VON NEUMANN ALGEBRA PREDUALS SATISFY THE LINEAR BIHOLOMORPHIC PROPERTY

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Abstract. We prove that for every JBW*-triple E of rank > 1, the symmetric part of its predual reduces to zero. Consequently, the predual of every infinite dimensional von Neumann algebra A satisfies the linear biholomorphic property, that is, the symmetric part of A_* is zero. This solves a problem posed by M. Neal and B. Russo [15, to appear in *Mathematica Scandinavica*].

1. Introduction

The open unit ball of every complex Banach space satisfies certain holomorphic properties which determine the global isometric structure of the whole space. An illustrative example is the following result of W. Kaup and H. Upmeier [13].

Theorem 1.1. [13] *Two complex Banach spaces whose open unit balls are biholomorphically equivalent are linearly isometric.* □

We recall that, given a domain U in a complex Banach space X (i.e. an open, connected subset), a function f from U to another complex Banach space F is said to be *holomorphic* if the Fréchet derivative of f exists at every point in U. When $f : U \to f(U)$ is holomorphic and bijective, $f(U)$ is open in F and $f^{-l} : f(U) \to U$ is holomorphic, the mapping f is said to be *biholomorphic*, and the sets U and $f(U)$ are *biholomorphically equivalent*. Theorem 1.1 gives an idea of the power of infinite-dimensional Holomorphy in Functional Analysis. A reviewed proof of Theorem 1.1 was published by J. Arazy in [1].

A consequence of the results established by Kaup and Upmeier in [13] gave raise to the study of the symmetric part of an arbitrary complex Banach space in the following sense: Let X be a complex Banach space with open unit ball denoted by D. Let $G = \mathrm{Aut}(D)$ denote the group of all biholomorphic automorphisms of D and let G^O stand for the connected component of the identity in G. Given a holomorphic function $h : D \to X$, we can define a holomorphic vector field $Z = h(z)\frac{\partial}{\partial z}$, which is a composition differential operator on the space $H(D, X)$ of all holomorphic functions from D to X, given by $X(f)(z) = (h(z)\frac{\partial}{\partial z})f(z) = f'(z)(h(z))$, $(z \in D)$. It is known that, for each z_0 the initial value problem $\frac{\partial}{\partial t}\varphi(t, z_0) = h(\varphi(t, z_0))$, $\varphi(0; z_0) = z_0$ has a unique solution $\varphi(t, z_0) : J_{z_0} \to D$ defined on a maximal open interval $J_{z_0} \subseteq \mathbb{R}$ containing 0. The holomorphic mapping h is called *complete* when $J_{z_0} = \mathbb{R}$, for every $z_0 \in D$. Denoting by $\mathrm{aut}(D)$ the Lie algebra of all complete,

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holomorphic vector fields on D, the *symmetric part of* D is $D_S = G(O) = G^O(O)$. The *symmetric part of* X, denoted by X_S or by $S(X)$, is the orbit of 0 under the set $\mathrm{aut}(D)$ of all complete holomorphic vector fields on D. Furthermore, X_S is a closed, complex subspace of X, $D_S = X_S \cap D$, and hence, D_S is the open unit ball of X_S, D_S is *symmetric* in the sense that for each $z \in D_S$ there exists a symmetry of D at z, i.e., a mapping $s_z \in \mathrm{Aut}(D_S)$ such that $s_z(z) = z$, $s_z^2 = identity$, and $s_z'(z) = -Id_E$; thus $D_S = E_S \cap D$ is a *bounded symmetric domain* (cf. [13], [4], and [1]).

A Jordan structure associated with the symmetric part of every complex Banach space X was also determined by W. Kaup and H. Upmeier in [13]. Namely, for every $a \in X_S$ there is a unique symmetric continuous bilinear mapping $Q_a : X \times X \to X$ such that $(a - Q_a(z,z))\frac{\partial}{\partial z}$ is a complete holomorphic vector field on D. A partial triple product is defined on $X \times X_S \times X$ by the assignment

$$\{.,.,.\} : X \times X_S \times X \to X,$$
$$\{x, a, y\} := Q_a(x, y).$$

It is known (cf. [13] and [4]) that the partial triple product satisfies the following properties:

(i) $\{.,.,.\}$ is bilinear and symmetric in the outer variables and conjugate linear in the middle one;
(ii) $\{X_S, X_S, X_S\} \subseteq X_S$;
(iii) The Jordan identity
$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\},$$
holds for every $a, b, y \in X_S$ and $x, z \in X$;
(iv) For each $a \in X_S$, the mapping $L(a, a) : X \to X$, $z \mapsto \{a, a, z\}$ is a hermitian operator;
(v) The identity $\{\{x, a, x\}, b, x\} = \{x, a, \{x, b, x\}\}$ holds for every $a, b \in X_S$ and $x \in X$.

