Strongly continuous one-parameter groups of holomorphic unit ball automorphisms

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PROBLEM HISTORY

J. Jamison - F. Botelho 2008

$\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(N)}$ complex Banach spaces

$\mathcal{B} := \{\text{bounded } N\text{-linear maps } \mathbf{X}^{(1)} \times \cdots \times \mathbf{X}^{(N)} \to \mathbb{C}\}$

$\mathbf{U} : \mathbb{R} \to \mathfrak{A} := \{\text{surjective linear isometries } \mathcal{B} \to \mathcal{B}\}$

strongly cont. 1-parameter group of surj. lin. isom.

$\mathbf{U}(t+h) = \mathbf{U}(t)\mathbf{U}(h) \qquad (t, h \in \mathbb{R})$

$t \mapsto \mathbf{U}(t)\Phi$ continuous $\qquad (\Phi \in \mathcal{B})$

Conjecture:

$$\mathbf{U}(t) = \underbrace{U_1^t \otimes \cdots \otimes U_N^t}_{\phi \mapsto \phi(U_1^t x_1, \ldots, U_N^t x_N)}$$

$t \mapsto U_k^t$ *str.cont. 1-par.grp. in*
$\mathfrak{A}_k := \{$*surj.lin.isom.* $\mathbf{X}^{(k)} \to \mathbf{X}^{(k)}\}$

NON-LINEAR ISSUE

$$\left\{\text{Surj.lin.isom. of } \mathbf{X}\right\} = \left\{F \in \underbrace{\mathrm{Aut}\big(\mathrm{Ball}(\mathbf{X})\big)}_{\textit{hol.aut. of unit ball}} \ : \ F(0)=0\right\}$$

$$\|\phi\| = \sup_{\|x_1\|=1,\dots,\|x_N\|=1} \big|\phi(x_1,\dots,x_N)\big| \qquad \text{in} \quad \mathcal{B}$$

Question: What about *strongly cont. 1-par. groups* in $\mathrm{Aut}(\mathcal{B})$?

Conjecture: $\mathrm{Aut}(\mathcal{B})$ LINEAR for $N > 2$ factors

↑

Stachó 1982: TRUE with HILBERT spaces of dim>1

Hilbert cases $N = 1, 2$: JB*-triples $\mathbf{H}$ resp. $\mathcal{L}\big(\mathbf{H}^{(1)}, \mathbf{H}^{(2)}\big)$

LINEAR CASE: HILBERT SPACES $N \geq 2$, [Stachó 2010]

$\mathbf{H}^{(1)}, \ldots, \mathbf{H}^{(N)}$ Hilbert spaces

$\mathrm{Aut}_0(\mathcal{B}) = \big\{[U_1 \otimes \cdots \otimes U_N] \circ I_\pi : \ U_k \in \mathcal{U}(\mathbf{H}^{(k)}), \ \pi \text{ adm.index-perm.}\big\}$

$\mathbf{U} : \mathbb{R} \to \mathrm{Aut}_0(\mathcal{B})$ str.cont. 1-par.grp.

Lemma. $\mathbf{U}(t) = U_{1,t} \otimes \cdots \otimes U_{N,t} \qquad (t \in \mathbb{R})$

THEOREM. *There are possibly* **unbounded self-adjoint** $A_k : \mathrm{dom}(A_k) \to \mathbf{H}^{(k)}$ *(defined on dense linear submanifolds) with*

$$\mathbf{U}(t) = \big[\exp(itA_1)\big] \otimes \cdots \otimes \big[\exp(itA_N)\big] \qquad (t \in \mathbb{R}).$$

Corollary. *If* $\mathbf{W} : \mathbb{R} \to \mathrm{Aut}\big(\mathcal{L}(\mathbf{H}^{(1)}, \mathbf{H}^{(2)})\big)$ *is a str.cont.* **lin.** *1-par.grp then* $\mathbf{W}(t)Z = \exp(tA_1)Z\exp(tA_2)$

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- Improvements and generalizations for the original proof
- Applications of the Thm for str-cont 1-prgs of lin auts in Cartan factors
- Str-cont 1-prgs of non-lin hol aut in Hilbert sp

