ORDERED INVOLUTIVE OPERATOR SPACES

DAVID P. BLECHER, KAY KIRKPATRICK, MATTHEW NEAL, AND WEND WERNER

Abstract. This is a companion to recent papers of the authors; here we consider the 'selfadjoint operator space' case of the universal property of the ordered noncommutative Shilov boundary'. We also discuss briefly 'maximal' and 'minimal' unitizations of nonunital ordered operator spaces.

1. Introduction

An operator space is a closed linear space of bounded operators between Hilbert spaces, or, equivalently, a subspace of a C^*-algebra. Although ordered operator spaces containing 1 are well understood and important (these are known as operator systems [5, 11]), *nonunital* ordered operator spaces have barely been studied at all. Indeed, we are only aware of [12, 13, 14, 4, 2] and a series of papers by Karn (see e.g. [7, 8] and references therein). In view of the importance of the notion of operator positivity, we offer, in this companion paper to [4, 2], a few results on this topic.

In 2001, as a 'Research Experience for Undergraduates' project, the second author began to develop a noncommutative Shilov boundary for 'selfadjoint operator spaces' [9]. It is well known (see [6, 3]) that the noncommutative analogue of the Shilov boundary of an operator space should be a *ternary ring of operators*, or TRO for short. This is a subspace Z of a C*-algebra which is closed under products of the form xy^*z. If X is selfadjoint then its enveloping TRO will also be selfadjoint. In [4] the first and last author developed the theory of '$*$-TROs', and the ways in which they can be ordered. If X is an ordered operator space, then one might hope that the 'arrows' in the universal diagram/property for the noncommutative Shilov boundary may be chosen to be (completely) positive. This was done in [2] in a general setting. In the present paper we give the technical details of how to adapt these results to the case of selfadjoint operator spaces. For example, the proof of the just mentioned universal property is necessarily quite different in our setting here. At the end of our paper, we discuss briefly 'maximal' and 'minimal' unitizations of nonunital ordered operator spaces.

Any unexplained notation can be found in [4, 2, 3]. A map $T : X \to Y$ between vector spaces which have involutions is called $*$-*linear* or *selfadjoint*, if it is linear and $i(x^*) = i(x)^*$ for all $x \in X$. A *selfadjoint operator space* is an operator space X with an involution $* : X \to X$, such that there exists a complete isometry $i : X \to B(H)$ which is $*$-linear. Alternatively, a selfadjoint operator space is an

Date: today.

*Blecher was partially supported by grant DMS 0400731 from the National Science Foundation. Neal was supported by Denison University. Werner was supported by the SFB 487 Geometrische Strukturen in der Mathematik, at the Westfälische Wilhelms-Universität, supported by the Deutsche Forschungsgemeinschaft.

operator space X with an involution $* : X \to X$, such that

$$\|[x_{ji}^*]\| = \|[x_{ij}]\|, \qquad n \in \mathbb{N},\ [x_{ij}] \in M_n(X).$$

(To see the nontrivial direction of this equivalence, suppose that X is a concrete (not necessarily selfadjoint) subspace of $B(H)$, and that $\tau : X \to X$ is an involution satisfying the last centered equation. Then the map

$$X \longrightarrow M_2(B(H)) : x \mapsto \begin{bmatrix} 0 & x \\ \tau(x)^* & 0 \end{bmatrix},$$

is a linear selfadjoint complete isometry of X onto a selfadjoint subspace of a C^*-algebra.)

