On Locally generated C1-spline surfaces covering triangular meshes of scanned 3D points

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Abstract.
Given a system of triangles with planar type graph in the 3-dimensional space obtained with scanned data from a 2-dimensional surface, we describe local linear procedures resulting in a surface consisting of curved 3D triangles parametrized in terms of at most 5-th degree rational polynomials of the barycenric weight functions of the given 3D triangles which passes through the vertices with C1-smooth coupling along the edges. We also investigate the degrees of freedom in case of non-planar graphs and establish the possibly non-polynomial generic form for the shape functions in the local steps of the algorihm.

AMS Classification: 65D07; 41A15; 65D15.

Keywords: $\mathcal{C}^1$-spline, triangular mesh, scanned data, planar graph

Recently [MDPI] we published a $\mathcal{C}^1$-spline interpolation method over 2-dimensional triangular meshes with polynomials of 5-th degree with low operational costs: using first order data (value and gradient) at the mesh vertices for input, it algorithm consists of less than 100 times the number of mesh triangles elementary operations.
Basic tool: given a non-degenerate triangle with vertices $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \in \mathbb{R}^2$ along with three values $f_1, f_2, f_2 \in \mathbb{R}$ respectively three linear functionals $A_1, A_2, A_3 \in \mathcal{L}(\mathbb{R}^2, \mathbb{R})$ and three vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbb{R}^2$ such that $\mathbf{u}_k \not\parallel \mathbf{p}_i - \mathbf{p}_j$ if i, j, k are different indices, we construct a polynomial $F : \mathbb{R}^2 \to \mathbb{R}$ of 5-th degree of the form

$$F = F_0 - H, \quad F_0(\mathbf{x}) = \sum_{i=1}^{3} \Big[\Phi\big(\lambda_i(\mathbf{x})\big) f_i + \Theta\big(\lambda_i(\mathbf{x})\big) A_i(\mathbf{x} - \mathbf{p}_1) \Big],$$
$$H(\mathbf{x}) = \lambda_1(\mathbf{x})^2 \lambda_1(\mathbf{x})^2 \lambda_1^2(\mathbf{x})^2 \sum_{k=1}^{3} \lambda_k(\mathbf{x})^{-1} \frac{M_k \mathbf{u}_k}{G_k \mathbf{u}_k}. \tag{F}$$

Here $\lambda_1, \lambda_2, \lambda_3 : \mathbb{R}^2 \to \mathbb{R}$ are the *barycentric weights* associated with the vertices $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ while the correction factors are defined in terms of the given data by means of the linear functionals $\mathbb{R}^2 \to \mathbb{R}$

$$G_k \mathbf{u} = \lambda_k' \mathbf{u} = \frac{d}{dt}\Big|_{t=0} \lambda_k(\mathbf{x} + t\mathbf{u}) = \lambda_k(\mathbf{x} + \mathbf{u}) - \lambda_k(\mathbf{x}), \quad G_k(\mathbf{p}_n - \mathbf{p}_m) = \delta_{kn} - \delta_{km},$$
$$M_k \mathbf{u} = [G_i \mathbf{u}]\big(30 f_i + 12 A_i(\mathbf{p}_j - \mathbf{p}_i)\big) + [G_j \mathbf{u}]\big(30 f_j + 12 A_j(\mathbf{p}_i - \mathbf{p}_j)\big) \quad \big(i \neq j \neq k \neq i\big)$$

and we used the *shape functions*

$$\Phi(t) = t^3(10 - 15t + 6t^2), \qquad \Theta(t) = t^3(4 - 3t).$$

It is straighforward that the interpolation function F_0 in (F) on the edge $[\mathbf{p}_i, \mathbf{p}_k]$ depends only on the terms f_i, f_j, A_i, A_j and the correction H results in the *reduced side derivatives* (RSD)

$$F'(\mathbf{x}_t)\mathbf{u_k} = \Theta(t)A_i\mathbf{u}_k + \Theta(1-t)A_j\mathbf{u}_k \quad \text{for} \;\; \mathbf{x}_t = t\mathbf{p}_i + (1-t)\mathbf{p}_j.$$

