Ternary rings of operators, affine manifolds and causal structure

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I. Finite Dimensions

Recall that the open unit ball of the C*-algebra M_p is a symmetric space: Let

$$\mathsf{U}(p,p) = \{A \in \mathsf{GL}(2p) \mid A^* I_{p,p} A = I_{p,p}\}\,,$$
$$I_{p,p} = \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix}.$$

Writing each element A in $\mathsf{U}(p,q)$ as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

we obtain an action on the open unit ball D_p of M_p by

$$AZ = (A_{11}Z + A_{22})(A_{21}Z + A_{22})^{-1}$$

The isotropy group of this action at 0 turns out to be

$$\mathsf{U}(p) \times \mathsf{U}(p) = \left\{ \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \mid A_{ii} \in \mathsf{U}(p) \right\},$$

and

$$D_p = \mathsf{U}(p,p)/\,\mathsf{U}(p) \times \mathsf{U}(p).$$

Since $\mathsf{U}(p) \times \mathsf{U}(p)$ is the fiexd point group of the involution

$$A \mapsto I_{p,p} A I_{p,p},$$

D_p is a symmetric space according to the Lie group definition.

II. The open unit ball of a C*-algebra

Denote by $D_{\mathfrak{A}}$ the open unit ball of a C*-algebra $\mathfrak{A}$ and let

$$G_{\mathfrak{A}} = \{\Phi : D_{\mathfrak{A}} \to D_{\mathfrak{A}} \,|\, \Phi \text{ is biholomorphic}\}$$

It is well known that any $\Phi \in G_{\mathfrak{A}}$ is of the form

$$a \longmapsto$$

$$\Psi\left[(1-bb^*)^{-1/2}(a+b)(1+b^*a)^{-1}(1-b^*b)^{1/2}\right]$$

where $b \in D_{\mathfrak{A}}$ and Ψ is the restriction of a Banach space isometry of $\mathfrak{A}$.

Furthermore, $G_{\mathfrak{A}}$ is a (Banach) Lie group acting transitively on $D_{\mathfrak{A}}$.

The isotropy group of this action is the group of all linear isometries, and $D_{\mathfrak{A}}$ again satisfies the (Lie group) definition for symmetric spaces.

There is also the notion of a *Riemannian* symmetric space, which is defined by requiring that the geodesic reflections around every point of a Riemannian manifold extend to globally defined isometries.

Both these definitions of symmetric spaces coincide when this formally makes sense.

The requirement of a Hilbertian structure on the tangent bundle of a Banach manifold is too restricitive in general. W.Kaup, H.Upmeier and others have studied various invariant Finsler structures instead, where, however, $G_{\mathfrak{A}}$ never seems to appear as the full group of isometries.

It is, however, possible to replace the Riemanniannian metric by an (invariant) affine connection, and then things look different.

II. Connections

Let $\pi : E \to B$ be a fibration of Banach manifolds.

A connection for π is a smooth vector subbundle H of TE with the property that:

H_p is for each $p \in E$ closed and complementary to the tangent space $\ker(d_p\pi)$ of the fibre $E_{\pi(p)}$ through p, i.e.

$$T_pE = H_p \oplus \ker(d_p\pi)$$

Since the fibres of π usually are considered to be vertical, H_p is called the *horizontal* subspace of T_pE at p.

In such a situation, the mapping

$$d_p\pi|_{H_p} : H_p \to T_{\pi(p)}B$$

is a continuous linear isomorphism, and so there is, for each vector field X on B a unique vector field X_h on E such that

$$d_p\pi(X_h(p)) = X(\pi(p)),$$

called the *horizontal lift* of X w.r.t. the connection H.

For a Banach space bundle $\pi : E \to B$, the covariant derivative corresponding to a connection H of a section s at $b \in B$ in direction $X \in T_bB$ can be defined by

$$\nabla_X s(b) = d_b s(X(b)) - X_h(s(b)),$$

which can be shown to be an element of

$$\ker d_{s(b)}\pi = T_{s(b)}E_b \cong E_b.$$

A connection is called *affine* if the covariant derivative is linear on vector fields.

A Banach manifold M is called *affine* if its tangent bundle carries an affine connection.

