Locally generated polynomial $\mathcal{C}^1$-splines over triangular meshes

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Abstract. We classify all possible local linear procedures over triangular meshes resulting in polynomial C^1-spline functions with affinely uniform shape for the basic functions at the edges, and fitting the 9 value- and gradient data at the vertices of the mesh members. There is a unique procedure among them with shape functions and basic polynomials of degree 5 and all other admissible procedures are its perturbations with higher degree.

1. Introduction

By a *triangular mesh* we mean a finite family of closed non-degenerate triangles on the plane $\mathbb{R}^2$ with pairwise non-intersecting interiors and admitting only common vertices or edges. As usually, we regard $\mathbb{R}^2$ as the set of all real couples $[\xi,\eta]$ considered also as 1×2 (row) matrices. We shall use the standard notations $x^{[1]} = x : [\xi,\eta] \mapsto \xi$, $x^{[2]} = y : [\xi,\eta] \mapsto \eta$ and $\langle \mathbf{u}|\mathbf{v}\rangle := \sum_{j=1}^{2} x^{[j]}(\mathbf{u})x^{[j]}(\mathbf{v})$ for the Cartesian coordinates and scalar product, respectively. We write $\|\mathbf{u}\| = \langle \mathbf{u}|\mathbf{u}\rangle^{1/2}$ for the norm of $\mathbf{u} \in \mathbb{R}^2$ and $\mathrm{Co}(\mathbf{S})$ for the convex hull of $\mathbf{S} \subset \mathbb{R}^2$ resp. $\det(\mathbf{u},\mathbf{v}) = x^{[1]}(\mathbf{u})x^{[2]}(\mathbf{v}) - x^{[1]}(\mathbf{v})x^{[2]}(\mathbf{u})$ for 2×2-determinants. Given a triangular mesh $\mathcal{T} = \{\mathbf{T}_1, \dots, \mathbf{T}_M\}$, in the sequel $\mathrm{Vert}(\mathbf{T}_k)$ and $\mathrm{Edge}(\mathbf{T}_k)$ will denote the sets of vertices resp. closed edges of the mesh members, furhermore $\mathrm{Dom}(\mathcal{T}) := \bigcup_{k=1}^{M} \mathbf{T}_k$, $\mathrm{Edge}(\mathcal{T}) := \bigcup_{k=1}^{M} \partial\mathbf{T}_k$, $\mathrm{Vert}(\mathcal{T}) := \bigcup_{k=1}^{M} \mathrm{Vert}(\mathbf{T}_k)$ will stand for the domain covered by $\mathcal{T}$, the line figure covered by all edges and the collection of all vertices, respectively. Recall that, given a *gradient-data*

$$F = \Big\{ \big(\mathbf{p}, f_{\mathbf{p}}, [f'_{x,\mathbf{p}}, f'_{y,\mathbf{p}}]\big) : \mathbf{p} \in \mathrm{Vert}\mathcal{T}) \Big\} \subset \mathrm{Vert}(\mathcal{T}) \times \mathbb{R} \times \mathbb{R}^2$$

on the set of the vertices in $\mathcal{T}$, a function $f : \mathbf{D} \to \mathbb{R}$ is a $\mathcal{C}^1$-*extension* of F on $\mathbf{D} := \mathrm{Dom}(\mathcal{T})$ if f has a continuous gradient $\mathbf{p} \mapsto \nabla f(\mathbf{p}) =$

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$\left[\frac{\partial}{\partial x}f(\mathbf{p}), \frac{\partial}{\partial y}f(\mathbf{p})\right]$ on Interior($\mathbf{D}$) which admits a continuous extension to $\mathbf{D}$ as well (denoted also by ∇f) such that

$$f(\mathbf{p}) = f_{\mathbf{p}}, \quad \nabla f(\mathbf{p}) = \left[f'_{x,\mathbf{p}}, f'_{y,\mathbf{p}}\right] \qquad \big(\mathbf{p} \in \mathrm{Vert}(\mathcal{T})\big). \tag{1.1}$$

A $\mathcal{C}^1$-extension $f : \mathbf{D} \to \mathbb{R}$ of F is said to be a $\mathcal{C}^1$-*spline interpolation* of F with respect to the mesh $\mathcal{T}$ if the restrictions $f|\mathbf{T}_k$ are polynomials of the coordinate functions x, y.

There exists a large variety of $\mathcal{C}^1$-splines for any admissible $\mathcal{T}$ and F which can be obtained e.g. as global polynomial extensions with Hermite type interpolation [5]. Obviously global polynomial fitting may primarily be interesting only from a pure theretical view point due to too large polynomial degree and hence high numerical instability. A better alternative could be an imitation of tensor product splines (e.g. with Catmull-Rom type hermition curves on edges developed for rectangular meshes [7,6]). This consists the construction of $\mathcal{C}^1$-splines as linear combinations on the rectangular mesh members from affine images of tensor products from only two special polynomials $\Phi, \Psi : [0,1] \to [0,1]$ (actually $\varphi(t) = t^2(3-2t)$, $\psi(t) = t^2(1-t)$). Some main features of tensor product spline procedures which can naturally be generalized even to procedures

$$\mathfrak{S} : (\mathcal{T}, F) \mapsto f_{\mathcal{T},F} \quad \big(\mathcal{T} \text{ triang. mesh, } F \text{ grad. data on } \mathrm{Vert}(\mathcal{T})\big) \tag{1.2}$$

furnishing $\mathcal{C}^1$-spline interpolation functions from gradient data at the vertices over triangular meshes can be formulated in Postulates A,B below.

Postulate A. (Linearity and being locally generated). *There are polynomial functions*

$$\varphi_{\mathbf{p},\mathbf{T}},\ \psi^{(1)}_{\mathbf{p},\mathbf{T}},\ \psi^{(2)}_{\mathbf{p},\mathbf{T}} : \mathbf{T} \to \mathbb{R} \quad \big(\mathbf{T} \textit{ non-deg. triangle}\},\ \mathbf{p} \in \mathrm{Vert}(\mathbf{T})\big)$$

depending only on the couple of the triangle $\mathbf{T}$ *with a distingvished vertex such that the restriction of* $\mathfrak{S}$ *to any mesh triangle* $\mathbf{T} \in \mathcal{T}$ *has the form*

$$f_{\mathcal{T},F}|\mathbf{T} = \sum_{\mathbf{p}\in\mathrm{Vert}(\mathbf{T})} \Big[f_{\mathbf{p}}\varphi_{\mathbf{p},\mathbf{T}} + f'_{x,\mathbf{p}}\psi^{(1)}_{\mathbf{p},\mathbf{T}} + f'_{y,\mathbf{p}}\psi^{(2)}_{\mathbf{p},\mathbf{T}}\Big]. \tag{1.3}$$

If Postulate A holds and $\mathrm{Vert}(\mathbf{T}) = \{\mathbf{a}, \mathbf{b}, \mathbf{p}\}$, in terms of the canonical frame vectors

$$\mathbf{e}^{[0]} := \mathbf{0} = [0,0],\ \mathbf{e}^{[1]} := [1,0],\ \mathbf{e}^{[2]} := [0,1]$$

we necessarily have

$$\begin{aligned}&\varphi_{\mathbf{p},\mathbf{T}}(\mathbf{p})=1,\ \nabla\varphi_{\mathbf{p},\mathbf{T}}(\mathbf{p})=\mathbf{0},\qquad \psi^{(j)}_{\mathbf{p},\mathbf{T}}(\mathbf{p})=0,\ \nabla\psi^{(j)}_{\mathbf{p},\mathbf{T}}(\mathbf{p})=\mathbf{e}^{[j]};\\ &\varphi_{\mathbf{p},\mathbf{T}}(\mathbf{x})=\psi^{(j)}_{\mathbf{p},\mathbf{T}}(\mathbf{x})=0,\ \nabla\varphi_{\mathbf{p},\mathbf{T}}(\mathbf{x})=\nabla\psi^{(j)}_{\mathbf{p},\mathbf{T}}(\mathbf{x})=\mathbf{0}\quad \big(\mathbf{x}\in\mathrm{Co}\{\mathbf{a},\mathbf{b}\}\big).\end{aligned}\tag{1.4}$$

The first statement in (1.4) is immediate from (1.3), while the second one is a consequence of the fact that given any point $\widetilde{\mathbf{p}}$ forming an adjacent triangle $\widetilde{\mathbf{T}}:=\mathrm{Co}\{\mathbf{a},\mathbf{b},\widetilde{\mathbf{p}}\}$, for the mesh $\mathcal{T}:=\{\mathbf{T},\widetilde{\mathbf{T}}\}$ with gradient data $F(\mathbf{q})=(0,\mathbf{0})$ for $\mathbf{q}=\mathbf{a},\mathbf{b},\widetilde{\mathbf{p}}$ we must have $f_{\mathcal{T},F}\equiv 0$ on $\widetilde{\mathbf{T}}$ and hence also $\nabla f_{\mathcal{T},F}\equiv 0$ on the common edge $\mathrm{Co}\{\mathbf{a},\mathbf{b}\}$ of the triangles $\mathbf{T},\widetilde{\mathbf{T}}$.

Locally generated linear spline procedures have the computational advantage that the resulting functions can be calculated on any mesh triangle regardless to what happens at vertices outside. A practical disadvantage is that in most cases only function values are available (mostly from scanned data) and convenient gradient values must be guessed or found by optimizing procedres.

Postulate B. (Uniform shape on edges). (1.3) *holds and there are polynomial functions* $\Phi,\Psi:[0,1]\to\mathbb{R}$ *such that*

$$\Phi(0)=\Psi(0)=\Phi'(0)=\Psi'(0)=\Psi(1)=0,\quad \Phi(1)=\Psi'(1)=1\tag{1.5}$$

and the graps of the basic functions $\varphi_{\mathbf{p},\mathbf{T}}$ *on the edges of the triangle* $\mathbf{T}$ *are affine images of the graph of* Φ*, and those of* $\psi^{(j)}_{\mathbf{p},\mathbf{T}}$ $(j=1,2)$ *are affine images of the graph of* Ψ*.*

That is, under Postulate B, for the generic points $\mathbf{y}_t:=t\mathbf{p}+(1-t)\mathbf{a}$ on the edge $\mathrm{Co}\{\mathbf{a},\mathbf{p}\}$, resp. $\mathbf{z}_t:=t\mathbf{p}+(1-t)\mathbf{b}$ on $\mathrm{Co}\{\mathbf{b},\mathbf{p}\}$ we have

$$\begin{aligned}&\varphi_{\mathbf{p},\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}}\big(\mathbf{y}_t\big)=\mathrm{const}_{\mathbf{a},\mathbf{p}}\Phi(t),\quad \psi^{(j)}_{\mathbf{p},\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}}\big(\mathbf{y}_t\big)=\mathrm{const}^{(j)}_{\mathbf{a},\mathbf{p}}\Psi(t),\\ &\varphi_{\mathbf{p},\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}}\big(\mathbf{z}_t\big)=\mathrm{const}_{\mathbf{b},\mathbf{p}}\Phi(t),\quad \psi^{(j)}_{\mathbf{p},\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}}\big(\mathbf{z}_t\big)=\mathrm{const}^{(j)}_{\mathbf{b},\mathbf{p}}\Psi(t)\end{aligned}\tag{1.6}$$

while for the points $\mathbf{x}_t:=(1-t)\mathbf{a}+t\mathbf{b}$ on the edge $\mathrm{Co}\{\mathbf{a},\mathbf{b}\}$ we simply have

$$\begin{aligned}&\varphi_{\mathbf{p},\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}}(\mathbf{x}_t)=\psi^{(j)}_{\mathbf{p},\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}}(\mathbf{x}_t)=0,\\ &\nabla\varphi_{\mathbf{p},\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}}(\mathbf{x}_t)=\nabla\psi^{(j)}_{\mathbf{p},\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}}(\mathbf{x}_t)=\mathbf{0}.\end{aligned}\tag{1.7}$$

In the sequel we call Φ,Ψ the *shape functions* of the spline procedure $(\mathcal{T},F)\mapsto f_{\mathcal{T},F}$ satisfying Postulate B. Notice that the requirements $\Phi(0)=\Phi'(0)=\Psi(0)=\Psi'(0)$ follow automatically from the order condition (1.4) on the edge $\mathrm{Co}\{\mathbf{a},\mathbf{b}\}$.

At first glance, shape uniformity may seem an artificial requirement. However, for a procedure satisfying Postulate A, the geometrically natural property of being invariant with respect to homothetic transformations (maps $\mathbb{R}\leftrightarrow\mathbb{R}$ of the form $\mathbf{x}\mapsto\mu\mathbf{x}\mathbf{S}+\mathbf{w}$ with some orthogonal matrix $\mathbf{S}$) implies Postulate B trivially. In our context we understand invariance as follows: given a surjective affine transformation $\mathbf{G}(\mathbf{x})=\mathbf{x}\mathbf{A}+\mathbf{w}$ with some invertible 2×2-matrix $\mathbf{A}$ of the plain, the spline procedure $\mathfrak{S}:(\mathcal{T},F)\mapsto f_{\mathcal{T},F}$ is $\mathbf{G}$-*invariant* if it transfers spline functions constructed with the gradient data of any smooth function h on $\mathrm{Vert}(\mathcal{T})$ from $\mathrm{Dom}(\mathcal{T})$ to the analogous objects with $h\circ\mathbf{G}^{-1}$ on $\mathrm{Dom}(\mathbf{G}(\mathcal{T}))=\mathbf{G}(\mathrm{Dom}(\mathcal{T}))$, that is

$$\begin{aligned} &f_{\mathcal{T},F}\circ\mathbf{G}^{-1}=f_{\mathbf{G}(\mathcal{T}),\mathbf{G}^\sharp(F)}\\ &\qquad\text{with}\quad \mathbf{G}^\sharp\big(\mathbf{x},\chi,[\mu_1,\mu_2]\big):=\big(\mathbf{G}(\mathbf{x}),\chi,[\mu_1,\mu_2]\mathbf{A}^{-1}\big). \end{aligned}\tag{1.8}$$

As we shall see (Lemma 4.1), if Postulate A holds, we can formulate $\mathbf{G}$-invariance in terms of the basic functions as follows:

$$\begin{aligned} &\varphi_{\mathbf{G}(\mathbf{p}),\mathbf{G}(\mathbf{T})}=\varphi_{\mathbf{p},\mathbf{T}}\circ\mathbf{G}^{-1},\\ &\big[\psi^{(1)}_{\mathbf{G}(\mathbf{p}),\mathbf{G}(\mathbf{T})},\psi^{(2)}_{\mathbf{G}(\mathbf{p}),\mathbf{G}(\mathbf{T})}\big]=\big[\psi^{(1)}_{\mathbf{p},\mathbf{T}}\circ\mathbf{G}^{-1},\psi^{(2)}_{\mathbf{p},\mathbf{T}}\circ\mathbf{G}^{-1}\big]\mathbf{A}. \end{aligned}\tag{1.9}$$

It is worth to notice (Corollary 4.4) that (1.9) cannot hold simultaneously for all invertible matrices $\mathbf{A}$ and $\mathbf{w}\in\mathbb{R}^2$. Thus there is no local linear spline procedure which is invariant under all invertible affine transformations and producing always $\mathcal{C}^1$-smooth functions (i.e. functions being continuously differentiable also over the edges of mesh triangles) functions.

Our aim in this paper is a parametric classification of the procedures satisfying Postulates A,B, resulting in $\mathcal{C}^1$-smooth functions. In particular we enumerate all the homothetically invariant linear local polynomial $\mathcal{C}^1$-spline interpolation procedues from gradient data over triangular meshes. It is remarkable that there is a unique one among them with lawest degree (degree 5) which turns out to be homothetically invariant. From the view point of applications, the results provide the complete list of hermition $\mathcal{C}^1$-splines with shape uniformity over edges from which one can choose the best fit one with respect to various aspects. It is worth to relate the latter fact to a celebrated alternative local linear polynomial spline interpolation procedure on the basis of Zlámal-Ženišek 2-nd order triangular spline equations [9]. This relies upon the fact that, given a triangular mesh with gradient and Hessian data at the vertices and normal derivative values at edge middle points, there is a unique fitting spline with 5th degree polynomials. The 21 polynomial coefficients over any mesh triangle can be

obtained as the unique solution of a system of 21 straightforward linear equations whose explicit formula was published recently [8]. Though not stated in the sources, easily seen this kind of procedure has some homothetical invariance properties. Hence it seems that our first order approach with the shape conditions of Postulate B provides a geometrically motivated alternative to several problems discussed in [8]. As mentioned earler and remarked also e.g. in [1], first order approches with a few (actually 9 in [1]) free parameters may have practical advantages versus higher oreder methods due to the fact that data sampling can rarely support e.g. Hessian data (or even adequate guesses for them).

