JORDAN MANIFOLDS

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Abstract. We introduce Jordan manifolds as Banach manifolds whose tangent spaces are endowed with Jordan triple products depending smoothly on the underlying points. As chief examples we study in detail the natural complex geometry bounded symmetric domains with their Harish-Chandra realizations as unit balls of JB*-triples and we extend the results to Jordan manifolds where the chart transition maps are locally generalized Möbius transformations.

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1. Introduction

Recently, in [9] the second author investigated the natural complex geometry of the unit ball of a complex Hilbert C*-module. Among others, he proved the existence of a unique symmetry invariant (Levi Civitá) connection in these structure along with its explicit form in terms of the non-commutative scalar product. Our first goal in Section 2 will be to show with a different Jordan theoretical approach that the results in [9] can be extended to the natural complex geometry of any bounded symmetric domain in a complex Banach space. Actually, such domains are biholomorphically equivalent to the unit balls of the so-called JB*-triples that is Banach spaces equipped with a three variable operation $(x, y, z) \mapsto \{xyz\}$ satisfying the generalized C*-condition $\|\{xxx\}\| = \|x\|^3$ along with the Jordan identity and the fact that its elementary inner derivations are positive Hermitian operators. Using the deep topological Jordan algebraic results in [3, Section 2] we achieve simple explicit formulas for the Levi Civitá connection in terms of the Bergman operator associated with the triple product $\{\ldots\}$. Taking into account that the context in [3, Section 2] extends to general real (not only complex) topological Jordan*-triples (even Jordan pairs) our techniques can also be applied in a fairly more general setting. Namely we establish analogous results in Section 3 for real Banach manifolds which admit an atlas consisting of maps into some Jordan*-triple such that the transitions between charts around closely situated couple of points are Möbius transformations of the product triple product (Jordan-Möbius manifolds in out terminology). These results apply automatically to all complex symmetric Hermitian Banach manifolds treated in [2] but go far beyond the complex setting. In Section 4 we start a more general approach. We introduce the concept of Jordan manifold as a real or complex Banach manifold equipped with smoothly varying Jordan*-triple products on the tangent spaces. This generality includes even classical Riemann spaces (with triple product $\{uvw\}_p := \frac{1}{2}\langle u|v\rangle w + \frac{1}{2}\langle w|v\rangle u$ in terms of the scalar product on the tangent space at the point p) in a natural manner. To approach the real background of the previous results, homogeneous and symmetric Jordan manifolds are more suited. These concepts are at hand because Jordan manifolds form a category with morphisms being smooth mappings whose derivatives are homomorphisms for the pointwise triple products. Namely a Jordan manifold is homogeneous if its automorphism group is transitive and symmetric if every point admits an automorphism whose derivative there is minus-identity. For the moment it seems to be an open question if the automorphisms of a symmetric Jordan manifold form a Banach-Lie group with an analogous construction as Upmeier's topology [8] for symmetric Hermitian Banach manifolds. We close the paper with examples of various Jordan manifolds.

3. The open unit ball of a complex JB*-triple

Throughout this work let $\mathbf{Z}$ denote an arbitrarily fixed complex Banach space with norm $\|\cdot\|$. We shall write $\mathrm{Ball}(\mathbf{Z})$ for its open unit ball $\mathrm{Ball}(\mathbf{Z}) := \{z \in \mathbf{Z} : \ \|z\| < 1\}$. By a *Jordan triple product* on $\mathbf{Z}$ we mean a continuous 3-variable operation $(x,y,z) \mapsto \{xyz\}$ being symmetric bilinear in the outer variables x,z and conjugate-linear in the inner variable y satisfying the *Jordan identity*

$$\text{(J)} \qquad \big\{ab\{xyz\}\big\} = \big\{\{abx\}yz\big\} - \big\{x\{bay\}z\big\} + \big\{xy\{abz\}\big\}$$

for all $a,b,x,y,z \in E$. We say that the Jordan triple $(\mathbf{Z},\{\ldots\})$ is a *JB*-triple* if the *generalized C*-axiom*

$$\big\|\{zzz\}\big\| = \|z\|^3 \ , \qquad z \in \mathbf{Z}$$

holds and all the operations

$$D(a) : z \mapsto \{aaz\} \ , \qquad a \in \mathbf{Z}$$

are $\mathbf{Z}$-Hermitian with non-negative spectrum, that is

$$\big\| \exp\big(\zeta D(a)\big)\big\| \le 1 \quad \text{for every} \quad \zeta \in \mathbb{C} \quad \text{with} \quad \mathrm{Re}\,\zeta \le 0 \ .$$

2.1 Remark. (1) The Jordan identity is equivalent with the fact that all the operators $iD(a)$ $(a \in E)$ are derivations of the triple product, that is we have the identities

$$iD(a)\{xyz\} = \{[iD(a)x]yz\} - \{x[iD(a)y]z\} + \{xy[iD(a)z]\}.$$

(2) C*-algebras are Jordan triples with the triple product

$$\{xyz\} := \frac{1}{2}xy^*x + \frac{1}{2}zy^*x \ ,$$

moreover complex C*-algebras are JB*-triples with their natural norm.

(3) Complex Hilbert C*-modules are also Jordan triples with

$$\{xyz\} := \frac{1}{2}\langle x|y\rangle z + \frac{1}{2}\langle z|y\rangle x.$$

(4) If the unit ball $\mathbf{D} := \mathrm{Ball}(\mathbf{Z})$ is symmetric (that is for each point $a \in \mathbf{D}$, there is a biholomorphic automorphism $S_a : \mathbf{D} \leftrightarrow \mathbf{D}$ with Fréchet derivative $S_a'(a) = -\mathrm{Id}_{\mathbf{Z}}$ at a) then E can be equipped with a *unique* JB*-triple product.

(5) Any bounded symmetric domain in a complex Banach space can be mapped biholomorphically onto some bounded balanced convex symmetric domain (a so-called *Harish-Chandra realization*), that is onto the unit ball of some equivalent JB*-norm.

In the sequel let $(\mathbf{Z}, \{\dots\})$ denote any given JB*-triple and let

$$\begin{aligned} &D(a,b) : z \mapsto \{a,b,z\}, \qquad D(d) := D(a,a), \\ &Q(a,b) : z \mapsto \{a,z,b\}, \qquad Q(a) := Q(a,a), \\ &B(a,b) := 1 - 2D(a,b) + Q(a)Q(b), \quad B(a) := B(a,a) \end{aligned}$$

be the usual *skew-derivations*, *quadratic representations* and *Bergman operators*, respectively. Recall that the transformation

$$T : c \mapsto \Big[\exp\Big(\big[c - Q(z)c\big]\frac{\partial}{\partial z}\Big)\Big]0$$

that is $T(x) = \big[$the value z_1 for the initial value problem $\frac{d}{dt}z_t = c - Q(z_t)c,\ z_0 = 0\big]$ is a well-defined real bianalytic mapping

$$T : \mathbf{Z} \longleftrightarrow \mathrm{Ball}(\mathbf{Z}) \ .$$

Given any point $a \in \mathrm{Ball}(\mathbf{Z})$, it is well-known [6, p. 27, 4] that the mapping

$$g_a := \exp\Big(\big[T^{-1}(a) - Q(z)T^{-1}(a)\big]\partial/\partial z\Big)$$

is a holomorphic automorphism of $\mathrm{Ball}(\mathbf{Z})$ and we have

$$g_a(z) = a + B(a)^{1/2}[1 + D(z,a)]^{-1}z, \qquad \|z\|, \|z\| < 1. \tag{2.2}$$

2.3 Definition. In the sequel we shall call the mappings $g_a \circ L$ composed with linear unitary operators of $\mathbf{Z}$ the *Möbius transformations* associated with the triple product $\{\dots\}$. It is well-known [3] the group Aut $\mathrm{Ball}(\mathbf{Z})$ of all holomorphic automorphisms of the unit ball of a complex JB*-triple coincides with the set of all Möbius transformations of the underlying triple product.