We end this introduction with the following remark which we will refer to later. One initial motivation to study nonunital ordered operator spaces comes from the fact that the dual of a C*-algebra has a very nice positive cone (namely, the positive linear functionals), and is an operator space. This leads one to hope that it may be treated as an ordered operator space. However the situation is certainly more difficult than one might first think:

Proposition 1.1. *If A' is the dual of a nontrivial C^*-algebra A, with its usual cone, then there exists no isometric positive map from A' into another C^*-algebra.*

Proof. First suppose that $A = \ell^\infty_2$. In this case this result was deduced in [2] from some rather intricate facts. However there is a simple direct proof. Suppose that $u : \ell^1_2 \to B(H)$ is an isometric positive map. If $u(e_i) = T_i$, then T_i are positive contractions. Since $T_1 - T_2$ is selfadjoint, a well known formula gives

$$\|T_1 - T_2\| = \sup\{|\langle (T_1 - T_2)\zeta, \zeta\rangle| : \zeta \in \mathrm{Ball}(H)\} \leq 1.$$

This contradicts the fact that $2 = \|(1, -1)\|_{\ell^1_2} = \|T_1 - T_2\|$.

We finish the proof by proving the more general fact that if M is any von Neumann algebra of dimension > 1, then there exists no positive isometry T from the predual M_* into another C*-algebra B. For if there did exist such T, then $T'' : M' \to B''$ would be a positive isometry into a C*-algebra. Let p be a nontrivial projection in M, and let $N = \mathrm{Span}\{p, p^\perp\}$, the injective 2 dimensional C*-algebra. By injectivity, there exists a completely positive unital projection $P : M \to N$. Then $P' : N' \to M'$ is a positive isometry. Thus $P' \circ T''$ induces a positive isometry from ℓ^1_2 into a C*-algebra, contradicting the previous paragraph. ☐

2. The selfadjoint and the ordered noncommutative Shilov boundary

The natural morphisms between TROs are the *ternary morphisms*, that is the maps satisfying $T(xy^*z) = T(x)T(y)^*T(z)$. As mentioned in the introduction of [4], the basic facts about TROs and their ternary morphisms have selfadjoint variants valid for $*$-TROs and $*$-linear ternary morphisms. We will use the term 'ternary $*$-morphism' for the latter. Similarly, as we shall see next, there is a selfadjoint variant of the noncommutative Shilov boundary of an operator space. That is, Hamana's theory of ternary envelopes (see [6, 3]) easily restricts to the context of selfadjoint subspaces of C*-algebras.

By a *ternary $*$-extension* of a selfadjoint operator space X, we mean a pair (Z, i) consisting of a $*$-TRO Z, and a linear selfadjoint complete isometry $i : X \to Z$ such that 'odd polynomials' with variables taken from $i(X)$ are dense in Z. Equivalently, there is no nontrivial subTRO of Z containing $i(X)$. We define a *ternary*

$*$-*envelope* of X to be any ternary $*$-extension (Y, i) with the universal property of the next theorem. This could also be called the 'selfadjoint noncommutative Shilov boundary'.

Theorem 2.1. *If X is a selfadjoint operator space, then there exists a ternary $*$-extension (Y, j) of X with the following universal property: Given any ternary $*$-extension (Z, i) of X there exists a (necessarily unique, surjective, and selfadjoint) ternary morphism $\pi : Z \to Y$, such that $\pi \circ i = j$.*

Proof. Let (Z, i) be a ternary $*$-extension of X. By the basic theory of the ternary envelope (see [6] or 8.3.10 in [3]), there exists a subspace N of Z such that Z/N is a copy of the ternary envelope of X. For any operator space X, the adjoint space $X^* = \{x^* : x \in X\}$ is a well defined operator space, independent of the representation of X [3, 1.2.25]. We have $(X/N)^* \cong X^*/N^*$ completely isometrically, using [3, 1.2.15], for example. Since the canonical map $X \to Z/N$ is completely isometric, so is the canonical map $X = X^* \to (Z/N)^* \cong Z^*/N^* = Z/N^*$. However N is the largest subbimodule W of Z such that the canonical map $X \to Z/W$ is completely isometric (see [6] or [1, Theorem A11]). Thus $N^* \subset N$, so that $N = N^*$. Hence the ternary envelope $Y = Z/N$ of X is a $*$-TRO, by observations in the introduction to [4], and hence is a ternary $*$-extension of X. If (W, k) is any ternary $*$-extension of X, then W is a ternary extension of X, and by the universal property of the ternary envelope there exists a (necessarily unique and surjective) ternary morphism $\pi : W \to Y$ with $\pi \circ k = j$. It is easy algebra to check that π is $*$-linear. $\square$

It is now routine to extend the basic properties of ternary envelopes (see e.g. [6, 1] or [3, 8.3.10 and 8.3.12]) to ternary $*$-envelopes. We omit the details. Indeed, part of the proof above shows, using routine arguments, that any ternary $*$-envelope of X is a ternary envelope of X.