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BACKGROUND OF PROOF DIFFICULTIES

$\mathbb{T} := \{\kappa : \ |\kappa| = 1\}$

$\mathbf{h}_1^* \otimes \cdots \otimes \mathbf{h}_N^* : \ (\mathbf{x}_1, \ldots, \mathbf{x}_N) \mapsto \prod_{k=1}^{N} \mathbf{h}_k^*(\mathbf{x}_k) = \prod_{k=1}^{N} \langle \mathbf{x}_k | \mathbf{h}_k \rangle$

Ambiguous representation: If $\mathbf{h}_k, \mathbf{u}_k \in \partial \mathrm{Ball}(\mathbf{H}^{(k)})$ then

$$\mathbf{h}_1^* \otimes \cdots \otimes \mathbf{h}_N^* = \mathbf{u}_1^* \otimes \cdots \otimes \mathbf{u}_N^*$$
$$\Updownarrow$$
$$\mathbf{h}_k = \kappa_k \mathbf{u}_k, \quad \kappa_k \in \mathbb{T}, \quad \prod_{k=1}^{N} \kappa_k = 1$$

PROOF STRATEGY

$\mathbf{U}(t) := U_{1,t} \otimes \cdots \otimes U_{N,t}$ str cont; unitary ops

1) **Adjusted continuity**: Find $\kappa_1, \widetilde{\kappa}_1, \ldots, \kappa_N, \widetilde{\kappa}_N : \mathbb{R} \to \mathbb{T}$ with

$$t \mapsto \kappa_k(t) U_{k,t} \text{ str. cont.}; \qquad t \mapsto \widetilde{\kappa}_k(t) U_{k,t} \text{ 1-prg}$$

2) **Local representations**: $\mathbf{H}^{(k)} = \oplus_{j \in J_k} \mathbf{H}_j^{(k)}$, $\mathbf{H}_j^{(k)}$ separ eigsp of $U_{k,t}$

$$U_{k,t} | \mathbf{H}_j^{(k)} \simeq \left[L^2(\Omega_{k,j}, \mu_{k,j}) \ni f \mapsto u_{k,t,j} f \right]$$

3) **Probabilistic arguments**: $t \mapsto u^t \in \mathcal{C}(\Omega)$ 1-prg., $t \mapsto \kappa(t) \mathbf{M}_{u^t}$ str.cont. $\Longrightarrow \exists\, \chi \in \mathcal{C}(\mathbb{R}, \mathbb{T})$ $\exists \widetilde{\alpha} : \Omega \to \mathbb{R}$ μ-mesurable

$$\kappa(t) u^t(\omega) = \chi(t) \exp\left(it \widetilde{\alpha}(\omega) \right) \quad (t \in \mathbb{R}) \qquad \forall_\mu\, \omega \in \Omega$$

Tools: $\Delta(h) := \sup_{|t| \leq h} \int_{\omega_1, \omega_2 \in \Omega} \left| u^t(\omega_1) - u^t(\omega_2) \right|^2 \mu(d\omega_1) \mu(d\omega_2) \searrow 0$

$\Omega_{t,r}^{(2)} := \left\{ (\omega_1, \omega_2) \in \Omega^2 : \; d\left(u^t(\omega_1), u^t(\omega_2) \right) < r \right\}$ $\quad d =$arc-length

$$\mu \otimes \mu \left(\bigcap_{n=1}^{\infty} \Omega_{t/n!, \pi/n!}^{(2)} \right) \geq 1 - \Delta(|t|)$$

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GENERALIZATIONS - adjusted continuity

$\mathbf{X}^{(1)},\dots,\mathbf{X}^{(N)}$ reflexive Banach spaces

Assume: $\big\|\mathbf{x}_j-\mathbf{x}\big\|\to 0$ if $\big\|\mathbf{x}_j\big\|\to\big\|\mathbf{x}\big\|,\ \langle\mathbf{x}_j-\mathbf{x},\phi\rangle\to 0\quad(\phi\in[\mathbf{X}^{(k)}]^*)$

