Két érdekes polinom

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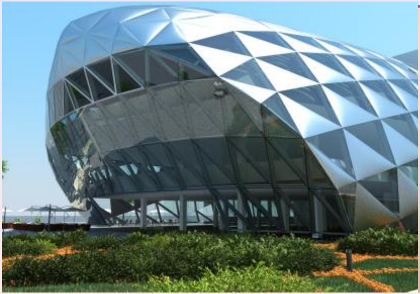
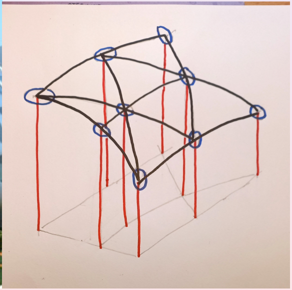
18/11/2023, Szeged

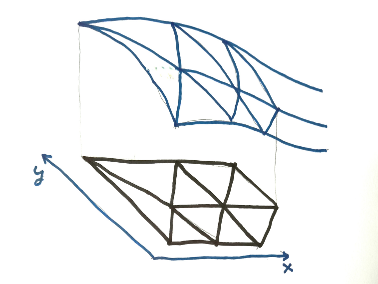
$$\Phi(t) := t^3(10 - 15t + 6t^2)$$

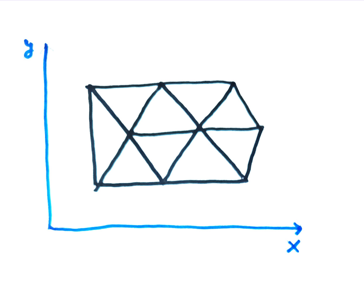
$$\Theta(t) := t^2(4 - 3t)$$

Szkennelt pontok → sima felület
KEVÉS SZÁMÍTÁSSAL

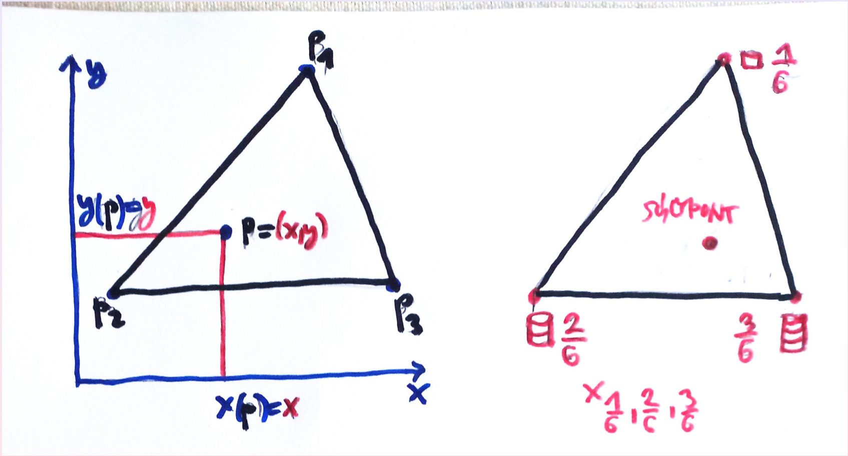
Spline fgv. háromszögeken





Külön-külön képlet háromszögenként
Egyesítve → $\mathcal{C}^1$-fgv (1x folyt. diff.)

Descartes-koord helyett SÚLYOK



$$t_1 + t_2 + t_3 = 1,$$

$$\mathbf{x}_{t_1,t_2,t_3} := t_1\mathbf{p}_1 + t_2\mathbf{p}_2 + t_3\mathbf{p}_3$$

Baricentrikus koordináták

$\mathbf{T} \subset \mathbb{R}^2$ háromszög (nem elfaj.)
Csúcsok: $\big[\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\big]$

