MR1245836 (94i:58010) 58B20 46L05

Corach, G. (RA-IAM); **Porta, H.** (1-IL);
Recht, L. **[Recht, Lazaro]** (YV-SBOL)

Convexity of the geodesic distance on spaces of positive operators.

Illinois J. Math. **38** (1994), *no.* 1, 87–94.

For a unital C^*-algebra A, the open convex cone A^+ of strictly positive elements has a natural transitive action of the group G of invertible elements in A and is thus a homogeneous space with an invariant Finsler metric. Using 1-parameter subgroups of G one can define geodesics in A^+. The authors prove various convexity theorems for functions defined by geodesics, for example the function $t \mapsto \operatorname{dist}(\gamma(t), \delta(t))$ of a real parameter. Here γ and δ are geodesics in A^+ and $\operatorname{dist}(a, b) = \| \ln(a^{-1/2} b a^{-1/2}) \|$ is the geodesic distance. As a corollary, geodesic spheres are shown to be convex sets.

Harald Upmeier (D-MRBG-MI)

MR1239452 (94h:46089) 46L05 58B20

Corach, G. (RA-IAM); **Porta, H.** (1-IL);
Recht, L. **[Recht, Lazaro]** (YV-SBOL)

The geometry of spaces of projections in C^*-algebras.

Adv. Math. **101** (1993), *no.* 1, 59–77.

Let A denote a C^*-algebra with identity, let Q be the set of all idempotent elements of A and let P be the set of selfadjoint elements of Q. The sets Q and P play the role of infinite-dimensional Grassmannians and important applications to operator theory and complex or differential geometry have already been found. However, much of the differential geometry of these Grassmannians is not completely understood today.

The authors of the paper under review began, several years ago, an extensive study of the geometry of the spaces Q and P and some similar idempotent varieties. The present paper investigates the natural fibration $Q \xrightarrow{\pi} P$ given by the polar decomposition of the associated symmetries; as a byproduct of this analysis the authors introduce a natural Finsler metric on Q and study the corresponding geodesics. They prove, for instance, the existence of a unique geodesic in Q joining two points of P and, respectively, the existence of a geodesic fully contained in the fibre of π joining any two points of this fibre.

Mihai Putinar (1-UCSB)

MR1217380 (94c:46114) 46L05 47A63 47B65 47D25 58B20

Corach, Gustavo (RA-IAM); **Porta, Horacio** (1-IL);
Recht, Lázaro (YV-SBOL)

Geodesics and operator means in the space of positive operators. (English. English summary)

Internat. J. Math. **4** (1993), *no.* 2, 193–202.

Let A be a unital C^*-algebra, A^+ the subset of positive invertible elements in A. Then A^+ has a natural Finsler structure and a natural connection for which the geodesic equation can be solved explicitly. It turns out that for any two points in A^+ there is a unique geodesic joining these two points. In this paper the authors study the notion of geodesic convexity with respect to this structure. They prove geodesic convexity of some subsets of A^+ and derive related operator inequalities. Finally they interpret the results in terms of operator means in the sense of Kubo and Ando and in terms of relative entropy.

Andreas Cap (A-WIEN)

MR1209304 (94d:58010) 58B20 46L05 47B15 47D25

Corach, Gustavo (RA-IAM); **Porta, Horacio** (1-IL);
Recht, Lázaro (YV-SBOL)

The geometry of the space of selfadjoint invertible elements in a C^*-algebra.

Integral Equations Operator Theory **16** (1993), *no.* 3, 333–359.

For a unital C^*-algebra A let G be the set of invertible elements in A and G^{s} the subset of selfadjoint elements in G. The authors show that for any $a \in G^{\mathrm{s}}$ the orbit $G^{\mathrm{s},a}$ through a of the action $g \cdot a = (g^{-1})^* a g^{-1}$ of G on G^{s} is open and closed in G^{s} and the map $g \mapsto g \cdot a$ defines a locally trivial principal bundle over the orbit. Next they construct a canonical connection on this principal bundle and study the differential equation which describes horizontal lifts of curves. Then they study connections on some G-equivariant bundles over G^{s} which are induced by this canonical connection, in particular on the tangent bundle TG^{s}, on which they obtain a canonical Finsler structure.

The polar decomposition defines a fibration from G^{s} to the space of orthogonal reflections in A. Using this fibration the authors study the geometry of G^{s}, in particular, geodesics with respect to the Finsler structure. The main result is that for two points in the same fiber there is a unique geodesic contained in the fiber which joins the two points, and this is the shortest curve in G^{s} with these endpoints.

Andreas Cap (A-WIEN)

MR1200155 (94a:46102) 46L87 58B20

Corach, G. (RA-IAM); **Porta, H.** (1-IL);
Recht, L. **[Recht, Lazaro]** (YV-SBOL)

Jacobi fields on space of positive operators. (English. English summary)

Linear Algebra Appl. **179** (1993), 271–275.

Summary: "Let A be a C^*-algebra with 1 and denote by A^+ the set of invertible positive elements of A with its canonical connection and Finsler structure [see G. Corach, H. Porta and L. Recht, Integral Equations Operator Theory **16** (1993), no. 3, 333–359]. Then a Jacobi field $J(t)$ along a geodesic in A^+ with initial conditions $J(0)=0$ or $(DJ/dt)|_{t=0}$ has increasing Finsler norm for $t\geq 0$."

MR1075945 (92h:46105) 46L99 58B20

Corach, G. (RA-IAM); **Porta, H.** (1-IL);
Recht, L. **[Recht, Lazaro]** (YV-SBOL)

A geometric interpretation of Segal's inequality $\|e^{X+Y}\|\leq \|e^{X/2}e^Ye^{X/2}\|$.

Proc. Amer. Math. Soc. **115** (1992), *no.* 1, 229–231.

Let A be a unital C^*-algebra. Suppose that A^+ is the set of positive invertible elements in A. Then A^+ is an open subset of A^s, the real Banach space of symmetric elements, and therefore the tangent space TA^+_a to the manifold A^+ at $a\in A^+$ can be identified with A^s. The manifold A^+ carries a natural Finsler metric defined by $||X||_a = ||a^{-1/2}Xa^{-1/2}||$ for $X\in TA^+_a$. The distance $d(a,b)$ in the Finsler metric is defined by $||X||_a$, where $b=e^{(1/2)Xa^{-1}X}ae^{(1/2)a^{-1}X}$.

The main result is proved by means of the Segal inequality. Theorem: For each $a\in A^+$, the exponential map $\exp_a\colon TA^+_a\to A^+$ increases distances in the sense that $d(\exp_a X,\exp_a Y)\geq ||X-Y||_a$ for all $X,Y\in TA^+_a$. *Liang Sen Wu* (PRC-ECNU)

MR1149695 (93j:46059) 46L05 46L99
Corach, Gustavo (RA-IAM); **Porta, Horacio** (1-IL);
Recht, Lázaro (YV-SBOL)
Splitting of the positive set of a C^*-algebra.
Indag. Math. (*N.S.*) **2** (1991), *no.* 4, 461–468.
Let A be a unital C^*-algebra, B a C^*-subalgebra of A containing the identity and $H \subset A$ a Banach space supplement of B in A; that is, $A = B \oplus G$. If H is closed under the $*$-operation in A, then $A^s = B^s \oplus H^s$ where "s" denotes the selfadjoint elements in the corresponding set. Denote the set of positive invertible elements in A and B by $A^\oplus$ and $B^\oplus$, respectively, and let $E = \{\exp(h)\colon h \in H^s\}$, the set of exponentials of elements in H^s. Consider the mapping $\Phi\colon B^\oplus \times E \to A^\oplus$ defined by $\Phi(b,e) = (be)^+$, where $(be)^+$ is the "positive part" of be, i.e., $(be)^+$ is the positive square root of $(be)(be)^+$. In the paper under review the authors prove that Φ is a diffeomorphism in several situations, most notably when A is finite-dimensional and $H = B^\perp$ for a suitable inner product in A. The proof relies on a theorem of R. Hermann [Indag. Math. **25** (1963), 47–56; MR0152969 (27 #2940)] and several of the previous results of the authors on the geometry of the space of selfadjoint invertible elements of a C^*-algebra.

Robert S. Doran (1-TXC)

MR1077981 (91m:47020) 47A63 47A30 47B15 47D25
Corach, G. (RA-IAM); **Porta, H.** (1-IL);
Recht, L. [Recht, Lazaro] (YV-SBOL)
An operator inequality.
Linear Algebra Appl. **142** (1990), 153–158.
From the text: "For a Hilbert space H consider the set G^s of bounded linear Hermitian invertible operators in H and the subset $P \subset G^s$ of unitary reflections (i.e., operators with $R^* = R = R^{-1}$). If we write $A \in G^s$ as $A = NR$ with N positive and R unitary (the polar decomposition of A), then $R \in P$, and $A \to R$ defines a map $\pi\colon G^s \to P$. The sets G^s and P are smooth submanifolds of the C^*-algebra of bounded linear operators in H, and $\pi\colon G^s \to P$ is a smooth fibration. Furthermore, we introduce on G^s a natural Finsler structure by assigning to a tangent vector $X \in T_A G^s$ the norm $\|X\|_A = \|N^{1/2} X N^{1/2}\|$ (operator norm). Elsewhere ["The geometry of the space of selfadjoint invertible elements of a C^*-algebra", to appear] we prove that the tangent map $T_A\pi\colon T_A G^s \to T_{\pi(A)} P$ decreases norms, together with some geometric consequences similar to those shown in another paper of ours ["The geometry of spaces of projections in C^*-algebras", Adv. in Math., to

appear]. The essential step in obtaining this result is the following operator inequality, whose proof is the objective of this note. Theorem: Let S, T be bounded linear operators in Hilbert space, with S invertible Hermitian or invertible skew-symmetric; then $\|STS^{-1} + S^{-1}TS\| \geq 2\|T\|$."

MR1073852 (93a:46099) 46H99 46M20 58B20

Corach, G. (RA-IAM); **Porta, H.** (1-IL);
Recht, L. [Recht, Lazaro] (YV-SBOL)

Differential geometry of spaces of relatively regular operators.

Integral Equations Operator Theory **13** (1990), *no.* 6, 771–794.

Let A be a Banach algebra with group of units G and with the set of idempotents Q. The authors study the topological and geometric properties of the space $S = \{(a,b) \in A \times A\colon\ ar = a,\ rb = b,\ ba = r\}$, where r is a fixed element in Q. In particular, they study principal fiber bundles $\tau\colon G \to S$, where $\tau(g) = (ga_0, b_0g^{-1})$, $(a_0, b_0) \in S$, and $\theta\colon S \to Q$, where $\theta((a,b)) = ab$. When A is a C^*-algebra and $r = r^*$, they also study a real analytic retraction of S onto $\{(a,b) \in S\colon\ b = a^*\}$. The authors announce further papers on similar topics.

W. Żelazko (Zbl 726:46028)

MR1051073 (91g:46056) 46H99 46M20 58B25

Corach, Gustavo (RA-IAM); **Porta, Horacio** (1-IL);
Recht, Lázaro (YV-SBOL)

Differential geometry of systems of projections in Banach algebras.

Pacific J. Math. **143** (1990), *no.* 2, 209–228.

