On the manifold of tripotents in JB*-triples*

LÁSZLÓ L. STACHÓ **

SZTE Bolyai Institute,
Aradi Vértanúk tere 1, 6725 Szeged,
Hungary

Abstract. Tripotents are natural generalizations of partial isometries in C*-algebras to the context of JB*-triples that is complex Banach spaces with symmetric unit ball. We give a survey on the main results papers [2,7,8,6] concerning the structure of the tripotents as a direct real-analytic submanifold in a JB*-triple. We also discuss some recent achievements.

1. Preliminaries: symmetry, JB*-triples, tripotents

Topological algebraic structures concerning spatial symmetry have their obvious importance in mathematical physics and they have independent mathematical interest as well. The underlying space in this paper will be a so-called JB*-triple, a complex Banach space $Z$ whose unit ball $B(Z) := \{ x \in Z : \|x\| < 1 \}$ is symmetric in the sense of holomorphy that is for every point $z \in B(Z)$ there is a biholomorphism $S_z : B(Z) \leftrightarrow B(Z)$ such that $S_z^2 = S_z \circ S_z = \text{Id}_{B(Z)}$, $S_z(z) = z$ and $S_z'(z) = -\text{Id}_Z$ for the Fréchet derivative of $S_z$. As result of a long development started with the Harish-Chandra realization of finite dimensional symmetric domains, in 1983 W. Kaup [9] established the following algebraic characterization. The Banach spaces with symmetric unit ball are exactly those admitting a Jordan-Banach *-triple product (JB*-triple product for short, hence the name JB*-triple).

By a JB*-triple product we mean an operation $\{, , ,\} : Z \times Z \times Z \rightarrow Z$ with three variables satisfying the axioms

\begin{align*}
\text{(J1)} & \quad \{ x, y, z \} \text{ is symmetric bilinear in } x, z \text{ and conjugate-linear in } y, \\
\text{(J2)} & \quad \| \{ x, x, x \} \| = \| x \|^3, \\
\text{(J3)} & \quad D(a) \{ x, y, z \} = \{ D(a)x, y, z \} - \{ x, D(a)y, z \} + \{ x, y, D(a)z \}, \\
\text{(J4)} & \quad \| \exp (\zeta D(a)) \| \leq 1 \quad \text{whenever} \quad \text{Re } \zeta \leq 0.
\end{align*}

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** email: stacho@math.u-szeged.hu

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As a typical example, each C*-algebra with its natural norm is a JB*-triple with the triple product \( \{x, y, z\} := [xy^*z + zy^*x]/2 \). It is remarkable that the JB*-triple product is unambiguously determined by the norm of the underlying space, furthermore any bounded symmetric domain is biholomorphically equivalent to the unit ball of some JB*-triple.

Henceforth \( Z \) will denote an arbitrarily fixed JB*-triple with norm \( \|\| \) and JB*-triple product \( \{, , , , \} \), respectively. We shall write

\[
\text{Der}(Z) := \{ \delta \in \mathcal{L}(Z) : \delta\{x, y, z\} = \{\delta x, y, z\} + \{x, \delta y, z\} + \{x, y, \delta z\} \}
\]

for the set of all derivations of the triple product and

\[
\text{Her}(Z) := \{ \alpha \in \mathcal{L}(Z) : \|\exp(it\alpha)\| = 1 \ (t \in \mathbb{R}) \}
\]

will stand for the set of all hermitian operators of the norm \( \|\| \). Axiom (J3) can be interpreted as the fact that all the operators \( iD(a) \) belong to \( \text{Der}(Z) \). In view of Sinclair’s theorem on the norm of hermitian operators (for an elementary proof see [6, p. 245]), axiom (J4) is an equivalent formulation of the fact that the operators \( D(a) \) are hermitian with non-negative spectra.\(^1\)

The link between complex geometry and Jordan structure in \( Z \) is established by the fact that the family \( \text{aut}\ B(Z) \) of all complete holomorphic vector fields of the unit ball is spanned by derivations and polynomials of second degree of the triple product. In this paper, by a vector field on a domain \( C \subset Z \) we simply mean a holomorphic mapping \( C \to Z \) and, by definition, the vector field \( V \) is complete in \( C \) if its flow is defined on the whole phase set \( D \times \mathbb{R} \). In particular \( V \in \text{aut}\ B(Z) \) if there is a necessarily real-analytic mapping \( F_V : B(Z) \times \mathbb{R} \to B(Z) \) such that \( F_V(p, 0) = p \) and \( \frac{d}{dt}F_V(p, t) = V(F_V(p, t)) \) for all \( p \in B(Z) \) and \( t \in \mathbb{R} \). Namely, in terms of the conjugate linear quadratic representation operators \( Q(a) : z \mapsto \{a, z, a\} \) we can write

\[
\text{aut}\ B(Z) = \{ [z \mapsto a - Q(z)a + \delta z] : a \in Z, \delta \in \text{Der}(Z) \}.
\]

The main objective of our work is the family

\[
\text{Tri}(Z) := \{ e \in Z : \{e, e, e\} = e \neq 0 \}
\]

\(^1\) Proof. Let \( \alpha \in \{ \beta \in \text{Her}(Z) : \text{Sp}(\beta) \geq 0 \} \) and assume \( \xi, \eta \in \mathbb{R} \) with \( \xi \leq 0 \). Then

\[
\|\exp(\xi + i\eta)\alpha\| = \|\exp(\xi)\alpha\| \quad \text{because the operator exp(\(i\eta\)) is unitary with respect to the norm } \|\|.
\]

Define \( \mu_1 := \text{max}\text{Sp}(\alpha) \) and \( \mu_2 := \text{min}\text{Sp}(\alpha) \) and consider the operator \( \beta := \alpha - 2^{-1}(\mu_1 + \mu_2)\text{Id} \). We have \( \beta \in \text{Her}(Z) \) and \( \alpha = \beta + 2^{-1}(\mu_1 + \mu_2)\text{Id} \). By Sinclair’s theorem, \( \|\beta\| = \text{max}\{\text{max}\text{Sp}(\beta), |\text{min}\text{Sp}(\beta)|\} = 2^{-1}(\mu_1 - \mu_2) \). Therefore

\[
\|\exp(\xi + i\eta)\alpha\| = \|\exp(\xi)\alpha\| = e^{2^{-1}\xi(\mu_1 + \mu_2)}\|\exp(\xi)\beta\| \leq e^{2^{-1}\xi(\mu_1 + \mu_2)}e^{\xi|\|\beta\|} = e^{2^{-1}\xi(\mu_1 + \mu_2)}e^{-\xi(\mu_1 + \mu_2)} = e^\lambda \leq 1.
\]