Our arguments are based on the use of baricentic coordinates associated with triangles instead of the usual Cartesian ones. Applying Remark 3.2 to the difference of the first order solution given in Theorem 2.3 a way is opened to develop a new geometric approach to the system of Zlámal-Ženišek equations and its alternative variants which may have further independent theoretical and educational interest.

2. Main results

Recall that given a non-degenerate triangle $\mathbf{T} \subset \mathbb{R}^2$ with $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\} = \mathrm{Vert}(\mathbf{T})$, the *normalized baricentric coordinates* of a point $\mathbf{x}$ are the terms of the necessarily unique triple $\big[\lambda^{\mathbf{a}}_{\mathbf{T}}(\mathbf{x}), \lambda^{\mathbf{b}}_{\mathbf{T}}(\mathbf{x}), \lambda^{\mathbf{c}}_{\mathbf{T}}(\mathbf{x})\big] \in \mathbb{R}^3$ such that

$$\mathbf{x} = \lambda^{\mathbf{a}}_{\mathbf{T}}(\mathbf{x})\mathbf{a} + \lambda^{\mathbf{b}}_{\mathbf{T}}(\mathbf{x})\mathbf{b} + \lambda^{\mathbf{c}}_{\mathbf{T}}(\mathbf{x})\mathbf{c}, \quad \lambda^{\mathbf{a}}_{\mathbf{T}}(\mathbf{x}) + \lambda^{\mathbf{b}}_{\mathbf{T}}(\mathbf{x}) + \lambda^{\mathbf{c}}_{\mathbf{T}}(\mathbf{x}) = 1.$$

We reserve the symbols $\lambda^{\mathbf{p}}_{\mathbf{T}}$ as standard notation. It is well-known from elementary analytic plain geomertry [2] that

$$\lambda^{\mathbf{p}}_{\mathbf{T}}(\mathbf{x}) = \mathrm{area}(\mathrm{Co}\{\mathbf{a}, \mathbf{b}, \mathbf{x}\})/\mathrm{area}(\mathbf{T}) \qquad \big(\mathbf{x} \in \mathbf{T}\big)$$

thus normalized baricentric coordinates can easily be calculated by means of determinants or inner products with a $(\pi/2)$-rotation:

$$\lambda^{\mathbf{p}}_{\mathbf{T}}(\mathbf{x}) = \frac{\det(\mathbf{x}-\mathbf{a}, \mathbf{x}-\mathbf{b})}{\det(\mathbf{p}-\mathbf{a}, \mathbf{p}-\mathbf{b})} \frac{\big\langle (\mathbf{b}-\mathbf{a})\mathbf{R} \big| \mathbf{x}-\mathbf{a} \big\rangle}{\big\langle (\mathbf{b}-\mathbf{a})\mathbf{R} \big| \mathbf{p}-\mathbf{a} \big\rangle} \quad \text{where} \quad \mathbf{R} := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{2.1}$$

For later use we also introduce the abbreviating notations

$$x^{[j]}_{\mathbf{p}} := x^{[j]} - x^{[j]}(\mathbf{p}), \ \xi^{\mathbf{v}}_{\mathbf{p},\mathbf{a}} := \frac{\langle \mathbf{v}-\mathbf{a} | \mathbf{p}-\mathbf{a} \rangle}{\|\mathbf{p}-\mathbf{a}\|^2}, \ \overline{\xi}^{\mathbf{v}}_{\mathbf{p},\mathbf{a}} := \frac{\langle \mathbf{v}-\mathbf{a} | (\mathbf{p}-\mathbf{a})\mathbf{R} \rangle}{\|\mathbf{p}-\mathbf{a}\|^2}. \tag{2.2}$$

As for geometric interpretation, $\xi^{\mathbf{v}}_{\mathbf{p},\mathbf{a}}$ resp. $\overline{\xi}^{\mathbf{v}}_{\mathbf{p},\mathbf{a}}$ are the affine coordinates of the point $\mathbf{v}$ with respect to the orthogonal frame $\big[\mathbf{a},\mathbf{p},\mathbf{a}+(\mathbf{p}-\mathbf{a})\mathbf{R}\big]$ with origin $\mathbf{a}$ so that $\mathbf{v}=\mathbf{a}+\xi^{\mathbf{v}}_{\mathbf{p},\mathbf{a}}(\mathbf{p}-\mathbf{a})+\overline{\xi}^{\mathbf{v}}_{\mathbf{p},\mathbf{a}}(\mathbf{p}-\mathbf{a})\mathbf{R}$.

Theorem 2.3. *There is a unique local linear polynomial $\mathcal{C}^1$-spline procedure acting on triagular meshes with the property of uniform shape on vertices* and having shape functions with minimal computational complexity. Its shape functions are*

$$^*\Phi(t)=t^3(10-15t+6t^2),\qquad {}^*\Psi(t)=t^3(t-1)(4-3t).$$

The corresponding basic functions (for a non-degenerate triangle $\mathbf{T}=\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}$ with distinguished vertex $\mathbf{p}$) have the form

$$^*\varphi_{\mathbf{p},\mathbf{T}}={}^*\Phi\big(\lambda^{\mathbf{p}}_{\mathbf{T}}\big)+30\,[\lambda^{\mathbf{p}}_{\mathbf{T}}]^2\lambda^{\mathbf{a}}_{\mathbf{T}}\lambda^{\mathbf{b}}_{\mathbf{T}}\Big[\xi^{\mathbf{b}}_{\mathbf{p},\mathbf{a}}\lambda^{\mathbf{b}}_{\mathbf{T}}+\xi^{\mathbf{a}}_{\mathbf{p},\mathbf{b}}\lambda^{\mathbf{a}}_{\mathbf{T}}\Big],$$
$$^*\psi^{(j)}_{\mathbf{p},\mathbf{T}}=\frac{{}^*\Psi\big(\lambda^{\mathbf{p}}_{\mathbf{T}}\big)}{\lambda^{\mathbf{p}}_{\mathbf{T}}-1}x^{[j]}_{\mathbf{p}}+12[\lambda^{\mathbf{p}}_{\mathbf{T}}]^2\lambda^{\mathbf{a}}_{\mathbf{T}}\lambda^{\mathbf{b}}_{\mathbf{T}}\Big[\xi^{\mathbf{b}}_{\mathbf{p},\mathbf{a}}x^{[j]}_{\mathbf{p}}(\mathbf{b})\lambda^{\mathbf{b}}_{\mathbf{T}}+\xi^{\mathbf{a}}_{\mathbf{p},\mathbf{b}}x^{[j]}_{\mathbf{p}}(\mathbf{a})\lambda^{\mathbf{a}}_{\mathbf{T}}\Big].$$

Theorem 2.4. *A spline procedure acting on triangular meshes and satisfying Postulates A,B produces $\mathcal{C}^1$-smooth splines if and only if its shape functions are of the form*

$$\Phi(t)={}^*\Phi(t)+t^3(1-t)^3\Phi_1(t),\qquad \Psi(t)={}^*\Psi(t)+t^3(1-t)^3\Psi_1(t)\quad(2.7)$$

and the basic functions (for a non-degenerate triangle $\mathbf{T}=\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}$ with distinguished vertex $\mathbf{p}$) can be written in terms of the modified shape function

$$\Theta(t):=\Psi(t)/(t-1)$$

and the rotation matrix $\mathbf{R}$ in (2.1) as

$$\begin{aligned}\varphi_{\mathbf{p},\mathbf{T}}&=\Phi\big(\lambda^{\mathbf{p}}_{\mathbf{T}}\big)+[\lambda^{\mathbf{p}}_{\mathbf{T}}]^2\lambda^{\mathbf{a}}_{\mathbf{T}}\lambda^{\mathbf{b}}_{\mathbf{T}}P^{\mathbf{p}}_{\mathbf{a},\mathbf{b}}\big(\lambda^{\mathbf{b}}_{\mathbf{T}},\lambda^{\mathbf{a}}_{\mathbf{T}}\big),\\ \psi^{(j)}_{\mathbf{p},\mathbf{T}}&=\Theta\big(\lambda^{\mathbf{p}}_{\mathbf{T}}\big)x^{[j]}_{\mathbf{p}}+[\lambda^{\mathbf{p}}_{\mathbf{T}}]^2\lambda^{\mathbf{a}}_{\mathbf{T}}\lambda^{\mathbf{b}}_{\mathbf{T}}Q^{j,\mathbf{p}}_{\mathbf{a},\mathbf{b}}\big(\lambda^{\mathbf{b}}_{\mathbf{T}},\lambda^{\mathbf{a}}_{\mathbf{T}}\big)\end{aligned}\qquad(2.5)$$

*That is satisfying Postulates A,B with $f_{\mathcal{T},F}\in\mathcal{C}^1\big(\mathrm{Dom}(\mathcal{T})\big)$.

where

$$\begin{aligned}
P^{\mathbf{p}}_{\mathbf{a},\mathbf{b}}(s,t) &= s\Big\{\xi^{\mathbf{a}}_{\mathbf{p},\mathbf{b}}\frac{\Phi'(1-s)}{(1-s)^2s^2}+\overline{\xi}^{\mathbf{a}}_{\mathbf{p},\mathbf{b}}k^{0,\mathbf{p}}_{\mathbf{b}}(s)\Big\}+\\
&\quad+t\Big\{\xi^{\mathbf{b}}_{\mathbf{p},\mathbf{a}}\frac{\Phi'(1-t)}{(1-t)^2t^2}+\overline{\xi}^{\mathbf{b}}_{\mathbf{p},\mathbf{a}}k^{0,\mathbf{p}}_{\mathbf{a}}(t)\Big\}+stR^{0,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(s,t),\\
Q^{j,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(s,t) &= s\Big\{\xi^{\mathbf{a}}_{\mathbf{p},\mathbf{b}}x^{[j]}_{\mathbf{p}}(\mathbf{b})\frac{\Theta'(1-s)}{s(1-s)^2}+\overline{\xi}^{\mathbf{a}}_{\mathbf{p},\mathbf{b}}k^{j,\mathbf{p}}_{\mathbf{b}}(s)\Big\}+\\
&\quad+t\Big\{\xi^{\mathbf{b}}_{\mathbf{p},\mathbf{a}}x^{[j]}_{\mathbf{p}}(\mathbf{a})\frac{\Theta'(1-t)}{t(1-t)^2}+\overline{\xi}^{\mathbf{b}}_{\mathbf{p},\mathbf{a}}k^{j,\mathbf{p}}_{\mathbf{a}}(t)\Big\}+stR^{j,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(s,t)
\end{aligned}\tag{2.6}$$

with the following free options in (2.5) *resp.* (2, 6)*:*

(i) $\Phi_1,\Psi_1:[0,1]\to\mathbb{R}$ *are arbitrary polynomial functions,*

(ii) $(\mathbf{p},\mathbf{q})\mapsto k^{i,\mathbf{p}}_{\mathbf{q}}$ $(i=0,1,2)$ *are arbitrary maps assigning polynomial functions* $\mathbb{R}\to\mathbb{R}$ *to pairs of distinct points,*

(iii) $(\mathbf{p},\mathbf{q},\mathbf{r})\mapsto R^{i,\mathbf{p}}_{\mathbf{q},\mathbf{r}}$ $(i=0,1,2)$ *are arbitrary maps assigning polynomial functions* $\mathbb{R}^2\to\mathbb{R}$ *to triples of non-collinear points with the symmetry* $R^{i,\mathbf{p}}_{\mathbf{q},\mathbf{r}}(s,t)\equiv R^{i,\mathbf{p}}_{\mathbf{r},\mathbf{q}}(t,s)$.

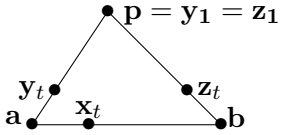
Remark 2.7. (i) Actually, Theorem 2.3 is simply a corollary of Theorem 2.4 by setting the options (i)−(iv) to 0. We emphasize it for its potential practical and educational use.

(ii) The formally rational expressions in $(2.5-2.6)$ are polynomials. Indeed, $\Phi'(1-t)/\big[t^2(1-t)^2\big]=30-3(1-2t)\Phi_1(1-t)+t(1-t)\Phi_1'(1-t)$, resp. $\Psi(t)/(t-1)=t^3[(4-3t)-(1-t)^2\Psi_1(t)]$, $\Theta'(1-t)/[t(1-t)^2]=12+(2-5t)\Psi_1(1-t)-t(1-t)\Psi_1'(1-t)$.

(iii) $\lambda^{\mathbf{p}}_{\mathbf{T}},\lambda^{\mathbf{a}}_{\mathbf{T}},\lambda^{\mathbf{b}}_{\mathbf{T}}$ are the affine functions determined by the properties $\mathrm{Line}\{\mathbf{a},\mathbf{b}\}=\big(\lambda^{\mathbf{p}}_{\mathbf{T}}=0\big)$, $\mathrm{Line}\{\mathbf{b},\mathbf{p}\}=\big(\lambda^{\mathbf{a}}_{\mathbf{T}}=0\big)$, $\mathrm{Line}\{\mathbf{a},\mathbf{p}\}=\big(\lambda^{\mathbf{b}}_{\mathbf{T}}=0\big)$, $\lambda^{\mathbf{p}}_{\mathbf{T}}(\mathbf{p})=\lambda^{\mathbf{a}}_{\mathbf{T}}(\mathbf{a})=\lambda^{\mathbf{b}}_{\mathbf{T}}(\mathbf{b})=1$. For the parametrized edge points in (1.6) we have On the other hand $x^{[j]}(\mathbf{y}_t)=(1-t)x^{[j]}(\mathbf{a}-\mathbf{p})=(1-t)x^{[j]}_{\mathbf{p}}(\mathbf{a})$ resp. $x^{[j]}(\mathbf{z}_t)=(1-t)x^{[j]}(\mathbf{b}-\mathbf{p})=(1-t)x^{[j]}_{\mathbf{p}}(\mathbf{b})$. Hence, with the formulas (2.5), the shape conditions (1.6) hold automatically with $\mathrm{const}_{\mathbf{a},\mathbf{p}}=\mathrm{const}_{\mathbf{b},\mathbf{p}}=1$ and $\mathrm{const}^{(j)}_{\mathbf{a},\mathbf{p}}=x^{[j]}(\mathbf{p}-\mathbf{a})$ resp. $\mathrm{const}^{(j)}_{\mathbf{b},\mathbf{p}}=x^{[j]}(\mathbf{p}-\mathbf{b})$, furthermore also (1.7) is fulfilled.

(iv) One can check with symbolic computer algebra that all the spline procedures described in Theorem 2.4 produce $\mathcal{C}^1$-functions. It suffices to

$$\lambda^{\mathbf{p}}_{\mathbf{T}}(\mathbf{x}_t) = \lambda^{\mathbf{a}}_{\mathbf{T}}(\mathbf{z}_t) = \lambda^{\mathbf{b}}_{\mathbf{T}}(\mathbf{y}_t) \equiv 0,$$
$$\lambda^{\mathbf{p}}_{\mathbf{T}}(\mathbf{y}_t) = \lambda^{\mathbf{p}}_{\mathbf{T}}(\mathbf{z}_t) = \lambda^{\mathbf{b}}_{\mathbf{T}}(\mathbf{x}_t) \equiv t,$$
$$\lambda^{\mathbf{a}}_{\mathbf{T}}(\mathbf{x}_t) = \lambda^{\mathbf{a}}_{\mathbf{T}}(\mathbf{y}_t) = \lambda^{\mathbf{b}}_{\mathbf{T}}(\mathbf{z}_t) \equiv 1 - t.$$



establish only that, given any two adjacent non-degenerate triangles $\mathbf{T} := \mathrm{Co}\{\mathbf{p},\mathbf{a},\mathbf{b}\}$ resp. $\widetilde{\mathbf{T}} := \mathrm{Co}\{\mathbf{p},\mathbf{a},\widetilde{\mathbf{p}}\}$ with common edge $\mathrm{Co}\{\mathbf{p},\mathbf{a}\}$ and distinguished point $\mathbf{p}$, the gradient vectors of the basic functions $\varphi_{\mathbf{p},\mathbf{T}}, \psi^{(j)}_{\mathbf{p},\mathbf{T}}$ coincide with those of $\varphi_{\mathbf{p},\widetilde{\mathbf{T}}}, \psi^{(j)}_{\mathbf{p},\widetilde{T}}$ at the points $\mathbf{y}_t = (1-t)\mathbf{a}+t\mathbf{b}$. Indeed, hence it follows that the unit spline functions $f_{\mathcal{T},F^i_{\mathbf{p}}}$ $\big(\mathbf{p}\in\mathrm{Vert}(\mathcal{T}),\ i=0,1,2\big)$ corresponding to the gradient data $F^0_{\mathbf{p}} := \big\{[\mathbf{p},1,\mathbf{0}],[\mathbf{q},0,\mathbf{0}] : \mathbf{q}\in\mathrm{Vert}(\mathcal{T})\setminus\{\mathbf{p}\}\big\}$ resp. $F^j_{\mathbf{p}} := \big\{[\mathbf{p},0,\mathbf{e}^{[j]}],[\mathbf{q},0,\mathbf{0}] : \mathbf{q}\in\mathrm{Vert}(\mathcal{T})\setminus\{\mathbf{p}\}\big\}$ $(j=1,2)$ are continuously differentiable.

