathbf{H}} = \frac{1}{2}\big(AB^* + B^*A\big)$.

By a result of [Upmeier], every J*-derivation of $\mathbf{S}$ is a weak*-limit of linear combinations

$$X \mapsto \sum_j i\big\{A_j A_j^* X\big\} = \frac{i}{2}\sum_j \big[A_j A_j^* X + X A_j^* A_j\big].$$

Since the left and right multiplication operators $L_Z : X \mapsto ZX$ resp. $X \mapsto XZ$ commute,

we have

$$\exp\Big[X \mapsto \sum_j i\big\{A_j A_j^* X\big\}\Big] = \exp\Big(\sum_j iA_j A_j^*\Big) X \exp\Big(\sum_j iA_j^* A_j\Big).$$

Since all surjective linear isometries of a JB*-triple are exponentials of J*-derivations

[Kaup], it follows that

$$\mathcal{U} \text{ is a surj. lin. } \mathbf{S}\text{-isometry} \iff \exists U, V\ \mathbf{H}\text{-unitary } U\mathbf{S}V = \mathbf{S}, \mathcal{U} = U \otimes V : X \mapsto UXV.$$

In particular, every holomorphic automorphism Φ of Ball($\mathbf{S}$) has the form

$$\Phi = M_A \circ \mathcal{U} = [X \mapsto UM_A(X)V].$$

Observe [Isidro-Stacho] that

$$M_A : X \mapsto (1 - AA^*)^{-1/2}(X + A)(1 - A^*X)^{-1}(1 - A^*A)^{1/2}$$

is of fractional linear form extending automatically to Ball($\mathcal{L}(\mathbf{H})$).

Question. Are the non-surjective linear isometries of $\mathbf{S}$ of the form $U \otimes V$?

We shall identify the operators in $\mathbf{S}$ with their matrices with respect to ortonormed basis in $(\mathbf{H})$. Actually this means that

....

....

Lemma. Suppose $\mathbf{K}$ is a Hilbert space and $R, S \in \mathcal{L}(\mathbf{K})$ are orthogonal reflections (self-adjoint operators with $R^2 = S^2 = 1$) such that $RS + SR = 0$. Then there exist a unitary operator $W \in \mathcal{L}(\mathbf{K})$ such that, in matrix form, we can write

$R = U \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} U^*, \quad S = U \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} U^*.$

Proof. The two eigensubspaces $\mathbf{K}^{(\varepsilon)} := \{\mathbf{x} : Rx = \varepsilon\mathbf{x}\}$ $(\varepsilon = \pm 1)$ or R span the underlying space orthogonally: $\mathbf{K} = \mathbf{K}^{(1)} \oplus \mathbf{K}^{(-1)}$ and hence R has the matrix form

$R = V \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} V^*$ with some unitary operator $U \in \mathcal{L}(\mathbf{K})$.

In terms of the decomposition $\mathbf{K} = \mathbf{K}^{(1)} \oplus \mathbf{K}^{(-1)}$, we can write $S = V \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} V^*$ where $s_{11} = s_{11}^*$, $s_{22} = s_{22}^*$ and $s_{21} = s_{12}^*$ because $S = S^*$.

Then the relation $RS + SR = 0$ means that we have

$$0 = (V^*RV)(V^*SV) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} + \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2s_{11} & 0 \\ 0 & 2s_{22} \end{bmatrix}$$

implying $\quad S = V \begin{bmatrix} 0 & s_{12} \\ s_{12}^* & 0 \end{bmatrix} V^*$. Since $S^2 = 1$ i.e. $(V^*SV)^2 = 1$, also

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & s_{12} \\ s_{12}^* & 0 \end{bmatrix}^2 = \begin{bmatrix} s_{12}s_{12}^* & 0 \\ 0 & s_{12}^*s_{12} \end{bmatrix} \quad \text{i.e.} \quad S \text{ is an isometry } \mathbf{K}^{(1)} \leftrightarrow \mathbf{K}^{(-1)}.$$

In matrix terms it follows that s_{12} is a unitary operator: $s_{12}s_{12}^* = s_{12}^*s_{12} = 1(= \mathrm{Id})$ and

we have the unitary equivalence

$$\begin{bmatrix} 0 & s_{12} \\ s_{12}^* & 0 \end{bmatrix} = \begin{bmatrix} s_{12} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s_{12}^* & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s_{12} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s_{12} & 0 \\ 0 & 1 \end{bmatrix}^{-1}.$$

Hence we obtain the statement of the lemma with the unitary operator $U := V \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Lemma. Let $\mathbf{H} = \mathbf{H}_1 \oplus \mathbf{H}_1$ be an orthogonal decomposition and let $A, B, C, D \in \mathrm{Re}(\mathbf{S})$

be an orthonormed set such that $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then we can find a

unitary operator $U = \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$ such that

$$UAU^* = A, \quad UBU^* = B, \quad UCU^* = \begin{bmatrix} 0 & \begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix} \\ \begin{smallmatrix} -i & 0 \\ 0 & i \end{smallmatrix} & 0 \end{bmatrix}, \quad UDU^* = \begin{bmatrix} 0 & \begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix} \\ \begin{smallmatrix} 0 & -i \\ -i & 0 \end{smallmatrix} & 0 \end{bmatrix}$$

with respect to some orthogonal decomposition $\mathbf{H}_1 = \mathbf{H}_2 \oplus \mathbf{H}_2$.

Proof. We can write $C = \left[\begin{smallmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{smallmatrix}\right], \quad D = \left[\begin{smallmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{smallmatrix}\right]$ with suitable operators $c_{k\ell}, d_{k\ell} \in$

$\mathcal{L}(\mathbf{H}_1)$. The relation $C \perp A$ means that

$$0 = 2\langle A|C\rangle = AC^* + C^*A = AC + CA = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]\left[\begin{smallmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{smallmatrix}\right] + \left[\begin{smallmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 2c_{11} & 0 \\ 0 & 2c_{22} \end{smallmatrix}\right]$$

implying $c_{11} = c_{22} = 0$. The operator C is self-adjoint as belonging to $\mathrm{Re}(\mathbf{S})$. Hence

$C = \left[\begin{smallmatrix} 0 & c_{12} \\ c_{12}^* & 0 \end{smallmatrix}\right]$. The consequence of the realtion $C \perp B$ is

$$0 = 2\langle B|C\rangle = BC^* + B^*C = BC + CB = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & c \\ c^* & 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 & c \\ c^* & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} c_{12}^*+c_{12} & 0 \\ 0 & c_{12}+c_{12}^* \end{smallmatrix}\right]$$

implying that $c_{12} = ic$ for some self-adjoint operator $c \in \mathcal{L}(\mathbf{H}_1)$.

Also, by assumption, we have $C^2 = 1(= \mathrm{Id}_{\mathbf{H}})$ that is $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & ic \\ -ic & 0 \end{smallmatrix}\right]^2 = \left[\begin{smallmatrix} c^2 & 0 \\ 0 & c^2 \end{smallmatrix}\right]$.

It follows $c^2 = 1(= \mathrm{Id}_{\mathbf{H}_1})$, thus is the operator c is an orthogonal reflection.

Similar arguments apply for D. Therefore

$$C = \begin{bmatrix} 0 & ic \\ -ic & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & id \\ -id & 0 \end{bmatrix} \quad \text{with} \quad c = c^*, d = d^*, c^2 = d^2 = 1.$$

Finally we proceed to the consequences of the relation $C \perp D$:

$$0 = 2\langle C|D\rangle = CD + DC = \begin{bmatrix} cd + dc & 0 \\ 0 & cd + dc \end{bmatrix}.$$

We can apply the previous lemma with $R := c$ and $S := d$ with the conclusion that

$c = u \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} u^*$, $d = u \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} u^*$ for some unitary $u \in \mathcal{L}(\mathbf{H}_1$.

We can check by immediate calculation that the statement of the lemma holds with the unitary operator matrix $U := \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$.

FRACTIONAL LINEAR FORMS

$\mathcal{A} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{L}(\mathbf{H}_1, \mathbf{H}_2),$

$\mathcal{F}(\mathcal{A}) : X \mapsto (AX + B)(CX + D)^{-1} = \big[\mathcal{A}(X\ 1)^{\mathrm{T}}\big]_1 \big[\mathcal{A}(X\ 1)^{\mathrm{T}}\big]_2^{-1}$

$\mathcal{F}(\mathcal{A}\mathcal{B}) = \mathcal{F}(\mathcal{A}) \circ \mathcal{F}(\mathcal{B})$

$M_a = \mathcal{F}(\mathcal{M}_a), \quad \mathcal{M}_a = \mathrm{diag}\begin{pmatrix} (1 - aa^*)^{-1/2} \\ (1 - a^*a)^{-1/2} \end{pmatrix} \begin{bmatrix} 1 & a \\ a^* & 1 \end{bmatrix}$

Surj. lin. isom: $X \mapsto UXV^*$, unitary $U \in \mathcal{L}(\mathbf{H}_1)$, unitary $V \in \mathcal{L}(\mathbf{H}_2)$

$\Phi^t := \mathcal{F}(\mathcal{A}_t), \qquad \big[\phi^t : t \in \mathbf{R}\big]$ str.cont,1prg.

$\mathcal{A}_t = \mathcal{M}_{a(t)} \mathrm{diag}(U_t, V_t)$

Attention: $U \otimes V^* = \mathcal{F}\big(\mathrm{diag}(U, V)\big) = \mathcal{F}\big(\kappa\, \mathrm{diag}(U, V)\big)$ with any $\kappa \in \mathbf{T}$

Adjusted str.cont.: [Stachó JMAA 2010, Cor. 2.6] can be applied with linear isomeries instead of unitary operators

$\exists\, t \mapsto \kappa(t) \in \mathbf{T} \quad t \mapsto \kappa(t)U_t, t \mapsto \kappa(t)V_t$ str.cont.

Case of $\mathbf{E} = \mathcal{L}(\mathbf{H}_1, \mathbf{H}_2)$ with $r := \dim(\mathbf{H}_2) < \infty$

We consider only str.cont.1-prgroups $[\Psi^t : t \in \mathbf{R}]$ in $\mathrm{Aut}(\mathbf{B})$

Recall. $\Psi^t = M_{a(t)} \circ U_t,\ \ a(t) = \Psi^t(0),$

$M_a : x \mapsto [1 - aa^*]^{-1/2}(x + a)[1 + a^*x]^{-1}[1 - a^*a]^{1/2}, \qquad U_t : X \mapsto u_t X v_t^*$ (u_t, v_t unitary)

Strong continuity: $\Psi^t(x) = x + o^{\mathrm{norm}}(1) = x + g_t,\ \ g_t \to 0$ ($t \to 0$)

Remark. If $[\Psi^t : t \in \mathbf{R}_+]$ is a str.cont.1-prsemigroup in of Carathéodory isometries of $\mathbf{B}$ then, by [Vesentini (1994), Thm. 4.3 (p.539)], we have the same formula with each u_t

being a linear not necessarily surjective isometry.