2.4 Remark. In [3] one can find $g_a(z) = a + B(a)^{1/2}[1 - D(x,a)]^{-1}z$ which is obviously incorrect in the sign of the term $D(z,a)$ as on can see on the 1-dimensional example of the classical Möbius transformation $g_a(z) = (z + a)/(1 + \overline{a}z)$ (with $a, z \in \mathrm{Ball}(\mathbb{C})$).

Next we are going to consider $\mathrm{Ball}(\mathbf{Z})$ as a complex manifold equipped with the charts $\big\{g_a^{-1} :\ a \in \mathrm{Ball}(\mathbf{Z})\big\}$. In this manner we get a natural generalization of the complex Poincaré model on the unit disc $\mathrm{Ball}(\mathbb{C})$ for the real 2-dimensional hyperbolic geometry.

Due to the possible lack of non-trivial smooth functions vanishing outside a ball, for a Banach manifold M, it is no longer convenient to apply the usual definition of a connection as a mapping $\nabla : TM \times TM \to TM$ as being a derivation for the first and linear in the second variable, both with respect to multiplication with smooth functions.

2.5 Definition [5]. Let M be a manifold, modeled over the Banach space E, and denote the space of bounded bilinear mappings $E \times E \to E$ by $L^2(E,E)$. Then M is said to possess a connection if there is an atlas $\mathcal{U}$ for M so that for each $(U,\Phi) \in \mathcal{U}$ (where U is some open subset of M and Φ is a homeomorphism of U onto some open subset of E) there is a smooth mapping $\Gamma_\Phi : \Phi(U) \to L^2(E,E)$, called the *Christoffel symbol* of the connection on U, which under a change of coordinates $\psi : \Phi(U) \to E$ transforms according to

$$\Gamma_{\psi\circ\Phi}(\psi' u, \psi' v) = \Psi''(u,v) + \psi'\Gamma_\Phi(u,v)$$

for smooth vector fields u, v on $\Phi(U) \subset E$. The covariant derivative of a vector field Y in the direction of the vector field X is, locally, defined to be the principal part of

$$\nabla_X Y = dX(Y) - \Gamma(X,Y),$$

that is, if X, Y are smooth vector fields on M with $u := \Phi^{\#} X\big(: \Phi(U) \ni \Phi(p) \mapsto \Phi'(p)X(p)\big)$ and $v := \Phi^{\#} X$ then

$$\Phi^{\#}\nabla_X Y = \nabla_u v + \Gamma_\Phi(u,v) \ : \ \Phi(U) \ni q \mapsto \frac{d}{dt}\Big|_{t=0} v\big(q + tu(q)\big) + \Gamma_\Phi\big(u(q), v(q)\big) \ .$$

If a Banach Lie group G acts smoothly on M then, for each $g \in G$ a connection $g^*\nabla$ is defined by letting

$$g^*\nabla_X Y = \nabla_{g^*X} g^* Y, \qquad g^*X(gm) = d_m g X(m).$$

The Christoffel symbols then transform as in the definition above,

$$\begin{aligned}&\Gamma_{\Phi\circ g}(g(m))\big(g'(m)X(m), g'(m)Y(m)\big) = \\ &\qquad = g''(m)\big(X(m),Y(m)\big) + g'(m)\big[\Gamma_\Phi(m)\big(X(m),Y(m)\big)\big], \qquad m \in M\end{aligned}$$

and we call ∇ invariant under the action of G whenever $g^*\nabla = \nabla$ for all $g \in G$.

2.6 Theorem. *On* $U := \mathrm{Ball}(\mathbf{Z})$, *there exists exactly one Möbius invariant connection whose Christoffel symbol at* a *is given by*

$$\Gamma_{\mathrm{Id}}(a)(x,y) = 2B(a)^{1/2}\{B(a)^{-1/2}x, a, B(a)^{-1/2}y\} = 2\{x, g_a'(0)a, y\}_a,$$

Proof. Notice first the invariance of ∇ with respect to the symmetry $S_0(z) := -z$ ($z \in U$) at the origin implies $-\Gamma_{\mathrm{Id}}(0)(x,y) = S_0'(0)\Gamma_{\mathrm{Id}}(0)(x,y) = \Gamma_{\mathrm{Id}}(0)\big(S_0'(0)x, S_0'(0)y\big) = \Gamma_{\mathrm{Id}}(0)(x,y)$ that is

$$\Gamma_{\mathrm{Id}}(0)(x,y) = 0.$$

Then, for any Möbius transformation g leaving the origin fixed, $g'' = 0$ and so $\Gamma_g(0)$ remains zero under the transformation g. Recall [6,4] that the Fréchet derivative of g_a can be expressed as

$$g_a'(z) = B(a)^{1/2} B(z, -a)^{-1}, \qquad \|z\| < 1.$$

Using that $Q'(z)(h) : x \longmapsto 2\{z, x, h\}$, and writing the second derivative as a bilinear mapping, we similarly arrive at

$$\begin{aligned} g_a''(z)(u, v) &= -B(a)^{1/2} B(z, -a)^{-1} \partial_1 B(u, a) B(z, -a)^{-1}(v) \\ &= 2B(a)^{1/2} B(z, -a)^{-1} \big[D(u, a) - Q(u, z)Q(-a)\big] B(z, -a)^{-1}(v), \end{aligned}$$

applying the chain rule. If we evaluate this expression at $z = 0$, then

$$\begin{aligned} \Gamma_{\mathrm{Id}}(a)(x, y) &= g_a''(0)(g_{-a}'(0)x, g_{-a}'(0)y) \\ &= 2B(a)^{1/2} D(g_{-a}'(0)x, a)(g_{-a}'(0)y), \\ &= 2B(a)^{1/2} \{B(a)^{-1/2}x, a, B(a)^{-1/2}y\} \ . \end{aligned}$$

From the above calculations it is immediate that a Christoffel symbol thus defined gives rise to a connection being invariant under all Möbius transformations. Qu.e.d.

Next we proceed to the geodesic equation of the connection ∇. Recall that a smooth curve $\gamma : I \to \mathrm{Ball}(Z)$ on an open real interval I around 0 is a ∇-geodesic if its derivative satisfies the equation $\nabla_{\dot\gamma(t)} \dot\gamma(t) = 0$. According to Theorem 2.6, this means

$$\ddot\gamma(t) + 2B\big(\gamma(t)\big)^{1/2} D\Big(B\big(\gamma(t)\big)^{-1/2} \dot\gamma(t), \gamma(t)\Big) B\big(\gamma(t)\big)^{-1/2} \dot\gamma(t) = 0.$$

First we look for particular solutions with the property $\gamma(0) = 0$. Let $0 \neq v \in Z$ be any fixed vector and let F_v denote the real JB*-subtriple generated by v. That is $F_v = \overline{\mathrm{Span}_{\mathbb{R}}}\big\{D(v)^n v : \ n = 0, 1, \ldots\big\}$. Since $(\mathbf{Z}, \{\ldots\})$ is JB*-triple, the spectrum of the operator $D(v)|F_v$ is non-negative, and, by writing $\Omega_v := \big[\mathrm{Sp} D(v)^{1/2}|F_v| \setminus \{0\}$, the commutative Gelfand-Naimark Theorem of JB*-triples asserts that there is a real JB*-isomorphism $H_v : \mathrm{Re}\, \mathcal{C}_0(\Omega_v) \to F_v$ such that $H_v(\mathrm{Id}_{\Omega_v}) = v$ and $H_v(\varphi\psi\chi) = \{H_v(\varphi), H_v(\psi), H_v(\chi)\}$ for all functions $\varphi, \psi, \chi \in \mathrm{Re}\, \mathcal{C}_0(\Omega_v)$. In particular, given any solution $\varphi : I \to \mathrm{Re}\mathcal{C}_0(\Omega_v)$, of the equation

$$\frac{d^2}{dt^2}\varphi(t, \omega) + 2\big[1 - \varphi(t, \omega)^2\big]^{-1} \varphi(t, \omega) \Big[\frac{d}{dt}\varphi(t, \omega)\Big]^2, \qquad \omega \in \Omega_v,$$

the curve $\gamma(t) := H_v\varphi(t)$, $t \in I$ is a ∇-geodesic. Notice that, for any real constant α, the function $x(t) = \mathrm{artanh}(\alpha t) := (e^{\alpha t} - e^{-\alpha t})(e^{\alpha t} + e^{-\alpha t})^{-1}$ is the solution of the initial value problem $\ddot x + (1 - x^2)^{-1} x \dot x^2$, $x(0) = 0$, $\dot x(0) = \alpha$. Indeed, we have $\dot x = 1 - x^2$ and hence $\ddot x = -2x\dot x = -2x(1 - x^2)$ and $\ddot x + (1 - x^2)^{-1} x \dot x^2 = -2x\dot x = -2x(1 - x^2) + 2(1 - x^2)^{-1} x (1 - x^2)^2 = 0$. Therefore we get the following.