Hence, given any triangular mesh (a family of non-degenerate closed triangles with pairwise disjoint interiors whose intersections are common edges or vertices or empty, vertices in the relative interior of any edge are not admitted) with any given value and gradient data f_i, A_i at the vertices and any given vectors $\mathbf{u}_{ij} = \mathbf{u}_{ji} \not\parallel [\mathbf{p}_i, \mathbf{p}_j$ associated with mesh edges, an application of the union of the interpolation functions obtained with the construction (F) applied to the mesh triangles is necessarily continuously differentiable.

The shape functions Φ, Θ and also the (RSD) method described above appeared in [MDPI] without heuristical introcuction. Actually they arose from our earlier study [STA] with somewhat restrictive postulates on the possible polynomial $\mathcal{C}^1$-interpolations over triangular meshes.
Our first goal, in Section 2, is the description of (RSD) interpolation methods over 2D triangular complexes in 3D in terms of data given in *intrinsic coordinates* of the mesh and not necessarily polynomial shape functions. In the remaing parts, in Sections 3-5 Our main purpose will be to test the results in a classical area of applications: reconstruction of a smooth 2D suface in 3D from a family of coordiates and normal vectors at the vertices of a triangularization of the surface. The related literature started to grow since the 1960-s and recently it became enormous. Typical applications appear in computer animation and mechanical engineering with finite element methods. It seems that, with the rapid increase of computing capacity, less and less attention is paid to computational complexity. As we emphasized, our work [MDPI] focuses to "minimalist" approaches.

We are going to investigate three natural strategies:

(S1) Approximate the surface with the graph of a $\mathcal{C}^1$ (continuously differentiable) function;

(S2) If the surface is topologically equivalent (homeomorphic) to a plain figure then we can approximate it with a parametrized surface (possibly self-crossing) whose three coordinate functions are $\mathcal{C}^1$-splines over a suitable triangular mesh;

(S3) A local method for approximating a generic $\mathcal{C}^1$-submanifold (like spheres or tori) possibly with perimeter (like Möbius bands). As input data we use the *triangular complex* with vertices of a given non-linear triangularization of the surface along with given vectors associated to directed edges of the triangular complex. First we apply the main part F_0 of the (RSD) consreuction with the barycentric weights of each triangle. Automatically, this gives rise to non-linear $\mathcal{C}^1$-triangles with vertices from the triangular complex and with directional derivatives coinciding to the given vectors associated to the directed edges starting from the vertex and being such that neighboring pairs are coupled continuously. To finish the procedure, independently for each mesh edge, we can establish equations for determining directions (like $\mathbf{u}_k$ in (F)) for the correction tems to obtain $\mathcal{C}^1$-smooth coupling. Notice that the latter equations may admit no solution, thus (RSD)-methods have own limits.

It is worth to remark that the graph of a function constructed with (S1) has the form $\{[\mathbf{x}, F(\mathbf{x})] : \mathbf{x} \in \mathbf{T}\}$ over any mesh triangle $\mathbf{T} = \mathrm{Conv}\{\mathbf{p}_1, \mathbf{p}_1, \mathbf{p}_3\}$. On the other hand, if we apply (S2) over the same mesh for approximating the functions x, y (the canonical coordinates) resp f with X, Y resp. F, then we obtain a surface of the form $\{[X(\mathbf{x}), Y(\mathbf{x}), F(\mathbf{x})] : \mathbf{x} \in \mathbf{T}\}$ over the mesh triangles. Is it possible to find an (RSD) method where these two graps coincide? The answer is: YES. Namely by replacing (F) with

$$\widetilde{F} = \widetilde{F}_0 - \widetilde{H}, \qquad \widetilde{F}_0 = \sum_{(i,j,k)\in\mathcal{I}} \Phi(\lambda_k)\big[f_k + A_k(\mathbf{x} - \mathbf{p}_k)\big],$$