We now turn to *ordered operator spaces*, by which we mean a selfadjoint operator space X with specified cones $\mathfrak{c}_n \subset M_n(X)$ for each $n \in \mathbb{N}$, such that there exists a completely isometric $*$-linear map $T : X \to B(H)$ such that T is completely positive in the sense that $T_n(\mathfrak{c}_n) \subset M_n(B(H))_+$. (We warn the reader that this notation is nonstandard, in other papers T^{-1} is usually also required to be completely positive on $\mathrm{Ran}(T)$.) An ordered operator space is often written as a pair $(X, (\mathfrak{c}_n))$. There is another important variant of the theory, where we have only one cone $\mathfrak{c} \subset X$, and T above is positive. However since this variant works out almost identically, we will not often mention it.

We next consider an ordered version of the ternary envelope or noncommutative Shilov boundary. That is, we seek an ordered version of the universal property/diagram in Theorem 2.1. Except under extra hypotheses (for example, if the positive cone of X densely spans X), it is easy to show that the embeddings $i : X \to Z$ occurring in the universal property/diagram in the ordered case cannot be allowed in general to be arbitrary complete order embeddings, or even arbitrary completely positive complete isometries. We will usually need to limit the size of the cone of Z. Fortunately there is an appropriate bound for this cone, and this bound depends on the given cone of X.

More specifically, suppose that X is an ordered operator space, with matrix cones $(\mathfrak{c}_n)$. We assign a canonical cone $\mathfrak{d}$ to the ternary $*$-envelope $(\mathcal{T}(X), j)$, namely the intersection of all natural cones (in the sense of [4]) containing $(j(\mathfrak{c}_n))$. To see that there exists at least one such cone, note that if $i : X \to B$ is a completely positive complete isometry into a C*-algebra, and if W is the $*$-TRO generated by $i(X)$, then by Theorem 2.1 there is a ternary $*$-morphism $\theta : W \to \mathcal{T}(X)$ with $\theta \circ i = j$. Thus $\mathcal{T}(X)$ is ternary $*$-isomorphic to a quotient of W. By [4, Lemma 5.1], this quotient of W has a natural cone containing the image of $i(\mathfrak{c})$ in the quotient.

As mentioned in [4, Section 4], the intersection of natural cones (in the sense of [4]) is natural. Hence $\mathfrak{d}$ is a natural cone on $\mathcal{T}(X)$ containing $j(\mathfrak{c})$. We call $\mathcal{T}(X)$ equipped with this cone the *ordered ternary envelope* $\mathcal{T}^o(X)$. We remark that this may differ from the ordered ternary envelope of [2], since we are working in a different category, and in particular with a different notion of *natural cone*. We remark too that up to the obvious notion of equivalence, the ordered ternary envelope is well defined independently of any particular ternary $*$-envelope of X. This follows for example from the universal property below.

The following result is reminiscent of [2, Theorem 5.3], but seems to require a completely different proof. (Note that the proof in the setting considered in [2] needs to use the notion of 'range tripotents'. Since these may not be 'central' in the sense of [4], they will not work for us here.)