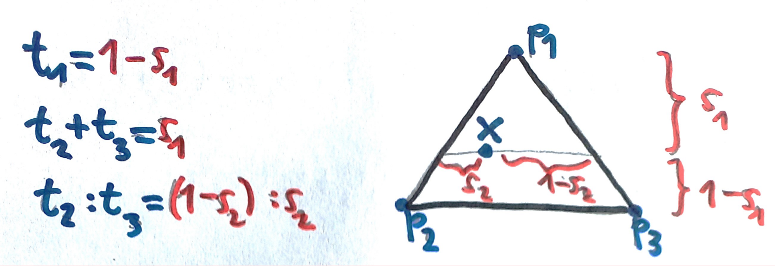
$\mathbf{x} \in \mathbb{R}^2$ súlyai $\lambda^{\mathbf{T}}_{\mathbf{p}_1}(\mathbf{x}), \lambda^{\mathbf{T}}_{\mathbf{p}_2}(\mathbf{x}), \lambda^{\mathbf{T}}_{\mathbf{p}_3}(\mathbf{x})$
$\big([\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3]$ szerinti *baricentr. koord.*$\big)$

$$\sum_{i=1}^{3} \lambda_i(\mathbf{x}) = 1, \quad \mathbf{x} = \sum_{i=1}^{3} \lambda_i(\mathbf{x})\mathbf{p}_i$$

$$\mathbf{x}_{t_1,t_2,t_3} = \sum_{k=1}^{3} t_k \mathbf{p}_k \quad \to \quad \lambda_k\big(\mathbf{x}_{t_1,t_2,t_3}\big) = t_k$$

$f : \mathbf{T} \to \mathbb{R}$ grafikonja

$$
\begin{aligned}
f &= \left\{ \left[x, y, f(x,y)\right] : [x,y] \in \mathbf{T} \right\} = \\
&= \left\{ \left[\mathbf{x}, f(\mathbf{x})\right] : \mathbf{x} \in \mathbf{T} \right\} = \\
&= \left\{ \left[\mathbf{x}_{t_1,t_2,t_3}, f(\mathbf{x}_{t_1,t_2,t_3})\right] : \Sigma_k t_k = 1,\ t_k \geq 0 \right\} = \\
&= \left\{ \begin{bmatrix} \mathbf{x}_{1-s_1, s_1(1-s_2), s_1 s_2} \\ f(\mathbf{x}_{1-s_1, s_1(1-s_2), s_1 s_2}) \end{bmatrix} : \begin{matrix} 0 \leq s_1 \leq 1, \\ 0 \leq s_2 \leq 1 \end{matrix} \right\}
\end{aligned}
$$



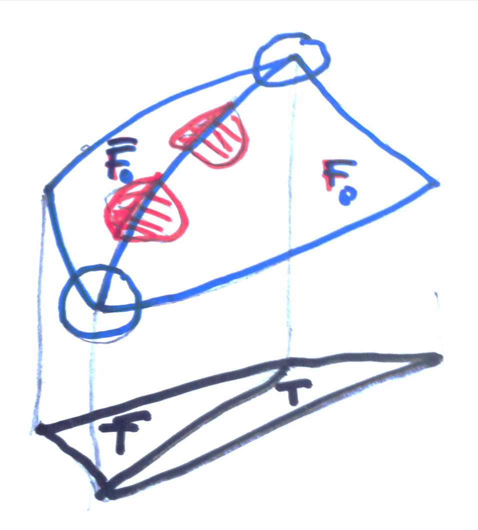
ILLESZTŐ ALAPFGV. Φ, Θ szerint

$f_1, f_2, f_3 \in \mathbb{R}, \quad A_1, A_2, A_3 : \mathbb{R}^2 \to \mathbb{R}$ lin.

$$A_i\big([x, y]\big) = \alpha_i x + \beta_i y$$

$$F_0(\mathbf{x}) := \sum_{i=1}^{3} \Phi\big(\lambda_i(\mathbf{x})\big) f_i + \sum_{i=1}^{3} \Theta\big(\lambda_i(\mathbf{x})\big) A_i(\mathbf{x} - \mathbf{p}_i)$$

Illeszkedés, de nem sima



$F_0(\mathbf{p}_i) = f_i, \quad F_0'(\mathbf{p}_i) = A_i$

$F'(\mathbf{p}) : \mathbb{R}^2 \ni \mathbf{u} \mapsto \frac{d}{dt}\Big|_{t=0} F(\mathbf{p} + t\mathbf{u})$