$$\widetilde{H} = \sum_{(i,j,k)\in\mathcal{I}} 30\lambda_i^2\lambda_j^2\lambda_k \left\{ \frac{[G_i\mathbf{u}_k]}{[G_k\mathbf{u}_k]}\big[f_i + \lambda_i A_i(\mathbf{p}_j - \mathbf{p}_i)\big] + \frac{[G_j\mathbf{u}_k]}{[G_k\mathbf{u}_k]}\big[f_j + \lambda_j A_i(\mathbf{p}_i - \mathbf{p}_j)\big] \right\}$$

we obtain a $\mathcal{C}^1$-spline interpolation procedure which leaves all affine functions invariant.

2. Generic algorithm of reduced side derivatives (RSD)

In this section we are looking for $\mathcal{C}^1$-spline interpolation procedures analogous to that described in [MDPI] with more general shape functions $\Psi_0, \Psi_1 \in \mathcal{C}^1[0,1]$ instead of Φ, Θ there. As for standard notations, througout this work let $\mathbf{T} = \mathrm{Conv}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} \subset \mathrm{I\!R}^2$ be a non-degenerate triangle with barycentric weights

$$\lambda_i = \lambda^{\mathbf{T}}_{\mathbf{p}_i} : \mathbf{x} \mapsto \frac{\det\big[\mathbf{x}-\mathbf{p}_j, \mathbf{x}-\mathbf{p}_k\big]}{\det\big[\mathbf{p}_i-\mathbf{p}_j, \mathbf{p}_i-\mathbf{p}_k\big]} \quad \text{for } \{i,j,k\} = \{1,2,3\}.$$

For the sake of brevity, in the sequel we omit the background parameters in most formulas (like writing $\lambda_i = \lambda^{\mathbf{T}}_{\mathbf{p}_i}$ above) without danger of confusion. It is well-known that $\lambda_1, \lambda_2, \lambda_3 : \mathrm{I\!R}^2 \to \mathrm{I\!R}$ are affine (linear+constant) functions with necessarily constant Fréchet derivative $G_i \in \mathcal{L}(\mathrm{I\!R}^2, \mathrm{I\!R})$, the linear functionals

$$G_i\mathbf{u} := \lambda_i'(\mathbf{x}) = \frac{d}{dt}\Big|_{t=0} \lambda_i(\mathbf{x}+t\mathbf{u}) \quad \text{independently of } \mathbf{x}.$$

Given any pair $\Psi[\Psi_0, \Psi_1]$ of continuosly differentialble functions $[0,1] \to \mathrm{I\!R}$ such that

$$(PSI) \qquad \Psi_0(0) = \Psi_1(0) = 0, \quad \Psi_0'(0) = \Psi_1'(0) = 0, \quad \Psi_0(1) = \Psi_1(1) = 1, \quad \Psi_1'(0) = 0$$

we are going to consider the *basic triangular interpolation of first order with shape functions* Ψ: Given any numbers $f_1, f_2, f_3 \in \mathrm{I\!R}$ along with any three linear functionals $A_1, A_2, A_3 \in \mathcal{L}(\mathrm{I\!R}^2, \mathrm{I\!R})$, we construct the continuously differentiable function

$$\big[\Psi, \mathbf{P}, \mathbf{f}, \mathbf{A}\big] \mapsto {}^{\Psi}F_{0}{}^{\mathbf{f},\mathbf{A}}_{\mathbf{P}} \in \mathcal{C}^1(\mathbf{T})$$

defined on the triangle $\mathbf{T}$ as

$$F_0 = {}^{\Psi}F_{0}{}^{\mathbf{f},\mathbf{A}}_{\mathbf{P}} := \sum_{i=1}^{3} \Big\{ \Psi_0\big(\lambda_i\big) f_i + \Psi_1\big(\lambda_i\big) A_i\big(\mathbf{x}-\mathbf{p}_i\big) \Big\}$$