Theorem 2.2. *Suppose that $(X, \mathfrak{c})$ is an ordered operator space. Suppose that ι is a $*$-linear completely positive complete isometry from X into a $*$-TRO Z such that $\iota(X)$ generates Z as a $*$-TRO. Let $\mathfrak{e}$ denote the intersection of all natural cones of Z containing $\iota(\mathfrak{c})$. Let $\mathfrak{d}$ be the natural cone defining the ordered ternary $*$-envelope $\mathcal{T}^o(X)$. Then the canonical ternary $*$-morphism $\theta : Z \to \mathcal{T}^o(X)$ from Theorem* 2.1*, takes $\mathfrak{e}$ onto $\mathfrak{d}$.*

Proof. This requires basic results and notation from [4]. Let u be the tripotent in $\mathcal{T}(X)''$ associated (by the correspondence from [4, Theorem 4.16]) with the natural cone $\mathfrak{d}$ in $\mathcal{T}(X)$. We write $J(u)$ for the 'Pierce 2-space' of u in $\mathcal{T}(X)''$. The inverse image under the ternary $*$-morphism θ'' of $J(u)$ is a weak*-closed ternary $*$-ideal I of Z''. Thus, $Z'' = I \oplus^\infty J$ for some ternary $*$-ideal J (see e.g. [4, Lemma 3.4 (i)]). Now let z be the tripotent associated with the natural cone $\mathfrak{e}$. Then $z = z_1 + z_2$ for orthogonal tripotents $z_1 \in I$ and $z_2 \in J$. Since z is central and selfadjoint, it follows easily that z_1 is central and selfadjoint. Because θ'' takes open central tripotents to open central tripotents (by a variant of [2, Proposition 3.5]), and hence natural cones to natural cones, $\theta''(z) \geq u$. Since $z_2 I = 0$, it is also clear that $\theta''(z_2)$ is orthogonal to u. Thus, $\theta''(z_1) \geq u$. But $\theta''(z_1) \in J(u)$, and so $\theta''(z_1) = u\theta''(z_1)u = u$. It follows that $\theta''(J(z_1)) \subset J(u)$, where $J(z_1)$ is the Pierce 2-space of z in Z''.

Recall that $\iota(\mathfrak{c})$ lies in $J(z)_+$, the positive part of the Pierce 2-space of z. Thus, for any $x \in \iota(\mathfrak{c})$, we have $x = z^2 x = z_1^2 x + z_2^2 x$. Since $x \in I$, we have $z_2^2 x = 0$, so that $x = z_1^2 x$. Hence, $\iota(\mathfrak{c})$ lies in $J(z_1)_+$, which is a natural cone in Z''. It follows that $\iota(\mathfrak{c})$ lies in the natural cone $\mathfrak{d}_{z_1}$ (see Lemma 4.15 (2) in [4]), which in turn is contained in $\mathfrak{d}_z = \mathfrak{e}$. Hence $\mathfrak{e} = \mathfrak{d}_{z_1}$. Since θ'' takes squares of selfadjoint elements in $J(z_1)$ to squares of selfadjoints in $J(u)$, we have $\theta(\mathfrak{e}) \subset \mathfrak{d}$. Since θ takes natural cones to natural cones (by a variant of [2, Proposition 3.5]), by the definition of $\mathfrak{d}$ we must have $\theta(\mathfrak{e}) = \mathfrak{d}$. □

The ordered noncommutative Shilov boundary is particularly nice in the case that X has a spanning cone: one may replace the cone $\mathfrak{e}$ defined in Theorem 2.2 by the entire positive cone of Z. One has:

Corollary 2.3. *Suppose that X is an ordered operator space with a cone $\mathfrak{c}$ which densely spans X, and that $i : X \to B$ is a positive complete isometry from X into a C*-algebra. Then the $*$-TRO A generated by $i(X)$ is a C*-subalgebra of B. Moreover, the ordered ternary envelope of X is a C*-algebra, and the canonical morphism $\theta : A \to \mathcal{T}^o(X)$ from the previous theorem, is a $*$-homorphism with $\theta(A_+)$ equal to the positive cone of $\mathcal{T}^o(X)$.*

This is proved just as in [2, Corollary 5.4]. Indeed most of the other results in [2, Section 5] carry over with almost identical proofs to our setting.