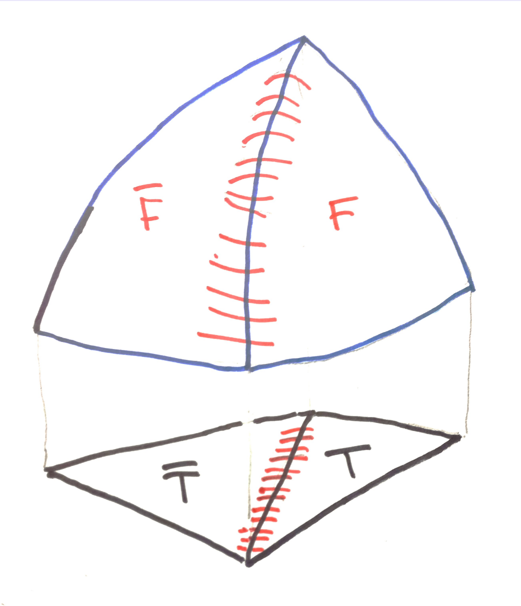
FŐ TÉTEL

Tfh. $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbb{R}^2$, $\mathbf{u}_k \nparallel [\mathbf{p_i}, \mathbf{p_j}]$.
Ekkor $\exists\ \zeta_1, \zeta_2, \zeta_3 \in \mathbb{R}$ *, amelyekkel az*

$$\begin{aligned} F(\mathbf{x}) := F_0(\mathbf{x}) &+ \zeta_1 \lambda_1(\mathbf{x}) \lambda_2(\mathbf{x})^2 \lambda_3(\mathbf{x})^2 + \\ &+ \zeta_2 \lambda_2(\mathbf{x}) \lambda_3(\mathbf{x})^2 \lambda_1(\mathbf{x})^2 + \\ &+ \zeta_3 \lambda_3(\mathbf{x}) \lambda_1(\mathbf{x})^2 \lambda_2(\mathbf{x})^2 \end{aligned}$$

polinom $F(\mathbf{x})$ *értékei ill.* $F'(\mathbf{x})\mathbf{u}_k$ *irány szerinti deriváltjai a* $[\mathbf{p}_i, \mathbf{p}_j]$ *oldalon*
független a $\mathbf{p}_k$ csúcs helyétől.

Sima illeszkedés az u-irányokban



A módosító együtthatók

$$\zeta_k = -\frac{1}{G_k \mathbf{u}_k}\Big[30\big([G_i\mathbf{u}_k]f_i + [G_j\mathbf{u}_k]f_j\big) + \\ + 12\Big([G_i\mathbf{u}_k]A_i + [G_j\mathbf{u}_k]A_j\Big)(\mathbf{p}_i - \mathbf{p}_j\Big]$$

ahol $(i, j, k) \in \mathrm{S}_3 = \big\{1, 2, 3 \text{ perm.}\big\}$

$$G_i := \lambda_i' : \mathbb{R}^2 \to \mathbb{R} \ \text{ lin.}$$

F' az éleken z u-irányokban

$$G_i\mathbf{u} := \lambda_i'(\mathbf{x})\mathbf{u} = \frac{d}{dt}\Big|_{t=0}\lambda_i(\mathbf{x}+t\mathbf{u}) =$$

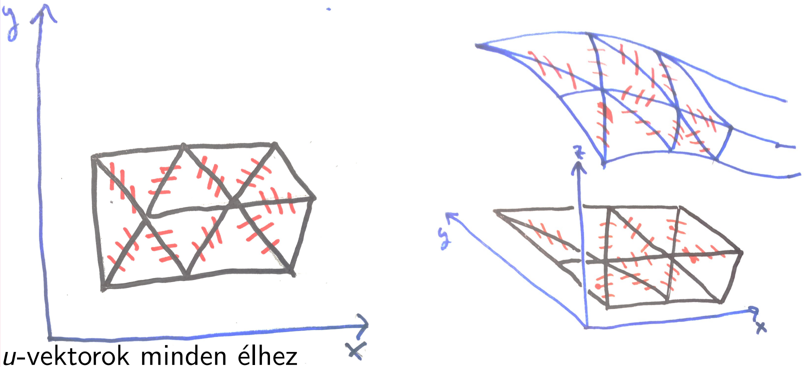
$$= \lambda_i(\mathbf{p}_i+\mathbf{u}) - 1 \quad \mathbf{x}\text{-től fgtl. jól-def;}$$

$$\mathbf{x}_t = t\mathbf{p}_i + (1-t)\mathbf{p}_j \Rightarrow$$

$$F'(\mathbf{x}_t)\mathbf{u}_k = \Theta(t)A_i\mathbf{u}_k + \Theta(1-t)A_j\mathbf{u}_k$$

$\mathcal{C}^1$-szplájn 5-ödfokú polinomokkal

2D háromszög-hálón



u-vektorok minden élhez

Pólya György
MI IS TÖRTÉNT ITT VOLTAKÉPPEN?