Theorem 2.8. *A $\mathcal{C}^1$-spline procedure $\mathfrak{S}$ described in Theorem 2.4 in the form* (2.7–9) *is isometry-invariant if and only if $k^{i,\mathbf{P}}_{\mathbf{c}}(t)\equiv 0$ for all $i=0,1,2$ and $\mathbf{p}\neq\mathbf{c}\in\mathbb{R}^2$ furthermore the higher terms $R^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}$ in* (2.9) *transform as $R^{0,\mathbf{G}(\mathbf{p})}_{\mathbf{G}(\mathbf{a}),\mathbf{G}(\mathbf{b})} = R^{0,\mathbf{p}}_{\mathbf{a},\mathbf{b}}$ resp. $\big[R^{1,\mathbf{G}(\mathbf{p})}_{\mathbf{G}(\mathbf{a}),\mathbf{G}(\mathbf{b})}, R^{2,\mathbf{G}(\mathbf{p})}_{\mathbf{G}(\mathbf{a}),\mathbf{G}(\mathbf{b})}\big] = \big[R^{1,\mathbf{p}}_{\mathbf{a},\mathbf{b}}, R^{2,\mathbf{p}}_{\mathbf{a},\mathbf{b}}\big]\mathbf{A}$ whenever $\mathbf{G}:\mathbf{x}\mapsto\mathbf{w}+\mathbf{x}\mathbf{A}$ is an isometry.*

3. Proof of Theorem 2.4

Henceforth we consider an arbitrarily fixed procedure $\mathfrak{S}:(\mathcal{T},F)\mapsto f_{\mathcal{T},F}$ which satisfies Postulates A,B and produces continuous but not necessarily continuously differentiable functions. We reserve the notations $\varphi_{\mathbf{p},\mathbf{T}}$, $\psi^{(j)}_{\mathbf{p},\mathbf{T}}$ resp. Φ,Ψ for the basic functions resp. shape functions as established in Section 1. In accordance with (1.5) we can write

$$\Phi(t) = t^2(3-2t)+t^2(1-t)^2\Phi_0(t), \qquad \Psi(t) = t^2(t-1)+t^2(1-t)^2\Psi_0(t) \quad (3.1)$$

and $\Theta(t) = t^2 + t^2(t-1)\Psi_0(t)$ with suitable polynomials Φ_0,Ψ_0.

Next we are going to express the constraints (1.4), (1.6–7) on the basic functions in terms of Φ,Ψ and baricentric coordinates. To this aim, we recall the following folklore fact from elementary algebraic geometry relating the root curves with a product decomposition of multivariate polynomials which is an easy consequence of Bézout's Theorem [3,4].

Remark 3.2(i) If $\mathbf{L}_0, \mathbf{L}_1, \dots, \mathbf{L}_m$ are distinct straight lines such that $\mathbf{L}_k = \big(\ell_k = 0\big)$ with the affine functions (i.e. polynomials of first degree) $\ell_k : \mathbb{R}^2 \to \mathbb{R}$ $(k = 1, \dots, m)$ then a polynomial $\mathbb{R}^2 \to \mathbb{R}$ is divisable with $\prod_{k=0}^m \ell_k^{\nu_k}$ if and only if, for any index k, it vanishes in order ν_k at the points of $\mathbf{L}_k$. In particular, given a non-degenerate triangle $\mathbf{T} := \mathrm{Co}\{\mathbf{a}, \mathbf{b}, \mathbf{p}\}$, a polynomial $Q : \mathbb{R}^2 \to \mathbb{R}$ of two variables has the form

$$Q = [\lambda^{\mathbf{p}}_{\mathbf{T}}]^{\nu_0}[\lambda^{\mathbf{a}}_{\mathbf{T}}]^{\nu_1}[\lambda^{\mathbf{b}}_{\mathbf{T}}]^{\nu_2}$$

for some polynomial $q : \mathbb{R}^2 \to \mathbb{R}$ if and only if it vanishes in order ν_0 at the points of $\mathrm{Line}\{\mathbf{a}, \mathbf{b}\}$, order ν_1 at $\mathrm{Line}\{\mathbf{p}, \mathbf{b}\}$ and order ν_2 at $\mathrm{Line}\{\mathbf{p}, \mathbf{a}\}$, respectively.[†]

(ii) If $Q : \mathbb{R}^2 \to \mathbb{R}$ is a polynomial of two variables, we can write

$$\begin{aligned} &Q(x,y) = Q(0,0) + xq_1(x) + yq_2(y) + xyq_3(x,y) \quad \text{where}\\ &\quad q_1(x) := [Q(x,0)] - Q(0,0)/x, \quad q_2(y) := [Q(0,y) - Q(0,0)]/y,\\ &\quad q_3(x,y) := \big[Q(x,y) - [Q(0,0) + xq_1(x) + yq_2(y)]\big]/(xy) \end{aligned}$$

are well-defined polynomials in one resp. two variables. We shall call the $\mathbb{R}^2$-polynomial $Q_0(x,y) := Q(0,0) + xq_1(x) + yq_2(y)$ of first degree the *principal part* of Q.

Lemma 3.4. *The basic functions* $\varphi_{\mathbf{p},\mathbf{T}}, \psi^{(j)}_{\mathbf{p},\mathbf{T}}$ *for* $\mathbf{T} = \mathrm{Co}\{\mathbf{a}, \mathbf{b}, \mathbf{p}\}$ *have the form*

$$\begin{aligned} \varphi_{\mathbf{p},\mathbf{T}} &= \Phi(\lambda^{\mathbf{p}}_{\mathbf{T}}) + [\lambda^{\mathbf{p}}_{\mathbf{T}}]^2\lambda^{\mathbf{a}}_{\mathbf{T}}\lambda^{\mathbf{b}}_{\mathbf{T}}\,\mathrm{Pol}(\lambda^{\mathbf{b}}_{\mathbf{T}}, \lambda^{\mathbf{a}}_{\mathbf{T}}),\\ \psi^{(j)}_{\mathbf{p},\mathbf{T}} &= \Theta(\lambda^{\mathbf{p}}_{\mathbf{T}})x^{[j]}_{\mathbf{p}} + [\lambda^{\mathbf{p}}_{\mathbf{T}}]^2\lambda^{\mathbf{a}}_{\mathbf{T}}\lambda^{\mathbf{b}}_{\mathbf{T}}\,\mathrm{Pol}(\lambda^{\mathbf{b}}_{\mathbf{T}}, \lambda^{\mathbf{a}}_{\mathbf{T}}) \end{aligned}$$

in terms of the baricentric coordinates (2.1)*, the shape functions* 3.1*,* $\Theta(t) := \Psi(t)/(t-1)$ *and with suitable polynomials of two variables.*

Proof. Fix any triangle $\mathbf{T} := \mathrm{Co}\{\mathbf{a}, \mathbf{b}, \mathbf{p}\}$. As mentioned, necessarily (3.*iiipi*) holds and Θ is a polynomial. Consider the functions

$$f := \Phi(\lambda^{\mathbf{p}}_{\mathbf{T}}), \qquad g^{(j)} := \Theta(\lambda^{\mathbf{p}}_{\mathbf{T}}) \cdot x^{[j]}_{\mathbf{p}}.$$

Along the edge $\mathrm{Co}\{\mathbf{a}, \mathbf{p}\}$, at the points $\mathbf{y}_t := (1-t)\mathbf{a} + t\mathbf{p}$ we have $\lambda^{\mathbf{p}}_{\mathbf{T}}(\mathbf{y}_t) = t$, $\lambda^{\mathbf{b}}_{\mathbf{T}}(\mathbf{y}_t) = 0$, $\lambda^{\mathbf{a}}_{\mathbf{T}}(\mathbf{y}_t) = [1 - \lambda^{\mathbf{a}}_{\mathbf{T}}(\mathbf{y}_t) - \lambda^{\mathbf{b}}_{\mathbf{T}}(\mathbf{y}_t) = 1 - t$. Observe that the functions $f, g^{(j)}$ suit the shape uniformity conditions because

$$\begin{aligned} f(\mathbf{y}_t) = \Phi(t), \quad g^{(j)}(\mathbf{y}_t) = \Theta(t)\big\langle \mathbf{e}^{[j]}\big|\mathbf{y}_t - \mathbf{p}\big\rangle = \Theta(t)(1-t)\big\langle \mathbf{e}^{[j]}\big|\mathbf{a} - \mathbf{p}\big\rangle =\\ = \big\langle \mathbf{e}^{[j]}\big|\mathbf{p} - \mathbf{a}\big\rangle\Psi(t) \end{aligned}$$

[†] Q vanishes in order ν at the point $[x_0, y_0]$ if $\frac{\partial^{k+m}}{\partial x^k \partial y^m}Q(x_0, y_0) = 0$ whenever $k+m < \nu$.

and since $f, g^{(j)}$ are polynomial multiples of $[\lambda^{\mathbf{p}}_{\mathbf{T}}]^2$. Also, since $\mathbf{y}_1 = \mathbf{p}$, $f(\mathbf{p}) = \Phi(\lambda^{\mathbf{p}}_{\mathbf{T}}(\mathbf{y}_1)) = \Phi(1) = 1$ and

$$\begin{aligned}\nabla f(\mathbf{p}) &= \Phi'\big(\lambda^{\mathbf{p}}_{\mathbf{T}}(\mathbf{y}_1)\big)\nabla\lambda^{\mathbf{p}}_{\mathbf{T}}(\mathbf{y}_1) = 0\cdot\nabla\lambda^{\mathbf{p}}_{\mathbf{T}}(\mathbf{y}_1) = \mathbf{0},\\ \nabla g^{(j)}(\mathbf{p}) &= \nabla_{\mathbf{x}=\mathbf{y}_1}\Big[\Theta\big(\lambda^{\mathbf{p}}_{\mathbf{T}}(\mathbf{x})\big)x^{[j]}_{\mathbf{p}}(\mathbf{x})\Big] =\\ &= x^{[j]}_{\mathbf{p}}(\mathbf{p})\nabla_{\mathbf{x}=\mathbf{y}_1}\Theta\big(\lambda^{\mathbf{p}}_{\mathbf{T}}(\mathbf{x})\big) + \Theta\big(\lambda^{\mathbf{p}}_{\mathbf{T}}(\mathbf{p})\big)\nabla_{\mathbf{x}=\mathbf{y}_1}x^{[j]}(\mathbf{p}) =\\ &= 0\cdot\Theta'(1)\nabla\lambda^{\mathbf{p}}_{\mathbf{T}}(\mathbf{y}_1) + \Theta(1)\mathbf{e}^{[j]} = \mathbf{e}^{[j]}.\end{aligned}$$

Therefore the difference functions $\varphi_{\mathbf{pT}} - f$ and $\psi^{(j)}_{\mathbf{p},\mathbf{T}} - g^{(j)}$ vanish on the edge $\mathrm{Co}\{\mathbf{a},\mathbf{p}\}$ of the triangle $\mathbf{T}$. Similar arguments with the points $\mathbf{z}_t := (1-t)\mathbf{b} + t\mathbf{p}$ show that $\varphi_{\mathbf{p},\mathbf{T}} - g$ and $\psi^{(j)}_{\mathbf{p},\mathbf{T}} - g^{(j)}$ vanish on $\mathrm{Co}\{\mathbf{b},\mathbf{p}\}$. By (1.7) their gradients also vanish on the edge $\mathrm{Co}\{\mathbf{a},\mathbf{b}\} = \big(\lambda^{\mathbf{p}}_{\mathbf{T}} = 0\big)$. Hence (cf. Remark 3.2) they are polynomial multiples of $[\lambda^{\mathbf{p}}_{\mathbf{T}}]^2\lambda^{\mathbf{a}}_{\mathbf{T}}\lambda^{\mathbf{b}}_{\mathbf{T}}$, say $\varphi_{\mathbf{pT}} = f + [\lambda^{\mathbf{p}}_{\mathbf{T}}]^2\lambda^{\mathbf{a}}_{\mathbf{T}}\lambda^{\mathbf{b}}_{\mathbf{T}}\Pi^{(0)}_{\mathbf{p},\mathbf{T}}$ and $\psi^{(j)}_{\mathbf{p},\mathbf{T}} = g^{(j)} + [\lambda^{\mathbf{p}}_{\mathbf{T}}]^2\lambda^{\mathbf{a}}_{\mathbf{T}}\lambda^{\mathbf{b}}_{\mathbf{T}}\Pi^{(j)}_{\mathbf{p},\mathbf{T}}$, respectively. Since $\lambda^{\mathbf{a}}_{\mathbf{T}}, \lambda^{\mathbf{b}}_{\mathbf{T}}$ are linearly independent affine functionals, the mapping $\Lambda^{\mathbf{p}}_{\mathbf{a},\mathbf{b}} : \mathbf{x} \mapsto \big[\lambda^{\mathbf{b}}_{\mathbf{T}}(\mathbf{x}), \lambda^{\mathbf{a}}_{\mathbf{T}}(\mathbf{x})\big]$ is an affine coordinatization on the plain $\mathbb{R}^2$. Thus we can express each term $\Pi^{(i)}_{\mathbf{p},\mathbf{T}}$ as a polynomial of the coordinates $\Lambda^{\mathbf{p}}_{\mathbf{a},\mathbf{b}}$ which completes the proof.

Notation 3.5. For later convenience, without danger of confusion, we introduce the unifying context-free notations

$$\lambda^{\mathbf{p}}_{\mathbf{a},\mathbf{b}} := \lambda^{\mathbf{p}}_{\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}}, \quad f^{0,\mathbf{p}}_{\mathbf{a},\mathbf{b}} := \varphi_{\mathbf{p},\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}}, \quad f^{j,\mathbf{p}}_{\mathbf{a},\mathbf{b}} := \psi^{(j)}_{\mathbf{p},\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}} \quad (j=1,2).$$

Furthermore, in view of Lemma 3.4, we shall write

$$f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}} = \Phi^{[i]}(\lambda^{\mathbf{p}}_{\mathbf{a},\mathbf{b}})x^{[i]}_{\mathbf{p}} + [\lambda^{\mathbf{p}}_{\mathbf{a},\mathbf{b}}]^2\lambda^{\mathbf{b}}_{\mathbf{a},\mathbf{p}}\lambda^{\mathbf{a}}_{\mathbf{b},\mathbf{p}}P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(\lambda^{\mathbf{b}}_{\mathbf{a},\mathbf{p}}, \lambda^{\mathbf{a}}_{\mathbf{b},\mathbf{p}}) \quad (i=0,1,2) \tag{3.6}$$

where

$$\Phi^{[0]} := \Phi, \quad \Phi^{[1]} := \Phi^{[2]} := \Theta, \quad x^{[0]}_{\mathbf{p}} : \mathbf{x} \mapsto 1 \ \text{ with } \ \mathbf{e}^{[0]} := \nabla x^{[0]}_{\mathbf{p}} = \mathbf{0}$$

and the terms $P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}$ $(i=0,1,2)$ are polynomials with coefficients depending on the ordered tuple $(i,\mathbf{p},\mathbf{a},\mathbf{b})$. Notice that necessarily

$$P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(s,t) \equiv P^{i,\mathbf{p}}_{\mathbf{b},\mathbf{a}}(t,s) \tag{3.7}$$

due to the trivial index symmetries $\lambda^{\mathbf{w}}_{\mathbf{u},\mathbf{v}} \equiv \lambda^{\mathbf{w}}_{\mathbf{v},\mathbf{u}}$ and $f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}} \equiv f^{i,\mathbf{p}}_{\mathbf{b},\mathbf{a}}$.