Let A be a unital Banach algebra. The authors focus on the refined structures of the set of all n-partitions of unity: $Q_n = \{(q_1, \cdots, q_n) \in A^n,\ q_iq_k = \delta_{ik}q_i,\ \sum_{i=1}^n q_i = 1\}$. The differential geometry and the algebraic topology of Q_n provide invariants of the algebra A itself.

The authors study the fibration $\pi\colon G \to Q_n$, $\pi(g) = gqg^{-1}$, $q \in Q_n$, by defining a natural connection on it. Here G stands for the group of invertible elements of A. The parallel transport equation with respect to this connection turns out to be identical to the transport of Yu. L. Daletskiĭ and S. G. Kreĭn [Dokl. Akad. Nauk SSSR **6** (1950), 433–436].

Mihai Putinar (1-UCSB)

MR1044801 (91b:47043) 47B15 47A55 58B20

Corach, G. (RA-IAM); **Porta, H.** (1-IL);
Recht, L. [Recht, Lazaro] (YV-SBOL)

A metric property of the polar decomposition of projections.

Analysis and partial differential equations, 417–426, *Lecture Notes in Pure and Appl. Math.*, 122, *Dekker, New York*, 1990.

The authors study a variational property of the embedding $P \subset Q$, namely the fact that for projections of rank 1, the distance (= geodesic distance) from a fixed element $q \in Q$ to a variable element $p \in P$ is attained at a unique $p = \pi(q)$. This element $\pi(q)$ can be characterized in a variety of ways. (Notation: If H is a Hilbert space, Q denotes the set of bounded linear $q\colon H \to H$ with $q^2 = q$ and $P \subset Q$ the selfadjoint q.)

{For the entire collection see MR1044775 (90j:00006)}

Themistocles M. Rassias (Athens)

MR1009189 (90h:46091) 46L05

Corach, Gustavo (RA-IAM); **Porta, Horacio** (1-IL);
Recht, Lázaro (YV-SBOL)

Two C^*-algebra inequalities.

Analysis at Urbana, Vol. II (*Urbana, IL*, 1986–1987), 141–143, *London Math. Soc. Lecture Note Ser.*, 138, *Cambridge Univ. Press, Cambridge*, 1989.

With a view to geometrical applications in a forthcoming paper ["The structure of projections in a C^*-algebra", to appear], the authors prove two inequalities of a technical nature for elements η and b of an arbitrary C^*-algebra A: (1) $\|\eta\| \le \|c\eta a \pm b\eta^* b\|$; (2) $\|c\eta b^* \pm b\eta^* c\| \le K\|c\eta a \pm b\eta^* b\|$. Here a and c are the positive square roots of $1 + b^*b$ and $1 + bb^*$, respectively, and K depends only on b. The proofs involve the study of the real-linear map $\Phi\colon A \to A$ given by $\Phi(\eta) = c\eta a - b\eta^* b$.

{For the entire collection see MR1009181 (90d:46003)}

Robert J. Archbold (4-ABER)

MR1108460 (92h:46068) 46G15 46H20 46M20 58B20

Corach, G. (RA-IAM); **Porta, H.** (1-IL);
Recht, L. [Recht, Lazaro] (YV-SBOL)

Multiplicative integrals and geometry of spaces of projections.

Conference in Honor of Mischa Cotlar (Buenos Aires, 1988).
Rev. Un. Mat. Argentina **34** (1988), 132–149 (1990).

Consider a Banach algebra A with identity, and set $Q_n = \{q = (q_1, \cdots, q_n) \in A^n\colon\ q_k^2 = q_k,\ q_i q_k = 0$ if $i \neq k$, and $\sum_{k=1}^n q_k = 1\}$. Let G denote the group of units of A, and define $\pi\colon G \to Q_n$ by $\pi(g) = g q_0 g^{-1}$, where $q_0 \in Q_n$ is given. Then π defines a principal fibre bundle over its image, and, in particular, any curve $\gamma\colon [0,1] \to Q_n$ with origin q_0 admits a lift to $\Gamma\colon [0,1] \to G$, that is, $\Gamma(t) q_0 \Gamma(t)^{-1} = \gamma(t)$ for each $t \in [0,1]$ [cf. the authors, Pacific J. Math. **143** (1990), no. 2, 209–228; MR1051073 (91g:46056)].

For a continuous and rectifiable $\gamma\colon [\alpha,\beta] \to Q_n$, the authors obtain an explicit lift Γ, which is the horizontal lift of γ for a connection on Q_n. The method is to construct, for every $t \in [\alpha,\beta]$, an element $M_\alpha^t(\gamma) \in G$ such that $M_\alpha^t(\gamma)\gamma(\alpha)M_\alpha^t(\gamma)^{-1} = \gamma(t)$ for all $t \in [\alpha,\beta]$. M turns out to be a multiplicative integral, in that $(M_u^v)(M_v^w) = M_u^w$ for all u, v and $w \in [\alpha,\beta]$; these are special cases of the multiplicative integrals of V. P. Potapov [Trudy Moskov. Mat. Obshch. **4** (1955), 125–236; MR0076882 (17,958f); translated in Amer. Math. Soc. Transl. (2) **15** (1960), 131–243; see MR0114915 (22 #5733)]. The lift Γ is then defined by $\Gamma(t) = M_\alpha^t(\gamma)$ for each $t \in [\alpha,\beta]$ and coincides with the lift mentioned earlier when γ is C^1.

An analogous result is obtained for spaces of the form $S_q = \{(a,b) \in A^2\colon\ aq = a,\ qb = b,\ ba = q\}$, where $q \in A$ is a fixed idempotent. Here, if $\gamma\colon [\alpha,\beta] \to S_q$ is rectifiable and continuous, then there is a lift $\gamma\colon [\alpha,\beta] \to G$ such that $\Gamma(t) \cdot \gamma(\alpha) = \gamma(t)$ for all $t \in [\alpha,\beta]$. In particular, if γ is C^1, then Γ is the unique solution of an initial value problem. This relates to the decomposition of Banach spaces studied by, e.g., A. Douady [Ann. Inst. Fourier (Grenoble) **16** (1966), no. 1, 1–95; MR0203082 (34 #2940)].

{For the entire collection see MR1108446 (91m:00041)}

David A. Robbins (1-TRC)

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CONVEXITY OF THE GEODESIC DISTANCE ON SPACES OF POSITIVE OPERATORS

G. CORACH, H. PORTA AND L. RECHT

Let A be a C^*-algebra with 1 and denote by A^+ the set of positive invertible elements of A. The set A^+ being open in $A^s = \{a \in A;\ a^* = a\}$ it has a C^∞ structure and we can identify TA_a^+ with A^s for each $a \in A^+$. We use G to denote the group of invertible elements of A. Notice that G operates on the left on A^+ by the rule

$$L_g a = (g^*)^{-1} a g^{-1} \quad (g \in G, a \in A^+).$$

This action allows us to introduce a natural reductive homogeneous space structure in the sense of [8] (for details see [2], [3], [4]).

The corresponding connection—which is preserved by the group action—has covariant derivative

$$\frac{DX}{dt} = \frac{dX}{dt} - \frac{1}{2}\left(\dot{\gamma}\gamma^{-1}X + X\gamma^{-1}\dot{\gamma}\right)$$

where X is a tangent field on A^+ along the curve γ and exponential

$$\exp_a X = e^{Xa^{-1}/2} a e^{a^{-1}X/2}, \quad a \in A^+, X \in TA_a^+.$$

The curvature tensor has the formula

$$R(X,Y)Z = -\tfrac{1}{4}a\left[\left[a^{-1}X, a^{-1}Y\right], a^{-1}Z\right]$$

for $X, Y, Z \in TA_a^+$. The manifold A^+ has also a natural Finsler structure given by

$$\|X\|_a = \|a^{-1/2}Xa^{-1/2}\| \text{ for } X \in TA_a^+$$

and the group G operates by isometries for this Finsler metric.

THEOREM 1. *If $J(t)$ is a Jacobi field along the geodesic $\gamma(t)$ in A^+ then $\|J(t)\|_{\gamma(t)}$ is a convex function of $t \in \mathbf{R}$.*

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Proof. The method of proof is based on a similar strategy used in [4]. By definition $J(t)$ satisfies the equation

$$\frac{D^2J}{dt^2} + R(J,V)V = 0 \tag{1}$$

where $V(t) = \dot{\gamma}(t)$.

Notice that by the invariance of the connection and the metric under the action of G we may assume that $\gamma(t) = e^{tX}$ is a geodesic starting at $\gamma(0) = 1 \in A$, where $X \in A^s$. Then for the field $K(t) = e^{-tX/2}J(t)e^{-tX/2}$ the differential equation (1) changes into

$$4\ddot{K} = KX^2 + X^2K - 2XKX, \tag{2}$$

(where the dots indicate ordinary derivative with respect to t). Since the group G acts by isometries, we have $\|J(t)\|_{\gamma(t)} = \|\gamma(t)^{-1/2}J(t)\gamma(t)^{-1/2}\| = \|K(t)\|$. Thus the proof reduces to showing that for any solution $K(t)$ of (2) the function $t \to \|K(t)\|$ is convex in $t \in \mathbf{R}$, where the norm is the *ordinary norm* in the C^* algebra A. So fix $u < v \in \mathbf{R}$ and let t satisfy $u \le t \le v$. We will prove that

$$\|K(t)\| \le \frac{v-t}{v-u}\|K(u)\| + \frac{t-u}{v-u}\|K(v)\|. \tag{3}$$

Consider first the case where the selfadjoint element $X \in A$ has the form

$$X = \sum_{i=1}^{n} \lambda_i p_i \tag{4}$$

with $\lambda_1, \lambda_2, \ldots \lambda_n$ real numbers and $p_1, p_2, \ldots p_n$ selfadjoint elements of A satisfying $p_i p_j = 0$ for $i \ne j$ and $p_1 + p_2 + \cdots + p_n = 1$.

Suppose that A is faithfully represented in a Hilbert space $\mathscr{H}$. For fixed $x \in A$ decompose $x \in \mathscr{H}$ as $x = \sum_{i=1}^n \xi_i x_i$ where x_i is a unit vector in the range of p_i and the ξ_i are appropriate scalars. Define next the matrix $k(t) = (k_{ij}(t))$ by $k_{ij}(t) = \langle K(t)x_i, x_j\rangle$ for all t. The differential equation (2) is equivalent to the equations

$$\ddot{k}_{ij}(t) = \delta_{ij}^2 k_{ij}(t) \tag{2ij}$$

where $\delta_{ij} = (\lambda_i - \lambda_j)/2$.