2.7 Theorem. *The ∇-geodesic curves passing through the origin have the form*

$$\gamma_{0,v}(t) = H_v\ \operatorname{artanh}\bigl(t\ \mathrm{Id}_{\Omega_v}\bigr), \qquad t \in \mathbf{R},\ v \in Z$$

in terms of the Gelfand-Naimark representations H_v.

2.8 Corollary. *The (unique) maximal ∇-geodesic $\gamma_{a,w}$ with the properties $\gamma_{a,w}(0) = a$ and $\dot\gamma(0) = w$ has the form*

$$\mathbf{R} \ni t \mapsto g_a\Bigl(H_{g_a'(0)^{-1}w}\ \operatorname{artanh}\bigl(t\ \mathrm{Id}_{\Omega_{g_a'(0)^{-1}w}}\bigr)\Bigr).$$

Proof. It is a well-known consequence of the Möbius invariance of the connection ∇ that $h \circ \gamma$ is ∇-geodesic whenever γ is a ∇-geodesic and $h \in \operatorname{Aut} \operatorname{Ball}(Z)$. The curve in the statement of the Corollary is thus a ∇-geodesic. It is straightforward to see that its starting point is a and its starting speed vector is w. The classical Piccard-Lindelöf Theorem ensures the uniqueness of the solution of the geodesic equation with given starting point and starting speed vector. Qu.e.d.

The expression for the geodesic in the above can be conveniently rewritten in terms of the power series for artanh. In fact, we have

$$\gamma_{0,v}(t) = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{2n+1} D(v,v)^{2n} v = \operatorname{artanh} tv.$$

In the same vein,

$$\gamma_{a,w}(t) = a + B(a)^{1/2}\bigl[1 + D(\operatorname{artanh} tB(a)^{1/2}v, a)\bigr]^{-1} \operatorname{artanh} tB(a)^{1/2}v.$$

3x. Complex Jordan manifolds

3x.1 Definition. We say a complex Banach-Jordan triple $(\mathbf{Z}, \{\ldots\})$ is *non-degenerate* if the quadratic representations $Q_a : x \mapsto \{xax\}$ do not vanish unless $a = 0$.

By definition, a tuple $(\mathcal{M}, \mathcal{A}, \mathcal{P})$ is a *complex Jordan manifold* complex manifold modeled over a fixed non-degenerate Jordan triple $(\mathbf{Z}, \{\ldots\})$ if $(\mathcal{M}, \mathcal{A})$ is a complex Banach manifold (with Atlas $\mathcal{A}$) modeled over $(\mathbf{Z}, \{\ldots\})$ and $\mathcal{P} = \bigl\{\{\ldots\}_p :\ p \in \mathcal{M}\bigr\}$ is a (real) smooth tensor field such that (following automatic???) $(T_pM, \{\ldots\}_p)$ is isomorphic to $(Z, \{\ldots\})$

Examples: Kaup/Upmeier (invariant Finsler structure) Lorentz manifolds (very special but important) Our example from previous chapter Porta/Recht

3. Local Möbius transformations in real Jordan-Banach triples and Jordan-Möbius manifolds

Throughout this section $\mathbf{E}$ denotes a real Jordan-Banach triple with the norm $\|.\|$ and triple product $\{\ldots\}$, respectively. Thus we only assume that $\mathbf{E}$ is a real Banach space and

$\{\ldots\}$ is a continuous real trilinear mapping $\mathbf{E}^3 \to \mathbf{E}$ satisfying the Jordan identity (J). We can take over the notations $D(a,b), Q(a,b), B(a,b)$ in the real setting without formal changes. In particular, all the operators $D(a,b) - D(b,a) : z \mapsto \{abz\} - \{baz\}$ belong to $\mathrm{Der}(\mathbf{E},\{\ldots\})$ the set of the derivations of the triple product. Though not explicitly stated, a straightforward inspection of [3, Corollary 2.20] establishes the existence of a constant $\varepsilon > 0$ such that the transformations

$$H_v := \exp V_v \quad \text{with the vector fields} \quad V_v := \big[v - \{zvz\}\big]\partial/\partial z$$

are well-defined on the ball $\varepsilon\mathrm{Ball}(\mathbf{E})$ whenever $\|v\| < \varepsilon$, moreover they have the fractional linear Möbius form

$$\begin{aligned} &H_v(z) = g_{H_v(0)}(z) \quad \text{where} \\ &g_a(z) := a + \lambda(a)[1 + D(z,a)]^{-1} z \qquad \text{where } \ \lambda(a) := H_v'(0) \in \mathcal{L}(\mathbf{E}) \ . \end{aligned} \tag{3.1}$$

Notice that in the case of JB*-triples we can write $\lambda(a) = B(a)^{1/2}$ in termd of the Bergman operator. Besides the vector fields V_v of polynomial degree 2, let us introduce also the linear vector fields

$$L_\ell := \big[\ell z\big]\partial/\partial z, \quad \ell \in \mathrm{Der}(\mathbf{E},\{\ldots\}).$$

For their Poisson commutators $\big[f(z)\partial/\partial z, g(z)\partial/\partial z\big] := \big(f'(z)g(z) - g'(z)f(z)\big)\partial/\partial z$, it is straightforward to check that we have

$$[L_\ell, V_v] = V_{\ell v}, \quad [L_\ell, L_m] = L_{[\ell,m]}, \quad [V_u, V_v] = L_{D(v,u)-D(u,v)}.$$

Therefore the real linear space

$$\big\{V_v + L_\ell : \ v \in Z, \ \ell \in \mathrm{Der}(\mathbf{E},\{\ldots\})\big\}$$

equipped with the norm $\|V_v + L_\ell\| := \sup\{\|v - \{zvz\} + \ell z\| : \ \|z\| \le 1\}$ is a Banach-Lie algebra with the Poisson commutator. Thus, according to the Campbell-Hausdorff formula, for some sufficiently small constant $\delta > 0$ we have

$$\big[\exp(V_u + L_\ell)\exp(V_v + L_m)\big]z = \big[\exp(V_{w(u,v,\ell,m)} + L_{\Delta(u,v,\ell,m)}\big]z$$

with two suitable real-analytic mappings

$$\begin{aligned} &w : \big[\delta\,\mathrm{Ball}(\mathbf{E})\big]^2 \times \big[\delta\,\mathrm{Ball}\big(\mathrm{Der}(\mathbf{E},\{\ldots\})\big)\big]^2 \longrightarrow \mathbf{E}, \\ &\Delta : \big[\delta\,\mathrm{Ball}(\mathbf{E})\big]^2 \times \big[\delta\,\mathrm{Ball}\big(\mathrm{Der}(\mathbf{E},\{\ldots\})\big)\big]^2 \longrightarrow \mathrm{Der}(\mathbf{E},\{\ldots\}) \end{aligned}$$

whenever $\|z\|, \|u\|, \|v\|, \|\ell\|, \|m\| < \delta$.