$a(t+h) = a(t) + o^{\mathrm{norm}}(1), \quad M_{a(t+h)}(x) = M_{a(t)}(x) + g_{t,h,x}, \quad \sup_{\|x\|\le 1} \|g_{t,h,x}\| = o(1)$ for $h \to 0$

$\forall\, \varepsilon > 0 \;\; \exists\, \delta > 0 \quad \Psi^t : (1+\delta)\mathbf{B} \to (1+\varepsilon)\mathbf{B}$ well-defined $(|t| < \delta)$

$M_a^{-1} = M_{-a}, \quad t \mapsto U_t = M_{-a(t)} \circ \Psi^t$ str.cont.

[Stachó JMAA 2010, Cor.2.6] $\Rightarrow \exists\, t \mapsto \kappa(t) \in \mathbf{T} \;\; t \mapsto \kappa(t) u_t, \kappa(t) v_t$ str.cont. (pointwise cont)

$\mathcal{F}\begin{pmatrix} A & B \\ C & D \end{pmatrix} : x \mapsto (Ax+B)(Cx+D)^{-1}$

$\Psi^t = \mathcal{F}\mathrm{diag}\begin{bmatrix} (1-a(t)a(t)^*)^{-1/2} \\ (1-a(t)^*a(t))^{-1/2} \end{bmatrix} \begin{bmatrix} 1 & a(t) \\ a(t)^* & 1 \end{bmatrix} \mathrm{diag}\begin{bmatrix} \kappa(t)u_t \\ \kappa(t)v_t \end{bmatrix}$

$\Psi^t = \mathcal{F}\begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix}, \quad t \mapsto A_t, B_t, C_t, D_t$ str.cont. determined up to a cont. factor $t \mapsto \kappa(t) \in \mathbf{T}$

$\Psi^{t+h} = \Psi^t \circ \Psi^h \Longrightarrow \begin{bmatrix} A_{t+h} & B_{t+h} \\ C_{t+h} & D_{t+h} \end{bmatrix} = \lambda(t,h) \begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} \begin{bmatrix} A_h & B_h \\ C_h & D_h \end{bmatrix} \quad \exists!\; \lambda(t,h) \in \mathbf{T}$

Assumptions without loss of gen. *up to Möbius equ.*:

(0) $0 \in \mathrm{dom}(\Psi')$ i.e. $t \mapsto a(t) = \Psi^t(0)$ diff.

(1) $\mathcal{A}_t \mathcal{A}_h = \lambda(t,h) \mathcal{A}_{t+h}, \qquad \lambda(t,h) \in \mathbf{T} = \{\zeta \in \mathbf{C} : |\zeta| = 1\}$

(2) $\mathcal{A}_t = \begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} \qquad t \mapsto A_t, B_t, C_t, D_t$ str.cont. $\quad \mathcal{A}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(3)* $\exists$ common fixed point (by reflexivity): $\mathcal{F}(\mathcal{A}_t)E = E \quad (t \in \mathbf{R})$.

$\lambda(t,h) = \mathcal{A}_{-(t+h)} \mathcal{A}_t \mathcal{A}_h$ cont. in t, h (prod. of unif.bded. str.cont. lin. maps)

$\Psi^t(E) = E, \quad E = \mathcal{F}(\mathcal{A}_t)(E) = \mathcal{F}\begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix}(E) = (A_t E + B_t)(C_t E + D_t)^{-1}$

$A_t E + B_t = \left[\mathcal{A}_t \begin{pmatrix} E \\ 1 \end{pmatrix}\right]_1, \quad C_t E + D_t = \left[\mathcal{A}_t \begin{pmatrix} E \\ 1 \end{pmatrix}\right]_2$

$S_t := [\mathcal{A}_t (E\ 1)^{\mathrm{T}}]_2 = C_t E + D_t.$

$\mathcal{A}_t (E\ 1)^{\mathrm{T}} = \begin{bmatrix} A_t E + B_t \\ C_t E + D_t \end{bmatrix} = \begin{bmatrix} E S_t \\ S_t \end{bmatrix} = (E\ 1)^{\mathrm{T}} S_t,$

$S_t S_h = \lambda(t,h) S_{t+h}$

Lemma. $[S_t : t \in \mathbf{R}]$ Abelian family, $\quad \lambda(t,h) \equiv \lambda(h,t)$.

$\mathrm{trace}\, AB = \mathrm{trace}\, BA \quad$ in finite dim.

$\mathrm{trace}(S_t S_h) = \lambda(t,h)\mathrm{trace}(S_{t+h}), \quad \mathrm{trace}(S_h S_t) = \lambda(h,t)\mathrm{trace}(S_{t+h})$

$[\lambda(t,h) - \lambda(h,t)]\mathrm{trace}(S_{t+h}) = 0$

$\mathrm{trace}(S_t S_h) \to \mathrm{trace}(S_0) = \mathrm{trace}\, 1 = \dim(\mathbf{H}_2)\ (t,h \to 0)$.

$\exists\, \varepsilon > 0 \quad \lambda(t,h) = \lambda(h,t)\ (|t|,|h| < \varepsilon)$.

$S_t \smile S_h$ for $|t|,|h| < \varepsilon$.

$u, v \in \mathbf{R}$, $u/m, v/m \in (-\varepsilon, \varepsilon)$,

$S_u = \widetilde{\lambda} S_{u/m}^m,\ \ S_v = \widetilde{\mu} S_{v/m}^m \quad \exists\, \widetilde{\lambda}, \widetilde{\mu} \in \mathbf{T}, \Longrightarrow S_u \smile S_v \quad$ Q.e.d.

Remark: In infinite dimensions, $AB = \lambda BA \neq 0 \not\Rightarrow A \smile B$ even if $\lambda \in \mathbf{T}$.

Example: $A : e_n \mapsto e_{n+1}\ (n = 0, \pm 1, \ldots)$ bilateral shift, $\ B : e_n \mapsto \lambda^n e_n$.

Remark: Even in $r < \infty$ dimensions, with $\lambda^r = 1$, $\exists\ A, B \quad AB = \lambda BA \neq 0, \ \ A \not\smile B$.

Example: $e_0, \ldots, e_{r-1}$ orthn. basis, $A : e_0 \mapsto e_1 \mapsto e_2 \mapsto \cdots e_{r-1} \mapsto e_0, \ \ B : e_k \mapsto \lambda^k e_k$.

Proposition. $\exists\ t \mapsto \mu(t) \in \mathbf{C}_0 := \mathbf{C} \setminus \{0\}$ cont., $\mu(0) = 1$ such that

$[\mu(t) S_t : t \in \mathbf{R}]$, $[\mu(t)\mathcal{A}_t : t \in \mathbf{R}]$ str.cont.1prg.

Proof. Lemma $\Rightarrow \quad \mathcal{S} := \mathrm{Span}\{S_t : t \in \mathbf{R}_{(+)}\}$ Abelian algebra with unit $S_0 = 1$.

$M : \mathcal{S} \to \mathbf{C}$ nontriv. mult. functional. $\quad$ (actually $\exists\, 0 \neq x \in \mathbf{H}_2 \ \ Sx = M(S)x\ (x \in \mathcal{S})$).

$M(S_t) M(S_h) = M(S_t S_h) = \lambda(t,h) M(S_{t+h}), \quad M(S_t) \neq 0$ since S_t is invertible

Define $\quad \mu(t) := 1|/M(S_t)$ $\qquad$ (Triv: $t \mapsto \mu(t)$ cont. $\quad \mu(0) = 1$)

$$\mu(t)S_t\mu(h)S_h = \frac{1}{M(S_t)M(S_h)}S_tS_h = \frac{\lambda(t,h)}{M(S_t)M(S_h)}S_{t+h} =$$
$$= \frac{M(S_t)M(S_h)/M(S_{t+h})}{M(S_t)M(S_h)}S_{t+h} = \frac{1}{M(S_{t+h})}S_{t+h} = \mu(t+h)S_{t+h}$$

Assumptions (by passing to $\mu(t)S_t, \mu(t)\mathcal{A}_t = \mathcal{M}_{a(t)}\mathrm{diag}\left[\begin{smallmatrix}\mu(t)u_t \\ \mu(t)v_t\end{smallmatrix}\right]$ for $S_t, \mathcal{A}_t$) : (**1**),(**2**),(**3**)+

(4) $[S_t : t \in \mathbf{R}_{(+)}]$ cont. 1prsg in $\mathcal{L}(\mathbf{H}_2)$ $\quad$ for $S_t := C_tE + D_t$

$$\mathcal{A}' := \frac{d}{dt}\Big|_{t=0}\mathcal{A}_t = \left\{\begin{bmatrix} x \\ y\end{bmatrix} \mapsto \frac{d}{dt}\Big|_{t=0}\begin{bmatrix} A_t & B_t \\ C_t & D_t\end{bmatrix}\begin{bmatrix} x \\ y\end{bmatrix}\right\} = \left\{\begin{bmatrix} x \\ y\end{bmatrix} : t \mapsto u_tx, v_ty \text{ diff.}\right\}$$

$\mathbf{D} := \mathrm{dom}(\mathcal{A}') = \mathrm{dom}(u') \oplus \mathrm{dom}(v') = \mathrm{dom}(U') \oplus \mathbf{H_2}$ since $\dim(\mathbf{H}_2) < \infty$.

$\mathcal{A}'$ is of $\mathbf{H}_1 \oplus \mathbf{H}_2$-split matrix form since $\mathbf{D} = \mathbf{D}_1 \oplus \mathbf{D}_2$ $\quad$ (by def.)

Observation: $t \mapsto \Phi^t(X)$ diff. $\;$ whenever $\left[\begin{smallmatrix} Xy \\ y\end{smallmatrix}\right] \in \mathbf{D}$ $\;\forall y \in \mathbf{H}_2$.

Proof: $\;X \in \mathcal{L}(\mathbf{H}_1, \mathbf{H}_2) \implies$ $\;$ since $\dim(\mathbf{H}_2) < \infty$,

$t \mapsto \mathcal{A}_t\left[\begin{smallmatrix} X \\ 1\end{smallmatrix}\right]$ diff. $\iff$ $t \mapsto \mathcal{A}_t\left[\begin{smallmatrix} X \\ 1\end{smallmatrix}\right]y = \mathcal{A}_t\left[\begin{smallmatrix} Xy \\ y\end{smallmatrix}\right]$ diff. $\forall y \in \mathbf{H}_2$. $\quad$ Qu.e.d.

Remark. From the general theory we know: if $0 \in \mathrm{dom}(\Psi')$ then

$\mathrm{dom}(\Psi') = \big\{X : t \mapsto U_t(X) \text{ differentiable}\big\} = \big[\text{dense Jordan*-subtriple}\big] \cap \mathbf{B}$.

Since $U_t : X \mapsto u_tXv_t^*$, all the operators $x \otimes y^*$ $\;(x \in \mathrm{dom}(u'), y \in \mathbf{H}_2)$ belong to $\mathrm{dom}(\Psi')$.