Lemma 3.8. *We have $f_{\mathcal{T},F} \in \mathcal{C}^1\big(\mathrm{Dom}(\mathcal{T})\big)$ for every triangular mesh with arbitrary gradient data if and only if*

$$\mathbf{b} \mapsto \nabla f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(\mathbf{y}) \equiv \mathrm{const}_{\mathbf{p},\mathbf{a},\mathbf{y}} \quad \text{for fixed } \mathbf{p} \neq \mathbf{a} \text{ and } \mathbf{y} \in \mathrm{Co}\{\mathbf{p},\mathbf{a}\}. \quad (NUM)$$

Proof. Given any non-degenerate triangle $\mathbf{T} = \mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}$, By construction, for the points $\mathbf{x}_t := (1-t)\mathbf{a} + t\mathbf{b}$, $\mathbf{y}_t := (1-t)\mathbf{a} + t\mathbf{p}$ and $\mathbf{z}_t := (1-t)\mathbf{b} + t\mathbf{p}$ on the edges of the triangle $\mathbf{T}$ we have $f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(\mathbf{x}_t) = 0$ independently of $\mathbf{p}$, $f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(\mathbf{y}_t) = \Phi^{[i]}(t)$ independently of $\mathbf{b}$ and $f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(\mathbf{z}_t) = \Phi^{[i]}(t)$ independently of $\mathbf{a}$. Thus the shape conditions are automatic from (3.6). Moreover, given any triangle $\widetilde{\mathbf{T}}$ with a common edge but disjoint interior to $\mathbf{T}$, the functions pairs $\varphi_{\mathbf{p},\mathbf{T}}, \varphi_{\mathbf{p},\widetilde{\mathbf{T}}}$ resp. $\psi^{(j)}_{\mathbf{p},\mathbf{T}}, \psi^{(j)}_{\mathbf{p},\widetilde{\mathbf{T}}}$ touch continuosly. The analogous necessary and sufficent condition for a $\mathcal{C}^1$-smooth touching is that the gradient pairs $\nabla\varphi_{\mathbf{p},\mathbf{T}}, \nabla\varphi_{\mathbf{p},\widetilde{\mathbf{T}}}$ resp. $\nabla\psi^{(j)}_{\mathbf{p},\mathbf{T}}, \nabla\psi^{(j)}_{\mathbf{p},\widetilde{\mathbf{T}}}$ coincide on the common edge:

$$\begin{aligned}
&\text{(i)} \quad \nabla f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(\mathbf{x}) = \nabla f^{i,\widetilde{\mathbf{p}}}_{\mathbf{a},\mathbf{b}}(\mathbf{x}) && \text{if } \mathbf{x} \in \mathrm{Co}\{\mathbf{a},\mathbf{b}\} = \mathbf{T} \cap \mathrm{Co}\{\mathbf{a},\mathbf{b},\widetilde{\mathbf{p}}\},\\
&\text{(ii)} \quad \nabla f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(\mathbf{y}) = \nabla f^{i,\widetilde{\mathbf{b}}}_{\mathbf{a},\mathbf{p}}(\mathbf{y}) && \text{if } \mathbf{y} \in \mathrm{Co}\{\mathbf{a},\mathbf{p}\} = \mathbf{T} \cap \mathrm{Co}\{\mathbf{a},\widetilde{\mathbf{b}},\mathbf{p}\},\\
&\text{(iii)} \quad \nabla f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(\mathbf{z}) = \nabla f^{i,\widetilde{\mathbf{a}}}_{\mathbf{b},\mathbf{p}}(\mathbf{z}) && \text{if } \mathbf{z} \in \mathrm{Co}\{\mathbf{b},\mathbf{p}\} = \mathbf{T} \cap \mathrm{Co}\{\widetilde{\mathbf{a}},\mathbf{b},\mathbf{p}\}.
\end{aligned} \quad (3.10)$$

Observe that (3.10(i)) holds automatically with the trivial value $\mathbf{0}$. Furthermore conditions (3.10(i)) and (3.10(ii)) are equivalent (by changing the roles of $\mathbf{a}$ and $\mathbf{b}$). Finally we observe that, in (3.iiipxiii(i)), for fixed $\mathbf{a},\mathbf{p}$ and $\mathbf{y} \in \mathrm{Co}\{\mathbf{a},\mathbf{p}\}$ we can choose the points $\mathbf{b}$ and $\widetilde{\mathbf{b}}$ on different half plain components of $\mathbb{R}^2 \backslash \mathrm{Line}\{\mathbf{a},\mathbf{p}\}$ arbitrarily. This implies that all the vectors $\nabla f^{i,\mathbf{b}}_{\mathbf{a},\mathbf{p}}(\mathbf{y}), \nabla f^{i,\widetilde{\mathbf{b}}}_{\mathbf{a},\mathbf{p}}(\mathbf{y})$ with $\mathbf{b},\widetilde{\mathbf{b}} \in \mathbb{R}^2 \backslash \mathrm{Line}\{\mathbf{a},\mathbf{p}\}$ must be the same. Due to the construction (1.3), the fact that all the pairs $\varphi_{\mathbf{p},\mathbf{T}}, \varphi_{\mathbf{p},\widetilde{\mathbf{T}}}$ resp. $\psi^{(j)}_{\mathbf{p},\mathbf{T}}, \psi^{(j)}_{\mathbf{p},\widetilde{\mathbf{T}}}$ of basic functions touch $\mathcal{C}^1$-smoothly in case of adjacent triangles $\mathbf{T},\widetilde{\mathbf{T}}$, ensures that the splines $f_{\mathcal{T},F}$ are all $\mathcal{C}^1$-smooth as well.

Notation 3.11. Given any ordered triple $(\mathbf{u},\mathbf{v},\mathbf{w})$ of non-collinear points, we shall write $\mathbf{g}^{\mathbf{w}}_{\mathbf{u},\mathbf{v}} := \nabla\lambda^{\mathbf{w}}_{\mathbf{u},\mathbf{v}}$ for the constant gradient vectors of the baricentric coordinate functions. Notice that, by (2.1),

$$\mathbf{g}^{\mathbf{w}}_{\mathbf{u},\mathbf{v}} := \frac{(\mathbf{u}-\mathbf{v})\mathbf{R}}{\big\langle (\mathbf{u}-\mathbf{v})\mathbf{R} \big| \mathbf{w}-\mathbf{u} \big\rangle} = \frac{\sigma^{\mathbf{w}}_{\mathbf{u},\mathbf{v}}(\mathbf{u}-\mathbf{v})\mathbf{R}}{\mathrm{area}(\mathrm{Co}\{\mathbf{u},\mathbf{v},\mathbf{w}\})}. \quad (3.12)$$

where $\sigma^{\mathbf{w}}_{\mathbf{u},\mathbf{v}} = \pm 1$ according as $(\mathbf{u},\mathbf{v},\mathbf{w})$ are oriented anticlockwise or clockwise. In particular, if $\mathbf{T} = \mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}$ is a non-degenerate triangle,

we have

$$\mathbf{g}^{\mathbf{p}}_{\mathbf{a},\mathbf{b}} + \mathbf{g}^{\mathbf{a}}_{\mathbf{b},\mathbf{p}} + \mathbf{g}^{\mathbf{b}}_{\mathbf{a},\mathbf{p}} = \nabla\big[\lambda^{\mathbf{p}}_{\mathbf{a},\mathbf{b}} + \lambda^{\mathbf{a}}_{\mathbf{b},\mathbf{p}} + \lambda^{\mathbf{b}}_{\mathbf{a},\mathbf{p}}\big] = \nabla 1 = \mathbf{0},$$
$$\mathbf{g}^{\mathbf{p}}_{\mathbf{a},\mathbf{b}} \perp \mathbf{b}-\mathbf{a}, \quad \mathbf{g}^{\mathbf{b}}_{\mathbf{a},\mathbf{p}} \perp \mathbf{a}-\mathbf{p}, \quad \mathbf{g}^{\mathbf{a}}_{\mathbf{b},\mathbf{p}} \perp \mathbf{b}-\mathbf{p}.$$

Lemma 3.13. *If* $\mathbf{T} = \mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}$ *is a non-degenerate triangle, at the points* $\mathbf{y}_t := (1-t)\mathbf{a} + t\mathbf{p}$ *of the edge* $\mathrm{Co}\{\mathbf{a},\mathbf{p}\}$ *we have*

$$\begin{aligned}\nabla f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(\mathbf{y}_t) =& x^{[i]}\big((1-t)(\mathbf{a}-\mathbf{p})\big)[\Phi^{[i]}]'(t)\,\mathbf{g}^{\mathbf{p}}_{\mathbf{a},\mathbf{b}} + \\ &+ \Phi^{[i]}(t)\mathbf{e}^{[i]} + t^2(1-t)P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(0,1-t)\,\mathbf{g}^{\mathbf{b}}_{\mathbf{a},\mathbf{p}}.\end{aligned} \tag{3.14}$$

Proof. With the abbreviations

$$\ell_0 := \lambda^{\mathbf{p}}_{\mathbf{a},\mathbf{b}},\ \ell_1 := \lambda^{\mathbf{b}}_{\mathbf{a},\mathbf{p}},\ \ell_2 := \lambda^{\mathbf{a}}_{\mathbf{b},\mathbf{p}}, \quad P^{[i]} := P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}, \quad G^{[i]} := \ell_0^2\ell_2 P^{[i]}(\ell_1,\ell_2)$$

we can write

$$\begin{aligned}\nabla f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}} &= \nabla\Big[x^{[i]}_{\mathbf{p}}\Phi^{[i]}(\ell_0) + \ell_1 G^{[i]}\Big] = \\ &= x^{[i]}_{\mathbf{p}}\,\nabla\big[\Phi^{[i]}(\ell_0)\big] + \Phi^{[i]}(\ell_0)\nabla x^{[i]}_{\mathbf{p}} + \ell_1\nabla G^{[i]} + G^{[i]}\nabla\ell_1 = \\ &= x^{[i]}_{\mathbf{p}}\Phi^{[i]\prime}(\ell_0)\nabla\ell_0 + \Phi^{[i]}(\ell_0)\mathbf{e}^{[i]} + \ell_1\nabla G^{[i]} + G^{[i]}\nabla\ell_1.\end{aligned}$$

We complete the proof with the observations that

$$\ell_0(\mathbf{y}_t) = t, \quad \ell_1(\mathbf{y}_t) = 0, \quad \ell_2(\mathbf{y}_t) = 1-t,$$
$$x^{[i]}_{\mathbf{p}}(\mathbf{y}_t) = x^{[i]}\big((1-t)(\mathbf{a}-\mathbf{p})\big), \quad \nabla x^{[i]}_{\mathbf{p}} \equiv \mathbf{e}^{[i]}.$$

Remark 3.15. To prove Theorem 2.4, we need a precise description for the coefficients of the polynomials $P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}$ in terms of the variables $\mathbf{a},\mathbf{b},\mathbf{p}$ such that (3.9) should hold.

According to Lemma 3.8, the procedure $\mathfrak{S} : (\mathcal{T},F) \mapsto f_{\mathcal{T},F}$ produces $\mathcal{C}^1$-splines for every admissible data if and only if, for any $t\in[0,1]$ and for any fixed pair $\mathbf{a},\mathbf{p}$ of distinct points, the gradient expressions (3.14) are independent of the variable $\mathbf{b}$ ranging in $\mathbb{R}^2\setminus\mathrm{Line}\{\mathbf{a},\mathbf{p}\}$. This latter condition can be formulated in terms of the $\mathbf{b}$-independent affine coordinates (2.2) as follows. By (3.12) we have

$$\mathbf{g}^{\mathbf{b}}_{\mathbf{a},\mathbf{p}} = \frac{(\mathbf{a}-\mathbf{p})\mathbf{R}}{\langle(\mathbf{a}-\mathbf{p})\mathbf{R}|\mathbf{b}-\mathbf{a}\rangle} = \|\mathbf{p}-\mathbf{a}\|^{-2}(1/\overline{\xi}^{\,\mathbf{b}}_{\mathbf{p},\mathbf{a}})(\mathbf{p}-\mathbf{a})\mathbf{R},$$
$$\mathbf{g}^{\mathbf{p}}_{\mathbf{a},\mathbf{b}} = \frac{(\mathbf{a}-\mathbf{b})\mathbf{R}}{\langle(\mathbf{a}-\mathbf{b})\mathbf{R}|\mathbf{p}-\mathbf{a}\rangle} = \frac{\xi^{(\mathbf{a}-\mathbf{b})\mathbf{R}}_{\mathbf{p},\mathbf{a}}(\mathbf{p}-\mathbf{a}) + \overline{\xi}^{(\mathbf{a}-\mathbf{b})\mathbf{R}}_{\mathbf{p},\mathbf{a}}(\mathbf{p}-\mathbf{a})\mathbf{R}}{\|\mathbf{p}-\mathbf{a}\|^2\overline{\xi}^{\mathbf{b}}_{\mathbf{p},\mathbf{a}}} =$$
$$= \|\mathbf{p}-\mathbf{a}\|^{-2}\Big[(\mathbf{p}-\mathbf{a}) + (\xi^{\mathbf{b}}_{\mathbf{p},\mathbf{a}}/\overline{\xi}^{\,\mathbf{b}}_{\mathbf{p},\mathbf{a}})(\mathbf{p}-\mathbf{a})\mathbf{R}\Big].$$

Thus we can rewrite (3.14) in the form

$$\nabla f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(\mathbf{y}_t) = \Big[\mathbf{b}\text{-independent terms}\Big] + \\ + \frac{x^{[i]}\big((1-t)(\mathbf{a}-\mathbf{p})\big)[\Phi^{[i]}]'(t)\xi^{\mathbf{b}}_{\mathbf{p},\mathbf{a}} + t^2(1-t)P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(0,1-t)}{\|\mathbf{p}-\mathbf{a}\|^2\overline{\xi}^{\ \mathbf{b}}_{\mathbf{p},\mathbf{a}}}(\mathbf{p}-\mathbf{a})\mathbf{R}. \tag{3.16}$$

Hence we conclude immediately the following.

Lemma 3.17. *We have* (3.9) *if and only if for every pair* $\mathbf{p},\mathbf{a}$ *of distinct points there exist polynomials* $K^{i,\mathbf{p}}_{\mathbf{a}}$ $(i=0,1,2)$ *of one variable such that*

$$K^{i,\mathbf{p}}_{\mathbf{a}}(t) = x^{[i]}\big((1-t)(\mathbf{a}-\mathbf{p})\big)[\Phi^{[i]}]'(t)\frac{\xi^{\mathbf{b}}_{\mathbf{p},\mathbf{a}}}{\overline{\xi}^{\ \mathbf{b}}_{\mathbf{p},\mathbf{a}}} + \frac{t^2(1-t)}{\overline{\xi}^{\ \mathbf{b}}_{\mathbf{p},\mathbf{a}}}P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(0,1-t) \tag{3.18}$$

independently of the choice of $\mathbf{b}$ *outside* $\mathrm{Line}\{\mathbf{a},\mathbf{p}\}$.