A diffeomorphism Φ of the manifold M is called an *affine automorphism* for (M, ∇), iff

$$\Phi_* \nabla_X Y = \nabla_{\Phi_* X} \Phi_* Y$$

for all vector fields X and Y. Here,

$$\Phi_* X(p) = d_{\Phi^{-1}p} \Phi X(\Phi^{-1} p)$$

If a Banach Lie group acts on M, then a connection ∇ is called invariant iff the subbundle H of $T(TM)$ is G-equivariant.

III. Affine symmetric spaces

An affine manifold is called an affine symmetric space, if for each point $p \in M$ there is an affine automorphism s_p such that $s_p^2 = \text{id}$, p is an isolated fixed point for s_p, and $ds_p(p) = -\text{id}_{T_pM}$.

If M is Riemannian and ∇ the Levi-Civita connection then M is affine symmetric iff it is Riemannian symmetric, i.e. iff s_p can be chosen to be an isometry.

All these notions coincide with the Lie group definition: M is symmetric, iff $M = G/H$, where G is a Lie group, and H a (closed) subgroup for which there is an involutive automorphism σ of G so that H is essentially the subgroup of fixed points.

If M is affine symmetric, then G can be chosen to be the group of affine automorphisms. If, on the other hand, $M = G/H$ there is a 'canonical connection' for which $G = \text{Aut}(M, \nabla)$.

IV. Invariant connections

Using the way we have defined affine connections, it ist possible to extend results of Nomizu, Kostant, Wang and others to the infinite dimensional set-up.

Theorem *Suppose $M = G/H$ is a symmetric Banach manifold and denote by $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ the decomposition given by the symmetry s_e. Then there is a bijection between the set of G-invariant connections and the set of*

- Ad H*-invariant closed subspaces of $\mathfrak{g}$, complementing $\mathfrak{h}$*

- *continuous bilinear mappings $S : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ so that for all $m_{1,2} \in \mathfrak{m}$ and $h \in H$*

$$S(\mathrm{Ad}(h)m_1, \mathrm{Ad}(h)m_2) = \mathrm{Ad}(h)S(m_1, m_2).$$

In the case of the open unit ball of the C*-algebra $\mathfrak{A}$ the above result reads

Theorem *On the open unit Ball $D_{\mathfrak{A}}$ of a C*-algebra $\mathfrak{A}$ the set of all $G_{\mathfrak{A}}$-invariant connections is in 1-1 correspondance with the continuous bilinear mappings $S : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ so that $S(\Psi a_1, \Psi a_2) = \Psi S(a_1, a_2)$ for all Banach space isometries Ψ of $\mathfrak{A}$.*

IV. A more appropriate category

C*-algebras actually do not form the most natural category for this kind of results.

In finite dimensions, an exact 1-1 correspondance of hermitian symmetric spaces of non-compact type (to which the open unit balls of the spaces M_p belong) exists with the open unit balls of the so called JB*-triple systems.

These are, up to some very few exceptional cases, the self-adjoint subspaces T of C*-algebras for which for all $x, y, z \in T$

$$\{xyz\} := \frac{xy^*z + zy^*x}{2} \in T.$$

The *canonical connection* ∇^0 is the connection that belongs to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, induced by $s_0 : x \mapsto -x$. It has the following properties:

(i) The automorphism group of (M, ∇^0) coincides with G.

(ii) Since G contains so many reflections, a great number of tensor fields have to vanish. Among them is the covariant derivative of the ternary structure on D, obtained through the action of G. ∇^0 can thus be considered as a noncommutative version of the Levi-Civita connection on a hyperbolic manifold.

(iii) The geodesics are precisely the subgroups $\exp(tX)$, $X \in T_0 D$, and parallel transport along them is induced by the exponential map as well.

V. Causality

Why all this?

It seems that the manifold on which we are living is Lorentzian.

Quantum field theory is much harder on Lorentzian manifolds than on Riemannian manifolds, and so one often tries to avoid the Lorentzian case by passing to an underlying Riemannian manifold (‘Wick Rotation’).

One of the byproducts of the Lorentzian structure is the existence of the so called *light cones* in each tangent space.

These cones help in deciding which points in space-time are allowed to interact and thus are responsible for any form of ‘causality’.

It is therefore important to maintain this information under the passage from the Lorentzian to the Riemannian situation.

Causality also shows the following behavior: Whenever a tangent vector is being moved along a geodesic γ from one point on γ to another by parallel transport, it must remain in the light cones all the time.

Guided by classical quantum theory, we will furthermore suppose, that ‘space’ itself corresponds to selfadjoint operators.