It should be remarked here that property (v) appears only implicitly in [4]. A complete substantiation is included in [16] (compare also [20]).

The extreme possibilities for the symmetric part X_S (i.e. $X_S = X$ or $X_S = \{0\}$) define particular and significant classes of complex Banach spaces. The deeply studied class of *JB*-triples*, introduced by W. Kaup in [12], is exactly the class of those complex Banach spaces X for which $X_S = X$. In the opposite side, we find the complex Banach spaces satisfying the *linear biholomorphic property* (LBP, for short). A complex Banach space X with open unit ball D satisfies the LBP when its symmetric part is trivial (cf. [1, page 145]).

The symmetric part of some classical Banach spaces was studied and determined by R. Braun, W. Kaup and H. Upmeier [4], L.L. Stachó [18], J. Arazy [1], and J. Arazy and B. Solel [2]. The following list covers the known cases:

(i) For $X = L_p(\Omega, \mu)$, $1 \leq p < 1$, $p \neq 2$, and $\dim(X) \geq 2$, we have $X_S = 0$;
(ii) For $X = H_p$ the classical Hardy spaces with $1 \leq p < 1$, $p \neq 2$, we have $X_S = 0$;
(iii) For $X = H_\infty$ or the disk algebra, $X_S = \mathbb{C}$;
(iv) When X is a uniform algebra $A \subseteq C(K)$, $A_S = A \cap \overline{A}$.
(v) When A is a subalgebra of $B(H)$ containing the identity operator I, then A_S is the maximal C*-subalgebra $A \cap A^*$ of A;

(*vi*) Let X be a complex Banach space with a 1-unconditional basis. Then $X = X_S$ if and only if X is the c_0-sum of a sequence of Hilbert spaces. Moreover, if X is a symmetric sequence space (i.e. the unit vector basis form a 1-symmetric basis of E) then either $X_S = \{0\}$ or $X_S = X$. In the last case, either $X = \ell_1$ or $X = c_0$.

In a very recent contribution, M. Neal and B. Russo stated the following problem:

Problem 1.2. [15, Problem 2] *Is the symmetric part of the predual of a von Neumann algebra equal to 0? What about the predual of a JBW***-triple which does not contain a Hilbert space as a direct summand?*

In this note we give a complete answer to the questions posed by Neal and Russo [15] in the above problem. Our main result proves that for every JBW*-triple W which is not a Hilbert space, the symmetric part of its predual reduces to zero. In particular the symmetric part of the predual of an infinite-dimensional von Neumann algebra is equal to $\{0\}$. Unfortunately, there exist examples of JBW*-triples W containing a Hilbert space as a direct summand for which $S(W_*) = (W_*)_S = \{0\}$.

2. Computing the symmetric part of a JBW*-triple predual

We recall that a JB*-triple is a complex Banach space E satisfying that $E_S = E$. JB*-triples were introduced by W. Kaup in [12], where he also gave the following axiomatic definition of these spaces: A *JB***-triple* is a complex Banach space E equipped with a triple product $\{\cdot,\cdot,\cdot\} : E \times E \times E \to E$ which is linear and symmetric in the outer variables, conjugate linear in the middle one, satisfies the axioms (*iii*) and (*iv*) in (1) and the following condition:

(*vi*) $\|\{x, x, x\}\| = \|x\|^3$ for all $x \in E$.

Every C*-algebra is a complex JB*-triple with respect to the triple product $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$, and in the same way every JB*-algebra with respect to $\{a, b, c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$.

Elements a, b in a JB*-triple E are said to be *orthogonal* (denoted by $a \perp b$) whenever $L(a, b) = 0$. It is known that $a \perp b \Leftrightarrow \{a, a, b\} = 0 \Leftrightarrow \{b, b, a\} = 0$ (cf. [8, Lemma 1]). The rank, $r(E)$, of a real or complex JB*-triple E, is the minimal cardinal number r satisfying $card(S) \leq r$ whenever S is an orthogonal subset of E, i.e. $0 \notin S$ and $x \perp y$ for every $x \neq y$ in S.

We briefly recall that an element e in a JB*-triple E is said to be a *tripotent* whenever $\{e, e, e\} = e$. A tripotent $e \in E$ is said to be complete whenever $a \perp e$ implies $a = 0$. When the condition $\{e, e, a\} = a$ implies that $a \in \mathbb{C}e$, we shall say that e is a minimal tripotent. The symbol $\mathrm{Tri}(E)$ will stand for the set of all tripotents in E.