We can regard (3.18) as a partial algebraic condition on the polynomials $P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}$ of two variables as

$$P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(0,1-t) = \overline{\xi}^{\ \mathbf{b}}_{\mathbf{p},\mathbf{a}}\frac{K^{i,\mathbf{p}}_{\mathbf{a}}(t)}{t^2(1-t)} - \\ - \xi^{\ \mathbf{b}}_{\mathbf{p},\mathbf{a}}\frac{x^{[i]}\big((1-t)(\mathbf{a}-\mathbf{p})\big)[\Phi^{[i]}]'(t)}{t^2(1-t)} \qquad (0<t<1). \tag{3.19}$$

Since, for fixed $\mathbf{a},\mathbf{p}$, the coordinates $\big(\xi^{\ \mathbf{b}}_{\mathbf{p},\mathbf{a}},\overline{\xi}^{\ \mathbf{b}}_{\mathbf{p},\mathbf{a}}\big)$ may assume arbitrary values (r,s) with $s\neq 0$, from (3.19) we obtain the polynomial divisability relations $t^2(1-t)\big|K^{i,\mathbf{p}}_{\mathbf{a}}(t)$ and $t^2(1-t)\big|x^{[i]}\big((1-t)(\mathbf{a}-\mathbf{p})\big)[\Phi^{[i]}]'(t)$. Since $x^{[0]}\big((1-t)(\mathbf{a}-\mathbf{p})\big)\equiv 1$ and $x^{[0]}\big((1-t)(\mathbf{a}-\mathbf{p})\big)\equiv(1-t)x^{[j]}(\mathbf{a}-\mathbf{p})$ for $j=1,2$, with the aid of (3.22′) we can state (3.19) in the form

$$P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(0,1\text{–}t) = \frac{\big\langle\mathbf{b}-\mathbf{a}\big|\mathbf{p}-\mathbf{a}\big\rangle}{\|\mathbf{p}-\mathbf{a}\|^2}\,x^{[i]}_{\mathbf{p}}(\mathbf{a})\,\chi^{[i]}(t) + \frac{\big\langle\mathbf{b}-\mathbf{a}\big|(\mathbf{p}-\mathbf{a})\mathbf{R}\big\rangle}{\|\mathbf{p}-\mathbf{a}\|^2}\,\kappa^{i,\mathbf{p}}_{\mathbf{a}}(t) \tag{3.20}$$

with suitable polynomials $\chi^{[i]}$ and $\kappa^{i,\mathbf{p}}_{\mathbf{a}}$ $(i=0,1,2;\ \mathbf{a}\neq\mathbf{p}\in\mathbb{R}^2)$ of one variable. Actually

$$\kappa^{i,\mathbf{p}}_{\mathbf{a}}(t) = \frac{K^{i,\mathbf{p}}_{\mathbf{a}}(t)}{t^2(1-t)^2},\quad \chi^{[0]}(t) = \frac{[\Phi^{[0]}]'(t)}{t^2(1-t)} = \frac{\Phi'(t)}{t^2(1-t)},$$
$$x^{[i]}_{\mathbf{p}}(\mathbf{a})\chi^{[j]}(t) = \frac{x^{[j]}\big((1-t)(\mathbf{a}-\mathbf{p})\big)[\Phi^{[j]}]'(t)}{t^2(1-t)} = x^{[j]}_{\mathbf{p}}(\mathbf{a})\frac{[\Psi(t)/(t-1)]'}{t^2}$$

for $j{=}1,2$ on the basis of (3.19) In terms of the Kronecker-δ, we can write even

$$\chi^{[i]}(t) = t^{-2}(1-t)^{-\delta_{i,0}}[\Phi^{[i]}]'(t) \qquad (i=0,1,2).$$

Clearly, the polynomials $K^{i,\mathbf{p}}_{\mathbf{c}}$ cannot be chosen arbitrarily. There is a unique obstacle: we obtained Lemma 3.13 and hence (3.18) by an inspection of $\nabla f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}$ on one of the edges of a triangle $\mathbf{T} = \mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}$ at the distinguished point $\mathbf{p}$ (namely $\mathrm{Co}\{\mathbf{a},\mathbf{p}\}$ with the parametrization $\mathbf{y}_t := (1-t)\mathbf{a}+t\mathbf{b}$) while also the analogous conclusion should also be taken simultaneously in to account with the second edge (namely $\mathrm{Co}\{\mathbf{b},\mathbf{p}\}$ issued from $\mathbf{p}$. Applying a change $\mathbf{a}\leftrightarrow\mathbf{b}$ and taking into account the symmetry (3.7), we see that also

$$P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(1{-}t,0){=}\frac{\langle\mathbf{a}{-}\mathbf{b}|\mathbf{p}{-}\mathbf{b}\rangle}{\|\mathbf{p}{-}\mathbf{b}\|^2}x^{[i]}_{\mathbf{p}}(\mathbf{b})\ \chi^{[i]}(t){+}\frac{\langle\mathbf{a}{-}\mathbf{b}|(\mathbf{p}{-}\mathbf{b})\mathbf{R}\rangle}{\|\mathbf{p}{-}\mathbf{b}\|^2}\ \kappa^{i,\mathbf{p}}_{\mathbf{b}}(t). \quad (3.21)$$

We obtain the complete description for the families of polynomials $K^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}$ being admissible by Lemma 3.17 by the next obervation.

Lemma 3.22. *For any pair $\mathbf{p}\neq\mathbf{c}\in\mathbb{R}^2$, in (3.20−21) we have $\chi^{[i]}(1) = \kappa^{i,\mathbf{p}}_{\mathbf{c}}(1) = 0$.*

Proof. Fix $i,\mathbf{p}\in\mathbb{R}^2$ and $\rho>0$ arbitrarily. Consider (3.20−21) for pairs $\mathbf{a},\mathbf{b}$ with $\|\mathbf{a}-\mathbf{p}\| = \|\mathbf{b}-\mathbf{p}\| = \rho$ written in the form

$$\mathbf{a} := \mathbf{c}_\sigma,\quad \mathbf{b} := \mathbf{c}_\tau \quad \text{where} \quad \mathbf{c}_\tau := \mathbf{p}+\rho\mathbf{u}_\tau,\ \mathbf{u}_\tau := \cos\tau\mathbf{e}^{[1]}+\sin\tau\mathbf{e}^{[2]}.$$

Due to (3.7), with the abbreviations $\alpha := \chi^{[i]}(1)$ and $\beta(\tau) := \kappa^{i,\mathbf{p}}_{\mathbf{c}_\tau}(1)$ we get

$$\begin{aligned}
0 &= P^{i,\mathbf{p}}_{\mathbf{c}_\sigma,\mathbf{c}_\tau}(0,0) - P^{i,\mathbf{p}}_{\mathbf{c}_\tau,\mathbf{c}_\sigma}(0,0) = \\
&= \Big[\big(\langle\mathbf{u}_\tau|\mathbf{u}_\sigma\rangle{-}1\big)x^{[i]}(\mathbf{c}_\sigma)\alpha{+}\langle\mathbf{u}_\tau|\mathbf{u}_\sigma\mathbf{R}\rangle\beta(\sigma)\Big]- \\
&\quad - \Big[\big(\langle\mathbf{u}_\sigma|\mathbf{u}_\tau\rangle{-}1\big)x^{[i]}(\mathbf{c}_\tau)\alpha{+}\langle\mathbf{u}_\sigma|\mathbf{u}_\tau\mathbf{R}\rangle\beta(\tau)\Big] = \\
&= (\langle\mathbf{u}_\sigma|\mathbf{u}_\tau\rangle{-}1)\big[x^{[i]}(\mathbf{c}_\sigma) - x^{[i]}(\mathbf{c}_\tau)\big]\alpha + \langle\mathbf{u}_\tau|\mathbf{u}_\sigma\mathbf{R}\rangle\beta(\sigma) - \langle\mathbf{u}_\sigma|\mathbf{u}_\tau\mathbf{R}\rangle\beta(\tau)\big) = \\
&= \big[\cos(\sigma-\tau)-1\big]\big[x^{[i]}(\mathbf{c}_\sigma) - x^{[i]}(\mathbf{c}_\tau)\big]\alpha + \sin(\tau-\sigma)\big[\beta(\sigma)+\beta(\tau)\big].
\end{aligned}$$

Since $x^{[0]}\equiv 1$, in any case we have $x^{[i]}_{\mathbf{p}}(\mathbf{c}_\sigma){-}x^{[i]}_{\mathbf{p}}(\mathbf{c}_\tau) = \rho[x^{[i]}(\mathbf{u}_\sigma){-}x^{[i]}(\mathbf{u}_\tau)]$. It follows

$$\begin{aligned}
\beta(\sigma)+\beta(\tau) &= \alpha\rho\frac{\cos(\tau-\sigma)-1}{\sin(\tau-\sigma)}\big[x^{[i]}(\mathbf{u}_\sigma) - x^{[i]}(\mathbf{u}_\tau)\big], \\
\big|\beta(\sigma)+\beta(\tau)\big| &\le \rho|\alpha|\frac{1-\cos(\tau-\sigma)}{\sin(|\tau-\sigma|)}\big\|\mathbf{u}_\sigma-\mathbf{u}_\tau\big\| \le \\
&\le 2\rho|\alpha|[1-\cos(\tau-\sigma)].
\end{aligned} \qquad (3.23)$$

Suppose indirectly $\beta(\tau)\neq 0$ for some $\tau\in\mathbb{R}$. Let $\varepsilon:=|\beta(\tau)|$ and choose $\delta>0$ such that $2\rho|\alpha|(1-\cos\theta)<\varepsilon/4$ whenever $|\theta|\leq\varepsilon$. Then we have $|\beta(\tau)+\beta(\tau\pm\delta/2)|<\varepsilon/4$ that is $\beta(\tau\pm\delta/2)\in\big[-\varepsilon/4,\varepsilon/4\big]-\beta(\tau)$. Therefore $\beta(\tau+\delta/2)+\beta(\tau-\delta/2)\in\big[-\varepsilon/2,\varepsilon/2\big]-2\beta(\tau)\subset\big[-\varepsilon/2,\varepsilon/2\big]+\{-2\varepsilon,2\varepsilon\}=\big[-5\varepsilon/2,-3\varepsilon/2\big]\cup\big[3\varepsilon/2,5\varepsilon/2\big]$ i.e. $|\beta(\tau+\delta/2)+\beta(\tau-\delta/2)|\in\big[3\varepsilon/2,5\varepsilon/2\big]$ However, we also have $|\beta(\tau+\delta/2)+\beta(\tau-\delta/2)|<\varepsilon/4$ which leads to the contradiction $|\beta(\tau+\delta/2)+\beta(\tau-\delta/2)|\in\big[3\varepsilon/2,5\varepsilon/2\big]\cap\big[0,\varepsilon/4\big]=\emptyset$. By the arbitrariness of the radius ρ, the angle τ and the origin $\mathbf{p}$, we conclude that $\kappa^{i,\mathbf{p}}_{\mathbf{c}}(1)=0$ in any case.

For $i=1,2$ we get $\alpha=0$ i.e. $\chi^{[i]}(1)=0$ immediately by plugging $\beta(\tau)=\beta(\sigma)=0$ with $\sigma:=\tau+\pi/4$ in the first equation of (3.23). (3.29). (Remark: $x^{[0]}(\mathbf{u}_\sigma)-x^{[0]}(\mathbf{u}_\tau)=1-1=0$, thus the argument does not work for $i=0$). In the case $i=0$ we conclude $\alpha=0$ as follows. Consider the difference of equations $(3.20-21)$ for $t=1$ with $\mathbf{a}:=\mathbf{p}+\mathbf{e}^{[1]}$ and $\mathbf{b}:=\mathbf{p}+\mathbf{e}^{[1]}+\mathbf{e}^{[2]}$. Since $\kappa^{i,\mathbf{p}}_{\mathbf{c}}=0$ ($\mathbf{c}=\mathbf{a},\mathbf{b}$ is estabished already, we get simply $0=-(1/2)\chi^{[0]}(1)$ which completes the proof.

Corollary 3.24. *The relations* (3.9) *hold if and only if we have* (3.18) *with the symmetry* (3.7) *where the polynomials* $K^{i,\mathbf{p}}_{\mathbf{c}}(t)$ *respectively* $x^{[i]}\big((1-t)(\mathbf{a}-\mathbf{p})\big)[\Phi^{[i]}]'(t)$ *are all divisable by* $t^2(1-t)^2$.

Proof. The relation $\kappa^{i,\mathbf{p}}_{\mathbf{c}}(1)=0$ implies that there is a polynomial $\widetilde{\kappa}^{i,\mathbf{p}}_{\mathbf{c}}$ such that $\kappa^{i,\mathbf{p}}_{\mathbf{c}}(t)=(1-t)\widetilde{\kappa}^{i,\mathbf{p}}_{\mathbf{c}}(t)$ and $K^{i,\mathbf{p}}_{\mathbf{c}}(t)=t^2(1-t)\kappa^{i,\mathbf{p}}_{\mathbf{c}}=t^2(1-t)^2\widetilde{\kappa}^{i,\mathbf{p}}_{\mathbf{c}}(t)$ with some polynomial. Similarly, from $\chi^{[i]}(1)=0$ we conclude that $\chi^{[i]}(t)=(1-t)\widetilde{\chi}^{[i]}(t)$ and $(1-t)x^{[i]}(\mathbf{a})[\Phi^{[i]}]'(t)=((1-t)t^2\chi^{[i]}(t)=t^2(1-t)^2\widetilde{\chi}^{[i]}(t)$ with some polynomial $\widetilde{\chi}^{[i]}$.

Corollary 3.25. *We can write* $K^{i,\mathbf{p}}_{\mathbf{c}}(t)=t^2(1-t)^2k^{i,\mathbf{p}}_{\mathbf{c}}(t)$ ($\mathbf{p}\neq\mathbf{c}\in\mathbb{R}^2$) *and the admissible shape functions* Φ,Ψ *have the form*

$$\begin{aligned}&\text{(i)}\quad \Phi(t)=t^3(10-15t+6t^2)+t^3(1-t)^3\Phi_1(t),\\ &\text{(ii)}\quad \Psi(t)=t^3(t-1)(4-3t)+t^3(1-t)^3\Psi_1(t)\end{aligned}\tag{3.26}$$

with suitable polynomials $k^{i,\mathbf{p}}_{\mathbf{c}},\Phi_2,\Psi_2$.

Proof. The stated form of $K^{i,\mathbf{p}}_{\mathbf{c}}$ is clear from 3.24). By definition $\Phi^{[0]}(t)=\Phi(t)$ and $x^{[0]}\big((1-t)(\mathbf{a}-\mathbf{p})\big)\equiv 1$. Furthermore $\Phi^{[j]}(t)=\Psi(t)/(t-1)$ and $x^{[j]}\big((1-t)(\mathbf{a}-\mathbf{p})\big)=(1-t)x^{[j]}_{\mathbf{p}}(\mathbf{a})$ for $j=1,2$. Thus, taking (3.1) into acount, the relation that $t^2(1-t)^2$ is a divisor of $x^{[0]}\big((1-t)(\mathbf{a}-\mathbf{p})\big)[\Phi^{[0]}]'(t)=\Phi'(t)=6t(1-t)+2t(1-t)(1-2t)\Phi_0(t)+t^2(1-t)^2\Phi_0'(t)$ means simply that $t(1-t)\big|6+2(1-$

$2t)\Phi_0(t)$ i.e. $6+2(1-2t)\Phi_0(t)|_{t=0,1}=0$ implying $\Phi_0(0)=-3$, $\Phi_0(1)=3$. Therefore $\Phi_0(t)=-3+6t+t(1-t)\Phi_1(t)$ with a polynomial Φ_1 and the generic form of Φ is $\big(3.1(\mathrm{i})\big)$. Also according to (3.1) in the cases $j=1,2$ we can write $\Psi(t)=-t^2(1-t)+t^2(1-t)^2\Psi_0(t)$ with some polynomial Ψ_0. Thus the relation that $t^2(1-t)^2$ is a divisor of $x^{[j]}\big((1-t)(\mathbf{a}-\mathbf{p})\big)[\Phi^{[0]}]'(t)\equiv(1-t)x^{[j]}_{\mathbf{p}}(\mathbf{a})\big[\Psi(t)/(1-t)\big]'$ means that $t^2(1-t)\big|\big[\Psi(t)/(1-t)\big]'\equiv -2t+t(2-3t)\Psi_0(t)+t^2(1-t)^2\Psi_0'(t)$ is equivalent to saying $t(1-t)\big|-2+(2-3t)\Psi_0(t)$ i.e. $-2+(2-3t)\Psi_0(t)|_{t=0,1}=0$ implying $\Psi_0(0)=1$ and $\Psi_0(1)=-2$. Therefore $\Psi_0(t)=1-3t+t^2(1-t)\Psi_1(t)$ with some polynomial Ψ_1 and the generic form of Ψ is $\big(3.26(\mathrm{i})\big)$.