A simple verification (or Bernoulli's formula) shows that all solutions of $\ddot{f}(t) = c^2 f(t)$ satisfy

$$f(t) = \phi(u,v,c;t)f(u) + \psi(u,v,c;t)f(v)$$

where

$$\phi(u,v,c;t) = \begin{cases} \dfrac{\operatorname{Sinh} c(v-t)}{\operatorname{Sinh} c(v-u)} & \text{for } c \neq 0, \\ \dfrac{(v-t)}{(v-u)} & \text{for } c = 0, \end{cases}$$

$$\psi(u,v,c;t) = \begin{cases} \dfrac{\operatorname{Sinh} c(t-u)}{\operatorname{Sinh} c(v-u)} & \text{for } c \neq 0, \\ \dfrac{(t-u)}{(v-u)} & \text{for } c = 0. \end{cases}$$

Then each $k_{ij}(t)$ satisfies

$$k_{ij}(t) = \phi_{ij}(t)k_{ij}(u) + \psi_{ij}(t)k_{ij}(v)$$

where $\phi_{ij}(t) = \phi(u,v,\delta_{ij};t)$ and $\psi_{ij}(t) = \psi(u,v,\delta_{ij};t)$. This can be written in matrix form as

$$k(t) = \Phi(t) \circ k(u) + \Psi(t) \circ k(v)$$

where $\Phi(t) = \{\phi_{ij}(t)\}$ and $\Psi(t) = \{\psi_{ij}(t)\}$,and the symbol $\circ$ denotes the Schur product $\{a_{ij}\} \circ \{b_{ij}\} = \{a_{ij}b_{ij}\}$ of matrices. It follows that

$$\|k(t)\| \leq \|\Phi(t) \circ k(u)\| + \|\Psi(t) \circ k(v)\|. \tag{5}$$

The final step is to prove the inequalities

$$\begin{aligned} \|\Phi(t) \circ k(u)\| &\leq \frac{v-t}{v-u}\|k(u)\|, \\ \|\Psi(t) \circ k(v)\| &\leq \frac{t-u}{v-u}\|k(v)\|. \end{aligned} \tag{6}$$

Notice that both $\Phi(t)$ and $\Psi(t)$ are positive semidefinite. This follows from Bochner's theorem [1] applied to $\phi(u,v,c;t)$ and $\psi(u,v,c;t)$ considered as functions of c. In both cases the matrix is of the form $\{F(\lambda_i - \lambda_j)\}$ where $F(c)$ is the Fourier transform of a positive function (see [7], formula 1.9.14, page 31).

Next we apply a theorem of Davis (see [6] and the generalization in [9]) according to which for $n \times n$-matrices A and P with P positive semidefinite we have

$$\|P \circ A\| \leq \Big(\max_{1 \leq i \leq n} P_{ii}\Big)\|A\|.$$

Taking $P = \Phi(t)$ and $P = \Psi(t)$ we get inequalities (6). Using now (5) and (6) we also get

$$\|k(t)\| \leq \frac{v-t}{v-u}\|k(u)\| + \frac{t-u}{v-u}\|k(v)\|. \tag{7}$$

Since the element x and the representation space $\mathscr{H}$ were not specified, we may assume without loss of generality that for a given t between u and v we have $\|K(t)x\| = |\langle K(t)x, x\rangle|$. Then writing $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ we conclude that

$$\begin{aligned}
|\langle k(t)\xi, \xi\rangle| &= |\langle K(t)x, x\rangle| = \|K(t)\| \\
|\langle k(u)\xi, \xi\rangle| &= |\langle K(u)x, x\rangle| \leq \|K(t)\| \\
|\langle k(v)\xi, \xi\rangle| &= |\langle K(v)x, x\rangle| \leq \|K(t)\|
\end{aligned}$$

and then (3) follows from (7) for X of the special form (4).

Let us go then to the general case—when X is an arbitrary selfadjoint element of A. The spectral theorem allows us to approximate X (in operator norm) by elements of the form (4). From the well-possedness of problem (2) we conclude that $(t, X) \to K(t)$ is norm continuous, and the inequality (3) for arbitrary X follows from the same inequality for X of the form (4). This completes the proof of Theorem 1.

For $a, b \in A^+$ let $\operatorname{dist}(a, b)$ denote the geodesic distance from a to b in the Finsler metric $\|X\|_a$ of A. It is not hard to prove (using the invariance of the metric) that

$$\operatorname{dist}(a, b) = \|\ln(a^{-1/2}ba^{-1/2})\|. \tag{8}$$

THEOREM 2. *If $\gamma(t)$ and $\delta(t)$ are geodesics in A^+ then $t \to \operatorname{dist}(\gamma(t), \delta(t))$ is a convex function of $t \in \mathbf{R}$.*

Proof. Suppose the geodesics $\gamma(t)$ and $\delta(t)$ are defined for $u \leq t \leq v$. Define $h(s, t)$ by the properties:

(a) the function $s \to h(s, u)$, $0 \leq s \leq 1$ is the geodesic joining $\gamma(u)$ and $\delta(u)$;

(b) the function $s \to h(s, v)$, $0 \leq s \leq 1$ is the geodesic joining $\gamma(v)$ and $\delta(v)$;

(c) for each s, the function $t \to h(s, t)$, $u \leq s \leq v$ is the geodesic joining $h(s, u)$ and $h(s, v)$.

In particular $h(0, t) = \gamma(t)$ and $h(1, t) = \delta(t)$. Define also $J(s, t) = \partial h(s, t)/\partial s$. Then, for each $s, t \to J(s, t)$ is a Jacobi field along the geodesic

$t \to h(s,t)$. Finally define

$$f(t) = \int_0^1 \| J(s,t) \|_{h(s,t)} \, ds.$$

From Theorem 1, $t \to \|J(s,t)\|$ is convex for each s. Hence $t \to f(t)$ is also convex for $u \le t \le v$. But $f(u) = \int_0^1 \|J(s,u)\|_{h(s,u)} \, ds$ is the length of the geodesic $s \to h(s,u)$ and therefore $f(u) = \operatorname{dist}(\gamma(u), \delta(u))$. Similarly, $f(v) = \operatorname{dist}(\gamma(v), \delta(v))$. Now for $u \le t \le v$, the value $f(t) = \int_0^1 \|J(s,t)\|_{h(s,t)} \, ds$ is the length of the curve $s \to h(s,t)$ joining $\gamma(t)$ and $\delta(t)$ and then we have $\operatorname{dist}(\gamma(v), \delta(v)) \le f(t)$. Convexity of $\operatorname{dist}(\gamma(v), \delta(v))$ follows and Theorem 2 is proved.

Corollary 2.1. *For any fixed $y \in A^+$ the function $f\colon A^+ \to \mathbf{R}$, $f(x) = \operatorname{dist}(x, y)$ is “convex in the geometric sense”, that is, each geodesic $\gamma(t)$ satisfies*

$$f(\gamma(t)) \le (1-t) f(\gamma(0)) + t f(\gamma(1)).$$

In particular geodesic spheres are convex sets.

Proof. Take $\delta(t) = y$ for all t and apply Theorem 2.

Corollary 2.2. *For any a_0, a_1, b_0, and b_1 in A^+ we have*

$$\begin{aligned} &\left\| \left(a_0^{1/2} \left(a_0^{-1/2} a_1 a_0^{-1/2} \right)^t a_0^{1/2} \right)^{1/2} \left(b_0^{1/2} \left(b_0^{-1/2} b_1 b_0^{-1/2} \right)^t b_0^{1/2} \right)^{1/2} \right\| \\ &\qquad \le \| a_0^{1/2} b_0^{1/2} \|^{1-t} \| a_1^{1/2} b_1^{1/2} \|^t. \end{aligned} \tag{9}$$

Proof. Take two geodesics $\gamma(t)$ and $\delta(t)$ and write them as

$$\begin{aligned} \gamma(t) &= a_0^{1/2} \left(a_0^{-1/2} a_1 a_0^{-1/2} \right)^t a_0^{1/2}, \\ \delta(t) &= b_0^{1/2} \left(b_0^{-1/2} b_1 b_0^{-1/2} \right)^t b_0^{1/2} \end{aligned}$$

where $a_0 = \gamma(0)$, $a_1 = \gamma(1)$, $b_0 = \delta(0)$, $b_1 = \delta(1)$. Then for each $0 \le t \le 1$ we have, by convexity,

$$\operatorname{dist}(\gamma(t), \delta(t)) \le (1-t) \operatorname{dist}(a_0, b_0) + t \operatorname{dist}(a_1, b_1)$$

or

$$\begin{aligned} &\left\| \ln\left(\gamma(t)^{-1/2} \delta(t) \gamma(t)^{-1/2} \right) \right\| \\ &\qquad \le (1-t) \left\| \ln\left(a_0^{-1/2} b_0 a_0^{-1/2} \right) \right\| + t \left\| \ln\left(a_1^{-1/2} b_1 a_1^{-1/2} \right) \right\|. \end{aligned}$$

Next we apply this formula to the geodesics $\gamma(t)$ and $k\delta(t)$ where $k > 0$. By choosing k large enough we can assume that

$$\gamma(t)^{-1/2}(k\delta(t))\gamma(t)^{-1/2} > 1$$
$$a_0^{-1/2}(kb_0)a_0^{-1/2} > 1$$
$$a_1^{-1/2}(kb_1)a_1^{-1/2} > 1$$

and therefore using $\|\ln x\| = \ln\|x\|$ for $x > 1$ and canceling out k, the last inequality for norms becomes

$$\left\|\gamma(t)^{-1/2}\delta(t)\gamma(t)^{-1/2}\right\| \leq \|a_0^{-1/2}b_0a_0^{-1/2}\|^{1-t}\|a_1^{-1/2}b_1a_1^{-1/2}\|^{t}.$$

Notice that $\gamma(t)^{-1}$ is also a geodesic so that the last formula gives also:

$$\left\|\gamma(t)^{1/2}\delta(t)\gamma(t)^{1/2}\right\| \leq \|a_0^{1/2}b_0a_0^{1/2}\|^{1-t}\|a_1^{1/2}b_1a_1^{1/2}\|^{t}$$

or equivalently

$$\left\|\gamma(t)^{1/2}\delta(t)^{1/2}\right\| \leq \|a_0^{1/2}b_0^{1/2}\|^{1-t}\|a_1^{1/2}b_1^{1/2}\|^{t}.$$

which is another way to write (9).

This inequality has many variations. For example, replacing a_i by a_i^2 and b_i by b_i^2 and using the definition of the geodesics, we get

$$\left\|\left(a_0\left(a_0^{-1}a_1^2a_0^{-1}\right)^t a_0\right)^{-1/2}\left(b_0\left(b_0^{-1}b_1^2b_0^{-1}\right)^t b_0\right)^{-1/2}\right\| \leq \|a_0b_0\|^{1-t}\|a_1b_1\|^{t}$$

or using $|z| = (zz^*)^{1/2}$:

$$\left\|\,\left|a_0\left(a_0^{-1}a_1^2a_0^{-1}\right)^{t/2}\right|\left|b_0\left(b_0^{-1}b_1^2b_0^{-1}\right)^{1/2}\right|\right\| \leq \|a_0b_0\|^{1-t}\|a_1b_1\|^{t}.$$

As special cases of (9) we can also get $\|ab^ta\| \leq \|aba\|^t$ and $\|a^tb^t\| \leq \|ab\|^t$ for any $a, b \in A^+$ and $0 \leq t \leq 1$.

THEOREM 3 (see [3]). *The exponential function in A^+ increases distances.*

Proof. By invariance it suffices to show that the exponential function increases distances at the identity $1 \in A^+$. Consider two geodesics of the form $\gamma(t) = e^{tX}$ and $\delta(t) = e^{tY}$. Then according to Theorem 2 the function

$$f(t) = \operatorname{dist}(\gamma(t), \delta(t)) = \left\|\ln(e^{-tX/2}e^{tY}e^{-tX/2})\right\|$$

is convex. Since $f(0) = 0$ this implies that $f(t)/t \le f(1)$ for each $0 < t \le 1$. Taking limits we have $\lim_{t \to 0} f(t)/t \le f(1)$.