Ex.: $\mathbf{X}^{(k)}$ uniformly conv. $\ (\iff [\mathbf{X}^{(k)}]^*$ unif. smooth$)$

Lemma. $\mathbf{e}_j^{(1)}\otimes\cdots\otimes\mathbf{e}_j^{(N)}\xrightarrow{\ \mathrm{w}\ }\mathbf{e}^{(1)}\otimes\cdots\otimes\mathbf{e}^{(N)}$ with 1-vectors, $\Rightarrow$
$\big\|\kappa_j^{(k)}\mathbf{e}_j^{(k)}-\mathbf{e}^{(k)}\big\|\to 0\ \ (k=1,\dots,N)\quad\exists\ \big[\kappa_j^{(k)}\big]$ in $\mathbb{T}$, $\prod_{\ell=1}^N\kappa_j^{(\ell)}=1$

Cor.: $t\mapsto\mathbf{x}_t^{(1)}\otimes\cdots\otimes\mathbf{x}_t^{(N)}\neq 0$ w-cont, $t\mapsto\prod_{k=1}^N\big\|\mathbf{x}_t^{(k)}\big\|$ cont, $\Rightarrow$
$\exists\ \big[\kappa_t^{(k)}\big]$ in $\mathbb{T}$ with $t\mapsto\kappa_t^{(k)}\big\|\mathbf{x}_t^{(k)}\big\|^{-1}\prod_{j=1}^N\big\|\mathbf{x}_t^{(j)}\big\|^{1/N}\mathbf{x}_t^{(k)}$ norm-cont

$*$**Lemma.** Assume $\mathbf{Z}$ top v-space with separating dual,
$t\mapsto\underbrace{\rho(t)}_{\neq 0}\underbrace{U_t}_{\in\mathrm{GL}(\mathbf{Z})}\mathbf{z}\ \ (\mathbf{z}\in\mathbf{Z})$ cont, $\quad U_{s+t}=\underbrace{\lambda(s,t)}_{\neq 0}U_sU_t\ (s,t\in\mathbb{R})$;
$\Longrightarrow\quad\exists\ \widetilde{\rho}:\mathbb{R}\to\mathbb{C}\setminus\{0\}$ making $\big[\widetilde{\rho}(t)U_t:\ t\in\mathbb{R}\big]$ 1-prg

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ADJUSTED CONTINUITY - conclusion

Recall: $\|\mathbf{x}_j - \mathbf{x}\| \to 0$ if $\|\mathbf{x}_j\| \to \|\mathbf{x}\|, \langle \mathbf{x}_j - \mathbf{x}, \phi\rangle \to 0 \quad (\phi \in [\mathbf{X}^{(k)}]^*)$

PROPOSITION. $t \mapsto \mathbf{U}(t) = \underbrace{U_{1,t}}_{\in \mathrm{GL}(\mathbf{X}^{(k)})} \otimes \cdots \otimes \underbrace{U_{1,t}}_{\in \mathrm{GL}(\mathbf{X}^{(k)})}$ *1-prg with*

$t \mapsto \|\mathbf{U}(t)(\mathbf{x}^{(1)} \otimes \cdots \otimes \mathbf{x}^{(N)})\|$ *cont* $\forall\ [\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(N)}]$ *fixed;*

$\Longrightarrow \exists\ \rho_1, \widetilde{\rho}_1, \ldots, \rho_N, \widetilde{\rho}_N : \mathbb{R} \to \mathbb{C} \setminus \{0\}$ *with*

$\left[\rho_k(t) U_{k,t} : \ t \in \mathbb{R}\right]$ *is a strongly continuous family*

$\left[\widetilde{\rho}_k(t) U_{k,t} : \ t \in \mathbb{R}\right]$ *is a one-parameter group*

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PROOF SHORTCUT - representation

- *Countably generated von Neumann algebras are singly generated* (∼**Beneduci, 2012**)

Cor. *Immediate representation in Hilbert case (without localization)*

$$U_{k,t} \simeq \left[L^2(\Omega_k, \mu_k) \ni f \mapsto u_{k,t} f \right]$$

with probab Radon measures μ_k *on comp* Ω_k *and cont* $u_{k,t} : \Omega_k \to \mathbb{T}$.