3.2 Definition. By a *Möbius transformation* of $(\mathbf{E},\{\ldots\})$ we mean a bianalytic mapping $\Phi : U \to \mathbf{E}$ defined in a neighborhood U of the origin in $\mathbf{E}$ such that for some

$\ell \in \mathrm{Der}(\mathbf{E},\{\ldots\})$ we have $\Phi(z) = g_a(\exp \ell z)$, $z \in U$ with the customary notation in $g_a(z) := a + \lambda(a)[1 - D(z,a)]^{-1}$ established in (3.1).

The next two auxiliary results 3.3-4 establish in particular that composition preserve Möbius tranformations.

3.3 Proposition. *There exists $\delta' > 0$ such that*

$$\|v\|, \|\ell\| < \delta' \text{ and } [\exp(V_v + L_\ell)]0 = 0 \quad \Longrightarrow \quad v = 0.$$

Proof. We can choose δ' to be so small that the mapping $u \mapsto [\exp(V_u)]0$ be injective on $\delta' \,\mathrm{Ball}(\mathbf{E})$ and the terms

$$[\exp(tL_m)][\exp(V_u + L_m)][\exp(-tL_m)]z, \quad |t|, \|z\|, \|u\|, \|m\| < \delta'$$

be all well-defined. Consider any couple v, ℓ with $\|v\|, \|\ell\| < \delta'$ and $[\exp(V_v + L_\ell)]0 = 0$. Then, for all $-\delta' < t < \delta'$,

$$\begin{aligned} 0 &= [\exp(tL_\ell)][\exp(V_v + L_\ell)][\exp(-tL_\ell)]0 = \\ &= \Big[\exp\Big(\big[\exp(tL_\ell^{\#})\big](V_v + L_\ell)\Big)\Big]0 = \\ &= \Big[\exp\big(V_{\exp t\ell)v} + L_\ell\big)\Big]0. \end{aligned}$$

Notice that the points

$$x_{s,t} := \Big[\exp\big(s[V_{(\exp t\ell)v} + L_\ell]\big)\Big]0, \quad |s| < 1 + \varepsilon', \ |t| < \delta'$$

are well-defined for some $\varepsilon' > 0$ and they satisfy the differential equation

$$\frac{\partial}{\partial s} x_{s,t} = (\exp t\ell)v - \{x_{s,t}[(\exp t\ell)v]x_{s,t}\} + \ell x_{s,t}.$$

Moreover, since $\big[\exp\big(V_{\exp t\ell)v} + L_\ell\big)\big]0 = 0$ for $|t| < \delta'$, it follows

$$\begin{aligned} 0 &= \frac{\partial^2}{\partial s \partial t}\Big|_{s=1,\ t=0} x_{s,t} = \frac{\partial^2}{\partial t \partial s}\Big|_{s=1,\ t=0} x_{s,t} = \\ &= \ell v - 2\Big\{\Big(\frac{\partial}{\partial t}\Big|_{t=0,\ s=1} x_{s,t}\Big) v x_{s,t}\Big\} + \ell \frac{\partial}{\partial t}\Big|_{t=0,\ s=1} x_{s,t}. \end{aligned}$$

However, here we have $\frac{\partial}{\partial t}\Big|_{t=0,\ s=1} x_{s,t} = 0$ which implies that necessarily $\ell v = 0$. It follows $[L_\ell, V_v] = V_{\ell v} = 0$ and therefore

$$0 = [\exp(V_v + L_\ell)]0 = [\exp V_v][\exp L_\ell]0 = [\exp V_v]0.$$

The choice of δ' with $\|v\| < \delta'$ ensures that $v = 0$. Qu.e.d.

3.4 Corollary. *There exists* $0 < \delta''(< \delta')$ *such that given any* $v \in Z$ *and* $\ell \in \mathrm{Der}(\mathbf{E}, \{\ldots\})$ *with* $\|v\|, \|\ell\| < \delta''$, *for some* $m \in \mathrm{Der}(\mathbf{E}, \{\ldots\})$ *we have*

$$\big[\exp(V_v + L_\ell)\big] z = g_{a(v,\ell)}\big([\exp m] z\big), \quad \|z\| < \delta''$$

where $a(v,\ell) := \big[\exp(V_v + L_\ell)\big] 0$. *In particular, for sufficiently small vectors* $p, q \in \mathbf{E}$,

$$g_p \circ g_q = g_{g_p(q)} \circ [\exp m_{p,q}]$$

with a real-analytic mapping $(p,q) \mapsto m_{p,q} \in \mathrm{Der} Z, \{\ldots\})$.

3.5 Definition. By a *Jordan-Möbius manifold* modeled with $(\mathbf{E}, \{\ldots\})$ we mean a a real-analytic manifold M with an *inverse atlas* $\mathcal{X} = \{X_p : \ p \in M\}$ such that for each point $p \in M$ we have

(3.6) $\quad X_p : U_p \to M, \quad X_p(0) = p, \quad X_p^{-1} \circ X_q$ Möbius transformation for $q \in X_p(U_p)$

where U_p is an open connected neighborhood of the origin in $\mathbf{E}$ the

A Jordan-Möbius manifold M with an inverse atlas $\mathcal{X} = \{X_p : \ p \in M\}$ satisfying (3.6) is said to be a *uniform Jordan-Möbius manifold* if there exists a common constant $\varepsilon > 0$ such that

$$\mathrm{dom}(X_p \circ g) \supset \varepsilon\,\mathrm{Ball}(\mathbf{E}) \quad \text{if} \quad g \in \{\text{Möbius transformations}\} \text{ and } \|g(0)\| < \varepsilon.$$

In particular the region $\bigcup_{L \in \mathrm{Aut}(\mathbf{E},\{\ldots\})} \varepsilon L \mathrm{Ball}(\mathbf{E})$ is contained in $\mathrm{dom}(X_p)$ for all $p \in M$.

3.7 Example. If $\mathbf{E} := \mathbf{Z}$ is a JB*-triple then its unit ball $M := \mathrm{Ball}(\mathbf{Z})$ is a uniform Jordan manifold with the charts $X_p := g_p | M$, $p \in M$. In this case we may choose $\varepsilon = 1$.

3.8 Example. Let $(\mathbf{E}, \{\ldots\})$ be any Jordan-Banach triple. Then we can find a constant $\varrho > 0$ such that the sections of the mapping $[\varrho \mathrm{Ball}(\mathbf{E})]^2 \ni (p,z) \mapsto g_p(z)$ are real-bianalytic for any fixed p and z, respectively. Then the ball $M := \varrho\mathrm{Ball}(\mathbf{E})$ with the topology from $\mathbf{E}$ and with the charts $X_p(z) := g_p(z)$ defined for $z \in U_p := \{u \in \mathbf{E} : \ \|u\|, \|g_p(u)\| < \varrho\}$ is a Jordan manifold which is not uniform in general.

3.8a Example. (Special case of 3.8 with Lorenz space). $E := \mathrm{Mat}(1,2,\mathbb{R})$, $S := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$; indefinite scalar product $\langle x|y\rangle_S := xSy^* \in \mathbb{R}$ $(x, y \in E)$, tripe product $\{xyz\}_S := \frac{1}{2}\langle x|y\rangle_S z + \frac{1}{2}\langle z|y\rangle_S x = \frac{1}{2}xy^*Sz + \frac{1}{2}zy^*Sx$ on E. Then the local Möbius transformations have the form

$$\begin{aligned} M_a(x) &= (1 - aSa^*)^{-1/2}(x+a)(1+Sa^*x)^{-1}(1-Sa^*a)^{1/2} = \\ &= (1-aSa^*)^{-1/2}(1+xSa^*)^{-1}(x+a)(1-Sa^*a)^{1/2}\ . \end{aligned}$$

Proof. By definition, $M_a(x) = a + B(a)^{1/2}[1 + D(x,a)]^{-1}x$. Here we have

$$\begin{aligned} B(a)z &= z - 2D(a)z + Q(z)^2 = z - aSa^*z - zSa^*a + aSa^*zSa^*a = \\ &= (1-aSa^*)z(1-Sa^*a), \\ D(x,a)^n x &= (xSa^*)^n x = x(Sa^*x)^n \quad (n = 0, 1, \ldots). \end{aligned}$$