The converse is an easy consequence of the spectral mapping theorem. Let \( \|\exp(\xi)\beta\| \leq 1 \) for \( \text{Re}\xi \leq 0 \). Then, given any \( \lambda \in \text{Sp}(\alpha) \), we have \( |e^\lambda| \leq 1 \) for \( \text{exp}(\xi) \alpha \| \leq 1 \) that is \( \text{Re}\xi \lambda \leq 0 \) whenever \( \text{Re}\xi \leq 0 \) which is possible only if \( \lambda \geq 0 \).
of the tripotents that is the idempotent elements of the triple product in \( Z \). In case of \( Z \) being a \( C^* \)-algebra \( \text{Tri}(Z) = \{ e : ee^*e = e \neq 0 \} \) is the set of all partial isometries. It is a well-known consequence of axioms (J1),(J3) that the operators \( D(e) \) and \( Q(e) \) are semisimple and commute if \( e \in \text{Tri}(Z) \). Namely we have \( D(e)(D(e) - 2^{-1}\text{Id})(D(e) - \text{Id}) = 0 \) and \( Q(e)^3 = Q(e) \) and hence the Peirce decomposition

\[
Z = Z_0(e) \oplus Z_{1/2}(e) \oplus Z_1(e), \quad Z^0(e) = Z_0(e) \oplus Z_{1/2}(e), \quad Z_1(e) = Z^1(e) \oplus Z^{-1}(e)
\]

with the eigenspaces

\[
Z_\lambda(e) := \{ z \in Z : D(e)z = \lambda z \}, \quad Z^\varepsilon(e) := \{ z \in Z : Q(e)z = \varepsilon z \}.
\]

It is also a well-known consequence of axioms (J1),(J3) that \( Q(e) \) acts as an involutive automorphism of the triple product on \( Z_1(e) \): \( Q(e)x = x \) and \( Q(e)\{x,y,z\} = \{Q(e)x,Q(e)y,Q(e)z\} \) for \( x,y,z \in Z_1(e) \). This fact along with \( iD(e) \in \text{Der}(Z) \) entails the so-called Peirce rules

\[
\{Z_\xi(e), Z_\alpha(e), Z_\eta(e)\} \subset Z_{\xi-\alpha+\eta}(e), \quad \{Z^\epsilon(e), Z^\rho(e), Z^\psi(e)\} \subset Z^{\epsilon+\rho+\psi}(e),
\]

\[
\{Z_0(e), Z_1(e), Z\} = \{Z_1(e), Z_0(e), Z\} = \{0\}.
\]

As a typical example, if \( Z \) is the \( C^* \)-algebra of all complex \( (m+n) \)-square matrices then \( e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \text{Tri}(Z) \) with the \( m \times m \) identity matrix \( I \) and, in terms of the \( (m,n) \) matrix decomposition we have \( Z_1(e) = \{ (a_0^0) : a \text{ hermitian} \}, \ Z^{-1}(e) = iZ^1(e), Z_{1/2}(e) = \{ (0_0^0) : x,y \text{ arbitrary} \}, \ Z_0(e) = \{ (0_0^0) : b \text{ arbitrary} \} \).

By the \( C^* \)-axiom (J2), tripotents have norm one. In finite dimensions their geometric importance as distinguished boundary objects relies upon the fact [11, 12] that the holomorphic boundary components (faces in holomorphic sense)\(^3\) of the unit ball have the form \( e + B(Z_0(e)), e \in \text{Tri}(Z) \) and the boundary \( \partial B(Z) \) is their disjoint union. In infinite dimensions there may be no tripotents at all as e.g. in the case of the commutative \( C^* \)-algebra \( Z := C_0(0,1) \) of the continuous functions \( f : (-1,1) \rightarrow \mathbb{C} \) with \( \lim_{|\omega| \rightarrow 1} f(\omega) = 0 \). However, using the canonical embedding of \( Z \) into its bidual \( Z^{**} \), we can regard \( B(Z) \) as a weak*-dense norm-closed subset of \( B(Z^{**}) \). Actually \( Z^{**} \) is always a JB*-triple whose triple product admits

\(^2\) We include a short simultaneous proof which cannot be found in the literature. Let \( e \in \text{Tri}(Z) \) and \( \delta := D(e), \mu := Q(e) \). Using only axioms (J1) and (J3), we have \( \{ x, e, e \} = \{ x, e, \mu \} = 2\{ x, e, e \} - 2\{ e, e, e \} \). This means the relation \( \delta = 2\delta^2 - \mu^2 \) or which is the same as \( (1) \mu^2 = 2\delta(\delta - 2^{-1}\text{Id}) \). Similarly, from the three term expansion of \( \{ x, e, \{ x, e, e \} \} \) we get \( (2) \delta \mu = 2\mu - \mu \delta \). Expanding \( \{ x, x, e \} \) we also get \( (3) \mu = 2\delta \mu - \mu \delta \). Equations (2),(3) imply immediately that \( (4) \mu = \delta \mu = \mu \delta \). Hence \( \mu^2(\delta - \text{Id}) = 0 \). In view of (1) this entails the first Peirce equation \( 2\delta(\delta - 2^{-1}\text{Id})(\delta - \text{Id}) = 0 \). The second Peirce equation has the form \( \mu^3 = \mu = 0 \). This is immediate from (1) and (4). Indeed, \( \mu^3 = \mu = \mu(\mu^2 - \text{Id}) = \mu \delta(2\delta^2 - \delta - \text{Id}) = \mu(2\delta^3 - \delta^2 - \delta) = 2\mu - \mu - \mu = 0 \).

\(^3\) The holomorphic boundary component of a point \( p \in \partial B(Z) = \{ z \in Z : ||z|| = 1 \} \) is the union of all finite sequences \( F_0, \ldots, F_n \), by holomorphic images of the unit disc \( \mathbb{D} := \{ \zeta \in \mathbb{C} : ||\zeta|| < 1 \} \) such that \( F_0, \ldots, F_n \subset \partial B(Z), p \in F_0 \) and \( F_{j-1} \cap F_j \neq \emptyset \).
plenty of tripotents and extends the triple product from $Z$ in a separately weak*-continuous manner [1]. Though the sets $e + B(Z^*_0(e)), e \in \text{Tri}(Z^*)$ do not cover $\partial B(Z)$ in general, we have [3]

$$\{\text{norm-exposed faces of } \overline{B}(Z^*)\} = \{e + \overline{B}(Z^*): e \in \text{Tri}(Z^*)\}$$

where $\overline{B}$ denotes closed unit ball. For more on JB*-triples see [16,12,13].