Notation: $b := a' = \frac{d}{dt}\big|_{t=0}a(t)$, $A' := \frac{d}{dt}\big|_{t=0}A_t$ with $\mathrm{dom}(A') := \big\{x : \frac{d}{dt}\big|_{t=0}A_t \text{ exists}\big\}$,

$B' := \frac{d}{dt}\big|_{t=0}B_t$, $C' := \frac{d}{dt}\big|_{t=0}C_t$, $D' := \frac{d}{dt}\big|_{t=0}D_t$ analogously

$\Psi^t(0) = a(t) = (A_t \cdot 0 + B_t)(C_t \cdot 0 + D_t)^{-1} = B_tD_t^{-1}$

$S_t = C_tE + D_t$, $\quad S' := C'E + D'$ well-def. in finite dim.

$\mathcal{A}_t = \left[\begin{smallmatrix} A_t & B_t \\ C_t & D_t\end{smallmatrix}\right] = \mathrm{diag}\left[\begin{smallmatrix}(1-a(t)a(t)^*)^{-1/2} \\ (1-a(t)^*a(t))^{-1/2}\end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & a(t) \\ a(t)^* & 1\end{smallmatrix}\right]\mathrm{diag}\left[\begin{smallmatrix} u_t \\ v_t\end{smallmatrix}\right]$

$A_t = [1 - a(t)a(t)^*]^{-1/2} u_t, \quad B_t = [1 - a(t)^* a(t)]^{-1/2} a(t) v_t,$

$C_t = [1 - a(t)a(t)^*]^{-1/2} a(t)^* u_t, \quad D_t = [1 - a(t)^* a(t)]^{-1/2} v_t$

By assumption we consider the case $0 \in \mathrm{dom}(\Psi')$ i.e. if $b = a'$ is well-def.

$A' = u', \quad B' = a' v_0 + a(0) v' = b,\ C' = [a']^* u(0) + a(0)^* u' = b^*, \quad D' = v'$

Hence can summarize the concusion of assumptions $(0), \ldots, (4)$ as follows:

Theorem. Up to Möbius equivalence may assume that

$\Psi^t = \mathcal{F}(\mathcal{A}_t)$ where $[\mathcal{A}_t : t \in \mathbf{R}]$ is a str.conr.1-prg. in $\mathcal{L}(\mathbf{H}_1 \oplus \mathbf{H}_2) \equiv \mathcal{L}(\mathbf{H}_1, \mathbf{H}_2)$ such that

$\mathcal{A}' = \begin{bmatrix} u' & b \\ b^* & v' \end{bmatrix}$ $\mathbf{H}_1 \oplus \mathbf{H}_2$-split with $\mathrm{dom}(\mathcal{A}') = \mathrm{dom}(u') \oplus \mathbf{H}_2$; u', v' i-symm. (i-self-adj.).

We have $\mathcal{A}_t \left[\begin{smallmatrix} E \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} E \\ 1 \end{smallmatrix}\right] S_t$ where $[S_t : t \in \mathbf{R}]$ is a cont.1-prg in $\mathcal{L}(\mathbf{H}_2$ with $S' = b^* E + v'$.

Furthermore we recall

$$\mathcal{A}_t = \mathcal{M}_{a(t)} \mathrm{diag}(u_t, v_t) = \mathrm{diag}\begin{pmatrix} [1 - a(t)a(t)^*]^{-1/2} \\ [1 - a(t)^* a(t)]^{-1/2} \end{pmatrix} \begin{bmatrix} 1 & a(t) \\ a(t)^* & 1 \end{bmatrix} \mathrm{diag}(u_t, v_t).$$

$A_t = [1 - a(t)a(t)^*]^{-1/2} u_t, \quad B_t = [1 - a(t)a(t)^*]^{-1/2} a(t) v_t,$

$C_t = [1 - a(t)a(t)^*]^{-1/2} a(t)^* u_t, \quad D_t = [1 - a(t)a(t)^*]^{-1/2} v_t$

$\mathrm{dom}(\mathcal{A}') = \mathbf{D}_1 \oplus \mathbf{H}_2, \qquad \mathbf{D}_1 = \mathrm{dom}(A) = \mathrm{dom}\big(\tfrac{d}{dt}\big|_{t=0} u_t\big).$

$t \mapsto a(t) = B_t D_t^{-1}$ is differentiable, $\quad a(t) = tb + o(t)$ at $t = 0$

$$\mathcal{A}' = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} u' & b \\ b^* & v' \end{bmatrix}, \quad u' := \tfrac{d}{dt}\big|_{t=0} u_t,\ v' := \tfrac{d}{dt}\big|_{t=0} v_t.$$

$\Psi'(X) = \frac{d}{dt}\big|_{t=0} \Psi^t(X) = \frac{d}{dt}\big|_{t=0} (A_t X + B_t)(C_X + D_t)^{-1} =$

$= (A'X + B')(C_0 X + D_0)^{-1} - (A_0 X + B_0)(C_0 X + D_0)^{-1}(C'X + D')(C_0 X + D_0)^{-1} =$

$= A'X + B' - X(C'X + D') = u'X + b - Xb^*X - Xv' = = b - \{XbX\} + \frac{d}{dt}\big|_{t=0} U_t X$

$$\mathcal{A}_t\begin{bmatrix}E\\1\end{bmatrix}=\begin{bmatrix}A_t & B_t\\ C_t & D_t\end{bmatrix}\begin{bmatrix}E\\1\end{bmatrix}=\begin{bmatrix}ES_t\\ S_t\end{bmatrix}=\begin{bmatrix}E\\1\end{bmatrix}S_t$$

$[S_t : t\in\mathbf{R}]$ str.cont.1prg, $\quad S' := \frac{d}{dt}\big|_{t=0}S_t = \mathrm{gen}[S_t : t\in\mathbf{R}]$

$y\in\mathbf{H}_2 \;\Rightarrow\; t\mapsto \mathcal{A}_t\left[\begin{smallmatrix}Ey\\ y\end{smallmatrix}\right]=\left[\begin{smallmatrix}E\\ 1\end{smallmatrix}\right]S_ty$ diff., $\quad \left[\begin{smallmatrix}Ey\\ y\end{smallmatrix}\right]\in\mathrm{dom}(\mathcal{A}'),\quad Ey\in\mathbf{D}_1.$

Projective translation: $\mathcal{T} := \begin{bmatrix}1 & E\\ & 1\end{bmatrix},\quad \mathcal{T}^{-1} := \begin{bmatrix}1 & -E\\ & 1\end{bmatrix}$

$\mathcal{B}_t := \mathcal{T}^{-1}\mathcal{A}_t\mathcal{T},\quad \mathcal{B}' := \mathcal{T}^{-1}\mathcal{A}\mathcal{T}$

$\mathcal{A}' = \mathrm{gen}\big[\mathcal{A}_t : t\in\mathbf{R}\big],\quad \mathcal{B}' = \mathrm{gen}\big[\mathcal{B}_t : t\in\mathbf{R}\big],\quad \mathrm{dom}(\mathcal{B}') = \mathcal{T}^{-1}(\mathbf{D}_1\oplus\mathbf{H}_2).$

$\mathrm{dom}(\mathcal{B}') = \big\{[d-Ey]\oplus y : d\in\mathbf{D}_1,\ y\in\mathbf{H}_2)\big\} = \mathbf{D}_1\oplus\mathbf{H}_2(=\mathrm{dom}(\mathcal{A}')).$

$$\mathcal{T}^{-1}\begin{bmatrix}A_t & B_t\\ C_t & D_t\end{bmatrix}\mathcal{T}=\begin{bmatrix}1 & -E\\ & 1\end{bmatrix}\begin{bmatrix}A_t & A_tE+B_t\\ C_t & C_tE+D_t\end{bmatrix}=\begin{bmatrix}1 & -E\\ & 1\end{bmatrix}\begin{bmatrix}A_t & ES_t\\ C_t & S_t\end{bmatrix}=$$
$$=\begin{bmatrix}A_t-EC_t & \mathbf{0}\\ C_t & S_t\end{bmatrix}.$$

$$\mathcal{B}' = \mathcal{T}^{-1}\mathcal{A}'\mathcal{T} = \begin{bmatrix}A'-EC' & 0\\ C' & S'\end{bmatrix}=\begin{bmatrix}u'-Eb^* & 0\\ b^* & b^*E+v'\end{bmatrix}$$

$W_t := \big[\mathcal{B}_t\big]_{11}$ str.cont.1prg. $\quad W' = \mathrm{gen}[W_t : t{\in}\mathbf{R}] = A'-EC' = u'-Eb^*$

$S_t := \big[\mathcal{B}_t\big]_{22}$ str.cont.1prg. $\quad S' = \mathrm{gen}[S_t : t{\in}\mathbf{R}] = C'E+D' = b^*E+v'$

Triangular lemma [Stachó JMAA 2016, Lemma 3.8] $\Rightarrow$

$$\mathcal{B}' = \mathrm{gen}\Bigg[\underbrace{\begin{bmatrix}W_t & 0\\ \int_0^t S_{t-h}C'W_h\,dh & S_t\end{bmatrix}}_{\mathcal{B}_t} : t\in\mathbf{R}\Bigg]$$

$\Psi^t = \mathcal{F}(\mathcal{B}_t):\ X\mapsto W_tX\big[\int_0^t S_{t-h}C'W_hX\,dh + S_t\big]^{-1},$

$\mathcal{A}' = \mathcal{T}\mathcal{B}'\mathcal{T}^{-1} = \mathrm{gen}\big[\mathcal{A}_t : t\in\mathbf{R}\big],\quad T := \mathcal{F}(\mathcal{T}) : X\mapsto X+E$

$\Phi^t = \mathcal{F}\big(\mathcal{A}_t\big) = \mathcal{F}\big(\mathcal{T}\mathcal{B}_t\mathcal{T}^{-1}\big) = T\circ\Psi_t\circ T^{-1}$

Closed integrated form: $\quad$ For all $X\in\mathrm{Ball}\big(\mathcal{L}(\mathbf{H}_1,\mathbf{H}_2)\big),$

$$\Phi^t(X) = E + W_t(X-E)\Big[\int_0^t S_{t-h}\underbrace{C'}_{b^*}W_h(X-E)\,dh + S_t\Big]^{-1}.$$

$$\Phi^t = \mathcal{F}(\mathcal{A}_t), \quad \mathcal{A}_t = \begin{bmatrix} W_t + EJ_t & ES_t - (W_t + EJ_t)E \\ J_t & S_t - J_tE \end{bmatrix}, \quad J_t := \int_0^t S_{t-h}b^*W_h\,dh$$

Vector fields

$\Phi^t(X) \in \mathcal{L}(\mathbf{H}_1, \mathbf{H}_2)$ $\quad \big[\mathbf{H}_1 \to \mathbf{H}_2$ operators$\big]$

$t \mapsto \Phi^t(X)$ diff. $\quad \Longleftrightarrow \quad t \mapsto \Phi^t(X)y$ diff. $\forall\, y \quad (\Leftarrow \dim(\mathbf{H}_2) < \infty.)$

If $\operatorname{ran}(X) \subset \mathbf{D}_1\big(= \operatorname{dom}([\mathcal{A}']_{11})\big)$ then

$t \mapsto \Phi^t(X)y = [A_tX + B_t][C_tX + D_t]^{-1}y \quad$ diff. $\forall\, y$

$\Phi' := \frac{d}{dt}\big|_{t=0}\Phi^t, \qquad \underline{\operatorname{dom}(\Phi') = \{X : \operatorname{ran}(X) \subset \mathbf{D}_1\}}$

Kaup type formula up to Möbius equ.:

$\Phi'(X)y = \frac{d}{dt}\big|_{t=0}[A_tX + B_t][C_tX + D_t]^{-1}y = [A'X + B']y - X[C'X + D']y =$

$$= \big[b - Xb^*X + u'X - Xv'\big]y \quad (\operatorname{ran}(X) \subset \mathbf{D}_1,\ y \in \mathbf{H}_2)$$

Integration of Kaup's type vector fields

$\Omega : X \mapsto b - Xb^*X + u'X - Xv'$ vector field on $\mathcal{L}(\mathbf{H}_1, \mathbf{H}_2)$, $\dim(\mathbf{H}_2) < \infty$

$b \in \mathcal{L}(\mathbf{H}_1, \mathbf{H}_2)$, $u' : \mathbf{D}_1 \to \mathbf{H}_1$ densely def. i-self-adj., $v' \in \mathcal{L}(\mathbf{H}_2)$ i-self-adj.