Observe next that $\ln x$ can be approximated on any interval $[x_0, x_1]$ with $0 < x_0 < x_1$ uniformly in the C^1 sense by polynomials $p_n(x)$. In particular $\lim_{n \to \infty} p_n(x) = \ln x$ and $\lim_{n \to \infty} p_n'(x) = 1/x$. Then

$$\begin{aligned} &\lim_{t \to 0} \frac{1}{t} \ln(e^{-tX/2} e^{tY} e^{-tX/2}) \\ &\quad = \lim_{n \to \infty} \lim_{t \to 0} \frac{1}{t} p_n(e^{-tX/2} e^{tY} e^{-tX/2}) \\ &\quad = \lim_{n \to \infty} \frac{d}{dt} p_n(e^{-tX/2} e^{tY} e^{-tX/2}) \Big|_{t=0} = Y - X \end{aligned}$$

(the last inequality is justified below). Now from this equality and convexity we conclude that $f(t) \ge t \|Y - X\|$ and this means that

$$\operatorname{dist}(\exp_a(tX), \exp_a(tY)) \ge t\|Y - X\| \quad \text{for all } a \in A^+ \quad \text{and all } X, Y \in TA_a^+.$$

To finish the proof write the polynomials p_n explicitly as $p_n(x) = \Sigma r_{n,k} x^k$. Then

$$\begin{aligned} &\frac{d}{dt} \ln(e^{-tX/2} e^{tY} e^{-tX/2})\big|_{t=0} \\ &\quad = \lim_{n \to \infty} \frac{d}{dt} p_n(e^{-tX/2} e^{tY} e^{-tX/2}) \Big|_{t=0} \\ &\quad = \lim_{n \to \infty} \sum r_{n,k} \frac{d}{dt} (e^{-tX/2} e^{tY} e^{-tX/2})^k \Big|_{t=0} \\ &\quad = \lim_{n \to \infty} \sum r_{n,k} (Y - X)^k = \lim_{n \to \infty} p_n'(1)(Y - X) = (Y - X). \end{aligned}$$

As observed in [3] this property of the exponential is equivalent to Segal's inequality ($\|e^{X+Y}\| \le \|e^X e^{tY}\|$ for X, Y selfadjoint) which is therefore another consequence of the convexity of the distance function in A^+.

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for all C^∞ functions φ
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Sometimes we write a
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$$\frac{1}{\mu(Q)}\left(\int_Q \frac{w}{\mu}\,d\mu\right)^{1/p}$$

for all cubes Q in $\mathbf{R}^n$
$1 \le p \le \infty$, $k \in \mathbf{N}$, an

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The Geometry of Spaces of Projections in C^*-Algebras*

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1. INTRODUCTION

This is a study of the geometric sructure of the space Q of idempotent elements of a C^*-algebra. Particular attention is devoted to the way Q is fibered over the space P of all selfadjont elements of Q.

Throughout A denotes a C^*-algebra with identity represented as an algebra of operators in a Hilbert space E. The set Q inherits from A a differentiable structure with a natural connection (see [2]), and the global structure of Q can be described as follows: the restriction of the exponential map of Q to the normal bundle N (formed by the skewsymmetric tangent vectors) of the immersion $P \subset Q$ is a diffeomorphism from N onto Q (see Section 4 below).

The resulting retraction $\pi: Q \to P$ is given by the polar decomposition of the associated symmetries, and alternative characterizations of the fibers $Q_p = \pi^{-1}(p)$ are presented. For instance, each $p \in P$ determines a non-degenerate conjugate-bilinear symmetric form B_p on the Hilbert space E and the elements of Q_p are the "inertial decompositions" $E = E_+ \oplus E_-$ where E_+ and E_- are B_p-orthogonal subspaces, with B_p positive definite on E_+ and negative definite on E_-. It is also natural to study the groups U_p of B_p-unitary elements. The group U_p operates by inner automorphisms on P with orbit Q_p. The stability subgroup $U_p^{(0)} = \{u \in U_p : up = pu\}$

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consists of the $u \in U_p$ which are also unitary in A. The set $V_p = U_p \cap A_+$, where A_+ denotes the set of positive elements of A, provides a section to the fibration $U_p \to Q_p$, $u \mapsto upu^{-1}$. Some geometric properties of the embedding $V_p \subset U_p$ are given in Theorem 5.2.

We also consider metric properties. A Finsler structure naturally related to $Q \xrightarrow{\pi} P$ is introduced in Q and the final part is devoted to the study of geodesics in Q, including the proof of the minimality of their lengths in this metric. Since the metric is far from behaving like a Riemannian metric, *ad hoc* methods are developed to obtain the minimality of geodesics, which we consider as the main result of this study (cf. [3] for a special case).

The paper is organized as follows. In Section 2 we collect some preliminary facts concerning the differential geometry of Q. In Section 3 we give a full description of the map π defined by the polar decomposition and the structure of its fibers Q_p. The matrix decompositions of the elements of Q_p obtained here are extensively used in the other sections.

In Section 4 we study the diffeomorphism $N \to Q$, and the structure of the B_p-unitary group U_p is studied in Section 5. The Finsler metric on Q is introduced in Section 6. Several basic properties of the tangent map of π are studied in Section 7, especially those related to the metric. In Section 8 we define a "reduction map" which is the key for the proof of the minimality of the length of geodesics in Q, presented in Section 9.

The bibliography on idempotents and projectors in Banach and C^*-algebras is very extensive. The classical book of Rickart [13] contains the references until 1960. The topological and differentiable structure of Q and P have been considered in [1, 3, 5, 7–10, 12, 14]. We also mention Kato's book [6] which contains much information about Q and P, in particular what we call the "transport curve." The authors are grateful to The Instituto Argentino de Matematíca for the opportunity to present these results in a series of lectures in July 1988, and we also thank Mrs. Leticia Scoccia for typing, at that time, the final draft of this paper.

2. Preliminary Facts

Given a manifold M we denote $T_x M$ the tangent space at $x \in M$. Recall [3] that $T_q Q$ may be identified with the set of $X \in A$ satisfying $qX + Xq = X$; for $p \in P$ we have $T_p P = \{X \in T_p Q : X^* = X\}$. For $p \in P$ denote $N_p \subset T_p Q$ the skewsymmetric part of $T_p Q$, or $N_p = \{X \in T_p Q : X^* = -X\}$ so that $T_p Q = T_p P \oplus N_p$. We refer to the resulting vector bundle $N \to P$ as the *normal bundle* of the immersion $P \subset Q$.

A connection ∇ on Q is defined by

$$\nabla_Y = \prod\left(\left.\frac{dY(\gamma(t))}{dt}\right|_{t=0}\right),$$

where $X = (d\gamma/dt)(0) \in T_q Q$, Y is a local tangent field near q, and $\Pi: A \to A$ is the map $\Pi a = (1-q)\,aq + qa(1-q)$. For properties of this connection see [3]. The geodesics of Q starting at q with velocity $X \in T_q Q$ have the form $\gamma(t) = e^{t\tilde{X}} q e^{-t\tilde{X}}$ where $\tilde{X} = [X, q]$ $(= Xq - qX)$. Therefore the exponential map of Q is given by $\mathrm{Exp}_q(X) = e^{\tilde{X}} q e^{-\tilde{X}}$. In Section 6 we use the "transport curve" $\Gamma(t)$ of a curve $\gamma(t)$ in Q. It is defined as the solution of $\dot{\Gamma}(t) = [\dot{\gamma}(t), \gamma(t)]\,\Gamma(t)$ with $\Gamma(0) = 1$ (see [3, 10]).

3. Polar Decomposition

The polar decomposition provides a map from the group G of units of A onto the product $G_+ \times U$ of the positive units G_+ and the unitary group U of A. Operating on the symmetries $\varepsilon = 2q - 1$ associated to idempotents q we can use the polar decomposition to express Q as a "product." Here are the details.

3.1. Proposition. *Suppose $q \in Q$ and let $\varepsilon = \lambda^2 \rho$ be the polar decomposition ($\lambda > 0$ and ρ unitary) of $\varepsilon = 2q - 1$. Denote $p = (\rho + 1)/2$. Then:*

(1) $\rho^2 = 1$ *and p is an orthogonal projection;*

(2) $\lambda\rho = \rho\lambda^{-1}$ *and therefore $q = \lambda p \lambda^{-1}$.*

Proof. Write the polar decomposition of ε as $\varepsilon = \mu\rho$ where $\mu = \lambda^2$ and λ are positive and ρ is unitary. Then $1 = \varepsilon^2 = \mu\rho\mu\rho$ so that $\mu\rho\mu = \rho^{-1}$. Taking adjoints we get $\mu\rho^{-1}\mu = \rho$. Hence $\mu^2\rho\mu^2 = \mu(\mu\rho\mu)\mu = \rho$, or $\rho\mu^2\rho^{-1} = \mu^{-2}$. Taking square roots gives $\rho\mu\rho^{-1} = \mu^{-1}$ so that $\rho\mu\rho^{-1} = \mu^{-1} = \rho\rho^{-1}\mu^{-1} = \rho\varepsilon^{-1} = \rho\varepsilon = \rho\mu\rho$ and cancelling $\rho\mu$ we get $\rho^{-1} = \rho$, which proves (1). Also from $\rho\mu\rho = \mu_{-1}$ and using $\rho^2 = 1$ we get $\rho\mu^{-1}\rho = \mu$ and taking square roots $\rho\lambda^{-1}\rho = \lambda$, which proves (2).

In the sequel we use $\pi: Q \to P$ to denote the map defined by $\pi(q) = p$, where $\varepsilon = 2q - 1$, $\rho = 2p - 1$ and $\varepsilon = \lambda^2\rho$ is the polar decomposition of ε as in Proposition 3.1. The fiber $\pi^{-1}(q)$ will be denoted Q_p.

Each $p \in P$ determines a conjugate-bilinear symmetric form B_p on E defined by $B_p(x, y) = \langle \rho x, y \rangle$ for all x, y in E. Observe that B_p is non-degenerate because ρ is invertible, and that λ is B_p-unitary, since $B_p(\lambda x, y) = \langle \rho\lambda x, y \rangle = \langle \lambda^{-1}\rho x, y \rangle = \langle \rho x, \lambda^{-1} y \rangle = B_p(x, \lambda^{-1} y)$. Given an arbitrary non-degenerate conjugate-bilinear symmetric form B, a *decomposition* of B is an idempotent $q \in Q$ such that:

(1) *B is positive definite on the range of q and negative definite on the kernel of q;*

(2) *q is B-symmetric: $B(qx, y) = B(x, qy)$ for all x, y in E.*

The set of all decompositions of B is denoted $\mathscr{D}(B)$. A natural map identifies the fiber $Q_p = \pi^{-1}(p)$ with the decompositions of Q_p. More precisely:

3.2. Proposition. *For $q \in Q$ and $s \in P$ with $\varepsilon = 2q-1$ and $\zeta = 2s-1$ the following conditions are equivalent*:

(1) $q \in \mathscr{D}(B_s)$;

(2) $\zeta\varepsilon > 0$;

(3) $\pi(q) = s$;

(4) $q = \lambda s \lambda^{-1}$ *where λ is positive and B_s-unitary.*

Proof. $(1) \Rightarrow (2)$ because for $x \in E$ we can write $x = x_0 + x_1$ with $\varepsilon x_0 = x_0$ and $\varepsilon x_1 = -x_1$. Then, by B_s-symmetry of ε we get $B_s(x_0, x_1) = 0$ and therefore $B_s(\varepsilon x, x) = B_s(x_0, x_0) - B_s(x_1, x_1) > 0$ when $x \neq 0$. But $B_s(\varepsilon x, x) = \langle \zeta\varepsilon x, x\rangle$ so $\zeta\varepsilon > 0$.