where $\Psi = [\Psi_0, Psi_1]$, $\mathbf{P} = [\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3]$, $\mathbf{f} = [f_1, f_2, f_3]$, $\mathbf{A} = [A_1, A_2, A_3]$ for short. Notice that

$$(DER) \qquad F_0' := \sum_{i=1}^{3} \Big\{ \big[\Psi_0'\big(\lambda_i\big) f_i + \big[\Psi_1'\big(\lambda_i\big) A_i\big(\mathbf{x}-\mathbf{p}_i\big)\big] G_i + \Psi_1\big(\lambda_i\big) A_i \Big\}$$

Remark. In view of (DER), it is not hard to see that conditions (PSI) are suffiecient and necessary to establish that for $F_0 = {}^{\Psi}F_{0}{}^{\mathbf{f},\mathbf{A}}_{\mathbf{P}}$ we have

$$(POST) \qquad F_0\big(\mathbf{p}_i\big) = f_i, \quad F_0'\big(\mathbf{p}_i\big) = A_i \qquad (i = 1,2,3)$$

under any choice of $\Psi, \mathbf{f}$ and $\mathbf{A}$.

Recall that a function $h : \mathbf{T} \to \mathbb{R}$ is *Lipschitzian* ($h \in \mathrm{Lip}(\mathbf{T})$) if $\sup_{\mathbf{x}_1,\mathbf{x}_2 \in \mathbf{T}} |h(\mathbf{x}_1) - h(\mathbf{x}_2)| < \infty$. In particular if $h : \mathbf{T} \to \mathbb{R}$ is a function which is continuously differentiable on the interior $\mathbf{T}^o := \{\mathbf{x} : \min_i \lambda_i(\mathbf{x}) > 0\}$ of the triangle T then we have $h \in \mathrm{Lip}(\mathbf{T})$ if and only if $h \in \mathrm{C}(\mathbf{T}$ and $\sup_{\mathbf{x}\in\mathbf{T}^o} \|h'(\mathbf{x})\| < \infty$.

Lemma. *Assume $h \in \mathcal{C}(\mathbf{T})$ has bounded continuous Fréchet derivative on $\mathbf{T}^o$. Then*

$$H := \lambda_1\lambda_2\lambda_3 h$$

belongs to $\mathcal{C}^1(\mathbf{T})$ (that is h' extends continuously to the closed triangle $\mathbf{T}$ from $\mathbf{T}^o$) and given any three different indices we have

$$H'(\mathbf{x}) = \big[\lambda_i\lambda_j h\big](\mathbf{x})\, G_k \qquad \big(\mathbf{x} \in [\mathbf{p}_i, \mathbf{p}_j]\big) .$$

Proof. We may assume $i = 1, j = 2, k = 3$. Let $\mathbf{x} \in [\mathbf{p}_1, \mathbf{p}_2]$ consider any sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots \in \mathbf{T}^o$ such that $\mathbf{x}_n \to \mathbf{x}$. It suffices to see that

$$h(\mathbf{x}_n) \to 0 \quad \text{and} \quad H'(\mathbf{x}_n) \to \big[\lambda_1\lambda_1 h\big](\mathbf{x}) G_3.$$

Since $h, \lambda_1, \lambda_2, \lambda_3 \in \mathcal{C}(\mathbf{T})$ with $\lambda_3(\mathbf{x}) = 0$, the relation $H(\mathbf{x}_n) = [\lambda_1\lambda_2 h](\mathbf{x}_n)\,\lambda_3(\mathbf{x}_n) \to [\lambda_1\lambda_2 h](\mathbf{x}) \cdot \lambda_3(\mathbf{x}) = 0$ is immediate. On the other hand $h, \lambda_1, \lambda_2, \lambda_3$ are continuously differentiable on $\mathbf{T}^o$, thus we can write

$$h'(\mathbf{x}_n) = \big[\lambda_1\lambda_2\lambda_3\big](\mathbf{x})\, h'(\mathbf{x}_n) + \big[\lambda_1\lambda_2 h\big](\mathbf{x}_n)\, G_3 + \big[\lambda_2\lambda_3 h\big](\mathbf{x}_n)\, G_1 + \big[\lambda_3\lambda_1 h\big](\mathbf{x}_n)\, G_2.$$