The question we would like to answer then is this:

How can ‘reasonable’ causal structures be characterized, which come from an interpreting the points of D as bounded Hilbert space operators?

We will formulate this question more rigorously and answer it in the next section.

Cones and embeddings

In the following, E denotes a fixed (abstract) ternary ring of operators, D its unit ball. Such objects have been intrinsically characterized by Zettl.

Important for us here is also the Neal-Russo Theorem which states that a ternary ring of operators E is characterized by the fact that (not only the unit ball of the space itself but also) all the the matrix spaces

$$M_n(E) = \left\{ (e_{ij})_{i,j=1,\dots,n} \,\middle|\, e_{ij} \in E \right\}$$

carry norms which turn E into an operator space and for which the open unit balls are bounded symmetric domains.

This reveals why we, in the first place, had to restrict the attention to ternary rings of operators.

We need one more structural element which is connected to the notion of ‘selfadjointness’.

We will need, more precisely, the existence of a *real form*, compatible with the (almost) complex structure.

This boils down to requiring that E carries an involutory real automorphism ‘*’ so that $(ix)^* = -ix^*$ for all $x \in E$.

Since we will be studying embeddings into $L(H)$, it will be necessary to impose the additional condition that $\{xyz\}^* = \{z^*y^*x^*\}$.

An ternary ring of operators E that meets all these conditions, will be called a *-ternary ring of operators.

We will suppose in the following that E is a space of this kind.

The ‘space manifold’ now is the open unit ball D_{sa} of the selfadjoint part of E.

Denote by G the group of biholomorphic mappings of D.

D_{sa} is itself a symmetric space. Its underlying group of automorphisms, G_σ consists of all elements in G which leave D_{sa} invariant.

The isotropy subgroup at 0, denoted by H_σ contains all selfadjoint linear isometries of D, i.e. the selfadjoint ternary automorphisms of D.

In order to comply with the requirement that causality be invariant under parallel transport we have to impose the condition that the field of cones we fix in TD_{sa} must be invariant under the action of G_σ.

We now consider smooth embeddings $\Phi : D \to L(H)$ with the following properties:

- Φ is an affine isomorphism onto the open unit ball of a ternary subsystem of $L(H)$.

- Φ is equivariant w.r.t. the action of the group of biholomorphic mappings

- Φ respects the complex structure as well as the (canonical) real forms on both sides.

And we want to know:

What characterizes the causal structure that is pulled back to D via Φ?

Since the causal structure must be invariant under the action of G_{sa}, we may restrict our attention to cones in $T_0 D = E$.

Furthermore, any cone in E that gives rise to a G_{sa} invariant field of cones has to be Ad H_{sa} invariant.

It can also be shown that under the assumptions made, $d\Phi$ has to respect the ternary structure of each tangent space $T_p D$.

The question we are asking thus becomes:

What properties must an Ad H_σ-invariant cone in E_{sa} possess so that it is of the form $\Psi^{-1}(L(H)_+)$ for a *-ternary monomorphism $\Psi : E \to L(H)$?

If a cone does come from such an embedding we will call it *natural*.

Theorem (D. Blecher/WW) *Let E be a *-ternary ring of operators, which for the sake of brevity is supposed to be a dual Banach space. Then a cone $C \subseteq E_{sa}$ is natural iff there is a central, selfadjoint tripotent element $u \in E$ so that*

$$C = \{eue^* \mid e \in E\}$$

Here, the center of a *-ternary ring of operators is defined to be

$$Z(E) = \{e \in E \mid exy = xye \text{ for all } x, y \in E\}$$

Since $\operatorname{Ad} h(e) = h(e)$ for all $e \in H_\sigma$ and $e \in E = T_0 D$, it is clear that a cone C as above is $\operatorname{Ad} H_\sigma$-invariant and thus gives rise to a causal structure (in the real part) of D.

In the last result, we have refrained from putting everything into a more geometrical language.

Given the fact that a number of different theories overlap here, other characterizations are possible.

Causal structure on finite dimensional symmetric spaces (without being concerned with embeddings into $L(H)$) have extensively been studied by e.g. Faraut, Hilgert and Olafsson. In these investigations, the approach is purely Lie theoretic.

More general manifolds with causal structure have been taken up, among others, by A.D. Aleksandrov, J. Hofmann, Lawson and Neeb.