$(2) \Rightarrow (1)$. The proof that $(2) \Rightarrow (1)$ is similar.

$(2) \Rightarrow (3)$. Let $\beta = \zeta\varepsilon > 0$. Then $\varepsilon = \zeta\beta = \beta^{-1}\zeta$ so by definition $s = \pi(q)$.

$(3) \Rightarrow (4)$ follows from Proposition 3.1.

$(4) \Rightarrow (2)$, since $\lambda^* = \lambda$ and λ is B_s-unitary, whence $\zeta\lambda^{-1} = \lambda\zeta$ so $\varepsilon = \lambda\zeta\lambda^{-1} = \zeta\lambda^{-2}$ and $\zeta\varepsilon = \lambda^{-2} > 0$.

Corollary. $Q_p = \mathscr{D}(B_p) = \{\lambda p \lambda^{-1} : \lambda > 0 \text{ and } B_p\text{-unitary}\}$.

Fix $p \in P$, abbreviate $B = B_p$, and denote a^B the B-adjoint of $a \in A$: $B(ax, y) = B(x, a^B y)$ for all $x, y \in E$ (since $a^B = \rho a^* \rho$ it is clear that $a^B \in A$). Given $q \in Q_p$, write $q = \lambda p \lambda^{-1}$ as in 3.2.4 and set $k = \lambda p$. Then, observing that $k^B = p\lambda^{-1}$, we get $q = kk^B$. It follows that k is a partial isometry for B, being B-isometric (and equal to λ) on $\operatorname{im}(p)$ and zero on $\ker(p)$. In terms of the decomposition $E = \operatorname{im}(p) \oplus \ker(p)$ we can write k in matrix form as $k = \left(\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right)$ with $a : \operatorname{im}(p) \to \operatorname{im}(p)$ positive (since $\lambda > 0$) and $b : \operatorname{im}(p) \to \ker(p)$. Since $B = \langle\ ,\ \rangle$ on $\operatorname{im}(p)$ and $B = -\langle\ ,\ \rangle$ on $\ker(p)$ we get

$$k^B = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}^B = \begin{pmatrix} a & -b^* \\ 0 & 0 \end{pmatrix}.$$

Furthermore, the fact that

$$q = kk^B = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} a & -b^* \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a^2 & -ab^* \\ ba & -bb^* \end{pmatrix}$$

is idempotent yields $k^B k = p$, or

$$\begin{pmatrix} a & -b^* \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} a^2 - b^*b & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = p.$$

This means that $a = \sqrt{1 + b^*b}$. We can summarize these facts as follows:

3.4. PROPOSITION. *The fiber Q_p can be parametrized by the operators $b: \mathrm{im}(p) \to \ker(p)$ (equivalently, $b \in (1-p)\,Ap$), by means of the matrix representation*

$$q = \begin{pmatrix} a^2 & -ab^* \\ ba & -bb \end{pmatrix}$$

in terms of the decomposition $E = \mathrm{im}(p) \oplus \ker(p)$, where $a = \sqrt{1 + b^*b}$.

We close this section by giving the matrix form of the generic λ. Notice that $\lambda p = \left(\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right)$ and $\lambda = \lambda^* > 0$ imply that

$$\lambda = \begin{pmatrix} a & b^* \\ b & c \end{pmatrix} \tag{3.5}$$

for some $c > 0$. Now, the fact that λ is B_p-unitary (3.2.4) gives

$$\lambda^{-1} = \begin{pmatrix} a & -b^* \\ -b & c \end{pmatrix}. \tag{3.6}$$

Therefore

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \lambda\lambda^B = \lambda\lambda^{-1} = \begin{pmatrix} a^2 - b^*b & -ab^* + b^*c \\ ba - cb & bb^* + c^2 \end{pmatrix},$$

so that λ has the form (3.5), with $b: \mathrm{im}(p) \to \ker(p)$ arbitrary in A and a, c the positive roots $a = \sqrt{1 + b^*b}$, $c = \sqrt{1 + bb^*}$ (the conditions $-ab^* + b^*c = 0$ and $ba - cb = 0$ are then automatic).

4. THE STRUCTURE OF Q

This section considers the diffeomorphism from N onto Q mentioned in the Introduction.

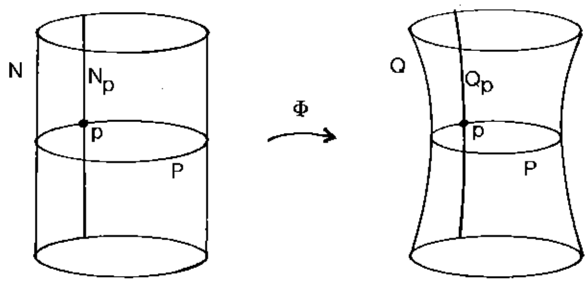

FIGURE 1

4.1. THEOREM. *Let $\Phi: N \to Q$ be the restriction to N of the exponential map of Q, so that $\Phi(p, X) = e^{\tilde{X}} p e^{-\tilde{X}}$ where for $X \in N_p$ we set $\tilde{X} = [X, p]$. Then*

(1) *Φ is a diffeomorphism from N onto Q.*

(2) *for each $p \in P$ we have $Q_p = \Phi(N_p)$ (see Fig. 1).*

Proof. Set again $\varepsilon = 2q - 1$, $p = \pi(q)$, $\rho = 2p - 1$, and let $Y = \log \lambda$ where λ has the same meaning as in Proposition 3.1. Then $Y^* = Y$ and $\rho\lambda = \lambda^{-1}\rho$ (see 3.1.2) implies $\rho \log \lambda = -(\log \lambda)\rho$, whence $Y = \log \lambda \in T_p P$ and therefore $X = [Y, p] \in N_p$. Clearly by definition $\Phi(p, X) = q$ and this provides the inverse $q \to (\pi(q), [\log \lambda, \pi(q)])$ of Φ, which proves (1).

To obtain (2) we first observe that the map $X \to e^{[X,p]}$ is a bijection from N_p onto the set of $\alpha \in A$ which are positive and B_p-unitary. Routine calculations give that $e^{[X,p]}$ is indeed of this type. Conversely, given such an α consider $\log \alpha$. Since $\alpha^B = \rho\alpha\rho = \alpha^{-1}$ we get $\rho(\log \alpha)\rho = -\log \alpha$ whence $\log \alpha \in TQ_p$ and since $\log \alpha$ is selfadjoint, the element $X = [\log \alpha, p]$ is in N_p. This settles the claim. But $\Phi(N_p)$ coincides with the set $\{\lambda p \lambda^{-1}; \lambda > 0$ and B_p-unitary$\}$ and (2) follows using Proposition 3.2. This concludes the proof of 4.1.

Given a non-degenerate form B we call the *Grassmanian* of B the submanifold $Gr(B)$ of Q consisting of all B-symmetric $q \in Q$. Similarly $Gr(B, *) = Gr(B) \cap P$ consists of the q which are simultaneously B-symmetric and $*$-symmetric. The tangent spaces are formed by the B-symmetric, and the B- and $*$-symmetric elements, respectively, of TQ. The *normal bundle* of the immersion $Gr(B, *) \subset Gr(B)$ is defined as $N(B) = N \cap TGr(B)$.

4.2. THEOREM. *Let $B = B_p$ where $p \in P$ is a fixed element. Then the map Φ of* 4.1 *induces a diffeomorphism from $N(B)$ onto $Gr(B)$.*

Proof. In view of 4.1 it suffices to show that $\Phi(N(B))\subset Gr(B)$ and $\Phi^{-1}(Gr(B))\subset N(B)$. Denote $\rho=2p-1$ and let $s\in Gr(B,*)$, $X\in N(B)_s$. From $s^B=s$, $X^B=X$, and $\tilde{X}=[X,s]$ we get $\tilde{X}^B=-\tilde{X}$ so that $\Phi(s,X)^B=e^{-\tilde{X}^B}s^B e^{\tilde{X}^B}=\Phi(s,X)$. Conversely if $q^B=q$ and $\varepsilon=2q-1=\lambda^2\sigma$ as in Proposition 3.1, then $\varepsilon=\varepsilon^{-1}=\varepsilon^B$ is B-unitary. From $\varepsilon=\lambda^2\sigma=\sigma\lambda^{-2}$ we get $\varepsilon=\varepsilon^B=(\lambda^{-2})^B\sigma^B$. But $(\lambda^{-2})^B=\tau\lambda^{-2}\tau>0$ and $(\sigma^B)^*\sigma^B=-\log\lambda$ and so $\Phi^{-1}(q)\in N(B)$ as claimed.

4.3. COROLLARY. (1) *P is a deformation retract of Q;*

(2) *$Gr(B)\cap P$ is a deformation retract of $Gr(B)$.*

5. THE UNITARY GROUP OF B_p

Concerning the structure of the "hyperbolic" unitary group we present the following two theorems.

Suppose $p\in P$ is fixed and let $\rho=2p-1$ and $B=B_p$. Denote by $U_p=U(B)$ the group of B-unitary elements of A, i.e., $U_p=\{u\in A: u^B=u^{-1}\}$. U_p operates by inner automorphism on Q_p and we denote $\kappa: U_p\to Q_p$ the map $\kappa(u)=upu^{-1}$. We use $U_p^{(0)}$ for the fiber $U_p^{(0)}=\kappa^{-1}(p)$ and V_p for the set of elements $v\in U_p$ which are positive $v^*=v>0$. Observe that $U_p^{(0)}$ consists of the elements u_0 that are both unitary and B-unitary. The Lie algebra $\mathscr{L}$ of U_p consists of the $X\in A$ which are B-skewsymmetric; it splits as $\mathscr{L}=\mathscr{L}_0\oplus\mathscr{L}_1$, where $\mathscr{L}_0=\{X\in\mathscr{L}: X\rho=\rho X\}$ and $\mathscr{L}_1=\{X\in\mathscr{L}: X\rho=-\rho X\}$. Of course, $\mathscr{L}_0=T_1U_p^{(0)}$ and $\mathscr{L}_1=T_1V_p$. The proof of the following result is straightforward.

5.1. THEOREM. (1) *Let $m: V_p\times U_p^{(0)}\to U_p$ denote the multiplication map $m(v,u_0)=vu_0$, $pr_1: V_p\times U_p^{(0)}\to V_p$ the first projection and $\nu: V_p\to Q_p$ the restriction of κ to V_p. Then m and ν are diffeomorphisms and the following diagram commutes*

$$\begin{tikzcd} V_p\times U_p^{(0)} \arrow[r, "m"] \arrow[d, "pr_1"'] & U_p \arrow[d, "\kappa"] \\ V_p \arrow[r, "\nu"'] & Q_p \end{tikzcd}$$

Moreover the inverse of m is the polar decomposition.

(2) *The map $X\to e^X$ is a diffeomorphism from $\mathscr{L}_1$ onto V_p.*

A feeling for the way V_p bends in U_p can be obtained from the following estimate.

5.2. THEOREM. *For* $0 \neq Z \in T_v V_p$ *decompose* $v^{-1}Z = W_0 + W_1$ *according to* $\mathscr{L} = \mathscr{L}_0 \oplus \mathscr{L}_1$. *Then* $\|W_0\| < \|W_1\|$.