Thus $B(a)^{1/2}z = (1-aSa^*)^{1/2}z(1-Sa^*a)^{1/2}$ and $[1+D(x,a)]^{-1}x = \sum_{n=0}^{\infty}(-1)^n D(x,a)^n x = (1-xSa^*)^{-1}x = x(1-Sa^*x)^{-1}$. Since also $(1-aSa^*)^{1/2}a = a(1-Sa^*a)^{1/2}$ as one can see from the Newtonian expansion, it follows

$$\begin{aligned}
M_a(x) &= a + (1-aSa^*)^{1/2}x(1+Sa^*x)^{-1}(1-Sa^*a)^{1/2} = \\
&= (1-aSa^*)^{-1/2}a(1-Sa^*a)^{1/2} + (1-aSa^*)^{1/2}x(1+Sa^*x)^{-1}(1-Sa^*a)^{1/2} = \\
&= (1-aSa^*)^{-1/2}\big[a + (1-aSa^*)x(1+Sa^*x)^{-1}\big](1-Sa^*a)^{1/2} = \\
&= (1-aSa^*)^{-1/2}\big[a(1+Sa^*x) + (1-aSa^*)x\big](1+Sa^*x)^{-1}(1-Sa^*a)^{1/2} = \\
&= (1-aSa^*)^{-1/2}(x+a)(1+Sa^*x)^{-1}(1-Sa^*a)^{1/2}.
\end{aligned}$$

The proof for the alternative expression of $M_a(x)$ is analogous.

3.9 Example [1,7]. Let $(\mathbf{Z},\{\ldots\})$ be any complex JB*-triple such that the family $\mathrm{Tri}(\mathbf{Z},\{\ldots\}) := \big\{e \in \mathbf{Z} :\ \{eee\} = e \neq 0\big\}$ of its non-trivial tripotents is not void. Two tripotents $e,f \in \mathrm{Tri}(\mathbf{Z},\{\ldots\})$ are said to be *equivalent* ($e \sim f$ in notation) if $D(e,e) = D(f,f)$. Actually $\sim$ is an equivalence relation on $\mathrm{Tri}(\mathbf{Z},\{\ldots\})$. Let M be a connected component of $\mathrm{Tri}(\mathbf{Z},\{\ldots\})$ (with respect to the topology inherited from $\mathbf{Z}$). Then the Peirce spaces $Z_{1/2}(e) := \{u \in Z :\ D(e,e)u = u/2\}$ $(e \in M)$ are all isomorphic and the set $\mathbb{M} := \big\{\mathbf{e} :\ e \in M\big\}$ $(\mathbf{e} := \{f \in M :\ f \sim e\})$ of its equivalence classes can be regarded as a complex hermitian symmetric manifold modeled on $Z_{1/2}(e_0)$ with any $e_0 \in M$ and being such that the automorphisms $\exp t[D(e,u) - D(u,e)]$ $\big(t \in \mathbb{R},\ u \in Z_{1/2}(e)\big)$ of $\mathbf{Z}$ acting on $\mathbb{M}$ form a continuous one-parameter subgroup of the Banach-Lie group $\mathrm{Aut}(\mathbb{M})$ of all biholomorphic automorphisms of $\mathbb{M}$ (with Upmeier's topology [8]). Given any $e \in M$, there is a neighborhood $\mathbf{U}$ of its equivalence class $\mathbf{e}$ in $\mathbb{M}$ along with a holomorphic chart map $\Phi : \mathbf{U} \to Z_{1/2}(e)$ such that each map $\Phi^{\#}\exp[D(e,u) - D(u,e)]$ $\big(u \in Z_{1/2}(e)\big)$ is a Möbius transformation with the triple product

$$\{u_1u_2u_3\} := \frac{\partial^3}{\partial\zeta_1\partial\overline{\zeta_2}\partial\zeta_3}\bigg|_{\zeta_1=\zeta_2=\zeta_3=0} \Phi^{\#}\Big(\Big[P_{\zeta_1u_1}, \Big[P_{\zeta_2u_2}, P_{\zeta_3u_3}\Big]\Big]_e\Big)$$

in terms of the vector fields $P_u(\mathbf{f}) := \big[t \mapsto \exp t[D(e,u) - D(u,e)]\mathbf{f}\big] \in \mathrm{T}_{\mathbf{e}}\mathbb{M}$.

3.10 Lemma. *Assume U,V are domains in a Banach space W and let $T : U \leftrightarrow V$ be a smooth diffeomorphism between them. Then given any couple $X,Y : U \to W$ of smooth vector fields on U, for their transforms $\widetilde{X} := T^{\#}X$ respectively $\widetilde{Y} := T^{\#}Y$ on V we have*

$$\widetilde{Y}'\widetilde{X}(v) = T''\big(T^{-1}(v)\big)X\big(T^{-1}(v)\big)Y\big(T^{-1}(v)\big) + T'\big(T^{-1}(v)\big)Y'\big(T^{-1}(v)\big)X\big(T^{-1}(v)\big), \quad v \in V.$$

Proof. By definition, $\widetilde{Y}'\widetilde{X}(v) = \frac{d}{dt}\big|_{t=0}\widetilde{Y}\big(v + t\widetilde{X}(v)\big)$. Therefore

$$\begin{aligned}
\widetilde{Y}'\widetilde{X}(v) &= \frac{d}{dt}\Big|_{t=0} T'\Big(T^{-1}\big(v + t\widetilde{X}(v)\big)\Big)Y\Big(T^{-1}\big(v + \widetilde{X}(v)\big)\Big) = \\
&= \Big[\frac{d}{dt}\Big|_{t=0} T'\Big(T^{-1}\big(v + t\widetilde{X}(v)\big)\Big)\Big]Y\big(T^{-1}(v)\big) + \\
&\quad + T'\big(T^{-1}(v)\big)\frac{d}{dt}\Big|_{t=0} Y\big(v + t\widetilde{X}(v)\big) = \\
&= T''\Big(T^{-1}(v)\big)\Big[\frac{d}{dt}\Big|_{t=0} T^{-1}\big(v + t\widetilde{X}(v)\big)\Big]Y\big(T^{-1}(v)\big) + \\
&\quad + T'\Big(T^{-1}(v)\big)Y'\big(T^{-1}(v)\big)\frac{d}{dt}\Big|_{t=0} T^{-1}\big(v + t\widetilde{X}(v)\big) = \\
&= T''\Big(T^{-1}(v)\big)\Big[T'\big(T^{-1}(v)\big)^{-1}\widetilde{X}(v)\Big]Y\big(T^{-1}(v)\big) + \\
&\quad + T'\Big(T^{-1}(v)\big)Y'\big(T^{-1}(v)\big)\big[T'\big(T^{-1}(v)\big)\big]^{-1}\widetilde{X}\big(T^{-1}(v)\big) = \\
&= T''\Big(T^{-1}(v)\big)X\big(T^{-1}(v)\big)Y\big(T^{-1}(v)\big) + \\
&\quad + T'\Big(T^{-1}(v)\big)Y'\big(T^{-1}(v)\big)X\big(T^{-1}(v)\big). \quad \text{Qu,e.d.}
\end{aligned}$$

In the sequel let $\big(M, \{X_p : \ p \in M\}\big)$ be a Jordan-Möbius manifold modeled on $(\mathbf{E}, \{\ldots\})$ with inverse charts $X_p : \mathbf{E} \supset U_p \to M$ satisfying $X_p(0) = p, p \in M$ and with Möbius transition maps $X_p^{-1} \circ X_q$ for couples of point lying sufficiently close together.