2. Tri($Z$) as a submanifold of $Z$

Recall that the tangent cone of a subset $S$ in a real Banach space $(X, \|\|)$ at the point $p \in S$ is the set $T_p(S)$ of all vectors $v \in X$ such that $v = \lim_n \xi_n(p_n - p)$ for some sequences $p_1, p_2, \ldots \in S$ and $\xi_1, \xi_2, \ldots \in \mathbb{R}_+$. Notice that $T_p(S)$ is always a closed cone in $X$. By definition, $S$ is a direct analytic submanifold in $X$ if for every point $p \in S$ there is a bianalytic mapping $\Phi_p : U_p \to V_p$ between some neighborhoods of the origin and the point $p$, respectively, along with a direct sum decomposition $X = X_p^{(0)} \oplus X_p^{(1)}$ into closed subspaces such that

$$V_p \cap S = \Phi_p(X_p^{(0)} \cap U_p).$$

It is a direct consequence of the inverse mapping theorem that $S$ is an analytic submanifold in $X$ if and only if, for any point $p \in S$, $T_p(S)$ is a closed complemented subspace of $X$ and there is an analytic mapping $\Psi_p$ from some neighborhood $U_p$ of the origin in $X$ into $X$ such that

$$\Psi_p(p) = \text{Id} \text{ and } \Psi_p(x) \in S \text{ if and only if } x \in T_p(S) \cap U_p.$$ 

Actually, in the latter case one can find a family $\{U_p: p \in S\}$ of 0-neighborhoods in $X$ such that the restricted mappings $\Psi_p|U_p, p \in S$ form an analytic atlas of $S$.

Given any tripotent $e \in \text{Tri}(Z)$, in the sequel we shall write $P_\lambda(e)$ for the Peirce projection onto $Z_\lambda(e)$ along the complementary sum $\oplus_{\mu \neq \lambda} Z_\mu(e)$. We also introduce the spaces $Z_\sigma^-(e) := Z_1(e) \cap Z^{\sigma}(e)$ and

$$Z^{(-)}(e) := Z_{1/2}(e) \oplus Z_{-1}(e), \quad Z^{(+)}(e) := Z_0(e) \oplus Z_1(e)$$

and write $P_\sigma(e) := 2^{-1}P_1(e)[\text{Id} + \sigma Q(e)]$ respectively $P^{(-)}(e) := P_{1/2}^-(e) + P_1^-(e)$ for the corresponding projections. Furthermore we shall keep fixed the notation $K(e, \cdot)$ for the operator

$$K(e, z) := D\left(2^{-1}P_1(e) + 2P_{1/2}(e)z\right) - D\left(2^{-1}P_1(e) + 2P_{1/2}(e)z\right).$$

Notice that $K(e, z)e = P^{(-)}(e)z$ for all $z \in Z$. Moreover, as being in the form $D(e, w) - D(w, e)$, we have $K(e, z) \in \text{Der}(Z)$, $z \in Z$. Hence $\exp K(e, w) \in \text{Aut}(Z)$ with the family of all (linear) automorphisms of the triple product $\{., ., .\}$.
(which coincides with the set of all surjective isometries $Z \to Z$). In particular
$\exp K(e, w)\text{Tri}(Z) = \text{Tri}(Z)$ and $Z\lambda(\exp K(e, w)e) = [\exp K(e, w)]Z\lambda(e)$, $\lambda = 0, 1/2, 1$. Regarding the complex JB*-triple $Z$ as a real Banach space and taking into account that $e + a + x$ cannot be a tripotent if $e \in \text{Tri}(Z)$, $a \in Z_1^1(e)$, $x \in Z_0(e)$ and $\|a\|, \|x\| < 2^{-1}$ we have the following observation.

2.1. Proposition [14]. $\text{Tri}(Z)$ is a real-analytic direct submanifold of $Z$. For any tripotent $e$ we have $T_e(Z) = Z_{1/2} \oplus Z_{1}^{-1}(e) = Z_{1/2} \oplus iZ_1^1(e)$. In terms of the operators $K$, the mappings $\Psi_e : Z \to Z$, $e \in \text{Tri}(Z)$ are well-defined by

$$
\Psi_e(x + v + a + ib) := [\exp K(e, v + ib)](e + x + a),
$$

$$
x \in Z_0(e), \ v \in Z_{1/2}(e), \ a, b \in Z_1^1(e).
$$

They are real-analytic with the properties $\Psi_e' = \text{Id}$ (Fréchet derivative in real sense) and, for $||x||, ||a|| < 2^{-1}$, we have $\Psi_e(x + v + a + ib) \in \text{Tri}(Z) \iff x = a = 0$.

A fundamental consequence of this fact is the possibility that we can establish a canonical one-to-one correspondence $E_e$ between the smooth curves in $Z_{1/2}(e) \oplus Z_{1}^{-1}(e)$ and those in $\text{Tri}(Z)$ with starting point $e$ as follows. Recall ([7] or [5]) that $\text{Aut}(Z)$ is an algebraic Banach-Lie subgroup of $L(Z)$ with $T\text{id}(\text{Aut}(Z)) = \text{Der}(Z)$. Hence each smooth function $F : \mathbb{R} \to \text{Der}(Z)$ admits a (unique) left multiplicative primitive function $L F : \mathbb{R} \to \text{Aut}(Z)$ such that

$$
\frac{d}{dt}L F(t) = [L F(t)]F(t), \quad L F(0) = \text{Id}.
$$

2.2. Theorem. Given a smooth curve $\gamma : \mathbb{R} \to Z^{(-)}$, the curve

$$
E_e(\gamma) := L K(e, \gamma(\cdot)) e
$$

ranges smoothly in $\text{Tri}(Z)$. Conversely, given any smooth curve $\varepsilon : \mathbb{R} \to \text{Tri}(Z)$, there is a unique $\gamma \in C^\infty(\mathbb{R}, Z_{1/2}(e) \oplus Z_{1}^{-1}(e))$ with $E_e(\gamma) = \varepsilon$.