Since $\mathbf{h}'$ is bounded on $\mathbf{T}^o$ by assumption, we get

$$[\lambda_1\lambda_2\lambda_3](\mathbf{x}_n)\, h'(\mathbf{x}_n) \to 0,\ \ \big[\lambda_1\lambda_2 h\big](\mathbf{x}_n) \to \big[\lambda_1\lambda_2 h\big](\mathbf{x}),\ \ \big[\lambda_2\lambda_3 h\big](\mathbf{x}_n) \to 0,\ \ \big[\lambda_3\lambda_1 h\big](\mathbf{x}_n) \to 0$$

implying that $H'(\mathbf{x}_n) \to \big[\lambda_1\lambda_2 h\big](\mathbf{x}) G_3$.

Lemma. *If $i, j, k \in \{1, 2, 3\}$ are theree different indices $F_0 = {}^{\Psi}F_{0}{}^{\mathbf{f},\mathbf{A}}_{\mathbf{P}}$ is a basic triangular interpolation procedure of first order, then at the generic point*

$$\mathbf{x}_t^k := t\mathbf{p}_i + (1-t)\mathbf{p}_j \qquad (0 \le t \le 1)$$

of the edge $[\mathbf{p}_i, \mathbf{p}_j]$ in the triangle $\mathbf{T}$ we have

$$\begin{aligned} F'\big(\mathbf{x}_t^k\big) = {} & \big[\Psi_0'(t) f_i + \Psi_1'(t)(1-t) A_i(\mathbf{p}_j - \mathbf{p}_i)\big] G_i + \Psi_1(t) A_i + \\ & + \big[\Psi_0'(1-t) f_j + \Psi_1'(1-t) t A_i(\mathbf{p}_j - \mathbf{p}_i)\big] G_i + \Psi_1(1-t) A_i. \end{aligned}$$

Proof. It is immediate from (DER) in view of the relations

$$(PAR) \qquad \begin{aligned} & \lambda_i\big(\mathbf{p}_j\big) = \delta_{ij}, \quad G_i\big(\mathbf{p}_j - \mathbf{p}_k\big) = \delta_{ij} - \delta_{ik}, \\ & \lambda_i(\mathbf{x}_t^k) = t, \quad \lambda_j(\mathbf{x}_t^k) = 1 - t, \quad \lambda_k(\mathbf{x}_t^k) = 0, \\ & \mathbf{x}_t - \mathbf{p}_i = (1-t)(\mathbf{p}_j - \mathbf{p}_i), \quad \mathbf{x}_t - \mathbf{p}_j = t(\mathbf{p}_i - \mathbf{p}_j). \end{aligned}$$

Remark. 1) The Postulates do not determine the value of $\Psi_1'(1)$.
2) If $H \in \mathcal{C}(\mathbf{T})$ admits a bounded continuous derivative on $\mathbf{T}^o$ then the the derivatives of function $\lambda_1\lambda_2\lambda_3 H \in \mathcal{C}^1(\mathbf{T})$ vanish at the vertices of the triangle $\mathbf{T}$ and hence the function $F := F_0 - \lambda_1\lambda_2\lambda_3 H$ has the also properties $F(\mathbf{p}_i) = f_i$, $F(\mathbf{p}_i) = A_i$ $(i = 1, 2, 3)$.