3.11 Proposition, *Let $p \in M$ be an arbitrarily given point, let $\mathbf{X}, \mathbf{Y}$ be two smooth vector fields on M and define $\mathbf{R} := \nabla_{\mathbf{X}}\mathbf{Y}$. Then, for any vector $w \in U_p$, the image $R := [X_p^{-1}]^{\#}\mathbf{R}$ of the vector field $\mathbf{R}$ by means of the local coordinate X_p can be expressed in terms of the image vector fields $X := [X_p^{-1}]^{\#}\mathbf{X}$ and $Y := [X_p^{-1}]^{\#}\mathbf{Y}$ as*

$$R(w) = H_w'(w)^{-1}H_w''(w)X(w)Y(w) + Y'(w)X(w) \quad \text{where} \quad H_w := X_{X_p(w)}^{-1} \circ X_p.$$

Proof. Let $w \in U_p$ be arbitrarily fixed and write $q := X_p(w)$. Then, with the mapping $T := H_w$ we have

$$T(w) = X_q^{-1} \circ X_p(w) = X_q^{-1}(q) = 0.$$

Also in terms of T we can write

$$\begin{aligned}
[X_q^{-1}]^{\#}\mathbf{X} &= [X_q^{-1}]^{\#}X_p^{\#}[X_p^{-1}]^{\#}\mathbf{X} = \\
&= [X_q^{-1} \circ X_p]^{\#}[X_p^{-1}]^{\#}\mathbf{X} = \\
&= T^{\#}X
\end{aligned}$$

and similarly $[X_q^{-1}]^{\#}\mathbf{Y} = T^{\#}Y$ and $[X_q^{-1}]^{\#}\mathbf{R} = T^{\#}R$. Hence

$$\begin{aligned}
[X_q^{-1}]^{\#}\mathbf{R}(0) &= \big[T^{\#}\big([X_p^{-1}]^{\#}\mathbf{R}\big)\big](0) = \\
&= T'\big(T^{-1}(0)\big)\big[[X_p^{-1}]^{\#}\mathbf{R}\big]\big(T^{-1}(0)\big) = \\
&= T'(w)\Big[\big[[X_p^{-1}]^{\#}\mathbf{R}\big](w)\Big] = \\
&= T'(w)R(w).
\end{aligned}$$

By axiomatic assumption we have $[X_q^{-1}]^\#\mathbf{R}(0) = \Big[[[X_q^{-1}]^\#\mathbf{Y}]'(0)\Big]\Big[[[X_q^{-1}]^\#\mathbf{X}](0)\Big]$. That is

$$\begin{aligned} R(w) &= [X_p^{-1}]^\#\mathbf{R}(w) = \\ &= T'(w)^{-1}\Big[[[X_q^{-1}]^\#\mathbf{Y}]'(0)\Big]\Big[[[X_q^{-1}]^\#\mathbf{X}](0)\Big] = \\ &= T'(w)^{-1}\big[(T^\# Y)'(0)\big]\big[T^\# X(0)\big] = \\ &= T'(w)^{-1}T''(w)X(w)Y(w) + Y'(w)X(w) \end{aligned}$$

in view of Lemma 3.7 and the fact that $w = T^{-1}(0)$. Qu.e.d.

Our technical results 3.3-4, 3.10-11, establish immediately that the calculations for the proof of Theorem 2.6 can be carried out locally even in the setting of general real Jordan-Möbius manifolds. As a first straightforward consequence we get the following.

3.10 Theorem. *Let M be a Jordan-Möbius manifold modeled on $(\mathbf{E}, \{\ldots\})$ with a system of inverse charts $\{X_p :\ p \in M\}$ having the properties (3.6). Then there exists a (necessarily unique) connection ∇ on M such that its Christoffel symbol Γ with the charts $\Phi_p := X_p^{-1}$ satisfies*

$$\Gamma_{\Phi_p}(0) = 0\ , \qquad p \in M.$$

Namely, if $p \in M$ is any point and the constant $\delta > 0$ is so chosen that $\delta\,\mathrm{Ball}(\mathbf{E}) \subset \mathrm{dom}(X_p)$ then for any couple of vectors $x, y \in \mathbf{E}$ and any point $a \in \delta\,\mathrm{Ball}(\mathbf{E})$ we have

$$\begin{aligned} \Gamma_{\Phi_p}(a)(x, y) &= g_a''(0)(g_{-a}'(0)x, g_{-a}'(0)y) = \\ &= 2\lambda(a)\big\{[\lambda(a)^{-1}x]a[\lambda(a)^{-1}y]\big\}\ . \end{aligned}$$

In particular $\Gamma_{\Phi_p}(a)(x, y) = 2B(a)^{1/2}\{[B(a)^{-1/2}x]a[B(a)^{-1/2}y]\}$ in the case of $(\mathbf{E}, \{\ldots\})$ being a complex JB-triple.*

In the above theorem we could not include a statement about symmetry invariant connections because, unlike the unit ball of a complex JB*-triple, Jordan-Möbius manifolds need not be necessarily symmetric. We close this section by showing that the assumption of uniformness implies this property.

3.11 Theorem. *Connected uniform Jordan-Möbius manifolds are symmetric and they admit a unique symmetry invariant connection.*

Proof. SKETCHED

Let M be a connected uniform Jordan-Möbius manifold modeled with $(\mathbf{E}, \{\ldots\})$ along with a constant $\varepsilon > 0$ and a system $\mathcal{X} = \{X_p :\ p \in M\}$ as in Definition 3.5. By Corollary 3.3, there exists $\delta \in (0, \varepsilon)$ such that $g_a\big(\delta\,\mathrm{Ball}(\mathbf{E})\big) \subset \varepsilon\mathrm{Ball}(\mathbf{E})$ for any vector $a \in \delta\,\mathrm{Ball}(\mathbf{E})$. Observe that, given any couple of points $p, q \in M$, we can find a finite sequence $v_1, \ldots, v_N \in \delta\,\mathrm{Ball}(\mathbf{E})$ such that the recursively defined sequence

$$p_0 := p, \quad p_{n+1} := X_{p_n}(v_n)$$

ends in $q = p_N$. Let us fix any point $p \in M$. We can see by induction that there exists a sequence $q_0 = p, q_1, \ldots, q_N \in M$ of points along with (linear) automorphisms $L_0 = \mathrm{Id}_{\mathbf{E}}, L_1, \ldots, L_N \in \mathrm{Aut}(\mathbf{E}, \{\ldots\})$ such that for the modified charts

$$Y_n := X_{q_n} \circ L_n, \qquad n = 1, \ldots n = 0, \ldots, N$$

we have

$$q_{n+1} = Y_n(-v_n), \quad Y_{n+1}^{-1} \circ Y_n = X_{q_n}^{-1} \circ X_{q_{n+1}}.$$

In view of 3,3-4, if we have another sequence $\widetilde{v}_1, \ldots, \widetilde{v}_{\widetilde{N}}$ and consider the corresponding points $\widetilde{p}_1, \ldots, \widetilde{p}_{\widetilde{N}}$ respectively $\widetilde{q}_1, \ldots, \widetilde{q}_{\widetilde{N}}$ with the above construction then the coincidence $p_N = \widetilde{p}_{\widetilde{N}}$ of the endpoints implies the coincidence $q_N = \widetilde{q}_{\widetilde{N}}$ as well. It is not hard to check that the thus well-defined transformation $S_p : p_N \mapsto q_N$ is a symmetry through the point p such that the maps $X_r^{-1} \circ S \circ X_p$ $(r \in M)$ are Möbius transformations. Finally we notice that $S_p \circ S_q$ is a Möbius transformation if its domain contains the origin of $\mathbf{E}$. Hence the statement of the theorem is immediate. Qu.e.d.

4. Jordan manifolds

4.1 Definition. A manifold M modeled on a (real) Jordan-Banach triple $(\mathbf{E}, \{\ldots\})$ is a *Jordan manifold* if its tangent spaces T_pM $(p \in M)$ are endowed with a triple product $\{\ldots\}$ being isomorphic to $\{\ldots\}$ on $\mathbf{E}$ and the mapping $(p, u, v, w) \mapsto \{uvw\}_p$ is continuously differentiable. In the sequel we shall write $(M, \overline{\mathcal{A}}, \mathcal{P})$ for the triple of the carrier space, atlas and system of triple products. A morphism $F : M \to \widetilde{M}$ between two Jordan manifolds is a smooth mapping such that its derivatives give rise to triple product homomorphisms on the tangent spaces.