Proof. Since $t \mapsto K(e, \gamma(t))$ ranges smoothly in $\text{Der}(Z)$, its left multiplicative primitive function is well-defined and ranges smoothly in $\text{Aut}(Z)$. Hence indeed $E_e(\gamma)e \in C^\infty(\mathbb{R}, \text{Tri}(Z))$. To prove the converse, we have to see that, given a smooth curve $\varepsilon : \mathbb{R} \to \text{Tri}(Z)$ with starting point $e = \varepsilon(0)$, there is a unique smooth curve $g : \mathbb{R} \to \text{Aut}(Z)$ such that $g(t)e = \varepsilon(t)$ and $\frac{d}{dt}g(t) = g(t)K(e, v(t))$ for some smooth curve $v : \mathbb{R} \to Z^{(-)}(e)$. According to Proposition 2.1, the maps $Z^{(-)}(f) \ni w \to \exp K(f, e)f$, $f \in \text{Tri}(Z)$ are real analytic local charts of $\text{Tri}(Z)$. Hence it readily follows that $\varepsilon(t) = h(t)e$, $t \in \mathbb{R}$ with some smooth curve $h : \mathbb{R} \to \text{Aut}(Z)$. Fixing such a curve $h$ (and regarding $g$ in the form $g = hk$), it suffices to see that there is a unique smooth curve $k : \mathbb{R} \to \text{Aut}(Z)$ such that $k(t)e = e$ and $\frac{d}{dt}h(t)k(t) = h(t)k(t)K(e, w(t))$.,
$t \in \mathbb{R}$ for some smooth curve $w : \mathbb{R} \to Z^{(-)}(e)$. By abbreviating $\frac{d}{dt}$ with $'$ as usually, this means the condition

$$k'(t) = k(t)K(e,w(t)) - \ell(t)k(t) \quad \text{where} \quad \ell(t) := h(t)^{-1}h'(t)$$
on $k(.)$ with suitable $w : \mathbb{R} \to Z^{(-)}(e)$. The requirement $k(t)e = e$ implies

$$0 = k(t)^{-1}(e)k'(t)e = K(e,w(t))e - k(t)^{-1}\ell(z)k(t)e = \triangledown_{X,Y}(e)w(t) - k(t)^{-1}\ell(z)e = w(t) - k(t)^{-1}\ell(z)e.$$

Thus necessarily $w(t) = k(t)^{-1}\ell(z)e = k(t)^{-1}\ell(z)k(t)e \in Z^{(-)}(e)$, $t \in \mathbb{R}$ if a required curve $k(.)$ exists. Since $h$ ranges in $\text{Aut}(Z)$, $\ell = h^{-1}h'$ ranges necessarily in the tangent of $\text{Aut}(Z)$ that is $\ell(t) \in \text{Der}(Z)$, $t \in \mathbb{R}$. As a consequence, also $k^{-1}\ell(k)e \in \text{Der}(Z)$ and $k^{-1}\ell(t)ke \in T_z\text{Tri}(Z) = Z^{(-)}(e)$ whenever $k \in \text{Aut}(Z)$. Therefore the initial value problem $k'(t) = k(t)K(e,k(t)^{-1}\ell(z)e) - \ell(t)k(t)k(t) = \text{Id}$ is wellposed in $\mathcal{L}(Z)$, with a unique solution ranging in the isotropy subgroup of the point $e$ in $\text{Aut}(Z)$. Its boundedness ensures that its (maximal) domain is the whole $\mathbb{R}$. \square

The model of curves in $\text{Tri}(Z)$ in the real vector space of curves in $Z^{(-)}(e)$ described by Theorem 2.2 is a powerful tool in the study of the natural differential geometry of $\text{Tri}(Z)$. In 2000 Chu and Isidro [2] have found an interesting generalization of the classical Riemannian connection on surfaces to $\text{Tri}(Z)$ by replacing the orthogonal projections to the tangent planes with the Peirce projections $P^{(-)}$. That is given two vector fields $X,Y$ on $\text{Tri}(Z)$ (functions $\text{Tri}(Z) \to Z$ such that $X(e),Y(e) \in T_e\text{Tri}(Z) = Z^{(-)}(e), e \in \text{Tri}(Z)$) we define

$$\nabla_X Y := P^{(-)}Y'X$$
i.e. $\nabla_X Y(e) = P^{(-)}(e)Y'(e)X(e) = P^{(-)}(e)\frac{d^2}{dt^2}Y(\exp K(e,tX(e)))$, $e \in \text{Tri}(Z)$. We shall refer to $\nabla$ as the \textit{algebraic connection} of $\text{Tri}(Z)$. In [2] one has established partial results on the algebraic form of the geodesics of finite rank tripotent in some JB*-triples. In 2005 in [7, Lemma 1] one achieved the solution of the geodesic equation

$$P^{(-)}(\varepsilon(t))\varepsilon''(t) = 0$$

for $\nabla$ with curves in the form $\varepsilon = E_e(\omega)$ with the following arguments. Let $g(t) = \text{L}K(e,\omega(t))$ and $\varepsilon(t) = g(t)e$. Then $\varepsilon' = gK(e,\omega)e = g\omega$ and $g'' = \varepsilon'\omega + g\omega' = g[K(e,\omega)\omega + \omega']$. Since $g(t) \in \text{Aut}(Z)$ for any $t$, $P^{(-)}(\varepsilon(t)) = g(t)P^{(-)}(e)g^{(-)}(t)$ and hence

$$P^{(-)}(\varepsilon(t))\varepsilon''(t) = g(t)P^{(-)}(e)[K(e,\omega(t))\omega(t) + \omega'(t)].$$

Given any vector $w = w_{1/2} + w_1 \in \mathbb{Z}_{1/2}(e) \oplus \mathbb{Z}^{-1}(e) = Z^{(-)}(e)$, from the Peirce rules it follows that $P^{-1}_{-1}(e)K(e,w)w = 0$ and $P_{1/2}(e)K(e,w)w = 2^{-1}\{w_1,e,w_{1/2}\} + 2\{w_{1/2},e,w_1\} - 2^{-1}\{e,w_{1/2}\} = 3\{w_1,e,w_{1/2}\}$. Thus for
the components $\omega_\lambda(t) := P_\lambda(e)\omega(t)$, $\lambda = 2^{-1}, 1$ we get the linear differential equations $\omega_1t = 0$ and $\omega_{1/2}' = 3D(\omega_1, e)\omega_{1/2}$. Hence Theorem 2.2 yields the following result.

