Work in progress

JORDAN MANIFOLDS

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MOTIVATION: THE UNIT BALL OF A C*-MODULE

W. WERNER, Cluj 2007 (Theorem 1.14.)

On the unit ball of a complex Hilbert C*-module
there is unique automorphism invariant connection.
Its Christoffel symbol is

$$\begin{aligned}\Gamma_a(x,y) = \quad & x\big[(1-\langle a|a\rangle)^{-1}a\big]y + \\ & + y\big[(1-\langle a|a\rangle)^{-1}a\big]x\end{aligned}$$

TOOLS OF PROOF

1) Ternary rings of operators (TRO)

$$\begin{aligned} \text{Hilbert C*-modules} &\simeq \text{TRO} \\ \langle x|y\rangle z &\simeq xy^*z \end{aligned}$$

2) Potapov's transformations (in TRO)

$$M_a(x) := (1 - aa^*)^{-1/2} \underbrace{(x + a)(1 - a^*x)^{-1}}_{\text{Möbius}} (1 - a^*a)^{-1/2}$$

Operator calculation for $M_a^{\#}\big[dY(X)\big]M_{-a}^{\#}$ at a

JORDAN STRUCTURE OF BOUNDED SYMMETRIC DOMAINS

Z complex Banach space, $0 \in D \subset Z$ bded symm domain

1983 Kaup: Renorm Z with γ_0^D (infinitesimal Carathéodory norm)
$D \sim \mathrm{Ball}(Z)$ (biholomorphic eqivalence)

JB*-triple $(Z, \{\ldots\})$

$\{xyz\}$ symm bilin. in x, z, conj-lin. in y

$\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$ (Jordan)

$\|\{xxx\}\| = \|x\|^3$ (C*)

$D(a) := [x \mapsto \{aax\}] \in \mathrm{Her}(Z, \|.\|)$

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BASIC INFINITESIMAL JB*-TRANSLATIONS

$S_a :=$ [Symmetry of $\mathrm{Ball}(Z)$ at a]
$X_a(x) := \lim_{\tau\to 0+} \big(S_{\tau a/2} S_0(x)\big)$

$$X_a = [a - \{xax\}]\frac{\partial}{\partial x} \quad \textbf{complete} \text{ vector field in } \mathrm{Ball}(Z)$$

$$\Phi_a := \exp(X_a) \in \mathrm{Aut}(\mathrm{Ball}(Z)) \quad \text{FLOW}$$

$$\frac{d}{d\tau}\Phi_{\tau a}(x) = X_a\big(\Phi_{\tau a}(x)\big)$$

Example: $Z := \mathbb{C}$, $\{xay\} := x\bar{a}y$,
Let $a \in [-1,1]$, $\quad X_a(x) = a - ax^2$

$$\Phi_{\tau a}(x) = \frac{x + \tanh \tau a}{1 - x \tanh \tau a}$$

JB*-MÖBIUS TRANSFORMATIONS

$c(a) := \Phi_a(0)$
$c : Z \leftrightarrow \mathrm{Ball}(Z)$ bianalytic

DEF. $M_{\Phi(a)} := \Phi_a$ [Example: $Z = \mathbb{C} \Rightarrow M_p(x) = \frac{x+p}{1-\overline{p}x}$]

$$\Psi \in \mathrm{Aut}(Z) \Rightarrow \exists!\ p \in \mathrm{Ball}(Z),\ U \in \mathrm{Unitary}(Z) \quad \Psi = M_p \circ U$$

Kaup 1983:

$$M_p(x) = p + B(p)^{1/2}[1 - D(x,p)]^{-1}x$$

$B(p) := 1 - 2D(p) + Q(p)^2$ BERGMAN op.
$D(x,p) : y \mapsto \{xpy\}$ polarized der.
$Q(p) : x \mapsto \{pxp\}$ quadratic repr.

INVARIANT CONNECTION ON A JB*-BALL

THEOREM. $(Z, \{\ldots\})$ complex JB*-triple $\Longrightarrow$
There is a unique Aut-invariant connection on $\mathrm{Ball}(Z)$.
Its Christoffel symbol is

$$\Gamma(p)(X,Y) = 2B(p)^{1/2}\{[B(p)^{-1/2}X(p)]p[B(p)^{-1/2}Y(p)]\}$$

Interpretation: $\nabla_X Y = dX(Y) - \Gamma(X,Y)$
If X, Y are represented by $f, g : G \to Z$ in a local coordinate $\xi : G \to \mathrm{Ball}(Z)$ then $\nabla_X Y$ is represented by

$$v \mapsto f'(v)g(v) - [\xi^{-1}]'(\xi(v))\Big[\Gamma(\xi(v))\Big(\xi'(v)f(v), \xi'(v)g(v)\Big)\Big]$$