The following characterization of complete holomorphic vector fields, which is originally due to L.L. Stachó (see [18], [19] and [21]), has been borrowed from [2, Proposition 2.5].

Proposition 2.1. *Let X be a complex Banach space whose open unit ball is denoted by D and let $h : D \to X$ be a holomorphic mapping. Then $h \in \mathrm{aut}(D)$ if and only if h extends holomorphically to a neighborhood of $\overline{D}$, and, for every $z \in X$, $\varphi \in X^*$ satisfying $\|z\| = \|\varphi\| = 1 = \varphi(z)$, we have $\Re\mathrm{e}\varphi(h(z)) = 0$.* □

In order to simplify the arguments, we recall some geometric notions. Elements s, y in a complex Banach space X are said to be *L-orthogonal*, denoted by $x \perp_L y$, (respectively, *M-orthogonal*, denoted by $x \perp_M y$) if $\|x \pm y\| = \|x\| + \|y\|$ (respectively, $\|x \pm y\| = \max\{\|x\|, \|y\|\}$). It is known that $x \perp_L y$ if, and only if, for all real numbers s, t, $sx \perp_L ty$ if, and only if, there exist elements $a, b \in X^*$ satisfying $a \perp_M b$, $\|x\| \, \|a\| = \|x\| = a(x)$, and $\|y\| \, \|b\| = \|y\| = b(y)$ (see, for example, [10, Lemma 3.1 and Corollary 4.3]). It is also known that for each pair of elements (a, b) in a JB*-triple E, the condition $a \perp b$ implies $a \perp_M b$ (cf. [8, Lemma 1] and [11, Lemma 1.3(a)]).

We also recall that a JBW*-triple is a JB*-triple which is also a dual Banach space. In this sense, JBW*-triples play an analogue role to that given to von Neumann algebras in the setting of C*-algebras. Every JBW*-triple admits a unique (isometric) predual and its product is separately weak*-continuous (see [3]).

We can proceed with a first technical result on the structure of the symmetric part of a JBW*-triple predual.

Proposition 2.2. *Let W be a JBW*-triple with predual $W_* = F$. Suppose, e_1, e_2 are two tripotent elements in W, $\varphi_1, \varphi_2 \in F$ with $\|\varphi_k\| = 1$, $e_1 \perp e_2$, and $e_j(\varphi_k) = \delta_{jk}$ $(j, k = 1, 2)$. Then $e_1(\phi) = e_2(\phi) = 0$, for every ϕ in F_S.*

Proof. Let ϕ be an element in F_S. Since $\phi \in F_S$, the holomorphic vector field $\big[\phi - Q_\phi(z, z)\big] \frac{\partial}{\partial z}$ is tangent to the unit sphere of F. Thus, by Proposition 2.1,

$$\Re\text{e}\big\langle e, \phi - Q_\phi(\varphi, \varphi)\big\rangle = 0,$$

for every $\varphi \in F$, $e \in W$ with $\|\varphi\| = \|e\| = 1 = \langle e, \varphi\rangle\big(= e(\varphi)\big)$.

Since $e_1 \perp e_2$ implies $e_1 \perp_M e_2$, it follows from the hypothesis that $\varphi_1 \perp_L \varphi_2$. In particular, for any weight $0 \le \lambda \le 1$ and $\kappa_1, \kappa_2 \in \mathbb{T} := \{\kappa \in \mathbb{C} : \ |\kappa| = 1\}$, $\kappa_1(1-\lambda)\varphi_1 + \kappa_2\lambda\varphi_2$ belongs to the unit sphere of F and $\overline{\kappa_1} e_1 + \overline{\kappa_2} e_2$ is a supporting functional for it. Therefore,

$$0 = \Re\text{e}\Big\langle \overline{\kappa_1} e_1 + \overline{\kappa_2} e_2, \phi - Q_\phi\big(\kappa_1(1-\lambda)\varphi_1 + \kappa_2\lambda\varphi_2, \kappa_1(1-\lambda)\varphi_1 + \kappa_2\lambda\varphi_2\big)\Big\rangle$$

$$= \Re\text{e}\Big(\overline{\kappa_1} e_1(\phi) + \overline{\kappa_2} e_2(\phi) + \kappa_1(1-\lambda)^2\alpha_1 + \kappa_2\lambda^2\alpha_2 + \kappa_1\lambda(1-\lambda)\beta_1 + \kappa_2\lambda(1-\lambda)\beta_2\Big)$$