2.3. Theorem. A curve $\varepsilon$ in $\text{Tri}(Z)$ is a $\nabla$-geodesic if and only if

$$
\varepsilon(t) = L K \left( e, w_1 + \exp[3t D(w_1, e)]w_{1/2} \right) e
$$

for some $e \in \text{Tri}(Z)$, $w_1 \in Z_1^{-1}(e)$ and $w_{1/2} \in Z_{1/2}(e)$.

As an immediate consequence, we get the following minor correction to [2, Thm. 2.7]: for fixed $e \in \text{Tri}(Z)$ and $w = w_{1/2} + w_1 \in Z_{1/2}(e) \oplus Z_1^{-1}(e)$, the curve $\varepsilon(t) := \exp K(e, tw)e$ is a $\nabla$-geodesic if and only if $\{w_1, e, w_{1/2}\} = \{e, w_1, w_{1/2}\} = 0$. For a nontrivial example let $E := \begin{bmatrix} 10 & 0 \\ 00 & 11 \end{bmatrix}$, $R := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $A := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $B := \begin{bmatrix} 01 & 01 \\ 01 & 01 \end{bmatrix}$ and let $Z$ be the C*-algebra of all $4 \times 4$ matrices. Then with $e := \begin{bmatrix} 00 \\ 00 \end{bmatrix}$, $w_1 := \begin{bmatrix} R0 \\ 00 \end{bmatrix}$, $w_{1/2} := \begin{bmatrix} 0A \\ 00 \end{bmatrix}$ we have $e \in \text{Tri}(Z)$, $w_1 \in Z_1^{-1}(e)$, $w_{1/2} \in Z_{1/2}(e)$ and $\{w_1, e, w_{1/2}\} = \{w_{1/2}, e, w_1\} = 0$.

Another issue for an effective application of Theorem 2.2 can be the investigation of minimal and stationary curves with respect to the distance in $\omega$ the components of the commutative C*-algebra $Z := C[0, 1]$, any curve $\varepsilon^\alpha(t) := [s \mapsto e^{i\alpha(s, t)}]$ is minimal joining the constant functions $\varepsilon^0$ and $\varepsilon^1$ whenever $\alpha$ is a smooth function $[0, 1]^2 \rightarrow [0, 1]$ such that each subfunction $\alpha(., t)$ maps increasingly the interval $[0, 1]$ onto itself. Disregarding the few cases where $\text{Tri}(Z)$ happens to be a Riemannian manifold, there seem to be no results in the literature on metric minimal and stationary curves of tripotents in general complex JB*-triples. Recently in [15] we achieved the following reformulation of the length variational equation for tripotents by the aid of the technique with multiplicative primitive functions.

2.4. Proposition. Let $\varepsilon : [0, 1] \rightarrow \text{Tri}(Z)$ be a smooth curve in the form $\varepsilon(t) = L K(e, \omega(t)) e$ where $e \in \text{Tri}(Z)$ and $\omega : [0, 1] \rightarrow Z^{(-)}(e)$ is a smooth curve. Then $\text{Length}(\varepsilon) = \int_0^1 \|\omega(t)\| \ dt$. If the curve $\varepsilon$ is stationary then we have

$$
\int_0^1 \left[ \xi(t)\delta(\omega(t), K(e, \omega(t))u) + \xi'(t)\delta(\omega(t), u) \right] dt = 0
$$
for any vector $u \in Z^{(-)}(e)$ and for any smooth function $\xi : [0,1] \to \mathbb{R}$ with
$\xi(0) = \xi(1) = 0$ where $\delta(z,v) := \lim_{s \to 0} s^{-1}\|z + sv\|$, $z, v \in Z$ denotes the subgradient of the norm in $Z$.

An immediate difficulty in the progress along these lines is the fact that the bad smoothness properties of the norm in most JB*-triples do not allow to carry out a routine partial integration in the latter formula. Hence the following problem is still open. In which JB*-triples are all $\nabla$-geodesics curves stationary?

3. The Grassmanian structure of the equivalence classes of tripotents

Since the tangent space $T_e \text{Tri}(Z) = Z_{1/2}(e) \oplus Z_1^{-1}(e)$ is not a complex subspace in $Z$ (in particular $ie \in Z_1^{-1}(e) = iZ_1^1(e)$ and $Z_1^{-1}(e) \cap Z_1^1(e) = \{0\}$), $\text{Tri}(Z)$ is not a complex submanifold of $Z$. Observe that if we ”go in the wrong directions” in $\text{Tri}(Z)$ in the sense that we consider curves in the form $\varepsilon(t) := g(t)$ with $g(t) := L K(e,\omega(t))$ and $\omega(t) \in Z_1^{-1}(e)$ then the operators $D(\varepsilon(t))$ determining the Peirce subspaces do not change. Indeed, it is well-known that $D(e,w) = \sigma D(w,e)$ whenever $w \in Z_1^1(e)$ whence $\frac{d}{dt}D(e) = D(\varepsilon',e) + D(e,\varepsilon') = D(g\omega,ge) + D(ge,gw) = g[D(\omega,e) + D(e,\omega)]g^{-1} = 0$. The equivalence of tripotents

$e \sim f \iff D(e) = D(f)$

was introduced and studied already in 1985 by E. Neher [13]. Originally he formulated this relationship as $\{e, e, f\} = f$ and $\{f, f, e\} = e$ and called it ”association” but established its equivalence with $D(e) = D(f)$ immediately. Since any automorphism of the triple product maps an equivalence class of $\sim$ onto another equivalence class and since the maps $\Psi_e : Z_{1/2} \oplus Z_{1}^{-1}(e) \ni w \mapsto \exp K(e,w)e$, $e \in \text{Tri}(Z)$ are local charts on $\text{Tri}(Z)$, it can be expected that the quotient manifold

$$\mathcal{M} := \text{Tri}(Z)/\sim := \{e^\sim : e \in \text{Tri}(Z)\} \quad \text{where} \quad e^\sim := \{f \in \text{Tri}(Z) : f \sim e\}$$

equipped with the maps

$$\Psi_e^\sim : Z_{1/2}(e) \ni w \mapsto \exp K(e,w)e^\sim = [\Psi_e(w)e]^\sim, \quad e \in \text{Tri}(Z)$$

becomes a real-analytic manifold. If so, $\mathcal{M}$ with the intrinsic metric of curve lengths must be symmetric in the following sense (cf. [2]). The Peirce reflections