For the following result, we recall that the classical Shilov boundary is usually shown to exist for function spaces which contain constant functions. A space not containing constant functions may not have a Shilov boundary in the usual sense. We prove that this boundary does exist in a case we have not seen discussed in the classical literature:

Corollary 2.4. *Suppose that X is a closed selfadjoint subspace of $C(K)$, for a compact Hausdorff K, and suppose that the cone $X \cap C(K)_+$ densely spans X. Then the Shilov boundary of X exists as a topological space in the following sense. Namely, there exists a locally compact topological space ∂X, and a positive linear isometry $j : X \to C_0(\partial X)$ such that $j(X)$ separates points of ∂X and does not vanish identically at any point, and such that for any other locally compact topological space Ω and positive linear isometry $i : X \to C_0(\Omega)$ such that $i(X)$ strongly separates points and does not vanish identically at any point, there exists a homeomorphism τ from ∂X onto a subset of Ω, such that $i(x) \circ \tau = j(x)$ for all $x \in X$.*

Proof. Suppose that X is completely order isomorphic to a subspace of $C(K)$. The TRO inside $C(K)$ generated by X is a C*-algebra A by Corollary 2.3, and it is commutative. By the universal property in Corollary 2.3, there is a $*$-homomorphism from A onto $\mathcal{T}^\circ(X)$. Thus $\mathcal{T}^\circ(X)$ is a commutative C*-algebra, so that $\mathcal{T}^\circ(X) = C_0(\partial X)$ for a locally compact topological space ∂X. Clearly the copy of X separates points of ∂X and does not vanish identically at any point. Given any (Ω, i) as stated, then $(C_0(\Omega), i)$ is a ternary $*$-extension of X, so that by Corollary 2.3 there is a surjective $*$-homomorphism $\pi : C_0(\Omega) \to C_0(\partial X)$ as in that Corollary, with $\pi \circ i = j$. By the well known dualities between categories of topological spaces and algebras of continuous functions, π induces a homeomorphic embedding $\tau : \partial X \to \Omega$ such that $i(x) \circ \tau = j(x)$. $\square$

The latter Shilov boundary will have the usual familiar properties of the classical Shilov boundary.

3. Unitizations

The obvious way to attempt to understand nonunital ordered operator spaces is to 'unitize' them; that is to embed them as a codimension one subspace of an operator system (in the usual sense of e.g. [5]). This was first done in [13]. That paper assigns to a 'matrix ordered operator space' X a unitization X^+, which is an operator system. Note X^+ is spanned by X and 1, and the embedding of X into X^+ is a completely contractive complete order embedding. Note that one cannot hope that the embedding of X into X^+ be isometric too. To see this, suppose that this embedding was isometric in the case that $X = A'$ is the dual of a nontrivial C*-algebra A, with its usual cone (which is a 'matrix ordered operator space').

Since X^+ is an operator system, and hence may be viewed as a subspace of $B(H)$ containing I_H, we have contradicted Proposition 1.1. (We remark in passing that the same argument shows that the main result in [7] is not correct as stated. After communicating this to the author of that paper, this led to the correction [8].)

It is shown in [13, Lemma 4.9 (c)] that X^+ has the following universal property: for any completely contractive completely positive map T from X into a (unital) operator system Y, the extension $x+\lambda 1 \mapsto T(x)+\lambda 1$, from X^+ into Y is completely positive. Since the latter map is unital, it is also completely contractive. From this property, it follows that X^+ possesses the smallest cones a unitization of X can have. To see this, take Y in the universal property above to be any other unitization of X.

In contrast, it may well be useful in certain cases to consider a unitization of X with the biggest possible cone. For ordered operator spaces this can be done by the results in Section 2. If X is an ordered operator space, let $W = \mathcal{T}^o(X)$ be its ordered ternary envelope. Since the positive cone of this is 'natural' in the sense of [4], by the results of that paper we may consider W as a $*$-TRO inside a C*-algebra, with the inherited cone. Then $B = W + W^2$ is a C*-algebra (see [4]), and we set X^1 be the span of X and the identity of the C^*-algebra unitization of B.