with the constants $\alpha_k := \big\langle e_k, Q_\phi(\varphi_k, \varphi_k)\big\rangle$, $\beta_k := 2\big\langle e_{3-k}, Q_\phi(\varphi_k, \varphi_{3-k})\big\rangle$. In particular, with the choice $\lambda = 1$ we get

$$\Re\text{e}\big(\overline{\kappa_1} e_1(\phi) + \overline{\kappa_2} e_2(\phi) + \kappa_2\alpha_2\big) = 0$$

for every $\kappa_1, \kappa_2 \in \mathbb{T}$. Replacing κ_2 with $-\kappa_2$ we have $\Re\text{e}\big(\kappa_1 e_1(\phi)\big) = 0 \quad (\kappa_1 \in \mathbb{T})$, and hence $e_1(\phi) = 0$. $\square$

Before dealing with our main result we shall review some results on JB*-triples of rank one. For a JB*-triple E, the following are equivalent:

(a) E has rank one;
(b) E is a complex Hilbert space equipped with the triple product given by $2\{a, b, c\} := (a|b)c + (c|b)a$, where $(.|.)$ denotes the inner product of E;
(c) The set of complete tripotents in E is non-zero and every complete tripotent in E is minimal;
(d) E contains a complete tripotent which is minimal.

The equivalence $(a) \Leftrightarrow (b)$ follows, for example, from [7, Proposition 4.5]. The implications $(b) \Rightarrow (c)$ and $(c) \Rightarrow (d)$ are clear. It should be commented here that a general JB*-triple might not contain any tripotent. However, since the complete tripotents of a JB*-triple E coincide with the real and complex extreme points of its closed unit ball (cf. [14, Proposition 3.5] and [5, Lemma 4.1]), by the Krein-Milman theorem, every JBW*-triple contains an abundant set of (complete) tripotents. In the setting of JBW*-triples, a tripotent e is minimal if and only if it cannot be written as an orthogonal sum of two (non-zero) tripotents (compare the arguments in [17, Proposition 2.2]). Back to the equivalences, the implication $(d) \Rightarrow (a)$ is established in [9, Proposition 3.7 and its proof].

Theorem 2.3. *Let W be a JBW*-triple of rank > 1 and let F denote its predual. Then $F_S = \{0\}$, that is, F satisfies the linear biholomorphic property.*

Proof. Let ϕ be an element in F_S. According to the Krein-Milman Theorem, the finite linear combinations of the extreme points of the closed unit ball, $D(W)$, of W form a weak*-dense subset in $D(W)$. Therefore, it suffices to prove that

$$e(\phi) = 0 \quad \text{for all} \quad e \in \operatorname{Ext}\big(\overline{D}(W)\big), \tag{1}$$

or equivalently, $e(\phi) = 0$ for every complete tripotent $e \in W$.

Let e be a complete tripotent in W. Since W has rank > 1, the comments preceding this theorem guarantee the existence of two non-zero tripotents e_1, e_2 in W such that $e_1 \perp e_2$ and $e = e_1 + e_2$. Let us notice that the JBW*-subtriple U of W generated by e_1 and e_2 coincides with $\mathbb{C}e_1 \bigoplus^\infty \mathbb{C}e_2$. We can easily define two norm-one functionals ψ_1, ψ_2 in U_* satisfying $\psi_j(e_k) = \delta_{jk}$. By [6, Theorem], there exists norm-one weak*-continuous functionals φ_1, φ_2 in W_* which are norm-preserving extensions of ψ_1 and ψ_2, respectively. Applying Proposition 2.2 we have $e_j(\phi) = 0$, for every $j = 1, 2$, and finally $e(\phi) = e_1(\phi) + e_2(\phi) = 0$ as we desired. $\square$

It is known that a von Neumann algebra, regarded as a JBW*-triple, has rank one if and only if it coincides with $\mathbb{C}$. We therefore have:

Corollary 2.4. *Let W be a von Neumann algebra of dimension < 1 and let $F = W_*$. Then $F_S = \{0\}$, that is, F satisfies the linear biholomorphic property.* $\square$

There is an additional aspect of Problem 1.2 that should be commented. Suppose H is a complex Hilbert space, W is a non-zero JBW*-triple, and consider the JBW*-triple $U = H \bigoplus^\infty W$ (the orthogonal sum of H and W). It is clear that U has rank > 1. Thus, Theorem 2.3 implies that $S(U_*) = \{0\}$. In other words, the predual of a JBW*-triple which does not contain a Hilbert space as a direct summand satisfies the linear biholomorphic property but the class of all JBW*-triples whose preduals satisfy the linear biholomorphic property is strictly bigger.

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