Φ, Θ helyett általánosabb fgv-ek

$$\Psi_0, \Psi_1 \in \mathcal{C}^1\big([0,1]\big)$$

Könnyű: F_0 "jól működjön" $\rightarrow$
$\Psi_0(0)=\Psi_1(0)=0,\quad \Psi_0'(0)=\Psi_1'(0)=0,$
$\Psi_0(1)=\Psi_1(1)=1,\quad \Psi_0'(1)=0,\quad \Psi_1'(1)=*.$

$$F_0 = \sum_{k=1}^{3} \Big[\Psi_0\big(\lambda_k\big) f_k + \Psi_1\big(\lambda_k\big) A_k(\mathbf{x} - \mathbf{p}_k)\Big]$$

Módosítás $\left(\zeta_k \lambda_i^2 \lambda_j^2 \lambda_k \to ? \right)$

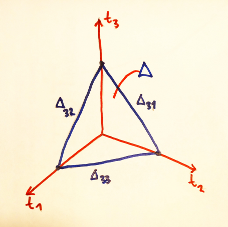
$$F := F_0 + H$$

$$H := \sum_{\{\ell,m,n\} \in \mathrm{S}_3} \left\{ f_\ell \frac{G_\ell \mathbf{u}_n}{G_n \mathbf{u}_n} \chi_0(\lambda_\ell, \lambda_m, \lambda_n) + \right.$$

$$\left. + A_\ell (\mathbf{p}_m - \mathbf{p}_\ell) \frac{G_\ell \mathbf{u}_n}{G_n \mathbf{u}_n} \chi_1(\lambda_\ell, \lambda_m, \lambda_n) \right\}$$

SOKFÉLE $\quad \chi_0, \chi_1 \in \mathcal{C}_0^1(\mathbb{R}_+^3)$

Alkalmas χ fgv-ek



$$\Delta_3 = \{[t_1, t_2, t_3] \in \mathbb{R}_+^3 : \\ t_1 + t_2 + t_3 = 1\}$$

(1) $\chi_m(\Delta_3 \text{ élein}) = 0,$

(2) $D_3\chi_0(t, 1-t, 0) = \Psi_0'(t),$

(3) $D_3\chi_1(t, 1-t, 0) = \Psi_1'(t) \cdot (1-t),$

a $\Delta_{3,k} = \{(t_1, t_2, t_3) \in \Delta_3 : t_k = 0\}$ éleken

(4) $\chi_r'(\mathbf{t}) = 0 \quad (\mathbf{t} \in \Delta_{3,1} \cup \Delta_{3,2}),$

(5) $D_m\chi_r(\mathbf{t}) = 0 \quad (\mathbf{t} \in \Delta_{3,3},\ m = 1, 2).$

PÉLDÁK

$$\Psi_0'(t) = w_{01}(t)w_{02}(1-t),$$

$$\Psi_1'(t)(1-t) = w_{11}(t)w_{12}(1-t)$$

$$w_{01}, w_{02}, w_{11}, w_{12} \in \mathcal{C}_0^1(\mathbb{R}_+),$$

$$w_{rk}(0) = w_{rk}'(0) = 0.$$

($\mathbf{a}$) Eredetileg: $\Psi_0 := \Phi$, $\Psi_1 := \Theta$,

$\Phi'(t) = 30t^2(1-t)^2$, $\Theta'(t) = 12t^2(1-t)$;

$\chi_0(t_1,t_2,t_3) = 30t_1^2 t_2^2 t_3$, $\chi_1(t_1,t_2,t_3) = 12t_1^2 t_2^2 t_3$,

$w_{01}(t) = 30t^2$, $w_{11}(t) = 12t^2$, $w_{02}(t) = w_{12}(t) = t^2$.