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GENERALIZATION - Hilbert space case

- $\mathcal{U} := \big[U^t : t \in \mathbb{R}\big] \subset \mathrm{GL}(\mathbf{H})$ *bded str-cont 1-prg,*
  $L_{R\to\infty}$ *Banach limit on* $\mathcal{C}_{\mathrm{b}}[0,\infty)$*;* $\Longrightarrow$

  $$\langle\!\langle \mathbf{x} | \mathbf{y} \rangle\!\rangle := L_{R\to\infty}\left(\frac{1}{2R}\int_{-R}^{R} \langle U^t\mathbf{x} | U^t\mathbf{y}\rangle \, dt\right)$$ *$\mathcal{U}$-inv equiv inner-prd*

Theorem. $|||\cdot|||_k$ *equiv norms on* $\mathbf{H}^{(k)}$*,*
$U_{k,t}$ *surjective* $|||\cdot|||_k$*-isometries* $(1 \le k \le N,\ t \in \mathbb{R})$*,*
$t \mapsto \mathbf{U}(t) := U_{1,t} \otimes \cdots \otimes U_{N,t}$ *1-prg;* $\Longrightarrow$

$\exists$ *equivalent inner products* $\langle\!\langle \cdot | \cdot \rangle\!\rangle_k$ *such that*

$$\mathbf{U}(t) = \big[\exp(itA_1)\big] \otimes \cdots \otimes \big[\exp(itA_N)\big] \quad (t \in \mathbb{R})$$

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$$\langle\!\langle \mathbf{x} | \mathbf{y} \rangle\!\rangle := L_{R\to\infty} \left(\frac{1}{2R} \int_{-R}^{R} \langle U^t \mathbf{x} | U^t \mathbf{y} \rangle \, dt \right) \quad \mathcal{U}\text{-inv equiv inner-prd}$$

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CARTAN FACTORS

Cartan factors: Types 1–6

Every JB-triple embeds into an ℓ^∞-product of Cartan factors*

Type 1: $\mathcal{L}(\mathbf{H}^{(1)}, \mathbf{H}^{(2)})$
Types 2,3: $\mathcal{L}^{(\mathrm{T},\pm)}(\mathbf{H}) = \{L \in \mathcal{L}(\mathbf{H}) : \ L = \pm L^{\mathrm{T}}\}$
Type 4: Spin factor on $\mathbf{H}$, $\quad \{\mathbf{xay}\} := \langle \mathbf{x}|\mathbf{a}\rangle \mathbf{y} + \langle \mathbf{y}|\mathbf{a}\rangle \mathbf{x} - \langle \mathbf{x}|\overline{\mathbf{y}}\rangle \overline{\mathbf{a}}$
Type 5: $\mathbf{0}^{1\times 2}$ $\quad 1\times 2$ complex octonion matrices (dim=16)
Type 6: $\mathcal{H}_3(\mathbf{0})$ $\quad 3\times 3$ complex Hermitian octonion matrices (dim=27)

$\mathbf{h} \mapsto \overline{\mathbf{h}}$ conjugation: $\overline{\mathbf{e}_\alpha} = \mathbf{e}_\alpha$, $\overline{i\mathbf{e}_\alpha} = -i\mathbf{e}_\alpha$ for a fixed ON basis
$L \mapsto L^{\mathrm{T}}$ transposition (wrt. conjugation $\overline{\cdot}$): $L^{\mathrm{T}}\mathbf{h} = \overline{L^*\overline{\mathbf{h}}}$

STR. CONT. 1PRGs of LIN. AUTs on FACTORS

Factors of type > 3 are not interesting (str. cont. wrt. Hilbert norm)

Type 1: $\mathcal{L}(\mathbf{H}^{(1)}, \mathbf{H}^{(2)}) \simeq \{\text{2-lin. funct. } \mathbf{H}^{(1)} \times \mathbf{H}^{(2)} \to \mathbb{C}\} \longrightarrow$ Case $N = 2$.