Definition. Given a non-degenerate triangle $\mathbf{T} := \mathrm{Conv}\big\{\mathbf{p}_1, \mathbf{p}_1, \mathbf{p}_1\big\}$ $F_0 \in \mathcal{C}^1(\mathbf{T})$ be a function satisfying the Postulates, and let $\mathbf{u}_1, \mathbf{u}_1, \mathbf{u}_1 \in \mathbb{R}^2$ be theree vectors such that $\mathbf{u}_k \not\parallel \mathbf{p}_i - \mathbf{p}_j$ whenever $\{i, j, k\} = \{1, 2, 3\}$. We say that a function of the form

$$(MOD) \qquad\qquad F := F_0 - H, \quad H \in \mathcal{C}_0^1(\mathbf{T})$$

is an *RSD modification* of F_0 (in the directions $\mathbf{u}_k$ along the edges $\big[\mathbf{p}_i, \mathbf{p}_j\big]$) if

$$F'\big(\mathbf{x}\big)\mathbf{u}_k = \Psi_1\big(\lambda_i(\mathbf{x})\big)A_i\mathbf{u}_k + \Psi_1\big(\lambda_j(\mathbf{x})\big)A_j\mathbf{u}_k \quad \big(\mathbf{x} \in \big[\mathbf{p}_i, \mathbf{p}_j\big],\ \{i, j, k\} = \{1, 2, 3\}\big).$$

Remark. By introducing the terms

$$g_{ij} = A_i(\mathbf{p}_j - \mathbf{p}_i) \quad (i, j = 1, 2, 3),$$

we can write

$$A_i = \sum_{j=1}^{2} g_{ij}\lambda_j G_j, \quad \mathbf{x} - \mathbf{p_i} = \sum_{j=1}^{3} \lambda_j(\mathbf{p_j} - \mathbf{p}_i) \quad (i = 1, 2, 3)$$

and hence

$$F_0 = \sum_{i=1}^{3} \Psi_0\big(\lambda_i\big) f_i + \sum_{i,j=1}^{3} \Psi_1\big(\lambda_i\big)\lambda_j g_{ij}.$$

$$\begin{aligned}
&\widehat{\bullet}\ : \mathcal{C}[0, 1] \to \mathcal{C}_0^1(\Delta), \qquad \Delta := \big\{[\xi, \eta] \in \mathbb{R}_+^2 : \xi + \eta \le 1\big\},\\
&\quad D_1\widehat{\psi}(0, t) = \psi(t),\ D_2\widehat{\psi}(0, t) = 0 \ \ (0 \le t \le 1),\\
&\quad D_m\widehat{\psi}(s, t) = 0 \ \text{ for } \ [s, t] \in (\partial\Delta)\setminus(\{0\} \times [0, 1]),\ m = 1, 2.
\end{aligned}$$

$$\begin{aligned}
H = \sum_{(i,j,k)\in\mathcal{I}} \Big\{ &\Big[\widehat{\psi_0}\big(\lambda_k, \lambda_i\big) f_i + \widehat{\psi_1}\big(\lambda_k, \lambda_i\big) g_{ij}\Big] \frac{[G_i\mathbf{u}_k]}{[G_k\mathbf{u}_k]} + \\
&+ \Big[\widehat{\psi_0}\big(\lambda_k, \lambda_j\big) f_j + \widehat{\psi_1}\big(\lambda_k, \lambda_j\big) g_{ji}\Big] \frac{[G_j\mathbf{u}_k]}{[G_k\mathbf{u}_k]} \Big\}
\end{aligned}$$

Lemma. *Let* $(i, j, k) \in \mathcal{I}$*,* $t \in [0, 1]$*,* $\mathbf{x}_t = t\mathbf{p}_i + (1 - t)\mathbf{p}_j$ *and* $\psi \in \mathcal{C}_0[0, 1]$*. Then*

$$\begin{aligned}
&\big[\widehat{\psi}\big(\lambda_k, \lambda_i\big)\big]'(\mathbf{x}_t) = \psi(t)G_k,\\
&\big[\widehat{\psi}\big(\lambda_k, \lambda_j\big)\big]'(\mathbf{x}_t) = \psi(1 - t)G_k,\\
&\big[\widehat{\psi}\big(\lambda_m, \lambda_n\big)\big]'(\mathbf{x}_t) = 0 \ \text{ for } \ m = i, j,\ n = 1, 2, 3.
\end{aligned}$$