Question. $\exists?$ $[\Phi^t : t \in \mathbf{R}]$ str.cont.1-prg. in $\operatorname{Aut}(B)$ such that $\Phi' = \Omega$?

Assumption. $E \in \operatorname{dom}(\Omega)$, $\|E\| = 1$, $\Omega(E) = 0$. With the earlier construction, let

$\Phi^t(X) := E + W_t(X-E)\Big[\int_0^t S_{t-h}b^*W_h(X-E)\,dh + S_t\Big]^{-1}$

Remark. $\Omega = \Phi'(= \frac{d}{dt}\big|_{t=0}\Phi^t)$

New condition. If $[\mathcal{A}_t : t \in \mathbf{R}]$ str.cont.1-prg and $\Phi^t = \mathcal{F}(\mathcal{A}_t) \in \mathrm{Aut}(\mathbf{B})$ $(t \in \mathbf{R})$ then,

with $c(t) := \Phi^t(0) = B_t D_t^{-1}$ we have $\Phi^t = M_{c(t)} \circ U_t$ with $U_t = u_t \otimes v_t^*$, u_t, v_t unitary.

Hence, with $t\lambda(t) \neq$ cont. and $t \mapsto u_t, v_t$ str.cont.,

$\mathrm{diag}\begin{bmatrix} [1-c(t)c(t)^*]^{-1/2} \\ [1-c(t)^*c(t)]^{-1/2} \end{bmatrix} \begin{bmatrix} 1 & -c(t) \\ -c(t)^* & 1 \end{bmatrix} \mathcal{A}_t = \lambda(t)\mathrm{diag}\begin{bmatrix} u_t \\ v_t \end{bmatrix}$, that is

(**5a**) $[1-c(t)c(t)^*]^{-1/2}[A_t - c(t)C_t] = \lambda(t)u_t$, (**5b**) $B_t - c(t)D_t = 0$,

(**5c**) $-c(t)^*A_t + C_t = 0$, (**5d**) $[1-c(t)^*c(t)]^{-1/2}[-c(t)^*B_t + D_t] = \lambda(t)v_t$

In particular (5b) is trivial and

$0 = -(B_t D_t^{-1})^* A_t + C_t$,

$[A_t - B_t D_t^{-1} C_t][A_e - B_t D_t^{-1} C_t]^* = |\lambda(t)|^2[1 - B_t D_t^{-1}(B_t D_t^{-1})^*]$,

$[-(B_t D_t^{-1})^* B_t + D_t][-(B_t D_t^{-1})^* B_t + D_t]^* = |\lambda(t)|^2[1 - (B_t D_t^{-1})^* B_t D_t^{-1}]$

Theorem. Given any b, E, u', v' satisfying $(1), \ldots, (4)$,

we have $\Phi^t \in \mathrm{Aut}(\mathbf{B}$ $(t \in \mathbf{R})$.

Proof. It suffices to see only that each Φ^t maps the unit ball $\mathbf{B}$ into itself. We have

$\mathcal{A}' = \mathrm{gen}[\mathcal{A}_t : t \in \mathbf{R}] = \begin{bmatrix} 0 & b \\ b^* & 0 \end{bmatrix} + \begin{bmatrix} u' & 0 \\ 0 & v' \end{bmatrix}$.

Since u', v' are i-self-adjoint (u' possibly unbded),

$\begin{bmatrix} u' & 0 \\ 0 & v' \end{bmatrix} = \mathrm{gen}[\widetilde{\mathcal{U}}^t : t \in \mathbf{R}]$, $\widetilde{\mathcal{U}}^t := \widetilde{u}^t \otimes \widetilde{v}^t$, $[\widetilde{u}^t : t \in \mathbf{R}]$, $[\widetilde{v}^t : t \in \mathbf{R}]$ str.cont.unitary 1-prg.

Recall [Engel-Nagel, p.230 Ex. 3.11] that pointwise we have

$\mathcal{A}_t = \lim_{n\to\infty} \left[\exp\left(\frac{t}{n}\begin{bmatrix} 0 & b \\ b^* & 0 \end{bmatrix}\right)\mathcal{U}^{t/n}\right]^n =$

$= \lim_{n\to\infty} \Big[\big[\text{Möbius matrix}\big]\big[\mathcal{L}(\mathbf{H}_1, \mathbf{H}_2) - \text{unitary matrix}\big]\Big]^n =$

$$= \lim_{n\to\infty} \Big[\big[\text{Möbius matrix}\big]\Big]^n = \big[\text{Möbius matrix}\big].$$

Hence each $\Phi^t = \mathcal{F}(\mathcal{A}_t)$ is a Möbius trf. mapping $\mathbf{B}$ onto itself. Qu.e.d.

Determining parameters (u', E, S')

We have seen: the integration of a vector field $x \mapsto b - \{xb^*x\} + u'x - xv'$ of Kaup's type with fixed point E in $\partial\mathbf{B}$ gives always rise to a str.cont.1-prsg. in $\mathrm{Iso}(d_{\mathbf{B}})$.

We shall see, it suffices to assume withot loss of generality that the fixed point E is a tripotent, i.e.

$$E = \sum_{k=1}^{m} f_k \otimes e_k^* \quad \{f_1, \ldots, f_r\} \text{ ORTN} \subset \mathbf{H}_1, \quad \{2_1, \ldots, 2_r\} \text{ ORTN} \subset \mathbf{H}_2.$$

Necessarily, algebraic relations hold between the parameters (b, u', E, v', S'). Namely

$$\begin{bmatrix} u' & b \\ b^* & v' \end{bmatrix} \begin{bmatrix} E \\ 1 \end{bmatrix} = \begin{bmatrix} ES' \\ S' \end{bmatrix}, \qquad u' = i\text{·symmetric-dense}, \ \ v' = i\text{·selfadjoint}.$$

We know that these conditions are sufficient already to give rise to a str.cont.1-prsg. in $\mathrm{Iso}(d_{\mathbf{B}})$. We are going to establish structural algebraic conditions to

$u'E + b = ES', \quad b^*E + v' = S', \quad u' = i\text{·symmetric-dense}, \ \ v' = i\text{·selfadjoint}.$

Equivalently we have

$b = ES' - u'E, \quad v' = -[v']^*$ i.e. $S' - b^*E = E^*b - [S']^*$ which is the same as

$(*) \quad S' - [S']^*E^*E + E^*[u']^*E = E^*ES' - E^*u'E - [S']^*.$

By the skew symmetry of u' we have $E^*[u']^*E = -E^*u'E$ and hence $(*)$ has the form

$(**) \quad [1 - E^*E]S' = -[S']^*[1 - E^*E]$ i.e. $[1 - E^*E]S'$ i·sefadjoint.

We investigate $(**)$ in matrix form. For some orthonormed systems $f_1,\dots,f_N \in \mathbf{H}_1$ resp. $e_1,\dots,e_n \in \mathbf{H}_2$ (being complete $\mathbf{H}_2$) we can write (by means of SVD decompostion)

$$E = \sum_{k=1}^{N} \lambda_k f_k \otimes e_k^*, \;\; 1 = \lambda_1 \ge \cdots \ge \lambda_N \ge 0, \qquad S' = \sum_{k,\ell=1}^{N} \sigma_{k\ell} f_k \otimes e_\ell^*.$$

The relation $(**)$ means that

$$(***) \qquad (1-\lambda_k)\sigma_{k\ell} = -\overline{\sigma_{k\ell}}(1-\lambda_\ell) \quad (k,\ell = 1,\dots,N).$$

We can write the sequence $[1-\lambda_k]_{k=1}^N$ in more details in the form

$$\big[1-\lambda_1,\dots,1-\lambda_N\big] = \Big[\underbrace{0,\dots,0}_{m_1},\underbrace{\mu_2,\dots,\mu_2}_{m_2},\dots,\underbrace{\mu_r,\dots,\mu_r}_{m_r}\Big], \quad 0<\mu_2<\cdots<\mu_r\le 1,\ m_1>0.$$

Then, with the partition $\sigma = \big[\sigma_{k\ell}\big]_{k,\ell=1}^N = \Big[\sigma^{(p,q)}\Big]_{p,q=1}^r$ into submatrices $\sigma^{(p,q)} \in \mathrm{Mat}(m_p,m_q)$, we can write $(***)$ into the form $\mu_p\sigma^{(p,q)} = -\mu_q[\sigma^{(q,p)}]^*$ $(p,q=1,\dots,r)$. This is possible if and only if

$\sigma^{(1,1)}$ is arbitrary, $\quad \sigma^{(p,p)} = -[\sigma^{(p,p)}]^*$, $\;\sigma^{(p,1)} = \sigma^{(1,p)} = 0 \quad (p>1)$,

$\sigma^{(p,q)}$ is arbitrary and $\;\sigma^{(q,p)} = -(\mu_p/\mu_q)[\sigma^{(p,q)}]^*$ $(1<q<p)$.

Proposition. Assume $[\Phi^t : t \in \mathbf{R}_+]$ has a Kaup type generator $\Phi'(x) = b - \{xb^*x\} + \mathrm{U}'x$ with $\mathrm{dom}(\Phi') = \mathrm{dom}(\mathrm{U}') \cap \mathbf{B}$ where is a (not necessaarily closed) Jordan subtriple of $\mathbf{E}$. Assume furthermore that F is a common fixed point of the continuous extensions $\overline{\Phi}^t$ to the closed unit ball $\overline{\mathbf{B}}$ of the maps Φ^t belonging to a finite dimensional face $\mathbf{F}$ of $\mathbf{B}$. Then

$$\mathrm{Span}(\mathbf{F}) \cap \mathbf{B} \subset \mathrm{dom}(\Phi').$$

Proof. We know [Peralta etc.] that there is a tripotent $E \neq 0$ (actually the middle point of $\mathbf{F}$) such that

$$\mathbf{F} = E + \left[\mathbf{B} \cap E^{\perp\mathrm{Jordan}}\right] = \{E + A : A \perp^{\mathrm{Jordan}},\ \|A\| < 1\}$$

where $(E^{\perp\mathrm{Jordan}})$ is a finite, say $N(<\infty)$ dimensional subtriple of $\mathbf{E}$.