$$S(e) := 2P_{1/2}(e) - \text{Id}, \quad e \in \text{Tri}(Z)$$

belong to $\text{Aut}(Z)$ with $S_e|Z_0(e) \oplus Z_1(e) = \text{Id}$ and $S_e|Z_{1/2}(e) = -\text{Id}$. Easily seen, $S(e)$ commutes with the chart map $\Psi_e$ that is $S(e)\Psi_e(w) = \Psi_e(-w)$ for all $w \in Z_{1/2}(e)$. Consequently, its quotient mapping $S^\sim(e)f^\sim := [S(e)f]^\sim$, $f \in \text{Tri}(Z)$ is a welldefined symmetry of $\mathcal{M}$ that is $S^\sim(e)$ is holomorphic with $S^\sim(e)e^\sim = e^\sim$ and $[S^\sim(e)]'(e^\sim) = -\text{Id}$. It is an open problem for the time being, in which cases
does $\mathcal{M}$ become with this atlas a complex manifold (i.e. all the coordinate changing maps $[\Psi^\sim]^{-1} \circ \Psi^\sim$ are holomorphic). In 2001, Kaup [10] published a paper on the Grassmanian manifold

$$\mathcal{IP} := \{ J_a : a \in Z, \exists V \subset Z \text{ subspace } J_a \oplus V = Z \}$$

of all principal inner ideals $J_a := \bigcap \{ J \subset Z : a \in J, \{ J, Z, J \} = J \}$ which are complemented in $Z$. One of its main conclusions is that the maps

$$\Theta_e : Z_{1/2}(e) \ni u \mapsto \exp D(u, e)J_e, \ e \in \text{Tri}(Z)$$

form an atlas on $\mathbb{IP}$ and $\mathbb{IP}$ becomes a complex symmetric manifold with them and a suitable metric. Notice that the equivalence $e \sim f$ of two tripotents can also be formulated in terms of their Peirce 1-subspaces as $Z_1(e) = Z_1(f)$ (as an easy consequence of $e \sim f \iff \{ e, e, f \} = f \& \{ f, f, e \} = e \iff D(e) = D(f)$). As it is also shown in [10], actually we have

$$\mathbb{IP} = \{ J_a : a \in \text{Reg}(Z) \} = \{ J_{s(a)} : a \in \text{Reg}(Z) \} = \{ Z_1(e) : e \in \text{Tri}(Z) \}$$

with the set $\text{Reg}(Z) := \{ a \in Z : \text{Sp}[D(a) \mathcal{C}(a)] > 0 \}$ of all von Neumann regular elements in $Z$ where $\mathcal{C}(a) := \text{Span}_{n=0}^\infty D(a)^n a$ denotes the closed subtriple generated by the element $a$ and

$$s(a) := \lim_{n \to \infty} \varphi_n(D(a))a, \quad a \in \text{Reg}(Z)$$

is the support tripotent of $a \in \text{Reg}(Z)$ well-defined with any sequence $(\varphi_n)$ of real polynomials such that $\varphi_n(x^2)x \to 1$ locally uniformly for $x > 0$. Thus, with the family $\mathcal{ID} := \{ iD(e) : e \in \text{Tri}(Z) \}$ of triple derivations, the diagram of mappings

$$\begin{array}{ccc}
\text{Tri}(Z) & \xleftarrow{\sim} & \mathcal{ID} \\
\downarrow & & \downarrow \\
\mathcal{IP} & \xleftarrow{\sim} & \mathcal{IM}
\end{array}$$

is commutative. Thus the complex structure of $\mathbb{IP}$ provided by the charts $\Theta_e$ on $\mathbb{IP}$ can be translated to $\mathcal{ID}$ and $\mathbb{IM}$ by its means. Henceforth we shall be concerned with the problem how to describe holomorphy in $\mathbb{IM}$ and $\mathbb{ID}$ in intrinsic manners, not involving principal ideals explicitly. Such kind of an approach may have interest from the following view point: the algebraically less sophisticated maps $u \mapsto \exp D(u, e)$ in the construction of the charts of $\mathbb{IP}$ apply to rather “big” objects such that we may have $J_e \cap J_f \neq \emptyset$ even if $e \neq f$ while $e^\sim \cap f^\sim = \emptyset$ and $D(e) \neq D(f)$ simply in the latter case. As a first natural question we can raise is how $\mathcal{ID}$ does behave topologically in the real-linear operator space $\text{Der}(Z)$. We gave the following answer in terms of decompositions with the projections

$$\pi_{k\ell}(e) : \mathcal{L}(Z) \ni L \mapsto P_{k/2}(e)L P_{\ell/2}(e), \quad H_m(e) := \sum_{|k-\ell| = m, \ k, \ell = 0,1,2} \pi_{k\ell}(e).$$
3.1. **Theorem** [8]. \(\Pi_0\) and \(\Pi_1\) map \(\text{Der}(Z)\) into itself and we have

\[
\text{Der}(Z) = \Delta_1(e) \oplus \Delta_0(e) \quad \text{where} \quad \Delta_m(e) := \Pi_m(e)\text{Der}(Z).
\]

\(\mathbb{I}D\) is a real-analytic direct submanifold of \(\text{Der}(Z)\). For any \(e \in \text{Tri}(Z)\), the map \(u \mapsto K(e, u)\) is a bijection \(Z_{1/2}(e) \leftrightarrow \Delta_1(e)\) and \(T_{iD}(e)\mathbb{I}D = \Delta_1(e)\). The families

\[
\{ [Z_{1/2}(e) \ni u \mapsto iD(\exp K(e, u)e)] : e \in \text{Tri}(Z) \},
\]

\[
\{ [Z_{1/2}(e) \ni u \mapsto (\exp K(e, u)e)^\sim] : e \in \text{Tri}(Z) \}
\]

are real-analytic atlases for \(\mathbb{I}D\) and \(\text{IM}\), respectively.

The main topological properties of the natural map \(\mathbb{I}D \leftrightarrow \text{IM}\) can be established by a fine estimate as follows.

3.2. **Proposition.** [8]. If \(e, f \in \text{Tri}(Z)\) and we have \(\|D(e) - D(f)\| < \frac{1}{10}\) then there exists \(f' \in f\sim\) such that \(\|e - f'\| \leq 16\|D(e) - D(f)\|\).