Given two Jordan manifolds $(M, \overline{\mathcal{A}}, \mathcal{P})$ and $(N, \overline{\mathcal{B}}, \mathcal{Q})$,
a continuously differentiable map $S : M \to N$ is a *Jordan morphism* if
$\forall\, p \in M \quad \forall\, u, v, w \in T_p(M) \qquad S'(p)\{uvw\}_p = \{[S'(p)u][S'(p)v][S'(p)w]\}_{S(p)}$.
Isomorphism and automorphisms can be defined as usually.
$\mathrm{Aut}(M) := \{S : M \leftrightarrow M \text{ with } S, S^{-1} \text{ Jordan morphisms}\}$.

(D3) Given $(M, \overline{\mathcal{A}}, \mathcal{P})$ and $(N, \overline{\mathcal{B}}, \mathcal{Q})$ with U open $\subset M$ and V open $\subset N$,
a continuously differentiable map $S : U \leftrightarrow V$ is a *local Jordan automorphism* if
S is a Jordan isomorphism between U and V as Jordan submanifolds.
$\mathrm{Aut}_{\mathrm{loc}}(M) := \{\text{local Jordan automorphism in } M\}$.
Notice that $S_1 \circ S_2 \in \mathrm{Aut}_{\mathrm{loc}}(M)$ whenever $S_1, S_2 \in \mathrm{Aut}_{\mathrm{loc}}(M)$ with $\mathrm{ran}(S_1) \cap \mathrm{dom}(S_2) \neq \emptyset$.

Convention. Henceforth $(M, \overline{\mathcal{A}}, \mathcal{P})$ stands for a Jordan manifold.

Example. (E0) Riemann spaces can be regarded as Jordan manifolds:
If $\langle .|. \rangle_p$ is the inner product on T_pM then we take
$\{uvw\}_p := \frac{1}{2}\langle u|v\rangle_p w + \frac{1}{2}\langle w|v\rangle_p u$.

Example. The triple products in $\mathcal{P}$ need not be isomorphic to each other:

(E1) $$M := \mathbb{C} \quad \text{with} \quad \{uvw\}_p := \mathrm{Re}(p) u\overline{v}w.$$

Conjecture. (HARD)
If we assume all the $\{\ldots\}_p$ to be JB^*-triple products then they are isomorphic.

Lemma. *If for every couple of points $p, q \in M$ there are neighborhoods $U, V \subset M$ along with a diffeomorphism $S : U \leftrightarrow V$ such that $q = S(p)$ and $S'(p)$ is an isomorphism between $\{\ldots\}_p$ and $\{\ldots\}_q$ then there is an atlas $\mathcal{A} \subset \overline{\mathcal{A}}$ with the properties* (N4).

Proof. Fix any $o \in M$ and a local coordinate $X \in \overline{\mathcal{A}}$ with $X(o) = 0$. For any $p \in M$ choose a local diffeomorhism S_p with $S_p(o) = p$ and the properties in (D2). We redefine the triple product $\{\ldots\}$ on Z as the image of $\{\ldots\}_o$ by $X'(0)$ and we set $X_p := X \circ S_p^{-1}$ $(p \in M)$.

Definition.
(D4) A Jordan manifold M is *homogeneous* if
$\mathrm{Aut}(M)$ is transitive on it: $\forall\, p, q \in M \quad \exists\, S \in \mathrm{Aut}(M) \qquad S(p) = q$.

(D5) A Jordan manifold is *locally homogeneous* if
$\mathrm{Aut}_{\mathrm{loc}}(M)$ is transitive on it.

Examples.
(E2) [Corach-Porta-Recht].
$M := \mathbf{A}^+$ the positive invertible elements of a unital C^*-algebra $\mathbf{A}$.
$E := \mathbf{A}$ with $\{xyz\} := \frac{1}{2}xy^*z + \frac{1}{2}zy^*y$ as usually.
$U_a := M$ and $X_a(b) := b - a$ for all $a, b \in M$.
$\{uvw\}_a = \{uvw\} \quad (a \in M;\ u, v, w \in T_aM \equiv E)$.
Then, for any invertible $g \in \mathbf{A}$ we have $L_g := [a \mapsto (g^*)^{-1}ag^{-1}] \in \mathrm{Aut}(M)$.
In particular M is homogeneous because $\{L_g(1) :\ e \in M\} = \{g^{-2} : e \in M\} = M$.

(E2′) [Corach-Porta-Recht$_2$].
$\mathbf{A}$ C^*-algebra with unit 1,
$E := \{a \in A :\ a = a^*\}, \quad \{xyz\} := \frac{1}{2}xyz + \frac{1}{2}zyx$
$\mathcal{M} := \{a \in A :\ a > 0\} = \{g^*g :\ g \text{ invertible}\}$
$L_g : \mathcal{M} \leftrightarrow \mathcal{M},\ x \mapsto g^*xg$ linear
$S : p \mapsto p^{-1}$ symmetry on $\mathcal{M}$
$S_{g^*g} := L_g \circ S \circ L_{g^{-1}} : p \mapsto g^*gp^{-1}g^*g$
We can define
$$\begin{aligned} \{uvu\}_{g^*g} :&= L_g\{(L_g^{-1}u)(L_g^{-1}u)(L_g^{-1}u)\} = \\ &= u(g^*g)^{-1}v(g^*g)^{-1}u. \end{aligned}$$

Lemma. *The transformations L_g, S_p are $\mathcal{P}$-automorphisms.*

Define $X_v := \frac{d}{d\tau}\big|_{\tau=0+} S_{\exp(\tau v/2)} \circ S_1$.
Notice that $\exp(X_v)p = \exp(v/2)p\exp(v/2)$.

Proposition. *In the chart* $Y : \big[\exp(v - \{xvx\})\frac{\partial}{\partial x}\big]0 \mapsto [\exp X_v]1$ *the transformations* $\exp(X_w)$ $(w \in E)$ *are not in general* $\{\ldots\}$*-Möbius type.*
Proof. We would have $Y : \tanh v \mapsto \exp v$ that is $Y(x) = \exp\big(\mathrm{areath}\, x\big)$ and $Y^{-1}(y) = \tanh \log y$. This implies

$$\begin{aligned} Y^{-1} \circ [\exp X_w] \circ Y : x &\overset{Y}{\mapsto} \\ &\exp(\mathrm{areath}\, x) \overset{\exp X_w}{\mapsto} \\ &\exp(w/2)\big[\exp(\mathrm{areath}\, x)\big]\exp(w/2) \overset{Y^{-1}}{\mapsto} \\ &\tanh\Big[\log\Big(\exp(w/2)\big[\exp(\mathrm{areath}\, x)\big]\exp(w/2)\Big)\Big]. \end{aligned}$$

Such mappings are of Möbius type only in the commutative case.

(E3) [Chu-Isidro-Kaup-Stachó]. A case with equivalence classes of tripotents.
$\mathbf{H}$ complex Hilbert space with inner product $\langle .|.\rangle$. $\mathbf{S}$:=[unit sphere].
Triple product $\{xyz\} := \frac{1}{2}\langle x|y\rangle z + \frac{1}{2}\langle z|y\rangle x$ $(x, y, z \in \mathbf{H})$.
$E := e_0^\perp = \{z \in \mathbf{H} :\ \langle z|e_0\rangle = 0$ with a fixed $e_0 \in \mathbf{S}$.
$M := \{\mathbb{T}e :\ e \in \mathbf{H},\ \langle e|e\rangle = 1\}$ with $\mathbb{T} := \{\zeta \in \mathbb{C} :\ |\zeta| = 1\}$.
For $e \in \mathbf{S}$, let us write $e^\sim := \mathbb{T}e$.
$T_{e^\sim}M \equiv e^\perp = \{z \in \mathbf{H} :\ \langle z|e\rangle = 0\}$.
$\{uvw\}_{e^\sim} := -\{uvw\} \in T_{e^\sim}M$ for $u, v, w \in T_{e^\sim}M$.
Given $e \in \mathbf{S}$, let $\mathbf{R}_e$ be a rotation of $\mathbf{H}$ with $\mathbf{R}_e(e) = e_0$.
$\Phi_{e^\sim}^{-1} : e \perp z \mapsto [\mathbb{C}\mathbf{R}_e(e + z)] \cap \mathbf{S}$ $(e \in \mathbf{S})$.
Then $\mathcal{X} := \{\Phi_{e^\sim} :\ e^\sim \in M\} = \{\Phi_{e^\sim} :\ e \in \mathbf{S}\}$ *is an atlas on* M *with holomorphic coordinate transitions.*
$\mathcal{G}^* \subset \mathrm{Aut}_{\mathrm{loc}}(M) \cap \{\mathcal{X}$*-holomorphic maps*$\}$ *is transitive on* M.
$\mathrm{Aut}(M) \cap \{\mathcal{X}$*-holomorphic maps*$\} = \{\mathrm{Identity}\}$.