Therefore $F = E + A$ where $A = \sum_{k=1}^{m} \lambda_k E_k$ for some Jordan-orthogonal family $E_1, \ldots, E_m$ with $m \leq N$ in $E^{\perp\mathrm{Jordan}}$ and $0 < \lambda_1 < \cdots < \lambda_m < 1$.

On the other hand,

$\{x \in \overline{\mathbf{B}} : t \mapsto \overline{\Phi}^t(x)$ is differentiable$\} = \{x \in \overline{\mathbf{B}} : t \mapsto \mathrm{U}^t$is differentiable$\} = \overline{\mathbf{B}} \cap \mathbf{J}$ with the Jordan subtriple $\mathbf{J} := \{x \in \mathbf{E} : t \mapsto \mathrm{U}^t$is differentiable$\}$. Since the orbit $t \mapsto F = \overline{\Phi}^t(F)$ is constant, trivially $F \in \mathbf{J}$ and hence

$\Delta F = \{\zeta F : |\zeta| < 1\} \subset \mathrm{dom}(\Phi') = \{x \in \mathbf{B} : t \mapsto \Phi^t(x)$is differentiable$\}$.

Thus, since $\mathrm{Span}(\mathbf{F}) = \mathbf{C}E + \bigoplus_{k=1}^{m} \mathbf{C}E_k$, it suffices to see that

$(*) \quad \oplus k = 0^m \mathbf{C}E_k \subset \mathbf{J} \quad$ where $E_0 := E$.

Since $F \in \mathbf{J}$ and $\mathbf{J}$ is a linear submanifold being closed to the triple product, we may establish $(*)$ by showing that $E_k \in \mathrm{Span} L(F,F)^k F \quad (k = 0, \ldots, m)$, or which is the same,

$(**) \quad \{L(F,F)^k F : k = 0, \ldots, F\}$ is a linearly independent family.

Notice that the vectors $E_0, \ldots, E_m$ are linearly independent as being pairwise Jordan ortogonal tripotents. Observe that, by setting $\lambda_0 := 1$, we have

$L(F,F)^n F = L\Big(\sum_{k=0}^{m} \lambda_k E_k, \sum_{k=0}^{m} \lambda_k E_k\Big)^n \sum_{k=0}^{m} \lambda_k E_k = \sum_{k=0}^{m} \lambda_k^{2n+1} E_k$.

Hence $(**)$ is equivalent to the statement that

$(***)\quad \det\left[\lambda_k^{2n+1}\right]_{k,n=0}^{m} \neq 0.$

However, $(***)$ is easy to see because

$\left[\lambda_k^{2n+1}\right]_{k,n=0}^{m} = \mathrm{diag}(\lambda_0,\dots,\lambda_m)\mathrm{VanderMonde}(\lambda_0^2,\dots,\lambda_m^2)$

with $0<\lambda_1<\lambda_2<\cdots<\lambda_m<1=\lambda_0$.

Corollary. If $\mathbf{E}=\mathcal{L}(\mathbf{H}_1,\mathbf{H}_2)$ with $r:=\dim(\mathbf{H}_2)<\infty$ and $[\Psi^t : t\in\mathbf{R}_+]$ is a C_0-SGR in $\mathrm{Iso}(d_{\mathbf{B}})$ then there is a C_0-SGR $[\Phi^t : t\in\mathbf{R}_+]$ in $\mathrm{Iso}(d_{\mathbf{B}})$ being Möbius equvalent to $[\Psi^t : t\in\mathbf{R}_+]$ such that its generator is of Kaup type and whose continuous extensions to the closed unit ball admit a common fixed point which is a tripotent.

Proof. We know [Stacho, RevRoum17] that any C_0-SGR in $\mathbf{E}=\mathcal{L}(\mathbf{H}_1,\mathbf{H}_2)$ with $r:=\dim(\mathbf{H}_2)<\infty$ whose 0-orbit is differentiable has a Kaup type generator (whose domain is the intersection of a not necessarily closed Jordan subtriple with the unit ball) and the continuous extensions of its mebers admit a common fixed point in the closed unit ball. Furthermore the boundary of the unit ball is a union of finite (at most r) dimensional faces. Let $F=E+A$ be a common fixed point of $[\overline{\Psi}^t : t\in\mathbf{R}_+]$ where E is a tripotent and $A\perp^{\mathrm{Jordan}} E$ with $\|A\|<1$. Consider the Möbius equivalent C_0-SGR $[\Phi^t : t\in\mathbf{R}_+]$ with $\Phi^t := M_{-A}\circ\Psi^t\circ M_A$. According to the Proposition, we have $\pm A\in\mathbf{B}\cap\sum_{k=0}^{r}\mathbf{C}L(F,F)F\subset \mathrm{dom}(\Psi')$. Hence the orbit $t\mapsto\Phi^t(0)=M_{-A}\big(\Psi^t(M_A(0))\big)=M_{-A}\big(\Psi^t(A)\big)$ is differentiable, that is $0\in\mathrm{dom}(\Phi^prime)$ implying also that Φ' is of Kaup type. Also we have

$$\overline{\Phi}^t\big(M_{-A}(F)\big)=M_{-A}\big(\overline{\Psi}^t(F)\big)=M_{-A}(F)\qquad (t\in\mathbf{R}_+)$$

that is the point $M_{-A}(F)$ is a common fixed point for $[\overline{\Phi}^t : t \in \mathbf{R}_+]$. However (since $E \perp^{\mathrm{Jordan}} A$),

$$M_{-A}(F) = M_{-A}(E+A) = -A + B(A)^{1/2}[1 - L(E+A, A)]^{-1}(E+A) =$$

$$= -A + B(A)^{1/2}[1 - L(A, A)]^{-1}(E+A) =$$

$$= -A + B(A)^{1/2}[1 - L(A, A)]^{-1}E + B(A)^{1/2}[1 - L(A, A)]^{-1}A =$$

$$= -A + B(A)^{1/2}E + B(A)^{1/2}[1 - L(A, A)]^{-1}A =$$

$$= -A + E + B(A)^{1/2}[1 - L(A, A)]^{-1}A = M_{-A}(A) + E = 0 + E = E$$

which completes the proof.

Lemma 1. Let $\mathbf{E} := \mathcal{L}(\mathbf{H}_1, \mathbf{H}_2)$ with $r := \dim(\mathbf{H}_2) < \infty$. Assume $[\Phi^t : t \in \mathbf{R}_+]$ is a C_0-SGR in $\mathrm{Iso}(d_{\mathbf{B}})$ such that $\Phi^t = M_{a(t)} \circ \mathrm{U}_t|\mathbf{B}$ where the orbit $t \mapsto a(t) = \Phi^t(0)$ is differentiable and $\mathrm{U}_t = \mathfrak{P}\mathcal{U}_t$, $t \mapsto \mathcal{U}_t = \left[\begin{smallmatrix} U_t & 0 \\ 0 & V_t \end{smallmatrix}\right]$ is such that U_t, V_t are linear isometries of $\mathbf{H}_1, \mathbf{H}_2$ respectively and there is a function $t \mapsto \mu(t) \in \mathbf{C} \setminus \{0\}$ such that $[\mu(t)\mathcal{M}_{a(t)}\mathcal{U}_t : t \in \mathbf{R}_+]$ is a C_0-SGR in $\mathcal{L}(\mathbf{H}_1 \oplus \mathbf{H}_2)$. Then

$$\mathrm{dom}([M \circ \mathrm{U}]')\big(:= \{X \in \mathbf{E} : [0, \varepsilon) \ni t \mapsto M_{a(t)(\mathrm{U}_t X)} \text{ diff. for some } \varepsilon > 0\}\big) =$$

$$= \{X \in \mathbf{E} : \mathrm{range}(X) \subset \mathrm{dom}(U')\}.$$

Proof. Since $\dim(\mathbf{H}_2) < \infty$,

$\mathrm{dom}([M \circ \mathrm{U}]') = \{X \in \mathbf{E} : t \mapsto \Phi^t(X) \text{ is differentiable at } 0+\} =$

$= \{X \in \mathbf{E} : t \mapsto \mathrm{U}_t(X) \text{ is differentiable at } 0+\} =$

$= \{X \in \mathbf{E} : t \mapsto U_t X V_t^{-1} \text{ is differentiable at } 0+\} =$

$= \{X \in \mathbf{E} : t \mapsto U_t X V_t^{-1} y$ is differentiable at $0+$ for all $y \in \mathbf{H}_2\}$.

By assumption (and since $a \mapsto \mathcal{M}_a$ is real-analytic), the orbit $t \mapsto \mu(t) V_t$ in the finite dimensional space $\mathcal{L}(\mathbf{H}_2)$ is differentiable, implying also the differentiability of $t \mapsto \mu(t)^{-1} V_t^{-1}$.

Let $e_1, \ldots, e_r$ be an orthonormed basis of $\mathbf{H}_2$ and consider any operator $X \in \mathbf{B}$. We have

$$\mu(t) U_t X e_k = U_t X V_t^{-1} \mu(t) V_t e_k = U_t X V_t^{-1} \sum_{\ell=1}^{r} \langle [\mu(t) V_t] e_k | e_\ell \rangle e_\ell =$$
$$= \sum_{\ell=1}^{r} \langle [\mu(t) V_t] e_k | e_\ell \rangle U_t X V_t^{-1} e_\ell,$$
$$U_t X V_t^{-1} e_k = [\mu(t) U_t] X [\mu(t)^{-1} V_t^{-1}] e_k = \sum_{\ell=1}^{r} \langle [\mu(t)^{-1} V_t] e_k | e_\ell \rangle \mu(t) U_t X e_\ell.$$

Thus the orbits $t \mapsto U_t X V_t^{-1} e_k$ and $t \mapsto \mu(t) U_t X e_\ell$ are differentiable in the same time. By passing to linear combinations we conclude that $X \in \mathrm{dom}(\Phi') \iff t \mapsto U_t X V_t^{-1} y$ is diff. for all $y \iff t \mapsto \mu(t) U_t X z$ is diff. for all z. Observe that the latter statement can be interpreted as $Xz \in \mathrm{dom}\big(\mu(t) U_t\big)'$ for all $z \in \mathbf{H}_2$ that is $\mathrm{range}(X) \subset \mathrm{dom}\big(\mu(t) U_t\big)'$.

Lemma 2. Let $(\mathbf{E}, \{..\cdot\})$ be a JB*-triple of finite rank, $\mathbf{J} \subset \mathbf{E}$ a dense linear subanifold being closed for the triple product and let e be a tripotent in $\mathbf{J}$. Then there is a tripotent f in $\mathbf{J}$ such that $f \perp^{\mathrm{Jordan}} e$ and $e + f$ is a maximal tripotent of $\mathbf{E}$ (i.e. $\{x \in \mathbf{E} : x \perp^{\mathrm{Jordan}} e + f\} = \{0\}$).