Proof. Set $\lambda = (v^{-1})^{1/2} > 0$ and $X = \lambda^{-1}(v^{-1}Z)\lambda = \lambda Z \lambda$. Clearly $\rho\lambda\rho = \lambda^{-1}$, because $\rho v \rho = v^{-1}$ so $\lambda^B = \lambda^{-1}$; also $Z^B = -v^{-1}Zv^{-1} = -\lambda^2 Z \lambda^2$, because $Z \in T_v V_p$, and then $X^B = (\lambda Z \lambda)^B = \lambda^B Z^B \lambda^B = -X$. Since also $X^* = X$ we get $X \in T_1 V_p = \mathscr{L}_1$. Writing matrix decompositions with respect to p (use 3.5. and 3.6)

$$X = \begin{pmatrix} 0 & \eta^* \\ \eta & 0 \end{pmatrix}, \qquad \lambda = \begin{pmatrix} a & b^* \\ b & c \end{pmatrix}, \qquad \lambda^{-1} = \begin{pmatrix} a & -b^* \\ -b & c \end{pmatrix},$$

so that $\lambda X \lambda^{-1} = W_0 + W_1$ with

$$W_0 = \begin{pmatrix} b^*\eta a - a\eta^* b & 0 \\ 0 & b\eta^* c - c\eta b^* \end{pmatrix},$$

$$W_1 = \begin{pmatrix} 0 & a\eta^* c - b^*\eta b^* \\ c\eta a - b\eta^* b & 0 \end{pmatrix}.$$

The result follows from the inequality $\|b^*\eta a - a\eta^* b\| < \|c\eta a - b\eta^* b\|$, a consequence of inequality (2) in [2].

6. THE FINSLER METRIC ON Q

We begin with the definition:

6.1. DEFINITION. The norm $\|X\|_q$ on the tangent space $T_q Q$ is defined by $\|X\|_q = \|\lambda^{-1} X \lambda\|$ where λ is the element of (3.1); thus Q with $\| \ \|_q$ is a Finsler space.

The norm $\| \ \|_q$ is actually a natural norm on all of A. In fact, for $q \in Q_p$, $q = \lambda p \lambda^{-1}$ we can define an inner product $\langle x, y \rangle_q = \langle \lambda^{-1}x, \lambda^{-1}y \rangle$ on E by transporting with λ the ordinary inner product. Equivalently

$$\langle x, y \rangle_q = B_p(qx, qy) - B_p((1-q)x, (1-q)y).$$

Then for each $a \in A$ the $\langle \ , \ \rangle_q$-operator norm of a coincides with $\|a\|_q$, i.e.,

$$\|a\|_q = \|\lambda^{-1} a \lambda\| = \sup\{\langle ax, ax \rangle_q^{1/2} : \langle x, x \rangle_q^{1/2} = 1\}.$$

We abbreviate $|x|_q = \langle x, x \rangle_q^{1/2}$ and when no ambiguity results we use $a^\#$ for the $\langle \ , \ \rangle_q$-adjoint of $a \in A$, or $\langle ax, y \rangle_q = \langle x, a^\# y \rangle_q$. This translates into $a^\# = \lambda^2 a^* \lambda^{-2} = \varepsilon\rho a^* \rho\varepsilon$ notation of 3.1).

Notice that the Finsler metric is U_p-invariant in the sense that $|ux|_q = |x|_r$ for $q = uru^{-1}$, $u \in U_p$, $q, r \in Q_p$, and $x \in E$.

7. The Tangent Map of $\pi: Q \to P$

In order to describe the tangent map of $\pi: Q \to P$ we begin with a few calculations. Suppose that $q \in Q$, $p = \pi(q)$, $Z \in T_q Q$, and $Y = T_q\pi(Z) \in T_p P$. Suppose further that $\delta(t)$ is a curve in Q with $\delta Z(0) = q$, $\dot{\delta}(0) = Z$, and denote $\gamma(t) = \pi(\delta(t))$ so that $\gamma(0) = p$ and $\dot{\gamma}(0) = Y$. According to 3.3 there is a curve $\lambda(t)$ with $\lambda(t) \in V_{\gamma(t)}$ and $\delta(t) = \lambda(t)\,\gamma(t)\,\lambda(t)^{-1}$. Also let $\Gamma(t)$ be the transport curve of $\gamma(t)$ (see Section 2) and define $u(t) = \Gamma(t)^{-1}\,\lambda(t)\,\Gamma(t)$, so $u(t) \in V_p$ and $u(0) = \lambda(0)$; this follows from the invariance of B_p, i.e., $B_{(sps^{-1})}(x, y) = B_p(s^{-1}x, s^{-1}y)$ for any s unitary, hence $U_{(sps^{-1})} = sU_p s^{-1}$ and $V_{(sps^{-1})} = sV_p s^{-1}$. Differentiating $\delta = \lambda\gamma\lambda^{-1}$ and $u = \Gamma\lambda\Gamma^{-1}$ at $t = 0$ we get

$$Z = \lambda Y \lambda^{-1} + \lambda[\lambda^{-1}\dot{\lambda}, p]\,\lambda^{-1} \tag{7.1}$$

$$\lambda^{-1}\dot{\lambda} = u^{-1}\dot{u} - u^{-1}[u, [Y, p]]. \tag{7.2}$$

With $\rho = 2p - 1$ and using $[Y, p] = Y\rho = -\rho Y$, $\rho u = u^{-1}\rho$ we obtain

$$u^{-1}[u, [Y, p]] = Y\rho - u^{-1}Y\rho u = (Y - u^{-1}Yu^{-1})\rho.$$

Then, setting $w = u^{-1}[u, [Y, p]]$ yields

$$\begin{aligned}[][w, p] &= \tfrac{1}{2}[w, p] = \tfrac{1}{2}[(Y - u^{-1}Yu^{-1})\rho, \rho] \\ &= \tfrac{1}{2}(Y - u^{-1}Yu^{-1} - \rho Y\rho + \rho u^{-1}Yu^{-1}\rho) \\ &= Y - \tfrac{1}{2}(u^{-1} + uYu)\end{aligned}$$

and then from (7.1) and (7.2),

$$Z = u[u^{-1}\dot{u}, p]\,u^{-1} + u\{\tfrac{1}{2}(u^{-1}Yu^{-1} + uYu)\}\,u^{-1}.$$

7.4. Proposition. *Let $Z = Z_s + Z_a$ be the decomposition of Z as a sum of a #-symmetric element Z_s and a #-skewsymmetric element Z_a. Then*

$$\begin{aligned} Z_s &= u\{\tfrac{1}{2}(u^{-1}Yu^{-1} + uYu)\}\,u^{-1} \\ Z_a &= u[u^{-1}\dot{u}, p]\,u^{-1}. \end{aligned}$$

Proof. Since u is B_p-unitary, as observed above, in view of 7.3 it suffices to show that $\tfrac{1}{2}(u^{-1}Yu^{-1} + uYu)$ is $*$-symmetric (which is obvious) and that $[u^{-1}\dot{u}, \rho]$ is skewsymmetric. For this, write $[u^{-1}\dot{u}, \rho]^* = \rho\dot{u}u^{-1} - \dot{u}u^{-1}\rho$ and use $u^{-1}\dot{u}\rho = -\rho\dot{u}u^{-1}$ (which follows differentiating $u(t)\rho = \rho u(t)^{-1}$).

The complete description of the tangent map of $\pi: Q \to P$ is given by the following result.

7.5. THEOREM. (1) *At each $p \in P$ the tangent map $T_p\pi : T_pQ \to T_pP$ is given by $T_p\pi(W) = W_s$, the $*$-symmetric part of $W \subset T_pQ$.*

(2) *At each $q \in Q$, $q = \lambda p \lambda^{-1}$ with $\lambda \in V_p$, the following diagram commutes*

$$\begin{tikzcd} T_qQ \arrow[r, "N_q"] \arrow[d, "T_q\pi"'] & T_qQ \arrow[d, "T_q\pi"] \\ T_pP \arrow[r, "K_q"'] & T_qP \end{tikzcd}$$

where $N_q(Z) = \lambda^{-1}Z\lambda$, $K_q(Y) = (1/2)(\lambda^{-1}Y\lambda^{-1} + \lambda Y \lambda)$, and $T_p\pi$ is described in (1).

(3) *N_q is an isometry of T_qQ with the norm $\| \ \|_q$ onto T_pQ with the ordinary norm ($= \| \ \|_p$) and $K_q : T_pP \to T_pP$ is invertible.*

Proof. Assume first $q = p$ in Proposition 7.4; then $\lambda = 1$ and so $Z_s = Y$, which proves (1). By invariance of the $\langle\ ,\ \rangle_q$ product, N_q is an isometry and preserves symmetric and skewsymmetric parts. Therefore (7.3) and (7.4) yield $K_q(T_q\pi(Z)) = K_q(Y) =$ symmetric part of $\lambda^{-1}Z\lambda = T_p\pi(N_q(Z))$, which proves (2). The invertibility of K_q is shown in the proof of the next theorem.

We close this section with the following metric property of $T_q\pi$.

7.6. THEOREM. *The tangent map of the projection $\pi : Q \to P$ does not increase norms.*

Proof. According to 7.5 it suffices to show that K_q is invertible and that $\|K_q(Y)\| \geqslant \|Y\|$. Observe that the commutativity of the above diagram implies that K_p is onto. So only the last estimate remains to be proven. Represent λ, λ^{-1}, and Y (notation of 7.4) in matrix form with respect to $p = \pi(q)$,

$$\lambda = \begin{pmatrix} a & b^* \\ b & c \end{pmatrix}, \qquad \lambda^{-1} = \begin{pmatrix} a & -b^* \\ -b & c \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & \eta^* \\ \eta & 0 \end{pmatrix}.$$

Then $K_q(Y) = \left(\begin{smallmatrix} 0 & \zeta^* \\ \zeta & 0 \end{smallmatrix}\right)$ where $\zeta = c\eta a + b\eta^* b$ (this is the same calculation used in the proof of 5.2). Since $\|Y\| = \|\eta\|$ and $\|K_q(Y)\| = \|\zeta\|$ the result follows from $\|\eta\| \leqslant \|c\eta a + b\eta^* b\|$, and this is inequality (1) in [2].

8. A REDUCTION PROCESS

If $H_1 \subset \operatorname{im}(p)$ is given (for a fixed $p \in P$), a projection p_1 in $H_1 \oplus \ker(p)$ with $\operatorname{im}(p_1) = H_1$ and $\ker(p_1) = \ker(p)$ results and a "reduction map"

$\mathscr{D}(B_p) \to \mathscr{D}(B_{p_1})$ can be defined. A detailed study of this situation in an abstract setting is contained in a forthcoming paper [4]. For our present needs only the following special results about operators in E and subspaces of E suffice.

Let us denote, for simplicity: $H = \mathrm{im}(p)$, $H_2 =$ orthogonal complement of H_1 in H, $H_3 = \ker(p)$, and $K = H_1 \oplus H_3$. Also denote B_1 the restriction of B to K.

8.1. DEFINITION. For $q \in Q_p$ define q_K to be the projection in K characterized by:

(1) q_K is B_1-symmetric, and

(2) $\mathrm{im}(q_K) = \mathrm{im}(q) \cap K$.