Case (E2)
$Z := \{z \in \mathbf{A} :\ z^* = z\}$ self-adjoint elements.
$\{xyz\} := \frac{1}{2}xyz + \frac{1}{2}zyx$ $(x, y, z \in Z)$.
$G := \{g \in Z :\ g$ invertible$\}$, $M = \mathbf{A}^+ = \{g^*g :\ g \in G\}$.
$\sigma : a \mapsto a^{-1}$ symmetry on M with fixed point 1.
$L_g(a) := g^*ag$ $(a \in M,\ g \in G)$; $L_g^{-1} = L_{g^{-1}},\ L_g : M \leftrightarrow M,\ 1 \mapsto g^*g$.
Definition. $\sigma_a := L_{a^{1/2}} \circ \sigma \circ L_{a^{-1/2}}$ symmetry at $e \in M$.
Remark. $a = g^*g \Rightarrow L_g \circ \sigma \circ L_{a^{-1}} = \sigma_a$ $(g \in G)$.
Definition. Infinitesimal translation on M with vector $h \in Z \equiv T_1M$ at 1:

$$X_h(a) := \frac{d}{d\tau}\Big|_{\tau=0} \sigma_{1+(\tau/2)h} \circ \sigma_1(a) \qquad (a \in M).$$

Lemma. *We have*

$$1)\ X_h(a) = \frac{1}{2}(ha + ah) = \{h1a\}, \qquad 2)\ X_h(a) = \frac{d}{d\tau}\Big|_{\tau=0} \sigma_{\exp(\tau h/2)} \circ \sigma(a).$$

Proof. 1) $\sigma_{1+(\tau h/2} \circ \sigma_1(a) =$
$= \sigma_{1+\tau h/2}(a^{-1}) = (1+\tau h/2)^{1/2}\big[(1+\tau h/2)^{-1/2} a^{-1} (1+\tau h/2)^{-1/2}\big]^{-1} (1+\tau h/2)^{1/2} =$
$= (1+\tau h/2) a (1+\tau h/2)$.
2) $\exp(\tau h/2) = \sum\limits_{n=0}^{\infty} \frac{1}{2^n n!} \tau^n h^n = 1 + \tau h/2 + o(\tau)$.

Lemma. $\exp X_h(a) = [\exp(h/2)]a[\exp(h/2)]$.

Proof. The left and right multiplications $\mathrm{Mult}_\ell^h : \mathbf{A} \ni x \mapsto hx$ and $\mathrm{Mult}_r^h : \mathbf{A} \ni x \mapsto hx$ commute. Observe that $H_h = \frac{1}{2}[\mathrm{Mult}_\ell^h + \mathrm{Mult}_r^h]$. Therefore $\exp X_{2h}(a) =$
$= \exp[\mathrm{Mult}_\ell^h + \mathrm{Mult}_r^h]a = \exp[\mathrm{Mult}_\ell^h]\exp[\mathrm{Mult}_r^h]a = [\mathrm{Mult}_\ell^{\exp h}][\mathrm{Mult}_\ell^{\exp h}]a$.

Remark. Since all elements from $M = \mathbf{A}^+$ admit a spectral resolution over some compact interval $0 < [\alpha, \beta] < \infty$, each element $a \in M = \mathbf{A}^+$ can be written as $a = \exp h$ with a *unique* self-adjoint $h \in Z$ such that $(\log \alpha) \cdot 1 \le h \le (\log \beta) \cdot 1$.

Definition. We write

$$\mathrm{Exp}_Z(h) := \exp(h) \quad (h \in Z), \qquad \mathrm{Log}_M a := \mathrm{Exp}_Z^{-1}(a) \quad (a \in M).$$

Let $h \in Z$ and set $a := \exp(h) = \mathrm{Exp}_Z(h) = \exp(X_h)1$. We call the mapping

$$M_a := \exp(X_h) = \exp X_{\mathrm{Log}_M a} : M \ni x \mapsto axa$$

the *pseudo-Möbius transformation* of the point a. Natural atlas on M:

$$\Phi_a := M_a \circ \mathrm{Log}_M \circ M_a^{-1} \qquad (a \in M).$$

Remark. $M_a^{-1} = M_{a^{-1}}$.

Our next aim is to calculate the image $\Phi_1^{\#} X_h : Z \to Z$ of the vector field in the atlas page of Φ_1. By definition,

$$\begin{aligned}\big[\Phi_1^{\#} X_h\big](z) &:= \big[\Phi_1'\big(\Phi_1^{-1}(z)\big)\big] X_h\big(\Phi_1^{-1}(z)\big) = \\ &= \frac{d}{d\tau}\Big|_{\tau=0} \Big[\Phi_1^{\#}\big[\sigma_{\exp(\tau h/2)} \circ \sigma\big]\Big](z) = \\ &= \frac{d}{d\tau}\Big|_{\tau=0} \Big[\Phi_1^{\#}\sigma_{\exp(\tau h/2)}\Big] \circ \Big[\Phi_1^{\#}\sigma\Big](z).\end{aligned}$$

Notice that $\Phi_1^{\#}\sigma(z) = -z$. Therefore

$$\begin{aligned}\Big[\Phi_1^{\#} X_h\Big](z) &= \frac{d}{d\tau}\Big|_{\tau=0} \Big[\Phi_1^{\#}\sigma_{\exp(\tau h/2)}\Big](-z) = \\ &= \frac{d}{d\tau}\Big|_{\tau=0} \mathrm{Log}_M\Big(\exp\big(\frac{1}{2}\tau h\big)\exp\big(z\big)\exp\big(\frac{1}{2}\tau h\big)\Big).\end{aligned}$$

Let us fix $z, h \in Z$ and define

$$w(\tau) := \mathrm{Log}_M\Big(\exp\big(\frac{1}{2}\tau h\big) \exp\big(z\big) \exp\big(\frac{1}{2}\tau h\big)\Big).$$

Since $w(0) = z$ trivially, we can write $w(\tau) = z + \tau b + o(\tau)$ with a unique vector $b \in Z$. In terms of b, we have

$$\frac{d}{d\tau}\Big|_{\tau=0} \exp\big(w(\tau)\big) = \frac{d}{d\tau}\Big|_{\tau=0} \exp\big(\frac{1}{2}\tau h\big) \exp\big(z\big) \exp\big(\frac{1}{2}\tau h\big).$$

On the right hand side we simply have

$$\frac{d}{d\tau}\Big|_{\tau=0} \exp\big(\frac{1}{2}\tau h\big) \exp\big(z\big) \exp\big(\frac{1}{2}\tau h\big) = \frac{1}{2} h \exp(z) + \frac{1}{2} \exp(z) h \ .$$

On the left hand side the derivation can be carried out with the power series of the exponential map termwise. That is

$$\frac{d}{d\tau}\Big|_{\tau=0} \exp\big(w(\tau)\big) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d}{d\tau}\Big|_{\tau=0} w(\tau)^n = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=1}^{n} z^{k-1} b z^{n-k} =$$
$$= \sum_{k,\ell=0}^{\infty} \frac{1}{(k+\ell+1)!} \big[\mathrm{Mult}_\ell^z\big]^k \big[\mathrm{Mult}_r^z\big]^\ell b \ .$$

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