INGREDIENTS OF PROOF

$\mathrm{Ball}(Z)$ symmetric $\Longrightarrow$ at most one Aut-invariant connection

$\Gamma(0) \equiv 0$

Trasformation rule: $M_p : 0 \mapsto p,\ [M_p]^{-1} = M_{-p},$

$$\begin{aligned}\Gamma(p)\big(x,y\big) &= M_p''(0)(M_{-p}'(0)x, M_{-p}'(0)y) \\ &= 2B(p)^{1/2}\{[M_{-p}'(0)x]p[M_{-p}'(0)y]\}, \\ &= 2B(p)^{1/2}\{[B(p)^{-1/2}x]a[B(p)^{-1/2}y]\}\end{aligned}$$

$$M_p'(x) = B(p)^{1/2}B(x,-p)^{-1}$$

$B(x,b) := 1 - 2D(x,b) + Q(x)Q(b)$ polarized Bergman op.

GEODESICS OF ∇

$\gamma : I \to \mathrm{Ball}(Z)$ is ∇-geodesic:

$$\ddot{\gamma}(t) + 2B\big(\gamma(t)\big)^{1/2} D\Big(B\big(\gamma(t)\big)^{-1/2}\dot{\gamma}(t), \gamma(t)\Big) B\big(\gamma(t)\big)^{-1/2}\dot{\gamma}(t) = 0$$

$\gamma_{p,v}$: ∇-geodesic with $\gamma_{p,v}(0) = p, \quad \dot{\gamma}(0) = v$

Theorem. $\gamma_{p,v}(t) = M_p\big(T\mathrm{artanh}(t\ \mathrm{id}_\Omega)\big)$

where

$T : \mathrm{Re}\mathcal{C}_0(\Omega) \to Z$ Gelfand repr.
$T(\mathrm{id}_\Omega) = M_p'(0)^{-1}v, \quad T(\varphi\psi\chi) = \{(T\varphi)(T\psi)(T\chi)\}$

LOCAL MÖBIUS TRF IN REAL JORDAN TRIPLES

E real Banach space, $\{\ldots\} : E^3 \to E$

$(E, \{\ldots\})$ **Jordan triple**: $\{\ldots\}$ trilinear $+$ Jordan identity

$D(a,b)x := \{abx\}$, $Q(a,b)x := \{axb\}$,
$B(a,b) := 1 - 2D(a,b) + Q(a)Q(b)$

$M_p(x) := p + B(p)^{1/2}[1 + D(x,p)]^{-1}x$,

$\Phi_v := \exp\left(X_v\right)$ flow of $X_v := [v - \{xvx\}]\partial/\partial x$

Lemma. $\exists\, \varepsilon > 0$ $\quad v \mapsto \Phi_v(0)$ bianal. on $\varepsilon\mathrm{Ball}(E)$;

$\forall\, x, v \in \varepsilon\mathrm{Ball}(E) \qquad M_{\Phi_v(0)}(x) = \Phi_v(x)$

LOCAL MÖBIUS TRF IN REAL JORDAN TRIPLES

$$\mathrm{Der}(E,\{\ldots\}) := \Big\{\ell\in\mathcal{L}(E):\ \ell\{xyz\} \equiv \{(\ell x)yz\}+\{x(\ell y)z\}+\{xy(\ell z)\}\Big\}$$

$$L_\ell := [\ell x]\tfrac{\partial}{\partial x} \quad (\ell \in \mathrm{Der}(E,\{\ldots\}))$$

Example: $D(a,b) - D(b,a) \in \mathrm{Der}(E,\{\ldots\})$

Definition. $\Phi : U \to E$ is a **local Möbius transformation** on E if U is a 0 nbh in E, Φ is bianalytic of the form $M_p \circ \exp(L_m)|U$

Proposition.
1) $\big\{X_v + L_\ell :\ v \in E,\ \ell \in \mathrm{Der}(E,\{\ldots\})\big\}$ Banach-Lie algebra
with $\big([f(x)\tfrac{\partial}{\partial x}, g(x)\tfrac{\partial}{\partial x}] := f'(x)g(x) - g'(x)f(x)\big)\tfrac{\partial}{\partial x}$.

2) $\exp(X_v + L_\ell)$ is a local Möbius trf for small v, ℓ.