Proof. Recall [Kaup81, Neher] that, as a consequence of the fact that only finite Jordan-orthogonal families of tripotents do exist in $\mathbf{E}$, every element $x \in \mathbf{E}$ admits a finite spectral decomposition of the form $x = \sum_{0 \neq \lambda \in \mathrm{Sp}(L(x))} \sqrt{\lambda} x_\lambda$ where the vectors x_λ are pairwise

Jordan-orthogonal tripotents being real-linear combinations from the family $\{L(x)^k x : k = 0, \ldots, r-1\}$ where $r := \mathrm{rank}(\mathbf{E}\{..^*.\})$. That is, every subtriple $\mathbf{K} \subset \mathbf{E}$ (even a non-closed one) is spanned algebraically by $\mathrm{Trip}(\mathbf{K})$. In particular, any non-trivial subtriple of $\mathbf{E}$ contains tripotents. Consider any maximal family $\mathbf{F}$ of pairwise orthogonal tripotents in $e^{\perp \mathrm{Jordan}} := \{z \in \mathbf{J} : z \perp^{\mathrm{Jordan}} e\}$. The set $\mathbf{F}$ contains at most $(r-1)$ elements and its sum $f := \sum_{g \in \mathbf{F}} g$ is a tripotent in $\mathbf{J} \cap e^{\perp \mathrm{Jordan}}$. Also $e + f \in \mathrm{Trip}(\mathbf{J})$. To complete the proof we show that the subtriple $\mathbf{E}_0 := [e+f]^{\perp \mathrm{Jordan}}$ of $\mathbf{E}$ is trivial (otherwise it would contain non-zero tripotents). By the well-known Peirce identity of tripotents [Neher],

$$L(e+f)^3 - \tfrac{3}{2} L(e+f)^2 + \tfrac{1}{2} L(e+f) = 0.$$

Hence $\mathbf{E}_0 = \mathrm{kernel}(L(e+f)) = \mathrm{range}(P)$ where $P := 2L(e+f)^2 - 3L(e+f) + \mathrm{Id}_{\mathbf{E}}$ is a projection ($P^2 = P$, the so-called Peirce-0 projection of $e+f$). Consider the the subtriple $\mathbf{J}_0 := \mathbf{J} \cap \mathbf{E}_0 = \{x \in \mathbf{J} : x \perp^{\mathrm{Jordan}} e + f\}$). Observe that $\mathbf{J}_0 = P\mathbf{J}$ because P preserves the subtriple $\mathbf{J}$. Since $\mathbf{J}$ is supposed to be (norm-)dense in $\mathbf{E}$, $\mathbf{J}_0 = P\mathbf{J}$ is necessarily dense in $P\mathbf{E} = \mathbf{E}_0$. However, since non-trivial subtriples contain non-zero tripotents, we have $\mathbf{J}_0 = \{0\}$ by the maximality of the family $\mathbf{F}$.

Proposition. Let $[\Psi^t : t \in \mathbf{R}_+]$ be a C_0-SGR in $\mathrm{Iso}(d_{\mathbf{B}})$ for the unit ball $\mathbf{B}$ of the TRO factor $\mathbf{E} := \mathcal{L}(\mathbf{H}_1, \mathbf{H}_2)$ with $\dim(\mathbf{H}_2) = r < \infty$. Then we can find a Möbius equivalent C_0-SGR $[\Phi^t : t \in \mathbf{R}_+]$ such that $\Phi^t = \mathfrak{P}\mathcal{A}^t \quad (t \in \mathbf{R}_+)$ where $[\mathcal{A}^t : t \in \mathbf{R}_+]$ is a C_0-SGR in

$\mathcal{L}(\mathbf{H}_1 \oplus \mathbf{H}_2)$ with generator of the form

$$\mathcal{A}' = \mathcal{T} \begin{bmatrix} U'_{11} - b^*_{11} & 0 & 0 & 0 & 0 \\ -b_{21} & U'_{22} & U'_{23} & 0 & 0 \\ -b_{31} & U'_{32} & U'_{33} & 0 & 0 \\ b^*_{11} & b^*_{21} & b^*_{31} & b_{11} + V'_{11} & b_{12} \\ b^*_{12} & 0 & 0 & 0 & V'_{22} \end{bmatrix} \mathcal{T}^{-1}, \quad \mathrm{dom}(\mathcal{A}') = \mathrm{dom}(U') \oplus \mathbf{H}_2$$

$$U' = \left[U'_{j,k} \right]_{j,k=1}^{3}, \quad U'_{j,k} \in \mathcal{L}(\mathbf{H}_{1,j}, \mathbf{H}_{1,k}), \quad \mathbf{H}_1 = \oplus_{j=1}^{3} \mathbf{H}_{1,j} \ ;$$

$$V' = \left[V'_{\ell,m} \right]_{\ell,m=1}^{2}, \quad V'_{\ell,m} \in \mathcal{L}(\mathbf{H}_{2,\ell}, \mathbf{H}_{2,m}), \quad \mathbf{H}_2 = \oplus_{\ell=1}^{2} \mathbf{H}_{2,\ell} \ ;$$

$$b := \left[b_{k,\ell} \right]_{\substack{j=1,2,3 \\ \ell=1,2}}, \quad b_{j,\ell} \in \mathcal{L}(\mathbf{H}_{1,j}, \mathbf{H}_{2,\ell}), \quad \dim(\mathbf{H}_{2,\ell}) = \dim(\mathbf{H}_{1,\ell})$$

where $U' = \mathrm{gen}[U^t : t \in \mathbf{R}_+]$ resp. $V' = \mathrm{gen}[V^t : t \in \mathbf{R}_+]$ are generators of C_0-SGRs of linear isometries of $\mathbf{H}_1$ resp. $\mathbf{H}_2$, furthermore

$$\mathcal{T} = \begin{bmatrix} \mathrm{Id}_{\mathbf{H}_{1,1}} & 0 & 0 & J & 0 \\ 0 & \mathrm{Id}_{\mathbf{H}_{1,2}} & 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id}_{\mathbf{H}_{1,3}} & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id}_{\mathbf{H}_{2,1}} & 0 \\ 0 & 0 & 0 & 0 & \mathrm{Id}_{\mathbf{H}_{2,2}} \end{bmatrix}, \mathcal{T}^{-1} = \begin{bmatrix} \mathrm{Id}_{\mathbf{H}_{1,1}} & 0 & 0 & -J^* & 0 \\ 0 & \mathrm{Id}_{\mathbf{H}_{1,2}} & 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id}_{\mathbf{H}_{1,3}} & 0 & 0 \\ 0 & 0 & 0 & \mathrm{Id}_{\mathbf{H}_{2,1}} & 0 \\ 0 & 0 & 0 & 0 & \mathrm{Id}_{\mathbf{H}_{2,2}} \end{bmatrix}$$

where $J : \mathbf{H}_{2,1} \to \mathbf{H}_{1,1}$ is a surjective linear isometry.

Proof. In [StachoRevRoum17, Cor.7.6] (as a completion with adjusted continuity arguments Vesentini's work [Ves94]) we estabished that $[\Psi^t : t \in \mathbf{R}_+]$ is Möbius equivalent to a C_0-SGR $[\Phi^t : t \in \mathbf{R}_+]$ of the form $\Phi^t = \mathfrak{P}\mathcal{A}_t$ where $[\mathcal{A}_t : t \in \mathbf{R}_+]$ is a C_0-SGR in $\mathcal{L}(\mathbf{H}_1 \oplus \mathbf{H}_2)$ with generator

$$\mathcal{A}' = \begin{bmatrix} U' & b \\ b^* & V' \end{bmatrix} = \mathcal{T} \begin{bmatrix} U' - Eb^* & 0 \\ b^* & b^*E + V' \end{bmatrix} \mathcal{T}^{-1}$$

where U', V' are generators of isometry C_0-SGR in $\mathbf{H}_1$ resp. $\mathbf{H}_2$, $\mathrm{dom}(\mathcal{A}') = \mathrm{dom}(U') \oplus \mathbf{H}_2$, $b, E \in \mathcal{L}(\mathbf{H}_1, \mathbf{H}_2)$ with $\|E\| = 1$ and $\Phi^t(E) = E$ $(t \in \mathbf{R}_+)$. We refine this representation by a choosing the common fixed point E to be a tripotent. According to the previous Corollary, this can be done without loss of generality. Furthermore, by Lemma 2, we can find a complementary tripotent F such that $F \perp^{\text{Jordan}} E$ and $E+F$ is a maximal tripotent of $\mathbf{E} = \mathcal{L}(\mathbf{H_1}, \mathbf{H}_2$. Actually we can write

$$E = \sum_{k=1}^{m} f_k \otimes e_k^*, \quad F = \sum_{k=m+1}^{r} f_k \otimes e_k^*$$

in terms of some orthonormed basis $\{e_k : k = 1, \dots, r\}$ of $\mathbf{H}_2$, an orthonormed system $\{f_k : k = 1, \dots, r\}$ in $\mathbf{H}_1$ and rank-1 $\mathbf{H}_2 \to \mathbf{H}_1$ operators $f \otimes e^* : x \mapsto e^*(x) f = \langle x | e \rangle f$.

Define

$$\mathbf{H}_{2,1} := \oplus_{k=1}^{m} \mathbf{C} e_k = \ker^{\perp}(E), \quad \mathbf{H}_{2,2} := \oplus_{k=m+1}^{r} \mathbf{C} e_k = \ker^{\perp}(F), \quad J := E | \mathbf{H}_{2,1},$$

$$\mathbf{H}_{1,1} := \oplus_{k=1}^{m} \mathbf{C} f_k = \mathrm{range}(E), \quad \mathbf{H}_{1,2} := \oplus_{k=m+1}^{r} \mathbf{C} f_k = \mathrm{range}(F),$$

$$\mathbf{H}_{1,3} := \mathbf{H}_1 \ominus [\mathbf{H}_{1,1} \oplus \mathbf{H}_{1,2}] = \mathbf{H}_1 \ominus \mathrm{range}(E+F).$$

Straightforward calculation yields

$$\mathcal{T}^{-1} \begin{bmatrix} U' & b \\ b^* & V' \end{bmatrix} \mathcal{T} =$$
$$= \begin{bmatrix} U'_{11} - J^* b^*_{11} & U'_{12} - J^* b^*_{21} & U'_{13} - J^* b^*_{31} & U'_{11} - J b^*_{11} J + b_{11} - J^* V'_{11} & b_{12} - J^* V'_{12} \\ U'_{21} & U'_{22} & U'_{23} & U'_{21} J + b_{21} & b_{22} \\ U'_{31} & U'_{32} & U'_{33} & U'_{31} J + b_{31} & b_{32} \\ b^*_{11} & b^*_{21} & b^*_{31} & b^*_{11} J + V'_{11} & V'_{12} \\ b^*_{12} & b^*_{22} & b^*_{22} & b^*_{12} J + V'_{21} & V'_{22} \end{bmatrix}.$$

The Kaup type vector field corresponding to the generator of $[\Phi^t : t \in \mathbf{R}]$ is

$$\Phi'(X) = b - X b^* X + U' X - X V', \quad \mathrm{dom}(\Phi') = \{X \in \mathbf{B} : \mathrm{ran}(X) \subset \mathrm{dom}(U')\}.$$

Moreover even

$$[M \circ \mathrm{U}]'(X) = b - X b^* X + U' X - X V', \quad \mathrm{dom}(\Phi') = \{X \in \mathbf{B} : \mathrm{ran}(X) \subset \mathrm{dom}(U')\}.$$

Taking into account that E is a common fixed point of the continuous extensions $\overline{\Phi}^t$ to the closed unit ball $\overline{\mathbf{B}}$, we have

$$(*) \qquad 0 = \overline{\Phi}'(E) = b - Eb^*E + U'E - EV', \qquad \mathrm{range}(E) \subset \mathrm{dom}(U').$$

In terms of the submatrices $b_{j,\ell}, U'_{j,k}, V'_{\ell,m}$, $(*)$ can be written as

$$0 = \begin{bmatrix} b_{11} - Jb_{11}^*J + U'_{11}JV'_{11} & b_{12} - JV'_{12} \\ b_{21} + U'_{21}J & b_{22} \\ b_{31} + U'_{31}J & b_{32} \end{bmatrix}.$$

Comparing the entries of $\mathcal{T}^{-1}\mathcal{A}'\mathcal{T}$ with the entries above, we get

$$\mathcal{T}^{-1}\mathcal{A}'\mathcal{T} = \begin{bmatrix} U'_{11} - b_{11}^* & 0 & 0 & 0 & 0 \\ -b_{21} & U'_{22} & U'_{23} & 0 & 0 \\ -b_{31} & U'_{32} & U'_{33} & 0 & 0 \\ b_{11}^* & b_{21}^* & b_{31}^* & b_{11} + V'_{11} & b_{12} \\ b_{12}^* & 0 & 0 & 0 & V'_{22} \end{bmatrix}$$

whence the statement is immediate.