3.27. Finish of the proof of Theorem 2.4

In view of (3.21−21) and Remark 3.2(ii) we can write

$$\begin{aligned}
P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(s,t) &= P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(0,0)+s\big[\big(P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(s,0)-P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(0,0)\big)/s\big]+\\
&\qquad +t\big[\big(P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(0,t)-P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(0,0)\big)/t\big]+st\mathrm{Pol}(s,t)=\\
&= s\big[\big(P^{i,\mathbf{p}}_{\mathbf{b},\mathbf{a}}(0,s)/s\big]+t\big[\big(P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(0,t)/t\big]+st\mathrm{Pol}(s,t)=\\
&= s\left[\bar{\xi}^{\,\mathbf{a}}_{\mathbf{p},\mathbf{b}}\frac{K^{i,\mathbf{p}}_{\mathbf{b}}(1-s)}{s^2(1-s)^2}-\xi^{\,\mathbf{a}}_{\mathbf{p},\mathbf{b}}\frac{x^{[i]}(\mathbf{b})\big[\Phi^{[i]}\big]'(1-s)}{s(1-s)^2}\right]+\\
&\qquad +t\left[\bar{\xi}^{\,\mathbf{b}}_{\mathbf{p},\mathbf{a}}\frac{K^{i,\mathbf{p}}_{\mathbf{a}}(1-t)}{t^2(1-t)^2}-\xi^{\,\mathbf{b}}_{\mathbf{p},\mathbf{a}}\frac{x^{[i]}(\mathbf{a})\big[\Phi^{[i]}\big]'(1-t)}{t(1-t)^2}\right]+stR^{\mathbf{p}}_{\mathbf{a},\mathbf{b}}(s,t)
\end{aligned}$$

with suitable polynomials $K^{i,\mathbf{p}}_{\mathbf{c}},\Phi^{[i]},R^{\mathbf{p}}_{\mathbf{a},\mathbf{b}}$ of one- resp. two variables such that $t^2(1-t)^2\big|K^{i,\mathbf{p}}_{\mathbf{c}}(t)$ and $t^2(1-t)\big|x^{[i]}_{\mathbf{p}}(\mathbf{a})[\Phi^{[i]}]'(t)$. It is straightforward to check that the functions $f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}$ are polynomials in these cases and $P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(s,t)=P^{i,\mathbf{p}}_{\mathbf{b},\mathbf{a}}(t,s)$ if and only if $R^{\mathbf{p}}_{\mathbf{a},\mathbf{b}}(s,t)=R^{\mathbf{p}}_{\mathbf{b},\mathbf{a}}(t,s)$. It remains to show that the expressions

$$\begin{aligned}
&\nabla f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(\mathbf{y}_t)\quad\text{with}\quad \mathbf{y}_t:=(1-t)\mathbf{a}+t\mathbf{p},\\
&f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}=\Phi^{[i]}(\lambda^{\mathbf{p}}_{\mathbf{a},\mathbf{b}})+[\lambda^{\mathbf{p}}_{\mathbf{a},\mathbf{b}}]^2\lambda^{\mathbf{b}}_{\mathbf{a},\mathbf{p}}\lambda^{\mathbf{a}}_{\mathbf{b},\mathbf{p}}P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(\lambda^{\mathbf{b}}_{\mathbf{a},\mathbf{b}},\lambda^{\mathbf{a}}_{\mathbf{b},\mathbf{p}})
\end{aligned}$$

are independent of the term $\mathbf{b}$ whenever

$$\begin{aligned}
&K^{i,\mathbf{p}}_{\mathbf{c}}(t)=t^2(1-t)^2k^{i,\mathbf{p}}_{\mathbf{c}}(t),\\
&\Phi^{[0]}(t)=\Phi(t),\quad \Phi^{[1]}(t)=\Phi^{[2]}(t)\equiv\big[\Psi(t)/(t-1)\big]'
\end{aligned}$$

with arbitrary polynomials $k^{i,\mathbf{p}}_{\mathbf{c}}$ and the polynomials Φ,Ψ have the form (3.26) with arbitrarily fixed polynomials Φ_1,Ψ_1 of one variable.

Repeating the calculations of Lemma 3.13, we see that (3.16) holds independently of the choice of $k^{i,\mathbf{p}}_{\mathbf{c}}, \Phi_1, \Psi_1, R^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}$. Notice that we have constructed the polynomials $P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(0,1-t) = P^{i,\mathbf{p}}_{\mathbf{b},\mathbf{a}}(1-t,0)$ in terms of $K^{i,\mathbf{p}}_{\mathbf{a}}$ in a manner such that (3.18) should be fulfilled. Thus the expression $\big[\overline{\xi}^{\,\mathbf{b}}_{\mathbf{p},\mathbf{a}}\big]^{-1}\Big[x^{[i]}\big((1-t)(\mathbf{a}-\mathbf{p})\big)[\Phi^{[i]}]'(t)\xi^{\mathbf{b}}_{\mathbf{p},\mathbf{a}} + t^2(1-t)P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(0,1-t)\Big]\ \big(=K^{i,\mathbf{p}}_{\mathbf{a}}(t)\big)$ is independent of $\mathbf{b}$ automatically which completes the proof in view of Lemma 3.17.

4. Invariance

Lemma 4.1. *Let $\mathbf{G}:\mathbf{x}\mapsto\mathbf{x}\mathbf{A}+\mathbf{w}$ be an invertible affine map $\mathbb{R}^2\leftrightarrow\mathbb{R}^2$. A spline procedure $\mathfrak{S}:(\mathcal{T},F)\mapsto f_{\mathcal{T},F}$ satisfying Postulate A is $\mathbf{G}$-invariant if and only if* (1.9) *holds for any non-degenerate triangle $\mathbf{T}$ with distinguished vertex $\mathbf{p}$.*

Proof. The $\mathbf{G}$-invariance of $\mathfrak{S}$ means that, given any triangular mesh $\mathcal{T}$, the unit functions $f_{\mathcal{T},F_{i,\mathbf{p}}}$ $\big(i=0,1,2;\ \mathbf{p}\in\mathrm{Vert}(\mathcal{T})\big)$ corresponding to the gradient data $F_{i,\mathbf{p}} := \big\{(\mathbf{p},1,\mathbf{0})$ if $i{=}0$, $(\mathbf{p},0,\mathbf{e}^{[i]})$ for $i=1,2\big\}\cup\big\{(\mathbf{q},0,\mathbf{0}) : \mathbf{p}\neq\mathbf{q}\in\mathrm{Vert}(\mathcal{T})\big\}$ are transformed by $\mathbf{G}$ as

$$\begin{aligned}
&f_{\mathcal{T},F_{i,\mathbf{p}}}\circ\mathbf{G}^{-1} = f_{\mathbf{G}(\mathcal{T}),\mathbf{G}^\sharp(F_{i,\mathbf{p}})}\quad (i=0,1,2)\\
&\text{where}\quad \mathbf{G}^\sharp(F_{i,\mathbf{p}}) =\\
&=\big\{\big(\mathbf{r},[f_{\mathcal{T},F_{i,\mathbf{p}}}\circ\mathbf{G}^{-1}](\mathbf{r}),\nabla[f_{\mathcal{T},F_{i,\mathbf{p}}}\circ\mathbf{G}^{-1}](\mathbf{r})\big) : \mathbf{r}\in\mathbf{G}(\mathrm{Vert}(\mathcal{T}))\big\} =\\
&\qquad=\big\{\big(\mathbf{G}(\mathbf{q}),f_{\mathcal{T},F_{i,\mathbf{p}}}(\mathbf{q}),[\nabla f_{\mathcal{T},F_{i,\mathbf{p}}}(\mathbf{q})][\mathbf{A}^{\mathrm{T}}]^{-1}\big) : \mathbf{q}\in\mathrm{Vert}(\mathcal{T})\big\}
\end{aligned}\tag{4.2}$$

with the gradient data of the transformed function on the transformed vertices. Consider any triangle $\mathbf{T}=\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}\in\mathcal{T}$. Notice that the basic functions over $\mathbf{T}$ are given as restrictions of the unit functions. In particular $f_{\mathcal{T},F_{0,\mathbf{p}}}|\mathbf{T} = \varphi_{\mathbf{p},\mathbf{T}}$ and $f_{\mathcal{T},F_{j,\mathbf{p}}}|\mathbf{T} = \psi^{(j)}_{\mathbf{p},\mathbf{T}}$ $(j=1,2)$. On the other hand, by Postulate A, for any gradient data G on $\mathrm{Vert}\big(\mathbf{G}(\mathcal{T})\big)$ of the transformed mesh, such that $\big(\mathbf{G}(\mathbf{p}),\omega,[\alpha,\beta]\big),\big(\mathbf{G}(\mathbf{a}),0,\mathbf{0}\big),\big(\mathbf{G}(\mathbf{b}),0,\mathbf{0}\big)\in G$, we have $f_{\mathbf{G}(\mathcal{T}),G} = \omega\varphi_{\mathbf{G}(\mathbf{p}),\mathbf{G}(\mathbf{T})} + \alpha\psi^{(1)}_{\mathbf{G}(\mathbf{p}),\mathbf{G}(\mathbf{T})} + \beta\psi^{(2)}_{\mathbf{G}(\mathbf{p}),\mathbf{G}(\mathbf{T})}$. We can apply this observation to (4.2) with $G := \mathbf{G}^\sharp(F_{i,\mathbf{p}})$ $(i=0,1,2)$ to conclude that

$$\begin{aligned}
&\varphi_{\mathbf{p},\mathbf{T}}\circ\mathbf{G}^{-1} = \varphi_{\mathbf{G}(\mathbf{p}),\mathbf{G}(\mathbf{T})},\\
&\psi^{(j)}_{\mathbf{p},\mathbf{T}}\circ\mathbf{G}^{-1} = \alpha_j\psi^{(1)}_{\mathbf{G}(\mathbf{p}),\mathbf{G}(\mathbf{T})} + \beta_j\psi^{(2)}_{\mathbf{G}(\mathbf{p}),\mathbf{G}(\mathbf{T})}\\
&\text{where}\ \ [\alpha_j,\beta_j] = [\nabla\phi^{(j)}_{\mathbf{p},\mathbf{T}}(\mathbf{p})][\mathbf{A}^{\mathrm{T}}]^{-1} = \mathbf{e}^{[j]}[\mathbf{A}^{\mathrm{T}}]^{-1}\quad (j=1,2).
\end{aligned}\tag{4.3}$$

Hence the matrix form in (1.9) is immediate: (4.3) implies that $[\mathbf{A}^{\mathrm{T}}]^{-1} = \begin{bmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{bmatrix}$ and $[\psi^{(1)}_{\mathbf{p},\mathbf{T}}\circ\mathbf{G}^{-1}, \psi^{(2)}_{\mathbf{p},\mathbf{T}}\circ\mathbf{G}^{-1}] = [\psi^{(1)}_{\mathbf{G}(\mathbf{p}),\mathbf{G}(\mathbf{T})}, \psi^{(2)}_{\mathbf{G}(\mathbf{p}),\mathbf{G}(\mathbf{T})}]\mathbf{A}^{-1}$.

Corollary 4.4. *There is no affine invariant $\mathcal{C}^1$-spline procedure. satisfying Postulate A.*

Proof. Proceed by contradiction. Assume the procedure $\mathfrak{S} : (\mathcal{T}, F)\mapsto f_{\mathcal{T},F}$ with basic functions $\varphi_{\mathbf{p},\mathbf{T}}, \psi^{(j)}_{\mathbf{p},\mathbf{T}}$ is affine invariant. Then, in particular, (1.9) holds for all transformations $\mathbf{G} : \mathbf{x}\mapsto \mathbf{x}\mathbf{A}+\mathbf{w}$ with $\det(\mathbf{A})\neq 0$ and $\mathbf{w}\in\mathbb{R}^2$. Consider the triangles

$$\mathbf{T}_\mathbf{b} := \mathbf{G}_\mathbf{b}(\mathbf{T}) \qquad \text{where} \quad \mathbf{G}_\mathbf{b} : \mathbf{x}\mapsto \mathbf{x}\mathbf{A}_\mathbf{b} \text{ with } \mathbf{A}_\mathbf{b} := \begin{bmatrix}1 & 0\\ x(\mathbf{b}) & y(\mathbf{b})\end{bmatrix}.$$

Then, according to (1.9), for the points $\mathbf{b}$ with $y(\mathbf{b})\neq 0$ we have

$$\big[\psi^{(1)}_{\mathbf{0},\mathbf{T}}\circ\mathbf{G}_\mathbf{b}^{-1}, \psi^{(2)}_{\mathbf{0},\mathbf{T}}\circ\mathbf{G}_\mathbf{b}^{-1}\big]\mathbf{A}_\mathbf{b} = \big[\psi^{(1)}_{\mathbf{G}_\mathbf{b}(\mathbf{0}),\mathbf{G}_\mathbf{b}(\mathbf{T})}, \psi^{(2)}_{\mathbf{G}_\mathbf{b}(\mathbf{0}),\mathbf{G}_\mathbf{b}(\mathbf{T})}\big]. \tag{4.5}$$

Since $\mathbf{G}_\mathbf{b}^{-1} : \mathbf{y}\mapsto \mathbf{y}\mathbf{A}_\mathbf{b}^{-1}$, in (4.5) we can write $\nabla[\psi^{(j)}_{\mathbf{0},\mathbf{T}}\circ\mathbf{G}_\mathbf{b}^{-1}](\mathbf{y}) = \big[\nabla\psi^{(j)}_{\mathbf{0},\mathbf{T}}(\mathbf{y}\mathbf{A}_\mathbf{b}^{-1})\big][\mathbf{A}_\mathbf{b}^{\mathrm{T}}]^{-1}$. Therefore, for any $\mathbf{y}\in\mathbf{T}_\mathbf{b}$ and $\mathbf{b}\in\mathbb{R}^2$ with $y(\mathbf{b})\neq 0$,

$$\nabla\psi^{(1)}_{\mathbf{0},\mathbf{T}_\mathbf{b}}(\mathbf{y}) = \big[\nabla\psi^{(1)}_{\mathbf{0},\mathbf{T}}(\mathbf{y}\mathbf{A}_\mathbf{b}^{-1})\big][\mathbf{A}_\mathbf{b}^{\mathrm{T}}]^{-1} + x(\mathbf{b})\big[\nabla\psi^{(2)}_{\mathbf{0},\mathbf{T}}(\mathbf{y}\mathbf{A}_\mathbf{b}^{-1})\big][\mathbf{A}_\mathbf{b}^{\mathrm{T}}]^{-1},$$
$$\nabla\psi^{(2)}_{\mathbf{0},\mathbf{T}_\mathbf{b}}(\mathbf{y}) = y(\mathbf{b})\big[\nabla\psi^{(2)}_{\mathbf{0},\mathbf{T}}(\mathbf{y}\mathbf{A}_\mathbf{b}^{-1})\big][\mathbf{A}_\mathbf{b}^{\mathrm{T}}]^{-1}.$$

Observe that the segment $\mathrm{Co}\{\mathbf{0}, \mathbf{e}^{[1]}\}$ is a common edge of all the triangles $\mathbf{T}_\mathbf{b}$. Hence, in view of Remark 3.15, the gradients $\nabla\psi^{(j)}_{\mathbf{0},\mathbf{T}_\mathbf{b}}(\mathbf{y}_t)$ with $\mathbf{y}_t := t\mathbf{e}^{[1]}$ must be independent of $\mathbf{b}$ for any fixed $t\in[0,1]$. Since $\mathbf{y}_t\mathbf{A}_\mathbf{b}^{-1} = \mathbf{y}_t$ ($t\in\mathbb{R}$, $y(\mathbf{b})\neq 0$), our indirect assumption leads to the conclusions that $\mathbf{0} = \nabla\psi^{(2)}_{\mathbf{0},\mathbf{T}_\mathbf{b}}(\mathbf{y}_t) = \nabla\psi^{(2)}_{\mathbf{0},\mathbf{T}}(\mathbf{y}_t)$ and $\big[\nabla\psi^{(2)}_{\mathbf{0},\mathbf{T}_\mathbf{b}}(\mathbf{y}_t)\big]\mathbf{A}_\mathbf{b}^{\mathrm{T}} = \nabla\psi^{(2)}_{\mathbf{0},\mathbf{T}}(\mathbf{y}_t)$ for all $t\in[0,1]$ and $\mathbf{b}\in\mathbb{R}^2$ with $y(\mathbf{b})\neq 0$. This latter identity means in particular that $x(\mathbf{b})\frac{\partial}{\partial x}\psi^{(2)}_{\mathbf{0},\mathbf{T}_\mathbf{b}}(\mathbf{y}_t) + y(\mathbf{b})\frac{\partial}{\partial y}\psi^{(2)}_{\mathbf{0},\mathbf{T}_\mathbf{b}}(\mathbf{y}_t) = \frac{\partial}{\partial y}\psi^{(2)}_{\mathbf{0},\mathbf{T}}(\mathbf{y}_t)$ which is possible with $\mathbf{b}$-independent $\nabla\psi^{(2)}_{\mathbf{0},\mathbf{T}_\mathbf{b}}(\mathbf{y}_t)$ only if $\frac{\partial}{\partial x}\psi^{(2)}_{\mathbf{0},\mathbf{T}_\mathbf{b}}(\mathbf{y}_t) = \frac{\partial}{\partial y}\psi^{(2)}_{\mathbf{0},\mathbf{T}_\mathbf{b}}(\mathbf{y}_t) = 0$ ($t\in[0,1]$). However, hence we get $\mathbf{0} = \nabla\psi^{(2)}_{\mathbf{0},\mathbf{T}_\mathbf{b}}(\mathbf{y}_0)$ which contradicts the defining relations (1.4) with $\nabla\psi^{(2)}_{\mathbf{0},\mathbf{T}_\mathbf{b}}(\mathbf{0}) = [0,1]$.