JORDAN-MÖBIUS MANIFOLDS

Definition. $(\mathcal{M}, \mathcal{A})$ is a **Jordan-Möbius manifold** on E if

$\mathcal{A} \supset \{Y_p : \ p \in \mathcal{M}\}$ with $Y_p : U_p \to \mathcal{M}$ such that

$Y_p(0) = p$, $U_p \subset E$ open 0-nbh;
each chart transition $Y_p^{-1} \circ Y_q$ is a local Möbius trf if $q \in Y_p(U_p)$

Example. 1) $\mathcal{M} := \text{Ball}(Z)$ of complex JB*-triple, $Y_p := M_p : \mathcal{M} \leftrightarrow \mathcal{M}$

2) $\mathcal{M} := \varrho\text{Ball}(E)$, $Y_p := M_p | U_p$ where ϱ is so small that
the sections of $(p, x) \mapsto M_p(x)$ are bianalytic on $[\varrho\text{Ball}(E)]^2$;
U_p is so small that $M_p(U_p) \subset \varrho\text{Ball}(E)$
E.g. $E := \{\text{real lower triag } n \times n \text{ Töplitz matrices}\}$ $\{xyz\} := xyz$, $\varrho = 1$

MAIN RESULTS ON JORDAN-MÖBIUS MANIFOLDS

$(\mathcal{M}, \mathcal{A})$ Jordan-Möbius manifold on $(E, \{\ldots\})$,
$Y_p : U_p \to \mathcal{M} \quad (p \in \mathcal{M})$ as before

Theorem. There is a unique connection $\nabla_X Y$ on $\mathcal{M}$ which is invariant under all local transformations of $\mathcal{M}$ whose repesentations in the charts Y_q $(q \in \mathcal{M})$ are local Möbius trf.

Definition. $(\mathcal{M}, \mathcal{A})$ is a **uniform** Jordan-Möbius manifold if
$\exists\ \varrho > 0 \qquad U_p \supset \varrho\mathrm{Ball}(E)$.

Proposition. Uniform Jordan-Möbius manifolds are *symmetric*.

Question. Is it possible to embed any JM-manifold into a *uniform* one?

MORE EXAMPLES

3) [Chu-Isidro-Kaup-Stachó]

$(\mathcal{H}, \langle .|. \rangle)$ complex Hilbert space, $\{xyz\} := \frac{1}{2}\langle x|y\rangle z + \frac{1}{2}\langle z|y\rangle x$

$E := e_0^{\perp} = \{z \in \mathcal{H} : \langle z|e_0\rangle = 0\}$, $\{\dots\}$ from $\mathcal{H}$, e_0 unit vector

$\mathcal{S}$:=[unit sphere], $e^{\sim} := \zeta e : |\zeta| = 1\}$

$R_e =$ [unitary op. on $\mathcal{H}$ with $R_e(e) = e_0$]

$$\mathcal{M} := \{e^{\sim} : e \in \mathcal{S}\},$$
$$Y_{e^{\sim}}(x) := [R_e(e_0 + x)/\|e_0 + x\|]^{\sim} \quad (e^{\sim} \in \mathcal{M}, \ x \in E)$$

4) Kaup's complex symmetric hermitian Banach manifolds are JM

$p_0 \in \mathcal{M}$ base point, $S_p =$ [symm at p], $E := T_{p_0}\mathcal{M}$

$X_v(p) := [\tau \mapsto S_{q(\tau)} \circ S_{p_0}(p)]^{\mathcal{A}} \in T_p\mathcal{M} \quad (q(.)^{\mathcal{A}} = v \in E)$

$\{xvy\} = \frac{\partial^3}{\partial x \partial \overline{v} \partial x}[X_x, [X_v, X_y]] \in E$

$$Y_p : [\exp(v - \{zvz\})\frac{\partial}{\partial z}]0 \mapsto [\exp X_v]p$$

JORDAN MANIFOLDS

Definition. $(\mathcal{M}, \mathcal{A}, \mathcal{P})$ is a **Jordan manifold** if

$(\mathcal{M}, \mathcal{A})$ connected real $\mathcal{C}^\infty$-Banach-manifold

$\mathcal{P} = \{\{\ldots\}_p : p \in \mathcal{M}\}$, $\{\ldots\}_p$ real J-triple product on $T_p\mathcal{M}$

$(p, x, y, z) \mapsto \{xyz\}_p$ smooth on $\mathcal{M} \times [T\mathcal{M}]^3$

Morphisms between Jordan manifolds:
smooth mappings whose derivatives are
triple product homomorphisms
on the respective tangent spaces.