Lemma. Let $p := \mathrm{Proj}_{\mathrm{ran}(E)} = EE^*$ and $q := 1 - p$. Then $p[U' - Eb^*]q = 0$.

Proof. We have $b - Eb^*E + U'E - EV' = 0$. Hence

$$U'E + b = EV' - Eb^*E,$$

$$\big[U'E - b\big]^* = \big[EV' + Eb^*E\big]^*,$$

$$E^*[U']^* - b^* = [V']^*E^* - E^*bE^*,$$

$$-E^*U' + b^* = -V'E^* + E^*bE^* = [-V' + E^*b]E^*,$$

$$[-E^*U' + b^*]q = \big[-EE^*V' - Eb^*\big]E^*(1 - EE^*) = 0,$$

$$-[EE^*U' - Eb^*]q = 0,$$

$$EE^*[U' - Eb^*]q = 0$$

since $E = EE^*E$ and $E^* = E^*EE^*$.

$$0 = b - Eb^*E + U'E - EV', \quad EE^*E = E, \quad \mathrm{Pr}_{\mathrm{ran}(E)}) = EE^*, \mathrm{Pr}_{\mathrm{ran}^\perp}(E) = E^*E$$

$$0 = (1 - EE^*)(b - Eb^*E + U'E - EV\prime) =$$

$$= (1 - EE^*)(b + U\prime E) \quad \Big|.^* \quad [U']^* \supset -U' \text{ antisymm.}$$

$$0 = (b^* - E^*[U']*)(1 - EE^*) =$$

$$= (b^* - E^*U')(1 - EE^*) \quad \Big|E\cdot$$

$$0 = (Eb^* - EE^*U')(1 - EE^*) =$$

$$= (EE^*Eb^* - EE^*U')(1 - EE^*) = EE^*(Eb^* - U')(1 - EE^*)$$

$$0 = \mathrm{Pr}_{\mathrm{ran}(E)}(U' - Eb^*)P_{\mathrm{ran}^\perp(E)}$$

$$\mathbf{H}_{1,1} := \mathrm{ran}(E), \ \mathbf{H}_{1,2} := \mathrm{ran}^\perp(E), \quad P_k := \mathrm{Pr}_{\mathbf{H}_{1,k}}$$

$$P_1(U' - Eb^*)P_2 = 0$$

$$\mathcal{T} := \begin{bmatrix} 1 & E \\ 0 & 1 \end{bmatrix}, \quad \mathcal{T}^{-1} = \begin{bmatrix} 1 & -E \\ 0 & 1 \end{bmatrix}, \quad \mathcal{A} := \begin{bmatrix} U' & b \\ b^* & V' \end{bmatrix}$$

$$0 = b - Eb^*E + U'E - EV'$$

$$\mathcal{T}^{-1}\mathcal{A}\mathcal{T} = \begin{bmatrix} U' - Eb^*) & 0 \\ b^* & V' - b^*E \end{bmatrix} =$$
$$= \begin{bmatrix} P_1(U' - Eb^*)P_1 & 0 & 0 \\ P_2(U' - Eb^*)P_1 & P_2(U' - Eb^*)P_2 & 0 \\ b^*P_1 & b^*P_2 & V' - b^*E \end{bmatrix}$$
$$P_2(U' - Eb^*)P_1 =^{P_2E=0}= (1 - EE^*)(U' - Eb^*)EE^* =$$

$$= (1 - EE^*)U'EE^* = P_2U'P_1$$

$$P_2(U' - Eb^*)P_2 = P_2U'P_2$$
$$\mathcal{T}^{-1}\mathcal{A}\mathcal{T} = \begin{bmatrix} U'_{1,1} - Eb^*P_1 & 0 & 0 \\ U'_{2,1} & U'_{2,2} & 0 \\ b^*P_1 & b^*P_2 & V' - b^*E \end{bmatrix}$$

FINITE DIM. HILBERT CASE: INVARIANT DISCS

$\mathbf{H}, \langle .|\rangle$ finite dim. complex Hilbert space. A unit vector $e \in \mathbf{H}$ is fixed point of a complete hol. vect. field of the unit ball

$$X(e) = 0, \text{where } \; X(z) := -\langle (iA - \lambda x - e)|e\rangle x + (iA + \lambda)(x - e), \quad A = A^* \in \mathcal{L}(\mathbf{H}), \quad \lambda \in \mathbf{R}.$$

Question. Does there exist an X-invariant disc passing in $\mathbf{B}$ touching e?

Equivalently: $\quad \exists ?\; v \not\parallel e \quad X(e + \zeta v) \| v \;\; (\zeta \in \mathbf{C})$.

$$X(e+\zeta v) = -\langle (iA - \lambda)\zeta v|e\rangle (e + \zeta v) + (iA + \lambda)\zeta v =$$

$$= -\zeta \langle (iA + \lambda)v|e\rangle e + \big[\|v\big] + \zeta iAv + \big[\|v\big] =$$

$$= \zeta\big[-P(iA - \lambda) + iA\big]v + \big[\|v\big] = (1 - P)(iA - \lambda)v + \big[\|v\big]$$

where $P :=$ [ort.proj. onto $\mathbf{C}e$] $= [x \mapsto \langle x|e\rangle e]$. Thus a disc $e + (1 + \Delta)v$ is X-invariant iff

$$\exists \; \mu \in \mathbf{C} \quad (1 - P)(iA - \lambda)v = \mu v.$$

Question. Is it possible that all the eigenvectors of $(1 - P)(iA - \lambda)$ are $\perp e$?

Observation: $(1-P)(iA-\lambda)|e^\perp = \big[(1-P)(iA)(1-P) + \lambda \mathrm{Id}\big]\big|\mathrm{ran}(1-P)$ is a *normal* operator $\mathrm{ran}(1-P) = e^\perp \to e^\perp$. Hence $e^\perp$ admits an orthonormed basis $f_1, \dots f_{N-1}$ consisting of eigenvectors of $(1 - P)A(1 - P)$ and, with some $\beta_1, \dots \beta_N \in \mathbf{R}$ and $\gamma_1, \dots, \gamma_{N-1} \in \mathbf{C}$, we can write the self-adjoint operator A in hermitian symmetric matrix form

$$\mathrm{Matrix}_{\{f_1,\dots,f_{N-1}\}}(A) = \begin{bmatrix} \beta_1 & & & & \overline{\gamma_1} \\ & \beta_2 & & & \overline{\gamma_2} \\ & & \ddots & & \vdots \\ & & & \beta_{N-1} & \overline{\gamma_{N-1}} \\ \gamma_1 & \gamma_2 & \cdots & \gamma_{N-1} & \beta_N \end{bmatrix}.$$

Thus a vector $v=\left[\zeta_1,\ldots,\zeta_{N-1},0\right]^{\mathrm{T}}\equiv\sum_k\zeta_k f_k$ is a μ-eigenvector of $(1-P)(iA-\lambda)$ if and only if $v\in\operatorname{Span}\left\{f_k : i\beta_k-\lambda=\mu\right\}$. Hence we have $(N-1)$ independent eigenvectors $\perp e$. At most one more eigendirection of $(1-P)(iA-\lambda)$ may remain which necessarily consists of multiples of a vector of the form $v=\left[\zeta_1,\ldots,\zeta_{N-1},1\right]^{\mathrm{T}}\equiv\sum_k\zeta_k f_k+e$. Then

$(1-P)(iA-\lambda)v=\sum\limits_{k<N}\left[\zeta_k(i\beta_k-\lambda)+i\overline{\gamma_k}\right]f_k$ and

$$(1-P)(iA-\lambda)v=\mu v \iff \quad \zeta_k(i\beta_k-\lambda)+i\overline{\gamma_k}=\mu \quad (k<N), \quad 0=\mu.$$

The latter system has no solution $(\zeta_1,\ldots,\zeta_{N-1})$ if and only if $\lambda=0$ and $\beta_k=0\neq\gamma_k$ for some index $k<n$. This is the case when all the eigenvectors of $(1-P)(iA-\lambda)$ are $\perp e$.

Example. $N=2,\ A:=\begin{bmatrix}0&1\\1&0\end{bmatrix},\ e:=\begin{bmatrix}0\\1\end{bmatrix},\ X(x):=-\langle iA(x-e)|e\rangle x+iA(x-e).$

Then $\{v : e+v+\Delta v\ X\text{-inv. disc}\}=\mathbf{C}f$ with $f:=[1\ 0]^{\mathrm{T}}$.

Proof. $P=\begin{bmatrix}1&0\\0&0\end{bmatrix}$, $(1-P)(iA)=i\begin{bmatrix}0&1\\0&0\end{bmatrix}$ nilpotent with eigenvectors only in $\mathbf{C}f$.