To obtain an expression $q_K = hh^{B_1}$ for q_K we start with $q = kk^B$, $k = \left(\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right)$ as in Section 3 and denote $i_1 : H_1 \to H$ the canonical injection $H_1 \to H_1 \oplus H_2$, and $f = ba^{-1} i_1 : H_1 \to H_3$. Observe that

$$\begin{aligned} \|ba^{-1}\|^2 &= \|(ba^{-1})^* (ba^{-1})\| = \|a^{-1} b^* b a^{-1}\| \\ &= \|a^{-2} b^* b\| = \|(1 + b^* b)^{-1} b^* b\| \\ &= (1 + \|b^* b\|)^{-1} \|b^* b\| < 1 \end{aligned}$$

so that $\|f\| < 1$ and we can also define

$$\alpha = (1 - f^* f)^{-1/2} : H_1 \to H_1, \qquad \beta = f\alpha : H_1 \to H_3.$$

Then $1 + \beta^* \beta = 1 + \alpha f^* f \alpha = 1 + \alpha^2 f^* = 1 + f^* f (1 - f^* f)^{-1} = (1 - f^* f)^{-1} = \alpha^2$.

Finally, let $h = \left(\begin{smallmatrix} \alpha & 0 \\ \beta & 0 \end{smallmatrix}\right) : K \to K$.

8.2. PROPOSITION. *With the above notations* $q_K = hh^{B_1}$.

Proof. First observe that $h^{B_1} = \left(\begin{smallmatrix} \alpha & -\beta^* \\ 0 & 0 \end{smallmatrix}\right)$ and therefore, from $\alpha^2 = 1 + \beta^* \beta$, we get $h^{B_1} h = p_1$. Hence, using $hp_1 = h$ we obtain $(hh^{B_1})^2 = hh^{B_1}$. This shows that hh^{B_1} is a B_1-symmetric projection in K. To obtain $q_K = hh^{B_1}$ it suffices to prove that $\mathrm{im}(hh^{B_1}) = \mathrm{im}(q) \cap K = \mathrm{im}(q_K)$. Now $\mathrm{im}(q)$ coincides with the graph of the map $ba^{-1} : H \to H_3$ and therefore $\mathrm{im}(q) \cap K$ coincides with the graph of $f : H_1 \to H_3$. It remains to show that the graph of f is the set of pairs $(\alpha x_1, \beta x_1)$, $x_1 \in H_1$. But, by definition, $\beta = f\alpha^{-1}$ so $(\alpha x_1, \beta x_1) = (x_1^1, f x_1^1)$ for $x_1^1 = \alpha x_1$, and we are done.

Thus $q \mapsto q_K$ can be interpreted as a smooth map $S : Q_p \to Q_{p_1}(K)$, where $Q_{p_1}(K)$ is the space of decompositions of B_1 in the algebra $\mathscr{L}(K)$ of all bounded linear operators in K. Here is the main property of S:

8.3. THEOREM. *The tangent map of S decreases norms.*

Before starting the proof some preliminary remarks are in order. The group $U(B_1)$ (resp., U_p) of B_1-unitary operators (resp., B-unitary operators) acts transitively and isometrically on $Q_{p_1}(K)$ (resp., Q_p) by conjugation. Moreover the inclusion $U(B_1) \to U_p$, $u \to \bar{u}$ given by $\bar{u} = u \oplus \mathrm{id}$ for the decomposition $E = K \oplus H_2$, is compatible with S, in the sense that $\lambda_1^{-1}(S_q)\,\lambda_1 = S((\overline{\lambda_1})^{-1}\,q\overline{\lambda_1})$ for all $\lambda_1 \in U(B_1)$. Therefore we may assume without loss of generality that $Sq = q_K = p_1$.

Next we determine the general form of all $q \in Q_p$ such that $Sq = p_1$. For this purpose (and also for the remainder of this paper) we use matrix representation in the decompossition $E = H_1 \oplus H_2 \oplus H_3$. Let $\lambda \in V_p$ satisfy $q = \lambda p \lambda^{-1}$, and let

$$q = kk^B = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix}$$

according to 3.5. Since $Sq = p_1$, we get $qx_1 = x_1$ for all $x_1 \in H_1$, which implies $q_{11} = 1$, $q_{21} = 0$, $q_{31} = 0$. But, using 3.4 and the B-symmetry

$$a^2 = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & q_{22} \end{pmatrix}.$$

Then, using $q_{31} = 0$, we obtain $(0 \;\; q_{32}) = ba = (b_1 \;\; b_2 q_{22})$, which proves that $b_1 = 0$.

$$a = \begin{pmatrix} 1 & 0 \\ 0 & q_{22} \end{pmatrix}, \qquad b = (0 \;\; b_2)$$

and this justifies the following

8.4. LEMMA. *Let $q \in Q_p$. Then $Sq = p_1$ if and only if $q = \lambda p \lambda^{-1}$ with*

$$\lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & d^* \\ 0 & d & c \end{pmatrix}, \tag{8.4.1}$$

where $z = \sqrt{1 + d^*d}$ and $c = \sqrt{1 + dd^*}$ for some $d\colon H_2 \to H_3$. Therefore

$$q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^2 & -zd^* \\ 0 & dz & -dd^* \end{pmatrix}. \tag{8.4.2}$$

Proof of Theorem 8.3. In view of the preceding remarks, it suffices to show that for $X \in T_q Q_p$, with $Sq = p_1$, we have

$$\|(T_q S)(X)\| \leqslant \|X\|_q.$$

Suppose that $\omega(t)$ is a curve in Q_p with $q = \omega(0)$ and $X = \dot{\omega}(0) \in T_q Q_p$. Then $S\omega(t) = \mu(t)\, p_1 \mu(t)^{-1}$ for certain $\mu(t) \in V(B_1)$. Set

$$X_1 = \frac{d}{dt}\left(\bar{\mu}(t)\, q \bar{\mu}(t)^{-1}\right)\Big|_{t=0}$$

$$X_2 = X - X_1,$$

so that $(T_q S)(X_2) = (T_q S)(X) - (T_q S)(X_1) = 0$ because $S(\bar{\mu}(t)\, q\bar{\mu}(t)^{-1}) = \mu(t)\, p_1 \mu(t)^{-1} = S\omega(t)$. Since $\|X\|_q = \|\lambda^{-1} X \lambda\|$ we only have to calculate $\lambda^{-1} X \lambda = \lambda^{-1} X_1 \lambda + \lambda^{-1} X_2 \lambda$. For that we use $X_1 = [\dot{\bar{\mu}}(0), q]$ and set

$$\bar{\mu}(t) = \begin{pmatrix} \xi(t) & 0 & \tau(t)^* \\ 0 & 1 & 0 \\ \tau(t) & 0 & \nu(t) \end{pmatrix}$$

so that

$$X_1 = \begin{pmatrix} 0 & \dot{\tau}(0)^*\, dz & -\dot{\tau}(0)^*\, c^2 \\ zd^* \dot{\tau}(0) & 0 & zd^* \dot{\nu}(0) \\ c^2 \dot{\tau}(0) & \dot{\nu}(0)\, dz & [dd^*, \dot{\nu}(0)] \end{pmatrix}$$

and

$$\lambda^{-1} X_1 \lambda = [\lambda^{-1} \dot{\bar{\mu}}(0) \lambda, p] = \begin{pmatrix} 0 & 0 & -\dot{\tau}(0)^*\, c \\ 0 & 0 & d^* \dot{\nu}(0) c \\ c\dot{\tau}(0) & c\dot{\nu}(0)\, d & 0 \end{pmatrix}.$$

On the other hand using (8.4.2) and the fact that $\lambda^{-1} X_2 \lambda \in T_p Q_p$, we conclude that $\lambda^{-1} X_2 \lambda$ has the form

$$\lambda^{-1} X_2 \lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\theta^* \\ 0 & \theta & 0 \end{pmatrix}.$$

Therefore

$$\lambda^{-1} X \lambda = \begin{pmatrix} 0 & 0 & -\dot{\tau}(0)^*\, c \\ 0 & 0 & d^* \dot{\nu}(0) c - \theta^* \\ c\dot{\tau}(0) & c\dot{\nu}(0)\, d + \theta & 0 \end{pmatrix}.$$

Denote $Y=(T_qS)(X)$. Then

$$Y=\frac{d}{dt}S\omega(t)\bigg|_{t=0}=\frac{d}{dt}(\mu(t)\,p_1\,\mu(t)^{-1}\bigg|_{t=0}$$

$$=[\dot{\mu}(0),p_1]=\begin{pmatrix}0 & -\dot{\tau}(0)^* \\ \dot{\tau}(0)^* & 0\end{pmatrix}$$

and finally $\|Y\|=\|\dot{\tau}(0)\|$.

Hence,

$$\|\lambda^{-1}X\lambda\|=\max\left\{\|(c\dot{\tau}(0)\;c\dot{\nu}(0)\,d+\theta)\|,\left\|\begin{pmatrix}-\dot{\tau}(0)^*\,c \\ d^*\dot{\nu}(0)-\theta^*\end{pmatrix}\right\|\right\}$$

$$\geqslant\|c\dot{\tau}(0)\|\geqslant\|\dot{\tau}(0)\|$$

(the last inequality holds because $c=\sqrt{1+bb^*}\geqslant 1$). Thus, $\|(T_qS)(X)\|\leqslant\|\lambda^{-1}X\lambda\|=\|X\|_q$, which proves Theorem 8.3.

9. Geodesics in Q

This section deals with geodesics in Q. Throughout lengths of curves in Q are calculated according to the metric described in Section 6. Our first result is the following minimality theorem.

9.1. Theorem. *Let $p_0, p_1 \in P$ at (geodesic) distance $d(p_0,p_1)<\pi$. Then there is a unique geodesic in P joining p_0 to p_1, it has length $d(p_0,p_1)$ and it is shorter than any other curve in Q joining p_0 and p_1.*

Proof. In view of [11] it suffices to show that for any curve in Q joining p_0 and p_1 there is a shorter curve in P with the same endpoints. But this is obvious because of Theorem 7.6 above.

We turn next to geodesics joining points in the same fiber Q_p:

9.2. Theorem. *Any pair of elements in a fiber Q_p can be joined by a geodesic of Q fully contained in Q_p. Such a geodesic is unique up to reparametrization.*

Proof. Let $q, s \in Q_p$ and suppose first that $s=p$. Then, Φ being the exponential map at s, the statement follows from 4.1. For an arbitrary pair $q, s \in Q_p$, endow E with the inner product $\langle\,,\,\rangle_s$. This produces the corresponding constructions and concepts: the space $P(\langle\,,\,\rangle_s)$ of $\langle\,,\,\rangle_s$-orthogonal projections, the adjoint $a^{\#}$ of an element $a \in A$, the new polar decomposition, and the resulting fibers $Q_w(\langle\,,\,\rangle_s)$ through each

$w \in P(\langle\ ,\ \rangle_s)$. Clearly $s \in P(\langle\ ,\ \rangle_s)$. Less obvious is the following identity: $Q_s(\langle\ ,\ \rangle_s) = Q_{\pi(s)}$, which can be proved using $a^{\#} = \zeta \rho a^* \rho \zeta$ and $\langle x, y\rangle_s = \langle \rho\zeta x, y\rangle$, where $\rho = 2\pi(s) - 1$ and $\zeta = 2s - 1$. This reduces the general case to the previous one and the theorem is proved.