As a consequence, by writing \(d(z, A) := \inf_{a \in A} \|z - a\|, z \in Z, A \subset Z\) for the point-set distance in \(Z\), the quotient topology of the equivalence classes in \(\text{IM}\) inherited from the norm topology of \(\text{Tri}(Z)\) coincides with the topology by the bias \(d_0(e^\sim, f^\sim) := \inf_{e^\sim \leq f^\sim} d(e, f^\sim)\). It coincides also with the topology by the Hausdorff metric \(d_H(e^\sim, f^\sim) := \max\{ \sup_{e^\sim \leq f^\sim} d(e, f^\sim), \sup_{f^\sim \leq e^\sim} d(f', e^\sim)\}\). Moreover the mapping \(e^\sim \mapsto iD(e)\) is bilipschitzian \(\text{IM} \leftrightarrow \mathbb{I}D\) with respect to \(d_H\).

Next we proceed to the question if the real-analytic structures given in Theorem 3.1 are compatible with those inherited from Kaup’s complex manifold structure on \(\mathbb{P}\). There is a natural candidate for a canonical technique to translate the coordinate map \(\Theta_e(u) := \exp(u, e)Z_1(e) = J_{\exp D(u)e}\) into \(\text{IM}\) and \(\mathbb{I}D\). Namely we can project the range of \(\exp D(\cdot, e)\) into \(\text{Tri}(Z)\) by using support tripotents resulting in the mappings

\[
\tilde{\Theta}_e(u) := [s(\exp D(u,e)e)]^\sim, \quad \tilde{\Theta}_e(u) := iD(s(\exp D(u,e)e))
\]

into from \(Z_{1/2}(e)\) into \(\text{IM}\) and \(\mathbb{I}D\), respectively. Are they real-analytic with respect to the atlases of \(\text{IM}\) and \(\mathbb{I}D\) given in Theorem 3.1?

3.3. **Theorem.** [8]. If \(Z\) is a JC*-triple, \(e \in \text{Tri}(Z)\) and \(u \in Z_{1/2}(e)\) then the function \(s_t := s([\exp tD(u,e)]e)\) is the solution of the initial value problem

\[
(3.4) \quad \frac{d}{dt}s_t = P_{1/2}(s_t)\{u, e, s_t\}, \quad s_0 = e.
\]

By a JC*-triple we mean a JB*-triple which is isomorphic to a subtriple of some C*-algebra \(\mathcal{A}(H)\) with a suitable Hilbert space \(H\). It is a well-known consequence of the Gelfand-Neumark theorem of JB*-triples due to Friedman and Russo [4] that
to establish Theorem 3.3 for general JB*-triples, it suffices to prove its statement additionally only in the special case $Z = H_3(O)$ of the 27-dimensional exceptional JB*-triple. It seems that any JB*-subtriple of $H_3(O)$ generated by a tripotent and an element from its Peirce $(1/2)$-subspace must be a JC*-triple. The solution of (3.4) passes in such a subtriple necessarily and the theorem is valid in general. However, we have no complete proof for the moment.

To bypass this difficulty, in [8] we construct holomorphic atlases on $\mathbb{I}M$ by means of the solutions of (3.4), leading to some results of independent interest. To this aim, first we have to understand the connection between the tangent vector fields of $\text{Tri}(Z)$ and those of $\mathbb{I}M$. Consider a flow $[\phi_t : t \in \mathbb{R}]$ of mappings $\phi_t : \text{Tri}(Z) \to \text{Tri}(Z)$ which preserve the equivalence classes of $\sim$ (i.e. $e \sim f \Rightarrow \phi_t(e) \sim \phi_t(f)$) such that $\phi_0 = \text{Id}$ and each curve $t \mapsto \phi_t(e)$ is smooth. Then the vector field $e \mapsto X(e) := \frac{d}{dt}|_{t=0}\phi_t(e) \in Z^{-1}(e)$ has the property

$$D(X(e), e) = D(X(f), f), \quad D(e, X(e)) = D(f, X(f)) \quad \text{whenever } e \sim f.$$  

We shall call such tangent vector fields equivariant. Different flows $[\phi_t], [\psi_t]$ may give rise to the same mappings of equivalence classes in the sense that $\phi_t(e^{-}) = \psi_t(e^{-})$ for all $t \in \mathbb{R}, e \in \text{Tri}(Z)$. Then, for the generator vector fields $X := \frac{d}{dt}|_{t=0}\phi_t$ and $Y := \frac{d}{dt}|_{t=0}\psi_t$, we have

$$D(X(e), e) = D(Y(e), e), \quad D(e, X(e)) = D(e, Y(e)), \quad e \in \text{Tri}(Z).$$

We call this property the equivalence of the fields $X, Y$ and write $X \approx Y$ for it.

**3.5. Proposition.** [8]. (1) Given an equivariant field $X$, its projection

$$P_{1/2}X : e \mapsto P_{1/2}(e)X(e)$$

is the unique equivariant field $Y$ with $X \approx Y$ and $Y(e) \in Z_{1/2}(e), e \in \text{Tri}(Z)$.

(2) A bounded locally Lipschitzian tangent vector field $X$ on $\text{Tri}(Z)$ is equivariant if and only if $\exp tX$ preserves the equivalence classes of $\sim$ for all $t \in \mathbb{R}$.

(3) The family of all smooth equivariant vector fields in $\text{Tri}(Z)$ is a Lie algebra with the operation $[X, Y]_* := \nabla_X Y - \nabla_Y X$ and we have $[X, Y]_* \approx [\hat{X}, \hat{Y}]_*$ whenever $X \approx \hat{X}$ and $Y \approx \hat{Y}$.

Define the auxiliary manifolds

$$S_\lambda := \{(e, x) : e \in \text{Tri}(Z), x \in Z_\lambda(e)\}, \quad \lambda = 1, 1/2, 0.$$  

Heuristically, $S_1$ can serve as a "disjointification" of the Grassmanian $\mathbb{I}P$. Its main features for the study of the solutions of (3.4) can be summarized as follows.

**3.6. Proposition.** [8]. $S_1$ is a real-analytic direct submanifold of $Z \times Z$ with

$$T_{(e,x)}S_1 = Z_1^{-1}(e) \times Z_1(e) \supset \left\{ \left( P_{1/2}(e)D(a, b)e, D(a, b)x \right) : a, b \in Z \right\}.$$
Given two smooth vector fields $C : \text{Tri}(Z) \to Z$, $D : Z \to Z$ being complete in $\text{Tri}(Z)$ and $Z$, respectively, the statements (1),(2),(3) below are equivalent.