($\mathbf{b}$) $\Psi_0 := \Psi_1 := \Phi$,
$\chi_0(t_1,t_2,t_3) = \chi_1(t_1,t_2,t_3) = 30t_1^2 t_2^2 t_3$,

$w_{01}(t) = w_{11}(t) = 30t^2$, $w_{02}(t) = t^2$, $w_{12}(t) = t^3$.

Geometriai tulajdonságok

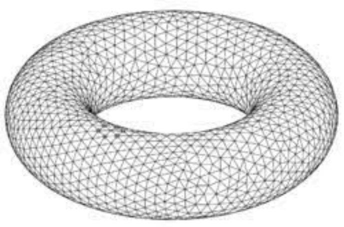
(a),(b)-nél az $\equiv 1$ fgv.-hez $F \equiv 1$

$\left[f_1 = f_2 = f_3 = 1, \quad A_1 = A_2 = A_3 \equiv 0\right]$;
$\to$ ELTOLÁS-INVARIANCIA (a),(b)-nél

(b) 6-odfokú, AZONBAN

$F \equiv x$ ill. $F \equiv y$ az x, y koord fgv-ek adataival,
$\to$ SÍKHÁROMSZÖG grafikon invarianciája

Az igazi kihívás: 3D



$\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_N \subset \mathbb{R}^3$ háromszögek

$\mathbf{T}_n = \mathrm{Conv}\left\{\mathbf{p}_{i_{n,1}}, \mathbf{p}_{i_{n,2}}, \mathbf{p}_{i_{n,3}}\right\}$

$$\text{FELÜLET} = \bigcup_{n=1}^{N} \mathbf{S}_n,$$

$\mathbf{S}_n = f_n(\mathbf{T}_n), \quad f_n : \mathbf{T}_n \to \mathbb{R}^3$

$$\mathbf{S}_n = \left\{ f_n\left(\sum_{m=1}^{3} t_m \mathbf{p}_{i_{k,m}}\right) : [t_1, t_2, t_3] \in \Delta_3 \right\}$$

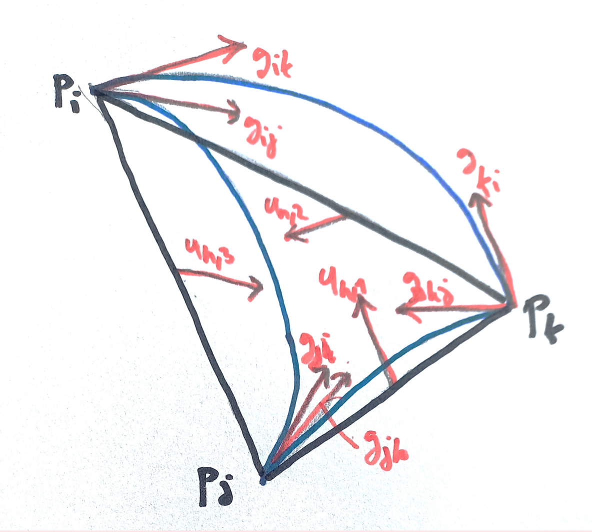
Adatok

(1) $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_M$ pontok,
(2) $\left[i_{k,1}, i_{k,2}, i_{k,1}\right]$ $(k = 1, \ldots, N)$,
(3) $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_M$ egységvektorok
$\mathbf{n}_i \approx \left[\text{FELÜLET normálisa } \mathbf{p}_i\text{-nél}\right]$.

(4) $\mathbf{g}_{i,j} \perp \mathbf{n}_i$ úgy hogy
$\mathbf{p}_i, \mathbf{p}_j$ vmely T_k háromszög csúcsai,
$\mathbf{g}_{i,j} \approx \frac{d}{dt}\Big|_{t=0} f\big(\mathbf{p}_i + t(\mathbf{p}_j - \mathbf{p}_i)\big)$.