Direct calculation:

$$x:=\begin{bmatrix}\xi\\\eta\end{bmatrix}\Rightarrow A(x-e)=\begin{bmatrix}\eta-1\\\xi\end{bmatrix},\ X(x)=-i\xi=\begin{bmatrix}\xi\\\eta\end{bmatrix}+=i\begin{bmatrix}\eta-1\\\xi\end{bmatrix}=i\begin{bmatrix}-1+\eta-\xi^2\\\xi-\xi\eta\end{bmatrix};$$
$$X(e+\zeta v)=X\left(\begin{bmatrix}\zeta\nu_1\\1+\zeta\nu_2\end{bmatrix}\right)=-i\zeta\begin{bmatrix}\zeta\nu_1^2+\nu_2\\\zeta\nu_1\nu_2\end{bmatrix}.$$
$$X(e+\zeta v)\|v \iff \det\begin{bmatrix}\zeta\nu_1^2+\nu_2&\nu_1\\\zeta\nu_1\nu_2&\nu_2\end{bmatrix}=0 \iff \nu_2^2=0 \iff v\|f.$$

Convolution of functions of the form $\mathrm{pol}(t)e^{\rho t}$

Let $p \in \mathrm{Pol}_n(\mathbf{R})$ that is $p^{(n+1)} \equiv 0$. Then

$$\int_{t=a}^{b} p(t)e^{\rho t}dt = \rho^{-1}e^{\rho t}p(t)\Big|_{t=a}^{b} - \int_{t=a}^{b} \rho^{-1}p'(t)\,dt =$$

$$= \rho^{-1}e^{\rho t}p(t)\Big|_{t=a}^{b} - \rho^{-2}e^{\rho t}p'(t)\Big|_{t=a}^{b} + \int_{t=a}^{b} \rho^{-2}p''(t)\,dt =$$

$$= \cdots = \sum_{k=0}^{n}(-1)^k\rho^{-(k+1)}p^{(k)}(t)e^{\rho t}\Big|_{t=a}^{b}.$$

Let $p_1 \in \mathrm{Pol}_n(\mathbf{R})$, $p_2 \in \mathrm{Pol}_m(\mathbf{R})$. $\left[p_1(t)e^{\rho_1 t}\right] * \left[p_2(t)e^{\rho_2 t}\right] =?$

$$\int_{s=0}^{t} \left[e^{\rho_1(t-s)}p_1(t-s)\right]\left[e^{\rho_2 s}p_2(s)\right] ds \stackrel{s=\frac{t}{2}+\frac{u}{2}}{=\!=}$$

$$= \int_{u=-t}^{t} e^{\rho_1\left(\frac{t}{2}-\frac{u}{2}\right)}p_1\left(\frac{t}{2}-\frac{u}{2}\right)e^{\rho_2\left(\frac{t}{2}+\frac{u}{2}\right)}p_2\left(\frac{t}{2}+\frac{u}{2}\right)\frac{1}{2}du =$$

$$= \frac{e^{\frac{\rho_1+\rho_2}{2}t}}{2}\int_{u=-t}^{t} e^{\frac{\rho_2-\rho_1}{2}u}\underbrace{p_1\left(\frac{t}{2}-\frac{u}{2}\right)p_2\left(\frac{t}{2}+\frac{u}{2}\right)}_{p(u)}du =$$

$$= \frac{e^{\frac{\rho_1+\rho_2}{2}t}}{2}\sum_{k=0}^{n_1+n_2}(-1)^k\left[\frac{\rho_2-\rho_1}{2}\right]^{-(k+1)}p^{(k)}(u)e^{\frac{\rho_2-\rho_1}{2}u}\Big|_{u=-t}^{t} =$$

$$= \sum_{k=0}^{n_1+n_2}\frac{(-1)\cdot 2^k}{(\rho_1-\rho_2)^{k+1}}\left[p^{(k)}(t)e^{\rho_2 t} - p^{(k)}(-t)e^{\rho_1 t}\right] =$$

$$= e^{\rho_1 t}\sum_{k=0}^{n_1+n_2}\frac{2^k}{(\rho_1-\rho_2)^{k+1}}p^{(k)}(-t) - e^{\rho_2 t}\sum_{k=0}^{n_1+n_2}\frac{2^k}{(\rho_1-\rho_2)^{k+1}}p^{(k)}(t).$$

Here we have

$$p^{(k)}(u) = \frac{d^k}{du^k}\left[p_1\left(\frac{t}{2}-\frac{u}{2}\right)p_2\left(\frac{t}{2}+\frac{u}{2}\right)\right] =$$

$$= \sum_{\ell=0}^{k}\binom{k}{\ell}\left[\frac{d^\ell}{du^\ell}p_1\left(\frac{t}{2}-\frac{u}{2}\right)\right]\left[\frac{d^{k-\ell}}{du^{k-\ell}}p_2\left(\frac{t}{2}+\frac{u}{2}\right)\right] =$$

$$= \sum_{\ell=0}^{k}\binom{k}{\ell}\left(-\frac{1}{2}\right)^{\ell}p_1^{(\ell)}\left(\frac{t}{2}-\frac{u}{2}\right)\left(\frac{1}{2}\right)^{k-\ell}p_2^{(k-\ell)}\left(\frac{t}{2}+\frac{u}{2}\right) =$$

$$= \sum_{\ell=0}^{k}(-1)^\ell\frac{1}{2^k}\binom{k}{\ell}p_1^{(\ell)}\left(\frac{t}{2}-\frac{u}{2}\right)p_2^{(k-\ell)}\left(\frac{t}{2}+\frac{u}{2}\right).$$

It follows

$$p^{(k)}(-t) = \sum_{\ell=0}^{k}(-1)^{\ell}\frac{1}{2^k}\binom{k}{\ell}p_1^{(\ell)}(t)\,p_2^{(k-\ell)}(0)\,,$$

$$p^{(k)}(t) = \sum_{\ell=0}^{k}(-1)^{\ell}\frac{1}{2^k}\binom{k}{\ell}p_1^{(\ell)}(0)\,p_2^{(k-\ell)}(t) \overset{\bar{\ell}=k-\ell,\ \binom{k}{\ell}=\binom{k}{\bar{\ell}}}{=}$$

$$= (-1)^k\sum_{\bar{\ell}=0}^{k}(-1)^{\bar{\ell}}\frac{1}{2^k}\binom{k}{\bar{\ell}}p_2^{(\bar{\ell})}(t)\,p_1^{(k-\bar{\ell})}(0)\,.$$

Hence

$$\left[p_1(t)e^{\rho_1 t}\right] * \left[p_2(t)e^{\rho_2 t}\right] =$$

$$= e^{\rho_1 t}\sum_{k=0}^{n_1+n_2}\frac{1}{(\rho_1-\rho_2)^{k+1}}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}p_1^{(\ell)}(t)\,p_2^{(k-\ell)}(0)+$$

$$+ e^{\rho_2 t}\sum_{k=0}^{n_1+n_2}\frac{1}{(\rho_2-\rho_1)^{k+1}}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}p_2^{(\ell)}(t)\,p_2^{(k-\ell)}(0)\,.$$

For later use we calculate the case with $p_k \equiv t^{n_k}$ $(k = 1, 2)$. In general, $\left[t^n\right]^{(m)} = \frac{n!}{(n-m)!}t^{n-m}$ with $\left[t^n\right]^{(m)}\big|_{t=0} = \delta_{mn}n!$ $(0 \le m \le n)$. In particular we have then $p_k^{(\ell)} \equiv 0$ for $\ell > n_k$ and $p_k^{(m)}(0) = 0$ for $m \neq n_k$. Therefore

$$\left[p_1(t)e^{\rho_1 t}\right] * \left[p_2(t)e^{\rho_2 t}\right] =$$

$$= e^{\rho_1 t}\sum_{k=0}^{n_1+n_2}\frac{1}{(\rho_1-\rho_2)^{k+1}}\sum_{\substack{\ell:0\le\ell\le n_1,\\ k-\ell=n_2}}(-1)^{\ell}\binom{k}{\ell}\frac{n_1!}{(n_1-\ell)!}t^{n_1-\ell}\,n_2!\,+$$

$$+ e^{\rho_2 t}\sum_{k=0}^{n_1+n_2}\frac{1}{(\rho_2-\rho_1)^{k+1}}\sum_{\substack{\ell:0\le\ell\le n_2,\\ k-\ell=n_1}}(-1)^{\ell}\binom{k}{\ell}\frac{n_2!}{(n_2-\ell)!}t^{n_2-\ell}\,n_1! =$$

$$= e^{\rho_1 t}\sum_{\ell=0}^{n_1}\frac{1}{(\rho_1-\rho_2)^{n_2+\ell+1}}(-1)^{\ell}\binom{n_2+\ell}{\ell}\frac{n_1!}{(n_1-\ell)!}t^{n_1-\ell}\,n_2!\,+$$

$$+ e^{\rho_2 t}\sum_{\ell=0}^{n_2}\frac{1}{(\rho_2-\rho_1)^{n_1+\ell+1}}(-1)^{\ell}\binom{n_1+\ell}{\ell}\frac{n_2!}{(n_2-\ell)!}t^{n_2-\ell}\,n_1! =$$

$$= e^{\rho_1 t}\sum_{d=0}^{n_1}\frac{1}{(\rho_1-\rho_2)^{n_1+n_2-d+1}}(-1)^{n_1-d}\binom{n_1+n_2-d}{n_1-d}\frac{n_1!n_2!}{d!}t^d\,+$$

$$+ e^{\rho_2 t}\sum_{d=0}^{n_2}\frac{1}{(\rho_2-\rho_1)^{n_1+n_2-d+1}}(-1)^{n_2-d}\binom{n_1+n_2-d}{n_2-d}\frac{n_1!n_2!}{d!}t^d\,.$$

Lemma. $\left[e^{\rho t}\right]^{*(n+1)} = \dfrac{t^n}{n!} e^{\rho t} \qquad (n = 0, 1, 2, \ldots).$

Proof. Induction by n with $\left[e^{\rho t}\right]^{*(n+1)} = p_n(t) e^{\rho t}$. The case $n = 0$ trivial with $p_0 \equiv 1$. On the other hand, $\left[\left[e^{\rho t}\right]^{*n}\right] * \left[e^{\rho t}\right] = \int_{s=0}^{t} p_n(s) e^{\rho s} e^{\rho(t-s)} ds = \left[\int_{s=0}^{t} p_n(s)\, ds\right] e^{\rho t}$, whence the statement is immediate.

$s(t) := \dfrac{\sin \lambda t}{\lambda}.$

$$s^{*n}(t) = \left[\frac{e^{i\lambda t} - e^{-i\lambda t}}{2i\lambda}\right]^{*n} = \frac{1}{(2i\lambda)^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left[e^{-i\lambda t}\right]^{*k} * \left[e^{i\lambda t}\right]^{*(n-k)} =$$

$$= \frac{1}{(2i\lambda)^n} \left[\sum_{k=1}^{n-1} (-1)^k \binom{n}{k} \left[e^{-i\lambda t}\right]^{*k} * \left[e^{i\lambda t}\right]^{*(n-k)} + \left[e^{i\lambda t}\right]^{*n} + (-1)^n \left[e^{-i\lambda t}\right]^{*n}\right] =$$

$$= \frac{1}{(2i\lambda)^n} \left[\sum_{k=1}^{n-1} \frac{(-1)^k n!}{k!(n-k)!} \left[\frac{t^{k-1}}{(k-1)!} e^{-i\lambda t}\right] * \left[\frac{t^{n-k-1}}{(n-k-1)!} e^{i\lambda t}\right] + \frac{t^{n-1}}{(n-1)!} \left(e^{i\lambda t} + (-1)^n e^{-i\lambda t}\right)\right] =$$

$$= \frac{1}{(2i\lambda)^n} \left[\sum_{k=1}^{n-1} \frac{(-1)^k n! \left[t^{k-1} e^{-i\lambda t}\right] * \left[t^{n-k-1} e^{i\lambda t}\right]}{k!(n-k)!(k-1)!(n-k-1)!} + \frac{t^{n-1}}{(n-1)!} \left(e^{i\lambda t} + (-1)^n e^{-i\lambda t}\right)\right] .$$