Lemma 4.6. (Reflection lemma). *Let $\mathbf{T}$ be a non-degenarate triangle of the form $\mathbf{T} := \mathrm{Co}\{\mathbf{0}, \rho\mathbf{e}^{[1]}, \mathbf{b}\}$ and assume $\mathfrak{S}$ is a spline procedure satisfying*

Postulate A. Then, for the fixed points $\mathbf{u}_t := t\mathbf{e}^{[1]}$ *of the reflection* $\mathbf{K} = \mathbf{K}^{-1} : \mathbf{x} \mapsto [x(\mathbf{x}, -y(\mathbf{x}] = \mathbf{x}\mathbf{U}$, $\mathbf{U} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ *through the x-axis we have*

$$\big\langle \nabla\varphi_{\mathbf{0},\mathbf{T}}(\mathbf{u}_t)\big|\mathbf{e}^{[2]}\big\rangle = 0,\ \big\langle \nabla\psi^{(1)}_{\mathbf{0},\mathbf{T}}(\mathbf{u}_t)\big|\mathbf{e}^{[2]}\big\rangle = 0,\ \psi^{(2)}_{\mathbf{0},\mathbf{T}}(\mathbf{u}_t) = 0 \quad (t \in [0,\rho]).$$

Proof. The triangles $\mathbf{T}$ and $\mathbf{K}(\mathbf{T})$ are adjacent, the segment $\mathrm{Co}\{\mathbf{0}, \rho\mathbf{e}^{[1]}\}$ is their common edge. According to Remark 3.15, the pairs $\varphi_{\mathbf{0},\mathbf{T}}, \varphi_{\mathbf{0},\mathbf{K}(\mathbf{T})}$ resp. $\psi^{(j)}_{\mathbf{0},\mathbf{T}}, \psi^{(j)}_{\mathbf{0},\mathbf{K}(\mathbf{T})}$ of basic functions must be coupled $\mathcal{C}^1$-smoothly along it: $\varphi_{\mathbf{0},\mathbf{T}}(\mathbf{u}_t) = \varphi_{\mathbf{0},\mathbf{K}(\mathbf{T})}(\mathbf{u}_t)$, $\nabla\varphi_{\mathbf{0},\mathbf{T}}(\mathbf{u}_t) = \nabla\varphi_{\mathbf{0},\mathbf{K}(\mathbf{T})}(\mathbf{u}_t)$ resp. $\psi^{(j)}_{\mathbf{0},\mathbf{T}}(\mathbf{u}_t) = \psi^{(j)}_{\mathbf{0},\mathbf{K}(\mathbf{T})}(\mathbf{u}_t)$, $\nabla\psi^{(j)}_{\mathbf{0},\mathbf{T}}(\mathbf{u}_t) = \nabla\psi^{(j)}_{\mathbf{0},\mathbf{K}(\mathbf{T})}(\mathbf{u}_t)$ for all $t \in [0,\rho]$. On the other hand, the transformation rules (1.9) require $\varphi_{\mathbf{0},\mathbf{K}(\mathbf{T})} = \varphi_{\mathbf{0},\mathbf{T}} \circ \mathbf{K}^{-1}$ resp. $\big[\psi^{(1)}_{\mathbf{0},\mathbf{K}(\mathbf{T})}, \psi^{(2)}_{\mathbf{0},\mathbf{K}(\mathbf{T})}\big] = \big[\psi^{(1)}_{\mathbf{0},\mathbf{T}} \circ \mathbf{K}^{-1}, \psi^{(2)}_{\mathbf{0},\mathbf{T}} \circ \mathbf{K}^{-1}\big]\mathbf{U}$ i.e. $\varphi_{\mathbf{0},\mathbf{K}(\mathbf{T})}(\mathbf{y}) = \varphi_{\mathbf{0},\mathbf{T}}(\mathbf{y}\mathbf{U})$ and $\psi^{(j)}_{\mathbf{0},\mathbf{K}(\mathbf{T})}(\mathbf{y}) = (-1)^{j-1}\psi^{(j)}_{\mathbf{0},\mathbf{T}}(\mathbf{y})$ for all $\mathbf{y} \in \mathbf{K}(\mathbf{T}) = \mathbf{T}\mathbf{U}$. By passing to gradients, since $\mathbf{K} = \mathbf{K}^{-1}$ and $\mathbf{U} = \mathbf{U}^{-1} = \mathbf{U}^{\mathrm{T}}$, we get $\nabla\varphi_{\mathbf{0},\mathbf{K}(\mathbf{T})}(\mathbf{y}) = \nabla[\varphi_{\mathbf{0},\mathbf{T}}(\mathbf{y}\mathbf{U})]\mathbf{U}$ and $\nabla\psi^{(j)}_{\mathbf{0},\mathbf{K}(\mathbf{T})}(\mathbf{y}) = (-1)^{j-1}[\nabla\psi^{(j)}_{\mathbf{0},\mathbf{T}}(\mathbf{y})$ for the points $\mathbf{y} \in \mathbf{K}(\mathbf{T})$. In particular on the common edge of $\mathbf{T}$ with $\mathbf{K}(\mathbf{T})$ we must have $\nabla\varphi_{\mathbf{0},\mathbf{T}}(\mathbf{u}_t)\nabla\varphi_{\mathbf{0},\mathbf{K}(\mathbf{T})}(\mathbf{u}_t) = \nabla[\varphi_{\mathbf{0},\mathbf{T}}((\mathbf{u}_t))]\mathbf{U}$ i.e. $\frac{\partial}{\partial x^{[k]}}\varphi_{\mathbf{0},\mathbf{T}}(\mathbf{u}_t) = (-1)^{k-1}\frac{\partial}{\partial x^{[k]}}\varphi_{\mathbf{0},\mathbf{T}}(\mathbf{u}_t)$ implying $0 = \frac{\partial}{\partial y}\varphi_{\mathbf{0},\mathbf{T}}(\mathbf{u}_t) = \big\langle \nabla\varphi_{\mathbf{0},\mathbf{T}}(\mathbf{u}_t)\big|\mathbf{e}^{[2]}\big\rangle$. Similarly we conclude that $\psi^{(j)}_{\mathbf{0},\mathbf{T}}(\mathbf{u}_t) = \psi^{(j)}_{\mathbf{0},\mathbf{K}(\mathbf{T})}(\mathbf{u}_t) = (-1)^{j-1}\psi^{(j)}_{\mathbf{0},\mathbf{T}}(\mathbf{u}_t)$ implying $\psi^{(2)}_{\mathbf{0},\mathbf{K}(\mathbf{T})}(\mathbf{u}_t) = 0$ and $\nabla\psi^{(j)}_{\mathbf{0},\mathbf{T}}(\mathbf{u}_t) = \nabla\psi^{(j)}_{\mathbf{0},\mathbf{K}(\mathbf{T})}(\mathbf{u}_t) = (-1)^{j-1}[\nabla\psi^{(j)}_{\mathbf{0},\mathbf{T}}(\mathbf{u}_t)]\mathbf{U}$ implying in particular $\frac{\partial}{\partial y}\psi^{(1)}_{\mathbf{0},\mathbf{T}}(\mathbf{u}_t) = 0$.

Proposition 4.7. *Homothetically invariant* $\mathcal{C}^1$*-spline procedures satisfying Postulate A are shape uniform on edges* (*i.e. they satisfy Postulate B automatically*).

Proof. Let $\mathbf{T} := \mathrm{Co}\{\mathbf{0}, \mathbf{e}^{[1]}, \mathbf{e}^{[2]}\}$ and define

$$\Phi(t) := \varphi_{\mathbf{0},\mathbf{T}}(\mathbf{u}_t), \quad \Psi(t) := \phi^{(1)}_{\mathbf{0},\mathbf{T}}(\mathbf{u}_t) \quad \text{where} \quad \mathbf{u}_t := (1-t)\mathbf{e}^{[1]}. \tag{4.8}$$

Cosider any other non-degenerate triangle $\widetilde{\mathbf{T}} := \mathrm{Co}\{\mathbf{p}, \mathbf{a}, \mathbf{b}\}$. Due to the arbitrariness of the choice of $\widetilde{\mathbf{T}}$, it suffices to see only that, for $j = 1, 2$ and $t \in [0,1]$,

$$\varphi_{\mathbf{0},\widetilde{\mathbf{T}}}(\mathbf{y}_t) = \Phi(t), \quad \psi^{(j)}_{\mathbf{0},\widetilde{\mathbf{T}}}(\mathbf{y}_t) = \mathrm{Const}^{(j)}_{\widetilde{\mathbf{T}}}\Psi(t) \quad \text{with} \quad \mathbf{y}_t := (1-t)\mathbf{a} + t\mathbf{p}. \tag{4.9}$$

It is a crucial fact that we can find a homothetic transformation

$$\mathbf{G} : \mathbf{x} \mapsto \mathbf{x}\mathbf{A} + \mathbf{p} \quad \text{such that } \mathbf{G}(\mathbf{e}^{[1]}) = \mathbf{a}, \quad \mathbf{G}(\mathbf{T}) \cap \widetilde{\mathbf{T}} = \mathrm{Co}\{\mathbf{a}, \mathbf{p}\}. \tag{4.10}$$

Actually $\mathbf{A} = \left[\begin{smallmatrix} x(\mathbf{a}-\mathbf{p}) & y(\mathbf{a}-\mathbf{p}) \\ -\sigma y(\mathbf{a}-\mathbf{p}) & \sigma x(\mathbf{a}-\mathbf{p}) \end{smallmatrix}\right]$ where $\sigma = 1$ if the points $\mathbf{a}, \mathbf{b}, \mathbf{p}$ are oriented clockwise and $\sigma = -1$ else. According to (1.9), $\varphi_{\mathbf{G(0)},\mathbf{G(T)}} = \varphi_{\mathbf{0},\mathbf{T}} \circ \mathbf{G}^{-1}$. Since $\mathrm{Co}\{\mathbf{p}, \mathbf{a}\}$ is a common edge of $\mathbf{T}$ and $\widetilde{\mathbf{T}}$, in view of Remark 3.15 we have

$$\varphi_{\mathbf{p},\widetilde{\mathbf{T}}}(\mathbf{y}_t) = \varphi_{\mathbf{p},\mathbf{G(T)}}(\mathbf{y}_t) = \varphi_{\mathbf{0},\mathbf{T}}\big(\mathbf{G}^{-1}(\mathbf{y}_t)\big) = \varphi_{\mathbf{0},\mathbf{T}}\big(\mathbf{u}_t\big) = \Phi(t) \tag{4.9}$$

which proves the first part of (3.12). To prove $\psi^{(j)}_{bf0,\widetilde{T}}(\mathbf{y}_t) = \mathrm{Const}^{(j)}_{\widetilde{\mathbf{T}}}\Psi(t)$, consider also the symmetry

$$\mathbf{H} : \mathbf{x} \mapsto [y(\mathbf{x}), x(\mathbf{x})] = \mathbf{x}\mathbf{S}, \quad \mathbf{v}_t := \mathbf{u}_t\mathbf{S} = t\mathbf{e}^{[2]} \text{ where } \mathbf{S} := \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \tag{4.10}$$

of the triangle $\mathbf{T}$. By (1.9) we have $\big[\psi^{(1)}_{\mathbf{0},\mathbf{T}}, \psi^{(2)}_{\mathbf{0},\mathbf{T}}\big] = \big[\psi^{(1)}_{\mathbf{0},\mathbf{T}} \circ \mathbf{H}, \psi^{(2)}_{\mathbf{0},\mathbf{T}} \circ \mathbf{H}\big]\mathbf{S}$ whence

$$\big[\psi^{(1)}_{\mathbf{0},\mathbf{T}}(\mathbf{u}_t), \psi^{(2)}_{\mathbf{0},\mathbf{T}}(\mathbf{u}_t)\big] = \big[\psi^{(1)}_{\mathbf{0},\mathbf{T}}(\mathbf{v}_t), \psi^{(2)}_{\mathbf{0},\mathbf{T}}(\mathbf{v}_t)\big]\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] = \big[\psi^{(2)}_{\mathbf{0},\mathbf{T}}(\mathbf{v}_t), \psi^{(1)}_{\mathbf{0},\mathbf{T}}(\mathbf{v}_t)\big].$$

Thus $\psi^{(2)}_{\mathbf{0},\mathbf{T}}(\mathbf{v}_t) = \Psi(t)$ while $\psi^{1)}_{\mathbf{0},\mathbf{T}}(\mathbf{v}_t) = \psi^{(2)}_{\mathbf{0},\mathbf{T}}(\mathbf{u}_t)$. On the other hand, by Lemma 4.ivpvii, $\psi^{(2)}_{\mathbf{0},\mathbf{T}}(\mathbf{u}_t) = 0$. Finally we apply Remark 3.15 and (1.9) to the points $\mathbf{y}_t$ of the common edge $\mathrm{Co}\{\mathbf{a}, \mathbf{p}\}$ between the triangles $\mathbf{G}(\mathbf{T})$ and $\widetilde{\mathbf{T}}$. Hence we conclude that

$$\begin{aligned} &\big[\psi^{(1)}_{\mathbf{p},\widetilde{\mathbf{T}}}(\mathbf{y}_t), \psi^{(2)}_{\mathbf{p},\widetilde{\mathbf{T}}}(\mathbf{y}_t)\big] = \big[\psi^{(1)}_{\mathbf{G(0)},\mathbf{G(T)}}(\mathbf{y}_t), \psi^{(2)}_{\mathbf{G(0)},\mathbf{G(T)}}(\mathbf{y}_t)\big] = \\ &= \big[\psi^{(1)}_{\mathbf{0},\mathbf{T}}(\mathbf{u}_t), \big[\psi^{(2)}_{\mathbf{0},\mathbf{T}}(\mathbf{u}_t)\big]\mathbf{A} = \big[\Psi(t), 0\big]\mathbf{A} = \Psi(t)\big[x(\mathbf{a}-\mathbf{p}, y(\mathbf{a}-\mathbf{p}\big]. \end{aligned} \tag{4.11}$$

Thus $\psi^{(j)}_{\mathbf{p},\widetilde{\mathbf{T}}}(\mathbf{y}_t) = x^{[j]}(\mathbf{a}-\mathbf{p})\Psi(t)$ $(j = 1, 2)$ which completes the proof.