(5) $\mathbf{u}_{n,j} \in \mathrm{Span}\{\mathbf{x} - \mathbf{y} : \mathbf{x}, \mathbf{y} \in \mathbf{T}_n\}$ vektor $\not\parallel$ $\mathbf{T}_n$ $\mathbf{p}_{n,j}$-vel szembeni oldalával

$$\mathbf{S} = \mathbf{S}_n, \quad \mathbf{T} = \mathbf{T}_n = \mathrm{Conv}\{\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k\}$$



$$\mathbf{S} \approx \{\mathcal{F}(\mathbf{x}) : \mathbf{x} \in \mathbf{T}\}, \quad \mathcal{F} = \mathcal{F}_0^{\mathbf{T}} + \mathcal{H}^{\mathbf{T}}$$

$$\begin{aligned}\mathcal{F} = \Big[F \,\Big|\, & \lambda_1 \to \lambda_{\mathbf{p}_i}^{\mathbf{T}}, \lambda_2 \to \lambda_{\mathbf{p}_j}^{\mathbf{T}}, \lambda_3 \to \lambda_{\mathbf{p}_k}^{\mathbf{T}}, \\ & f_1 \to \mathbf{p}_i, f_2 \to \mathbf{p}_j, f_2 \to \mathbf{p}_j, \\ & A_1(\mathbf{p_2} - \mathbf{p_1}) \to \mathbf{g}_{i,j}, \ldots, \\ & A_1(\mathbf{x} - \mathbf{p}_1) \to \lambda_{\mathbf{p}_j}^{\mathbf{T}}(\mathbf{x})\mathbf{g}_{i,j} + \lambda_{\mathbf{p}_k}^{\mathbf{T}}(\mathbf{x})\mathbf{g}_{i,k}, \ldots, \\ & \mathbf{u}_1 \to \mathbf{u}_{n,1}, \mathbf{u}_2 \to \mathbf{u}_{n,2}, \mathbf{u}_3 \to \mathbf{u}_{n,3}, \\ & G_1 \to \left[\lambda_{\mathbf{p}_i}^{\mathbf{T}}\right]', G_2 \to \left[\lambda_{\mathbf{p}_j}^{\mathbf{T}}\right]', G_3 \to \left[\lambda_{\mathbf{p}_k}^{\mathbf{T}}\right]' \Big]\end{aligned}$$

Szomszédos darabok illeszkednek,
de esetleg *nem simán*

TÉTEL: Sima illesztés közös éleknél

$T = \mathrm{Conv}\{\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k\}$, $\overline{T} = \mathrm{Conv}\{\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_{\overline{k}}\}$ szomszédos Δ-ek,

$$\mathcal{F} = \mathcal{F}^{\mathbf{T}} : \mathbf{T} \to \mathbb{R}^3, \quad \overline{\mathcal{F}} = \mathcal{F}^{\overline{\mathbf{T}}} : \overline{\mathbf{T}} \to \mathbb{R}^3.$$

Ha $\mathcal{F}, \overline{\mathcal{F}}$ bijektívek és $\mathcal{F}'(\mathbf{x})$ ill. $\overline{\mathcal{F}}'(\bar{\mathbf{x}})$ nem-elfajuók
$\mathbf{T}$ ill. $\overline{\mathbf{T}}$ minden $\mathbf{x}$ ill. $\bar{\mathbf{x}}$ pontjánál, akkor található olyan

$$U \in \mathrm{RacPol}_{16}^{12}(\mathbb{R}, \mathbb{R}^3),$$

max. 12-edfokú számlálóval ill. 16-odfokú nevezővel, hogy

$$\mathcal{U} := \left[\lambda_{\mathbf{p}_i}^{\mathbf{T}} \lambda_{\mathbf{p}_j}^{\mathbf{T}}\right]^2 U\left(\lambda_{\mathbf{p}_k}^{\mathbf{T}}\right),$$
$$\overline{\mathcal{U}} := -\left[\lambda_{\mathbf{p}_i}^{\overline{\mathbf{T}}} \lambda_{\mathbf{p}_j}^{\mathbf{T}}\right]^2 U\left(\lambda_{\mathbf{p}_{\overline{k}}}^{\overline{\mathbf{T}}}\right)$$

a $\mathrm{range}\left(\mathcal{F} + \mathcal{U}\right)$ ill. $\mathrm{range}\left(\overline{\mathcal{F}} + \overline{\mathcal{U}}\right)$ felületek simán illeszkednek.

BOLDOG
80. SZÜLETÉSNAPOT
KEDVES JÓSKA