Case of RSD procedures leaving affine functions invariant

During this subsection we study the case of interpolating the weight function $\lambda_1 : \mathbf{T} \to [0,1]$ with the RSD procedure using the first order data at the vertices of the triagle $\mathbf{T}$. That is let

$$f_1 = 1, \quad f_2 = f_3 = 0, \quad A_1 = A_2 = A_3 = G_1$$

and we are looking for explicit forms for the shape functions Ψ_0, Ψ_1 giving rise to an RSD procedure such that

$$\lambda_1 = F_0 - H = {}^{[\Psi_0,\Psi_1]}F^{[1,0,0],[G_1,G_1,G_1]}_{0,[\mathbf{p_1},\mathbf{p_2},\mathbf{p_3}]} - {}^{[\Psi_0,\Psi_1]}H^{[1,0,0],[G_1,G_1,G_1]}_{[\mathbf{p_1},\mathbf{p_2},\mathbf{p_3}],[\mathbf{u_1},\mathbf{u_2},\mathbf{u_3}]}$$

under any admissible choice of the side vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. Observe that, in this case, we simply have $g_{12} = g_{13} = -1$, $g_{21} = g_{31} = 1$ and $g_{11} = g_{22} = g_{33} = 0$ implying that

$$\begin{aligned} {}^{[\Psi_0,\Psi_1]}F^{[1,0,0],[G_1,G_1,G_1]}_{0,[\mathbf{p_1},\mathbf{p_2},\mathbf{p_3}]} &= \Psi_0(\lambda_1) - \Psi_1(\lambda_1)\lambda_2 - \Psi_1(\lambda_1)\lambda_3 + \Psi_1(\lambda_2)\lambda_1 + \Psi_1(\lambda_3)\lambda_1 = \\ &= \Psi_0(\lambda_1) - (\lambda_2+\lambda_3)\Psi_1(\lambda_1) + \lambda_1\big[\Psi_1(\lambda_2) + \Psi_1(\lambda_3)\big]. \end{aligned}$$

Therefore we have to study the identity

$$\begin{aligned} \lambda_1 = \Psi_0(\lambda_1) - (\lambda_2+\lambda_3)\Psi_1(\lambda_1) + \lambda_1\big[\Psi_1(\lambda_2) + \Psi_1(\lambda_3)\big] - \\ - \Big\{ \Big[\widehat{\psi_0}(\lambda_3,\lambda_1) + \widehat{\psi_1}(\lambda_3,\lambda_1)(-1)\Big] \frac{[G_1\mathbf{u}_3]}{[G_3\mathbf{u}_3]} + \widehat{\psi_0}(\lambda_3,\lambda_1)\frac{[G_2\mathbf{u}_3]}{[G_3\mathbf{u}_3]} + \\ + \widehat{\psi_1}(\lambda_2,\lambda_3)\frac{[G_3\mathbf{u}_2]}{[G_2\mathbf{u}_2]} + \Big[\widehat{\psi_0}(\lambda_2,\lambda_1) + \widehat{\psi_1}(\lambda_2,\lambda_1)(-1)\Big]\frac{[G_1\mathbf{u}_2]}{[G_2\mathbf{u}_2]} \Big\}. \end{aligned}$$

Definition. Given a non-degenerate triangle $\mathbf{T} := \mathrm{Conv}\{\mathbf{p}_1, \mathbf{p}_1, \mathbf{p}_1\}$ $F_0 \in \mathcal{C}^1(\mathbf{T})$ be a function satisfying the Postulates, and let $\mathbf{u}_1, \mathbf{u}_1, \mathbf{u}_1 \in \mathbb{R}^2$ be theree vectors such that $\mathbf{u}_k \not\parallel \mathbf{p}_i - \mathbf{p}_j$ whenever $\{i,j,k\} = \{1,2,3\}$. We say that a function of the form