Local morphisms: morphisms between submanifolds

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EXAMPLES OF JORDAN MANIFOLDS

1) Riemann spaces can be regarded as Jordan manifolds:
$\langle . | . \rangle_p$ inner product on $T_p M \longrightarrow \{uvw\}_p := \frac{1}{2}\langle u|v\rangle_p w + \frac{1}{2}\langle w|v\rangle_p u$

2) Jordan-Möbius manifolds are Jordan manifolds:
$$\{uvw\}_p = Y_p'(0)\Big\{ \big[[Y_p'(0)]^{-1}u\big]\big[[Y_p'(0)]^{-1}v\big]\big[[Y_p'(0)]^{-1}w\big] \Big\}$$

3) In general, the triple products in $\mathcal{P}$ need not be isomorphic:
$M := \mathbb{R}, \ \{uvw\}_p := puvw$
$\{\ldots\}_p \sim \{\ldots\}_q$ iff $\operatorname{sgn}(p) = \operatorname{sgn}(q)$

ISOMORPHISM OF LOCAL TRIPLE PRODUCTES

Theorem. If the members of $\mathcal{P}$ are finite dimensional JB*-triple products then they are isomorphic.

Proof. With convergent subsequences of **grids** in tangent spaces.

Conjecture. The condition $\dim(\mathcal{M}) < \infty$ can be dropped

Definition. $(\mathcal{M}, \mathcal{A}, \mathcal{P})$ is **homogeneous**:

$$\forall\ p, q \in \mathcal{M} \quad \exists\ \Psi : \mathcal{M} \leftrightarrow \mathcal{M} \text{ Jordan automorphism} \quad \Psi(p) = q$$

Locally homogeneous: $\forall\ p, q \in \mathcal{M} \quad \exists\ \mathcal{U}\ p\text{-nbh} \quad \exists\ \mathcal{V}\ q\text{-nbh}$

$$\exists\ \Theta : \mathcal{U} \leftrightarrow \mathcal{V} \text{ Jordan automorphism} \quad \Theta(p) = q$$

Example. $(1, \infty)$ as Jordan-Möbius manifold with $Y_p(x) := \frac{x+p}{1+xp}$
only locally homogeneous

COMPLEX HOMOGENEOUS JORDAN MANIFOLDS

Kaup 1977 $\longrightarrow$

$(\mathcal{M}, \mathcal{A}, \mathcal{P})$ complex symm. hom. Jordan manifold

all $\{\ldots\}_p$ are JB*-triples $\Longrightarrow$ $(\mathcal{M}, \mathcal{A}, \mathcal{P})$ **hermitian**

$\sim$ canonical manifold of a complex hermitian J*-triple

Example. Equivalence classes of tripotents in a JB*-triple $\big(Z, \{\ldots\}\big)$

$$\big(T_{e^\sim}\mathcal{M}, \{\ldots\}_{e^\sim}\big) \sim \big(Z_{1/2}(e), \{\ldots\}|_{Z_{1/2}(e)}\big)$$

Problem. Can symmetry be dropped?

REAL SYMMETRIC NON-JORDAN-MÖBIUS JORDAN MANIFOLD

Corach-Porta-Recht 1993

A C*-algebra with 1, $\quad E := \{a \in A : \; a = a^*\}, \quad \{xyz\} := \frac{1}{2}xyz + \frac{1}{2}zyx$

$\mathcal{M} := \{a \in A : \; a > 0\} = \{g^*g : \; g \text{ inv.}\}$

$L_g : \mathcal{M} \leftrightarrow \mathcal{M}, \; x \mapsto g^*xg$ linear

$S : p \mapsto p^{-1}$ symmetry on $\mathcal{M}$

$S_{g^*g} := L_g \circ S \circ L_{g^{-1}} : p \mapsto g^*gp^{-1}g^*g$

$$\begin{aligned}\{uvu\}_{g^*g} :&= L_g\{(L_g^{-1}u)(L_g^{-1}u)(L_g^{-1}u)\} = \\ &= u(g^*g)^{-1}v(g^*g)^{-1}u \qquad \text{well-def.}\end{aligned}$$

L_g, S_p $\mathcal{P}$-automorphisms,

$X_v := \frac{d}{d\tau}\Big|_{\tau=0+} S_{\exp(\tau v/2)} \circ S_1, \qquad \exp(X_v)p = \exp(v/2)p\exp(v/2)$

Proposition. In the chart $Y : \left[\exp(v - \{xvx\})\frac{\partial}{\partial x}\right]0 \mapsto [\exp X_v]1$ the transformations $\exp(X_w)$ $(w \in E)$ are not in general $\{\ldots\}$-Möbius type