4.12. Proof of Therem 2.8

It is clear that the coordinate values $\xi^{\mathbf{v}}_{\mathbf{p},\mathbf{a}} = \langle \mathbf{v}-\mathbf{a} | \mathbf{p}-\mathbf{a}\rangle / \|\mathbf{p}-\mathbf{a}\|^2$ are homothetic invariant i.e. $\xi^{\mathbf{G(v)}}_{\mathbf{G(p)},\mathbf{G(a)}} = \xi^{\mathbf{v}}_{\mathbf{p},\mathbf{a}}$ whenever the transformation $\mathbf{G} : \mathbb{R}^2 \leftrightarrow \mathbb{R}^2$ is of the form $\mathbf{G}(\mathbf{x}) = \mathbf{w} + \rho(\mathbf{x}-\mathbf{q})\mathbf{S}$ with a constant $\rho > 0$ and an orthogonal 2×2-matrix $\mathbf{S}$. The baricentric coordinates $\lambda^{\mathbf{p}}_{\mathbf{T}}$ are even affine invariant as it is well-known from classical Projective Geometry. Hence it suffices to see that the invariance relations

$$\varphi_{\mathbf{G(p)},\mathbf{G(T)}} = \varphi \circ \mathbf{G}^{-1}, \quad \big[\psi^{(1)}_{\mathbf{G(p)},\mathbf{G(T)}}, \psi^{(2)}_{\mathbf{G(p)},\mathbf{G(T)}}\big] = \big[\psi^{(1)} \circ \mathbf{G}^{-1}, \psi^{(1)} \circ \mathbf{G}^{-1}\big]\mathbf{S}$$

imply that $k^{i,\mathbf{p}}_{\mathbf{a}} \equiv 0$ $(i = 0,1,2)$ whenever $\mathbf{T} = \mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}$ is a non-degenerate triangle and $\mathbf{G} : \mathbb{R}^2 \leftrightarrow \mathbb{R}^2$ is the orthogonal reflection through $\mathrm{Line}\{\mathbf{a},\mathbf{p}\}$ i.e.

$$\mathbf{G}(\mathbf{a}) = \mathbf{a}, \quad \mathbf{G}(\mathbf{p}) = \mathbf{p}, \quad \xi^{\mathbf{G}(\mathbf{x})}_{\mathbf{p},\mathbf{a}} = \xi^{\mathbf{x}}_{\mathbf{p},\mathbf{a}}, \quad \overline{\xi}^{\mathbf{G}(\mathbf{x})}_{\mathbf{p},\mathbf{a}} = -\overline{\xi}^{\mathbf{x}}_{\mathbf{p},\mathbf{a}} \quad (\mathbf{x} \in \mathbb{R}^2),$$

so that $\mathbf{G}(\mathbf{x}) = \mathbf{a}+(\mathbf{x}-\mathbf{a})\mathbf{S}$ where $\mathbf{S} = \|\mathbf{p}-\mathbf{a}\|^{-2} \left[\begin{smallmatrix}\mathbf{p}-\mathbf{a}\\ (\mathbf{p}-\mathbf{a})\mathbf{R}\end{smallmatrix}\right]^{\mathrm{T}} \left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right] \left[\begin{smallmatrix}\mathbf{p}-\mathbf{a}\\ (\mathbf{p}-\mathbf{a})\mathbf{R}\end{smallmatrix}\right]$. Let us first investigate the relation $\varphi_{\mathbf{G}(\mathbf{p}),\mathbf{G}(\mathbf{T})} = \varphi_{\mathbf{p},\mathbf{T}} \circ \mathbf{G}^{-1}$. By pluging

$$\begin{aligned} t = t(\mathbf{x}) &:= \lambda^{\mathbf{a}}_{\mathbf{G}(\mathbf{T})}(\mathbf{x}) = \lambda^{\mathbf{G}(\mathbf{a})}_{\mathbf{G}(\mathbf{T})}(\mathbf{x}) = \lambda^{\mathbf{a}}_{\mathbf{T}}\big(\mathbf{G}^{-1}(\mathbf{x})\big),\\ s = s(\mathbf{x}) &:= \lambda^{\mathbf{G}(\mathbf{b})}_{\mathbf{G}(\mathbf{T})}(\mathbf{x}) = \lambda^{\mathbf{b}}_{\mathbf{T}}\big(\mathbf{G}^{-1}(\mathbf{x})\big)\end{aligned}$$

in the expressions of $\varphi_{\mathbf{G}(\mathbf{p}),\mathbf{G}(\mathbf{T})}$ resp. $\varphi_{\mathbf{p},\mathbf{T}} \circ \mathbf{G}^{-1}$ formed with (2.5), since $\lambda^{\mathbf{p}}_{\mathbf{T}} = 1 - \lambda^{\mathbf{a}}_{\mathbf{T}} - \lambda^{\mathbf{b}}_{\mathbf{T}}$ and $\mathbf{G}(\mathbf{a}) = \mathbf{a}$ resp. $\mathbf{G}(\mathbf{p}) = \mathbf{p}$, we get

$$\begin{aligned} 0 &= \varphi_{\mathbf{G}(\mathbf{p}),\mathbf{G}(\mathbf{T})}(\mathbf{x}) - \varphi_{\mathbf{p},\mathbf{T}} \circ \mathbf{G}^{-1}(\mathbf{x}) = \varphi_{\mathbf{p},\mathbf{G}(\mathbf{T})}(\mathbf{x}) - \varphi_{\mathbf{p},\mathbf{T}} \circ \mathbf{G}^{-1}(\mathbf{x}) =\\ &= \Big[\Phi(1-s-t)+(1-s-t)^2 st P^{\mathbf{p}}_{\mathbf{a},\mathbf{G}(\mathbf{b})}(s,t)\Big] -\\ &\qquad - \Big[\Phi(1-s-t)+(1-s-t)^2 st P^{\mathbf{p}}_{\mathbf{a},\mathbf{b}}(s,t)\Big],\\ 0 = P^{\mathbf{p}}_{\mathbf{a},\mathbf{G}(\mathbf{b})}(s,t) - P^{\mathbf{p}}_{\mathbf{a},\mathbf{b}}(s,t) &= \Big[s\Big\{\xi^{\mathbf{a}}_{\mathbf{p},\mathbf{G}(\mathbf{b})}\frac{\Phi'(1-s)}{s(1-s)^2} + \overline{\xi}^{\mathbf{a}}_{\mathbf{p},\mathbf{G}(\mathbf{b})} k^{0,\mathbf{p}}_{\mathbf{G}(\mathbf{b})}(s)\Big\} +\\ &\qquad + t\Big\{\xi^{\mathbf{G}(\mathbf{b})}_{\mathbf{p},\mathbf{a}}\frac{\Phi'(1-t)}{t(1-t)^2} + \overline{\xi}^{\mathbf{G}(\mathbf{b})}_{\mathbf{p},\mathbf{a}} k^{0,\mathbf{p}}_{\mathbf{a}}(t)\Big\} + st R^{0,\mathbf{p}}_{\mathbf{a},\mathbf{G}(\mathbf{b})}(s,t)\Big] -\\ &\qquad - \Big[s\Big\{\xi^{\mathbf{a}}_{\mathbf{p},\mathbf{b}}\frac{\Phi'(1-s)}{s(1-s)^2} + \overline{\xi}^{\mathbf{a}}_{\mathbf{p},\mathbf{b}} k^{0,\mathbf{p}}_{\mathbf{b}}(s)\Big\} +\\ &\qquad + t\Big\{\xi^{\mathbf{b}}_{\mathbf{p},\mathbf{a}}\frac{\Phi'(1-t)}{t(1-t)^2} + \overline{\xi}^{\mathbf{b}}_{\mathbf{p},\mathbf{a}} k^{0,\mathbf{p}}_{\mathbf{a}}(t)\Big\} + st R^{0,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(s,t)\Big]\end{aligned}$$

Comparing the coefficients of the monomials $s^m t^n$, in view of (2.9) we see that

$$\begin{aligned} &\text{(i)}\ 0 = \Big[\xi^{\mathbf{G}(\mathbf{b})}_{\mathbf{p},\mathbf{a}}\frac{\Phi'(1-t)}{t(1-t)^2} + \overline{\xi}^{\mathbf{G}(\mathbf{b})}_{\mathbf{p},\mathbf{a}} k^{0,\mathbf{p}}_{\mathbf{a}}(t)\Big] - \Big[\xi^{\mathbf{b}}_{\mathbf{p},\mathbf{a}}\frac{\Phi'(1-t)}{t(1-t)^2} + \overline{\xi}^{\mathbf{b}}_{\mathbf{p},\mathbf{a}} k^{0,\mathbf{p}}_{\mathbf{a}}(t)\Big],\\ &\text{(ii)}\ 0 = \Big[\xi^{\mathbf{a}}_{\mathbf{p},\mathbf{G}(\mathbf{b})}\frac{\Phi'(1-s)}{s(1-s)^2} + \overline{\xi}^{\mathbf{a}}_{\mathbf{p},\mathbf{G}(\mathbf{b})} k^{0,\mathbf{p}}_{\mathbf{G}(\mathbf{b})}(s)\Big] - \Big[\xi^{\mathbf{a}}_{\mathbf{p},\mathbf{b}}\frac{\Phi'(1-s)}{s(1-s)^2} + \overline{\xi}^{\mathbf{a}}_{\mathbf{p},\mathbf{b}} k^{0,\mathbf{p}}_{\mathbf{a}}(s)\Big],\\ &\text{(iii)}\ 0 = R^{0,\mathbf{p}}_{\mathbf{a},\mathbf{G}(\mathbf{b})}(s,t) - R^{0,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(s,t).\end{aligned}$$

Since $\xi^{\mathbf{G}(\mathbf{b})}_{\mathbf{p},\mathbf{a}} = \xi^{\mathbf{b}}_{\mathbf{p},\mathbf{a}}$ and $\overline{\xi}^{\mathbf{G}(\mathbf{b})}_{\mathbf{p},\mathbf{a}} = -\overline{\xi}^{\mathbf{b}}_{\mathbf{p},\mathbf{a}}$, from (i) we conclude that $k^{0,\mathbf{p}}_{\mathbf{a}}(t) = 0$. On the other hand, (iii) implies the isometry invariance of the non-principal

parts because the lines $\mathrm{Line}\{\mathbf{p},\mathbf{a})\}$ can be chosen arbitrarily and hence the corresponding reflections generate the whole group of self-isometries of $\mathbb{R}^2$.

The treatment of the relations

$$\big[\psi^{(1)}_{\mathbf{G(p)},\mathbf{G(T)}},\psi^{(2)}_{\mathbf{G(p)},\mathbf{G(T)}}\big]=\big[\psi^{(1)}\circ\mathbf{G}^{-1},\psi^{(1)}\circ\mathbf{G}^{-1}\big]\mathbf{S}$$

is analogous by using the vectorial forms $\boldsymbol{\psi}_{\mathbf{q},\mathbf{W}}:=\big[\psi^{(1)}_{\mathbf{q},\mathbf{W}},\psi^{(2)}_{\mathbf{q},\mathbf{W}}\big]$ for triangles $\mathbf{W}$ with distinguished vertex $\mathbf{q}$. With this formalism the above invariance relation can be written as $\boldsymbol{\psi}_{\mathbf{G(p)},\mathbf{G(T)}}(\mathbf{x})=\big[\boldsymbol{\psi}_{\mathbf{p},\mathbf{T}}\big(\mathbf{G}^{-1}(\mathbf{x})\big)\big]\mathbf{S}$ where $\mathbf{G}=\mathbf{G}^{-1}$, $\mathbf{G(p)}=\mathbf{p}$, $\mathbf{G(a)}=\mathbf{a}$ and

$$\begin{aligned}&\boldsymbol{\psi}_{\mathbf{G(p)},\mathbf{G(T)}}(\mathbf{x})=\Theta(1-s-t)(\mathbf{x}-\mathbf{p})+(1-s-t)^2stQ^{\mathbf{p}}_{\mathbf{a},\mathbf{G(b)}}(s,t),\\&\big[\boldsymbol{\psi}_{\mathbf{p},\mathbf{T}}\big(\mathbf{G}^{-1}(\mathbf{x})\big)\big]\mathbf{S}=\Theta(1-s-t)\big(\mathbf{G}(\mathbf{x})-\mathbf{p}\big)\mathbf{S}+(1-s-t)^2stQ^{\mathbf{p}}_{\mathbf{a},\mathbf{b}}(s,t)\mathbf{S}\end{aligned}$$

with the vector valued polynomials

$$\begin{aligned}\mathbf{Q}^{\mathbf{p}}_{\mathbf{a},\mathbf{w}}(s,t):=&\big[Q^{1,\mathbf{p}}_{\mathbf{a},\mathbf{w}}(s,t),Q^{2,\mathbf{p}}_{\mathbf{a},\mathbf{w}}(s,t)\big]=\\&=s\Big\{\xi^{\mathbf{a}}_{\mathbf{p},\mathbf{w}}\frac{\Theta'(1-s)}{s(1-s)^2}(\mathbf{w}-\mathbf{p})+\bar{\xi}^{\mathbf{a}}_{\mathbf{p},\mathbf{w}}\mathbf{k}^{\mathbf{p}}_{\mathbf{w}}(s)\Big\}+\\&+\quad t\Big\{\xi^{\mathbf{w}}_{\mathbf{p},\mathbf{a}}\frac{\Theta'(1-t)}{t(1-t)^2}(\mathbf{a}-\mathbf{p})+\bar{\xi}^{\mathbf{w}}_{\mathbf{p},\mathbf{a}}\mathbf{k}^{\mathbf{p}}_{\mathbf{a}}(t)\Big\}+st\mathbf{R}^{\mathbf{p}}_{\mathbf{a},\mathbf{w}}(s,t)\end{aligned}$$

for $\mathbf{w}:=\mathbf{b},\mathbf{G(b)}$ where $\mathbf{R}^{\mathbf{p}}_{\mathbf{a},\mathbf{w}}:=\big[\mathbf{R}^{1,\mathbf{p}}_{\mathbf{a},\mathbf{w}},\mathbf{R}^{2,\mathbf{p}}_{\mathbf{a},\mathbf{w}}\big]$ and $\mathbf{k}^{\mathbf{p}}_{\mathbf{u}}:=\big[\mathbf{k}^{1,\mathbf{p}}_{\mathbf{w}},\mathbf{k}^{2,\mathbf{p}}_{\mathbf{w}}\big]$. Clearly $(\mathbf{G}(\mathbf{x})-\mathbf{p})\mathbf{S}=(\mathbf{G}(\mathbf{x})-\mathbf{G}(\mathbf{p}))\mathbf{S}=\big((\mathbf{a}+(\mathbf{x}-\mathbf{a})\mathbf{S})-(\mathbf{a}+(\mathbf{p}-\mathbf{a})\mathbf{S})\big)\mathbf{S}=(\mathbf{x}-\mathbf{p})\mathbf{S}^{\mathbf{2}}=\mathbf{x}-\mathbf{p}$. Hence the comparison of the coefficients of the monomials s^mt^n yields

$$\begin{aligned}&\text{(i}')\quad 0=\Big[\xi^{\mathbf{G(b)}}_{\mathbf{p},\mathbf{a}}\frac{\Theta'(1-t)}{t(1-t)^2}\big(\mathbf{G(b)}-\mathbf{p}\big)+\bar{\xi}^{\mathbf{G(b)}}_{\mathbf{p},\mathbf{a}}\mathbf{k}^{\mathbf{p}}_{\mathbf{a}}(t)\Big]-\\&\qquad\qquad-\Big[\xi^{\mathbf{b}}_{\mathbf{p},\mathbf{a}}\frac{\Theta'(1-t)}{t(1-t)^2}(\mathbf{b}-\mathbf{p})+\bar{\xi}^{\mathbf{b}}_{\mathbf{p},\mathbf{a}}\mathbf{k}^{\mathbf{p}}_{\mathbf{a}}(t)\Big]\mathbf{S},\\&\text{(ii}')\quad 0=\Big[\xi^{\mathbf{a}}_{\mathbf{p},\mathbf{G(b)}}\frac{\Theta'(1-s)}{s(1-s)^2}(\mathbf{a}-\mathbf{p})+\bar{\xi}^{\mathbf{a}}_{\mathbf{p},\mathbf{G(b)}}\mathbf{k}^{\mathbf{p}}_{\mathbf{G(b)}}(s)\Big]-\\&\qquad\qquad-\Big[\xi^{\mathbf{a}}_{\mathbf{p},\mathbf{b}}\frac{\Theta'(1-s)}{s(1-s)^2}(\mathbf{a}-\mathbf{p})+\bar{\xi}^{\mathbf{a}}_{\mathbf{p},\mathbf{b}}\mathbf{k}^{\mathbf{p}}_{\mathbf{a}}(s)\Big]\mathbf{S},\\&\text{(iii}')\quad 0=\mathbf{R}^{\mathbf{p}}_{\mathbf{a},\mathbf{G(b)}}(s,t)-\mathbf{R}^{\mathbf{p}}_{\mathbf{a},\mathbf{b}}(s,t)\mathbf{S}.\end{aligned}$$

Considering again (i′), since $\mathbf{G(b)}-\mathbf{p}=\mathbf{G(b)}-\mathbf{G(p)}=(\mathbf{b}-\mathbf{p})S$ and since $\xi^{\mathbf{G(b)}}_{\mathbf{p},\mathbf{a}}=\xi^{\mathbf{b}}_{\mathbf{p},\mathbf{a}}$ resp. $\bar{\xi}^{\mathbf{G(b)}}_{\mathbf{p},\mathbf{a}}=-\bar{\xi}^{\mathbf{b}}_{\mathbf{p},\mathbf{a}}$, we conclude $\mathbf{k}^{\mathbf{p}}_{\mathbf{a}}(t)=0$.

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