$$(MOD) \qquad F := F - \lambda_1\lambda_2\lambda_3 H$$

is an *RSD modification of first order* of F_0 (in the directions $\mathbf{u}_k$ along the edges $[\mathbf{p}_i, \mathbf{p}_j]$) if $H \in \mathcal{C}(\mathbf{T})$ admits a bounded continuous derivative on $\mathbf{T}^o$ and

$$F'(\mathbf{x})\mathbf{u}_k = \Psi_1(\lambda_i(\mathbf{x}))A_i\mathbf{u}_k + \Psi_1(\lambda_j(\mathbf{x}))A_j\mathbf{u}_k \quad \big(\mathbf{x} \in [\mathbf{p}_i, \mathbf{p}_j],\ \{i,j,k\} = \{1,2,3\}\big)$$

Lemma. *The function $F_0 = {}^{\Psi}F_{0\mathbf{P}}^{\mathbf{f},\mathbf{A}}$ satisfying the Postulates with $f_1 = 1,\ f_2 = f_3 = 0$ and $A_1 = A_2 = A_3 = 0$ admits an RSD modification if*

Lemma. *Let be. Assume $H \in \mathcal{C}(\mathbf{T})$ has bounded continuous Fréchet derivative on $\mathbf{T}^o$. Then given any three different indices $i,j,k \in \{1,2,3\}$, for the functon $h := \lambda_1\lambda_2\lambda_3 H \in \mathcal{C}^(\mathbf{T})$ we have*

$$h'(\mathbf{x}) = \big[\lambda_i\lambda_j H\big](\mathbf{x})\, G_k \qquad \big(\mathbf{x} \in [\mathbf{p}_i, \mathbf{p}_j]\big) .$$

Proof. We may assume $i = 1, j = 2, k = 3$. Let $\mathbf{x} \in [\mathbf{p}_1, \mathbf{p}_2]$ consider a sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots \in \mathbf{T}^o$ such that $\mathbf{x}_n \to \mathbf{x}$. It suffices to see that $h(\mathbf{x}_n) \to 0$ and $h'(\mathbf{x}_n) \to \big[\lambda_1\lambda_1 H\big](\mathbf{x}_n)G_3$.
Since $H, \lambda_1, \lambda_2, \lambda_3 \in \mathcal{C}(\mathbf{T})$ with $\lambda_3(\mathbf{x}) = 0$, the relation $h(\mathbf{x}_n) = [\lambda_1\lambda_2 H](\mathbf{x}_n)\,\lambda_3(\mathbf{x}_n) \to [\lambda_1\lambda_2 H](\mathbf{x}) \cdot \lambda_3(\mathbf{x}) = 0$ is immediate. On the other hand $H, \lambda_1, \lambda_2, \lambda_3$ are continuously differentiable, thus we can write

$$h'(\mathbf{x}_n) = \big[\lambda_1\lambda_2\lambda_3\big](\mathbf{x})\,H'(\mathbf{x}_n) + \big[\lambda_1\lambda_2 H\big](\mathbf{x}_n)\,G_3 + \big[\lambda_2\lambda_3 H\big](\mathbf{x}_n)\,G_1 + \big[\lambda_3\lambda_1 H\big](\mathbf{x}_n)\,G_2.$$

Since $\mathbf{H}'$ is bounded on $\mathbf{T}^o$, we get

$$[\lambda_1\lambda_2\lambda_3](\mathbf{x}_n)\,H'(\mathbf{x}_n) \to 0,\ \ [\lambda_1\lambda_2 H](\mathbf{x}_n) \to [\lambda_1\lambda_2 H](\mathbf{x}),\ \ [\lambda_2\lambda_3 H](\mathbf{x}_n) \to 0,\ \ [\lambda_3\lambda_1 H](\mathbf{x}_n) \to 0$$

implying that $h'(\mathbf{x}_n) \to \big[\lambda_1\lambda_2 H\big](\mathbf{x})G_3$.