Types $k = 2, 3$:

$$\mathfrak{A}_k := \left\{\text{surjective lin. isom. of } \mathcal{L}^{(\mathrm{T},(-1)^k)}(\mathbf{H})\ \right\}$$

$$\mathcal{L}^{(\mathrm{T},(-1)^k)}(\mathbf{H}) \simeq \left\{ \begin{array}{c} \text{symm.} \\ \text{antisymm.} \end{array} \ \text{2-lin. } \mathbf{H} \times \mathbf{H} \to \mathbb{C} \text{ funct. } \right\}$$

$$\mathbf{U} \in \mathfrak{A}_2 \iff \exists\, U \in \mathcal{U}(\mathbf{H}) \qquad \underline{\mathbf{U} : L \mapsto U L U^{\mathrm{T}}}$$

TYPES 2,3

Theorem. *Let* $k = 2, 3$, $\mathbf{U}_k : \mathbb{R} \to \mathfrak{A}_k$ *str. cont. 1prg. Then*

$\exists\, A$ unbded. self-adj. **H**-op. $\mathbf{U}(t) = \big[L \mapsto \exp(itA)L\exp(it\underbrace{A^{\mathrm{T}}}_{\overline{A}})\big]$

Proof. $\mathbf{U}_k : \underbrace{U_t}_{unit.} \otimes U_t^{\mathrm{T}} \big| F_k$

$\exists \sigma : \mathbb{R} \to \{-1, 1\}$ $t \mapsto \sigma(t)U(t)$ str.cont.

SEP. CONT. + LOC. + PRB. $\longrightarrow$ Theorem.

GENERAL JB*-TRIPLES

E JB*-triple $\quad E \hookrightarrow \mathrm{Atomic}\big(E^{**}\big) = \bigoplus_{k\in\mathcal{K}} \underbrace{F_k}_{Cartan}$

$\mathbf{U} : \mathbb{R} \to \mathrm{Aut}(E)$ str. cont, 1prg.

$\Pi_k : E \to F_k$ canonical proj.

$t \mapsto \mathbf{U}(t)^{**} \circ \Pi_k \underbrace{\text{str. cont.}}_{???}$ 1prg. $\mathbb{R} \to \mathrm{Aut}(F_k)$

If YES , Gelfand-Neumark description for str.cont. 1prg of $\mathrm{Aut}(E)$.

Problems. (1) *E w*-dense JB*-subtriple in F (Cartan factor),*
$t \mapsto \mathbf{U}(t)$ str.cont. 1prg. $\mathbb{R} \to \mathrm{Aut}(E) \Longrightarrow^{?} t \mapsto \mathbf{U}^{w*}(t)$ str.cont.?
(2) *Category description of w*-dense JB*-subtriples of Cartan factors*

Conjecture. [∼2003 Isidro-Stacho] E w*-dense JB*-subtrp in factor Type 1 $\iff \exists E_0 \subset E$ $w*-dense$ TRO (ternary ring of operators)

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PROBLEMS IN NON-LINEAR CASE

$N < 3$: $\mathbf{H}, \mathcal{L}(\mathbf{H}^{(1)}, \mathbf{H}^{(2)})$ JB*-triples

$\mathbf{E}$ JB*-triple $\mathbf{B} := \mathrm{Ball}(\mathbf{E})$

$t \mapsto G^t \in \mathrm{Aut}(B)$ str.cont 1-prg: $t \mapsto G^t(x)$ norm.cont. $\forall x$

$$G^t = g_{G^t(0)} \circ \underbrace{U_t}_{\mathrm{LIN} \in \mathrm{Aut}(\mathbf{B})} \qquad g_a(x) = a + \underbrace{\beta(a)}_{I - 2D(a) + Q(a)^2}{}^{1/2}[I - D(x,a)]^{-1}x$$

Lemma. *$t \mapsto G^t$ str.cont. $\iff$ $t \mapsto G^t(0)$ cont., $t \mapsto U_t$ str.cont.*