1. $[\exp tD]x \in Z_1([\exp tC]e)$ for all $(e, x) \in S_1$ and $t \in \mathbb{R}$,
2. $D(x) = \{C(e), e, x\} + \{e, C(e), x\} + \{e, e, D(x)\}$ for all $(e, x) \in S_1$,
3. $[\exp(tD)]Z_1(e) = Z_1([\exp tC]e)$ for all $(e, x) \in S_1$ and $t \in \mathbb{R}$.

For any couple $(e, u) \in S_{1/2}$, let us introduce the tangent vector field

$$C^{(e)}_u(f) := P_{1/2}(f)D(u, e)f, \quad f \in \text{Tri}(Z).$$

On the basis of Propositions 3.5 and 3.6 we can complete the argument.

**3.7. Theorem.** [8]. For each $e \in \text{Tri}(Z)$ there exists a neighborhood $W$ of 0 in $Z_{1/2}(e)$ and a real-analytic map $T_e : W \to \text{Tri}(Z)$ such that

$$T_e(0) = e, \quad \exp D(u, e)e \in Z_1(T_e(u)), \quad u \in W.$$

**Proof.** Fix any $u \in Z_{1/2}(e)$ and set $x^u := \exp D(u, e)e$. Notice that the vector field $E_u(f) := P_{1/2}(f)D(u, f)f, f \in \text{Tri}(Z)$ is a tangent to $\text{Tri}(Z)$ and its exponential is a well-defined mapping $\text{Tri}(Z) \to Z$. Let

$$T_e(u) := (\exp E_u)e, \quad u \in Z_{1/2}(e).$$

Then the curve $t \mapsto e_t := T_e(tu), t \in \mathbb{R}$, is the solution of the initial value problem $e_0 = e, \frac{d}{dt}e_t = P_{1/2}(e_t)D(u, e_t)$. Consider the mapping

$$F(f, y) := (P_{1/2}(f)D(u, e)f, D(u, e)y), \quad (f, y) \in S_1.$$

From Proposition 3.6 we see that $F$ is a tangent vector field to $S_1$ and its exponential is a well-defined mapping $S_1 \to S_1$. In particular, there is a curve $t \mapsto (f_t, y_t) \in S_1, t \in \mathbb{R}$, such that $(f_0, y_0) = (e, e)$ and $\frac{d}{dt}(f_t, y_t) = F(f_t, y_t)$. Then we have $\frac{d}{dt}y_t = D(u, e)y_t, y_0 = e$ and $\frac{d}{dt}f_t = P_{1/2}(f_t)D(u, e)f_t, f_0 = e$. By the uniqueness of solutions of initial value problems, $y_t = (\exp tD(u, e))e$ and $f_t = e_t$ for all $t \in \mathbb{R}$. Since $(f_t, y_t) \in S_1$, we have $y_t \in Z_1(f_t)$ for all $t \in \mathbb{R}$. In particular $\exp D(u, e)e = y_1 \in Z_1(f_1) = Z_1(T_e(u))$ which completes the proof. □

On the basis of these results we can describe the holomorphic atlases corresponding to Kaup’s coordinatization for $\mathbb{P}$ both on $\mathbb{D}$ and $\mathbb{M}$ as follows. Consider the vector fields

$$C^{(e)}_u : \text{Tri}(Z) \ni f \mapsto P_{1/2}(f)D(u, e)f, \quad (e, u) \in S_{1/2}.$$

According to the results of Section 2, they are real-analytic (with respect to the coordinates $Z(-)(e) \ni w \mapsto \exp K(e, w))$. and tangent to $\text{Tri}(Z)$. Hence the
curves \( t \mapsto [\exp tC_u^{(e)}]e \) are well-defined on the whole \( \mathbb{R} \) and range in \( \text{Tri}(Z) \). By definition they are solutions of (3.4). Also the maps

\[
Y_e Z_{1/2}(e) \ni u \mapsto [\exp C_u^{(e)}]e, \quad e \in \text{Tri}(Z)
\]

are all well-defined, real-analytic and range in \( \text{Tri}(Z) \). Using Propositions 3.5, 3.6 and Theorem 3.7 we conclude the following.

**3.8. Theorem.** [8]. The vector fields \( C_u^{(e)} \) are equivariant and complete in \( \text{Tri}(Z) \).

For any tripotent \( e \), there exists a neighborhood \( W_e \) of the origin in \( Z_{1/2}(e) \) such that the restricted map \( Y_e|_{W_e} \) is real-analytic with

\[
Y_e(0) = e, \quad [\exp D(u,e)]J_e = J_{Y_e(u)} = Z_1(Y_e(u)), \quad u \in W_e.
\]

By setting \( \hat{Y}^{(e)}(u) := iD(Y_e(u)), \quad \tilde{Y}^{(e)}(u) := Y_e(u)^{\sim}, \quad Y^{(e)}(u) := J_{Y_e(u)} \)

the families

\[
\{\hat{Y}^{(e)} : e \in M\}, \quad \{\tilde{Y}^{(e)} : e \in M\}, \quad \{Y^{(e)} : e \in M\}
\]

are holomorphic atlases for \( \mathbb{D}, \mathbb{M} \) and \( \mathbb{P} \) with commuting diagram

\[
\begin{array}{ccc}
\mathbb{Z}_{1/2}(e) & \hookrightarrow & u \\
\downarrow & & \downarrow \\
\mathbb{D} & \leftrightarrow & \mathbb{M} & \leftrightarrow & \mathbb{P} \\
& & \hat{Y}^{(e)}(u) & & \tilde{Y}^{(e)}(u) & & Y^{(e)}(u)
\end{array}
\]

Since the points of \( \mathbb{M} \) are actually pairwise disjoint subsets in \( Z \), it is natural to ask how can we describe the holomorphy of a function \( \mathbb{M} \rightarrow \mathbb{C}^* \) (and hence holomorphy to general Banach spaces) in terms of holomorphy in \( Z \).

**3.9. Theorem** [8]. Let \( U \) be an open subset of \( \mathbb{M} \) and let \( U := \bigcup_{e^{\sim} \in U} e^{\sim} \) denote its trace in \( Z \). A function \( \Phi : U \rightarrow \mathbb{C}^* \) is holomorphic if and only if for any point \( e \in U \), there exists an open neighborhood \( V \) of \( e \) in \( Z \) along with a holomorphic function \( \phi : V \rightarrow \mathbb{C}^* \) such that \( \phi(f^{\sim}) = \Phi(f^{\sim}) \) whenever \( f \in U \cap V \).

**References**