Theorem 3.1. *Suppose that X is an ordered operator space with a densely spanning cone (resp. X is an ordered operator space), that H is a Hilbert space, and that $i : X \to B(H)$ is a $*$-linear complete isometry which is completely positive (resp. completely positive and such that there is no smaller natural cone in $\langle i(X)\rangle$ containing the image of the cone of X). Then there is a completely positive unital map from $i(X) + \mathbb{C}\, I_H \to X^1$ extending the canonical map $i(X) \to X$.*

Proof. Let θ be the canonical ternary $*$-morphism from the TRO A generated by $i(X)$ to $\mathcal{T}^o(X)$—see Theorem 2.2 and Corollary 2.3. If X has a densely spanning cone. Then by Corollary 2.3, A is a C*-algebra A, and θ is a $*$-homomorphism, and hence extends to a $*$-homomorphism π from a unitization of A, which we may take to be the span of A and I_H, into a unitization of the C*-algebra $\mathcal{T}^o(X)$. We then restrict π to the span of $i(X)$ and I_H.

In the 'respectively case', by Theorem 2.2 θ is positive, and hence is positive as a map into the C^*-algebra $B = W + W^2$ discussed above. By [4, Corollary 4.3] we can extend θ to a completely positive unital $*$-homomorphism from a unitization of $\langle i(X)\rangle + \langle i(X)\rangle^2$ into the unitization of $W + W^2$. Finally, restrict to the span of $i(X)$ and I_H as before. $\square$

In [7, 8] a unitization for 'matricial Riesz spaces' is introduced. We show next that this unitization coincides with the one in [13]. Thus there is a nice 'internal' description of the cone on X^+ in the case that X is a 'matricial Riesz space'.

WEND ADD

References

[1] D. P. Blecher, *The Shilov boundary of an operator space and the characterization theorems,* J. Funct. Anal. **182** (2001), 280–343.

[2] D. P. Blecher and M. Neal, *Open partial isometries and positivity in operator spaces* Preprint 2006.

[3] D. P. Blecher and C. Le Merdy, *Operator algebras and their modules—an operator space approach,* Oxford Univ. Press, Oxford (2004).

[4] D. P. Blecher and W. Werner, *Ordered C^*-modules*, Proc. London Math. Soc. **92** (2006), 682-712.
[5] M.-D. Choi and E. G. Effros, *Injectivity and operator spaces,* J. Funct. Anal. **24** (1977), 156–209.
[6] M. Hamana, *Triple envelopes and Silov boundaries of operator spaces,* Math. J. Toyama University **22** (1999), 77-93.
[7] A. K. Karn, *Adjoining an order unit to a matrix ordered space,* Positivity **9** (2005), 207-223.
[8] A. K. Karn, *Corrigedem to the paper "Adjoining an order unit to a matrix ordered space"*, Preprint 2006.
[9] K. Kirkpatrick, *The Shilov boundary and M-structure of operator spaces,* Research Experiences for Undergraduates paper, Houston (2001).
[10] M. Neal, *Inner ideals and facial structure of the quasi-state space of a JB-algebra,* J. Funct. Anal. **173** (2000), 284-307.
[11] V. I. Paulsen, *Completely bounded maps and operator algebras,* Cambridge Studies in Advanced Math., 78, Cambridge University Press, Cambridge, 2002.
[12] W. J. Schreiner, *Matrix regular operator spaces,* J. Funct. Anal. **152** (1998), 136–175.
[13] W. Werner, *Subspaces of $L(H)$ that are $*$-invariant,* J. Funct. Anal. **193** (2002), 207–223.
[14] W. Werner, *Multipliers on matrix ordered operator spaces and some K-groups,* J. Funct. Anal. **206** (2004), 356-378.

Department of Mathematics, University of Houston, Houston, TX 77204-3008
E-mail address, David P. Blecher: `dblecher@math.uh.edu`

Department of Mathematics, University of ADD
E-mail address, ADD: `ADD`

Department of Mathematics, Denison University, Granville, OH 43023
E-mail address: `nealm@denison.edu`

Mathematisches Institut, Einsteinstr. 62, D-48149 Münster, Germany
E-mail address, Wend Werner: `wwerner@math.uni-muenster.de`