Question: $G^t \stackrel{?}{=} \exp\left[t\big(a - \{aa^*x\} + Ax\big)\partial/\partial x\right]$ *A unbded.$\{\ldots\}$-der.*

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EXAMPLE IN HILBERT SPACE

$\mathbf{E} := \big[\text{Hilbert sp with equiv norm}\big], \quad \mathbf{D} := \mathrm{Ball}(\mathbf{E})$
$\big[G^t : \ t \in \mathbb{R}\big]$ str-cont 1-prg in $\mathbb{G} := \mathrm{Aut}(\mathbf{D})$

Remark. $\mathbf{E}_0 := \mathbb{G}(0)$ refl subsp, **Kaup 1981**: finite $\oplus^\infty$-sum of Cartan facts, w-cont hol **D**-aut; **Krein-Milman**$\Rightarrow \bigcap_t \mathrm{Fix}(\overline{G^t}) \neq \emptyset$

Theorem. *Assume* $G^t(\mathbf{a}) = \mathbf{a}$ $(t \in \mathbb{R})$ *with* $\mathbf{a} \in \mathbf{E}_0 \cap \mathbf{D}$.
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STR-CONT 1-PR AUTG with FIXED POINT in $\overline{\mathrm{Ball}}(\mathbf{H})$

$\mathbf{E} := \mathbf{H}$ Hilbert sp, $\quad \mathbf{B} := \mathrm{Ball}(\mathbf{H}), \quad \mathbf{e} \in \partial\mathbf{B}$

Assume: $\left[G^t : t \in \mathbb{R}\right]$ str-cont 1-prg in $\mathrm{Aut}(\overline{\mathbf{B}}), \quad \mathbf{e} \in \bigcap_t \mathrm{Fix}(G^t)$

- $G^t(x) = \dfrac{A_t x + b}{\langle x | c_t \rangle + d_t}\Big|_{x=e} \equiv 1 \qquad \exists!\ A_t \in \mathcal{L}(\mathbf{H});\ b_t, c_t \in \mathbf{H};\ d_t \in \mathbb{C}$

Proposition. *$t \mapsto \mathcal{G}^t := \begin{bmatrix} A_t & b_t \\ c_t^* & d_t \end{bmatrix}$ str-cont 1-prg in $\mathcal{L}(\mathbf{H} \oplus \mathbb{C})$*

$\exists\ K > 0 \ \ \exists\ \tau > 0 \quad \left\|\mathcal{G}^t\right\| \le e^{Kt} \quad (t \ge \tau)$

Remark. Hille-Yosida $\Rightarrow \mathcal{G}^t = \exp(t\mathcal{B}) \quad (\exists\, \mathcal{B}$ unbded gen$)$

STR-CONT 1-PR AUTG with FIXED POINT in $\overline{\mathrm{Ball}}(\mathbf{H})$

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Assume: $\left[G^t : t\in\mathbb{R}\right]$ str-cont 1-prg in $\mathrm{Aut}(\overline{\mathbf{B}})$, $\mathbf{e} \in \bigcap_t \mathrm{Fix}(G^t)$

- $G^t(x) = \dfrac{A_t x + b}{\langle x|c_t\rangle + d_t} \Big|_{x=e} \equiv 1$ $\quad\exists!\ A_t \in \mathcal{L}(\mathbf{H});\ b_t, c_t \in \mathbf{H};\ d_t \in \mathbb{C}$

Proposition. *$t \mapsto \mathcal{G}^t := \begin{bmatrix} A_t & b_t \\ c_t^* & d_t \end{bmatrix}$ str-cont 1-prg in $\mathcal{L}(\mathbf{H}\oplus\mathbb{C})$*

$\exists\ K>0\ \ \exists\ \tau>0\quad \left\|\mathcal{G}^t\right\| \le e^{Kt}\quad (t \ge \tau)$

Remark. Hille-Yosida $\Rightarrow \mathcal{G}^t = \exp(t\mathcal{B})$ $\quad (\exists\,\mathcal{B}$ unbded gen$)$

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