Finally we come to our main result:

9.3. Theorem. *Let $p \in P$ and $q \in Q_p$ and let γ be the unique geodesic contained in Q_p joining p and q. Then γ has minimal length among all curves in Q joining p and q.*

We set some notation before describing the strategy of the proof. Denote by $\gamma(t)$, $0 \leqslant t \leqslant 1$, the unique geodesic joining p and q in Q_p, so $\gamma(t) = e^{t\tilde{X}} p e^{-t\tilde{X}}$ where $X = \dot{\gamma}(0) \in T_p Q_p$ and $\tilde{X} = [X, p]$ (for this use 9.2 and the explicit formulas of Section 2). From $\tilde{X}^* = \tilde{X}$ we get $\tilde{X}^2 \geqslant 0$ and since $\tilde{X}^2$ commutes with p we can find (changing the representation space if necessary) an element $h \in E$ with $ph = h$, $|h| = 1$, $\tilde{X}^2 h = \|\tilde{X}^2\|\, h$. Define next $H_1 =$ one dimensional subspace generated by h, and all the associated objects described in Section 8, whose notation we use here unchanged. The minimality of γ will result from the following three facts.

9.4. Proposition. *For any $q \in Q_p$ and any curve in Q of length L joining p and q there is a curve fully contained in Q_p also joining p and q whose length does not exceed L.*

9.5. Proposition. *The map $S\colon Q_p \to Q_{p_1}(E)$ of Section* 8 *pushes curves in Q_p onto curves in $Q_{p_1}(E)$ without increasing their lengths and the image of the geodesic γ is a geodesic of the same length.*

9.6. Proposition. *$Q_{p_1}(E)$ is a Riemannian space with constant negative curvature and therefore its geodesics are minimal.*

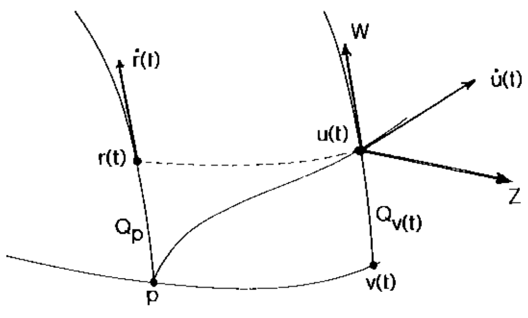

Figure 2

Proof of 9.4. Denote by $u(t)$, $0 \leqslant t \leqslant 1$, $u(0)=p$, $u(1)=q$, the given curve of length $L=\int_0^1 \|\dot{u}(t)\|_{u(t)}\, dt$. Let $v(t)$ be the projected curve in P, $v(t)=\pi(u(t))$, and $\Gamma(t)$ the transport curve of v, i.e., $\dot{\Gamma}=[\dot{v}, v]\Gamma$ for $0 \leqslant t \leqslant 1$ and $\Gamma(0)=1$, so that $\Gamma(t)$ is unitary (because $[\dot{\gamma}, \gamma]$ is skewsymmetric) and $\Gamma(t)\, p\Gamma(t)^{-1}=v(t)$. We claim that $r(t)=\Gamma^{-1}(t)\, u(t)\, \Gamma(t)$ is contained in Q_p and that its length is $\leqslant L$ (see Fig. 2).

Since conjugation by unitary elements commutes with (polar decomposition, hence with) $\pi: Q \to P$, we get $\pi(r(t))=\Gamma(t)^{-1}\, \pi(u(t))\, \Gamma(t)= \Gamma(t)^{-1}\, v(t)\, \Gamma(t)=p$ for all t and therefore $r(t) \in Q_p$. To show that length does not increase we use the following observation: for fixed t, if $\dot{u}(t)= Z+W$ is the decomposition of $\dot{u}(t)$ in its symmetric part Z and its skewsymmetric part W with respect to the $u(t)$-inner product $\langle\ ,\ \rangle_{u(t)}$, then

$$\|\dot{u}(t)\|_{u(t)} \geqslant \|W\|_{u(t)} \tag{9.7i}$$

$$\|W\|_{u(t)} \geqslant \|\dot{r}(t)\|_{r(t)}. \tag{9.7ii}$$

Of course (9.7i) is obvious: $W=\frac{1}{2}(\dot{u}-\dot{u}^{\#})$ so

$$\|W\|_{u(t)} \leqslant \tfrac{1}{2}(\|\dot{u}\|_{u(t)}+\|\dot{u}^{\#}\|_{u(t)})=\|\dot{u}\|_{u(t)}.$$

The second estimate will follow from explicit formulas for Z and W. First,

$$\begin{aligned} \dot{r} &= \Gamma^{-1}(\dot{u}-\dot{\Gamma}\Gamma^{-1}u+u\dot{\Gamma}^{-1})\Gamma \\ &= \Gamma^{-1}(\dot{u}-[[\dot{v}, v], u])\Gamma. \end{aligned}$$

Therefore $\dot{u}-[[\dot{v}, v], u]$ is $\langle\ ,\ \rangle_{u(t)}$-skewsymmetric because it is the conjugate by the unitary Γ of the $\langle\ ,\ \rangle_{r(t)}$-skewsymmetric tangent vector $\dot{r}(t) \in T_{r(t)}Q_p$. On the other hand, using $\varepsilon=2u-1$, $\rho=2v-1$, and denoting temporarily $Y=[[\dot{v}, v], u]$ we have

$$Y^*=[u^*, [v, \dot{v}]]=[[\dot{v}, v], u^*]$$

$$\rho Y^*\rho=[\rho[\dot{v}, v]\rho, \rho u^*\rho]$$

and, since $\rho v=v\rho=v$, $\rho\dot{v}=-\dot{v}\rho$, and $\rho u^*\rho=u$, we can simplify the last formula:

$$\rho Y^*\rho=[-[\dot{v}, v], u]=-Y.$$

But $Y \in T_{u(t)}Q$ so $\varepsilon Y=-Y\varepsilon$ and therefore the $u(t)$-adjoint $Y^{\#}$ of Y satisfies

$$Y^{\#}=\varepsilon\rho Y^*\rho\varepsilon=-\varepsilon Y\varepsilon=Y.$$

In other words, $Y=[[\dot{v},v],u]$ is $u(t)$-symmetric, which means

$$W=\dot{u}-[[\dot{v},v],u],$$
$$Z=[[\dot{v},v],u].$$

Now $\dot{r}=\Gamma^{-1}W\Gamma$ and the preservation of local norms under conjugation by unitary elements (Section 6) prove (9.7ii).

The proof of Proposition 9.4 is completed as follows. Notice that $r(t)$, $0\leqslant t\leqslant 1$, lies on the fiber Q_p and has length$(r)\leqslant$length(u), by (9.7i) and (9.7ii). But then the curve $\Gamma(1)\,r(t)\,\Gamma(1)^{-1}$ (which has the same length as r and also lies in Q_p) satisfies the required conditions because $\Gamma(1)\,r(0)\,\Gamma(1)^{-1}=\Gamma(1)\,p\Gamma(1)^{-1}=p$ and $\Gamma(1)\,r(1)\,\Gamma(1)^{-1}=\Gamma(1)\,\Gamma(1)^{-1}$ $r(1)\,\Gamma(1)\,\Gamma(1)^{-1}=u(1)=q$.

Proof of 9.5. In view of 8.3 only the statement about the geodesic needs proof. Write

$$\tilde{X}=\begin{pmatrix}0 & 0 & m^*\\ 0 & 0 & n^*\\ m & n & 0\end{pmatrix}$$

for $m\colon H_1\to H_3$ and $n\colon H_2\to H_3$. Therefore $\tilde{X}^2$ has the form

$$\tilde{X}^2=\begin{pmatrix}m^*m & * & *\\ n^*m & * & *\\ 0 & * & *\end{pmatrix}$$

and $\tilde{X}^2h=\|\tilde{X}\|^2\,h$ implies $m^*mh=\|\tilde{X}^2\|\,h$ and $n^*mh=0$, i.e., $n^*m=0$. Furthermore $\|m\|=\|\tilde{X}\|$. Let us decompose $\tilde{X}=R+T$ where

$$R=\begin{pmatrix}0 & 0 & m^*\\ 0 & 0 & 0\\ m & 0 & 0\end{pmatrix}\qquad T=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & n^*\\ 0 & n & 0\end{pmatrix}.$$

Then $RT=TR=0$, and therefore $e^{-tR}\gamma(t)\,e^{tR}=e^{tT}pe^{-tT}$. Observe that $R=R^*=-R^B$, $T=T^*=-T^B$, so that e^{tT}, $e^{tR}\in V_p$. Also, $p_1T=Tp_1$ $(=0)$ and then $\operatorname{im}(e^{tT}pe^{-tT})\supset\operatorname{im}(e^{tT}p_1e^{-tT})=\operatorname{im}(p_1)=H_1$. Since H_1 is a maximal positive subspace for B in K and $e^{tT}pe^{-tT}\in Q_p$, we conclude that $\operatorname{im}(e^{tT}pe^{-tT})\cap K=H_1$. On the other hand, $Rp_2=p_2R$ $(=0)$ so that $e^{tR}p_2=p_2e^{tR}$ also and therefore e^{tR} leaves kernel and image of p_2 invariant. Thus $e^{tR}K=K$ and finally

$$e^{tR}H_1e^{-tR}=e^{tR}(\operatorname{im}(e^{tT}pe^{-tT})\cap K)\,e^{-tR}$$
$$=\operatorname{im}(e^{t\tilde{X}}pe^{-t\tilde{X}})\cap K.$$

But $e^{tR}p_1e^{-tR}$ is B_1-symmetric and therefore $S(\gamma(t))=e^{t\tilde{Y}}p_1e^{-t\tilde{Y}}$ (where $Y=[R|_K, p_1]$) is a geodesic. The length of $S(\gamma(t))$ is $\|Y\|=\|m\|=\|X\|$, and 9.5 follows.

Proof of 9.6. Introduce in $Q_{p_1}(K)$ the Riemannian structure given by $\langle X, Y\rangle=\langle Xk, Yk\rangle_q$ for $X, Y\in T_qQ_{p_1}(K)$ and $k\in \operatorname{im}(p_1)$ with $|k|_q=1$. The connection of $Q_{p_1}(K)$ is the Levi–Civita connection of this metric. On the other hand it is clear that $\|X\|_q=\langle X, X\rangle^{1/2}$. It only remains to calculate the sectional curvature of $Q_{p_1}(K)$. For this purpose let V, W be tangent vectors at $q\in Q_{p_1}(K)$ and write their matrix representations in the decomposition $K=H_1\oplus H_3$

$$V=\begin{pmatrix} 0 & -v^{\#} \\ v & 0 \end{pmatrix}, \qquad W=\begin{pmatrix} 0 & -w^{\#} \\ w & 0 \end{pmatrix}.$$

We assume that V and W are orthogonal for $\langle\ ,\ \rangle$ in $T_qQ_{p_1}(K)$.

The curvature of $Q_{p_1}(K)$ is given by $R(X, Y)Z=[[X, Y], Z]$ for X, Y, $Z\in T_qQ_{p_1}(K)$, where $[X, Y]=XY-YX$ as operators (see [3]).

Accordingly,

$$R(W, V)V=\begin{pmatrix} 0 & -y^{\#} \\ y & 0 \end{pmatrix}$$

with $y=-wv^{\#}v+2vw^{\#}v-vv^{\#}w$, so

$$(R(W, V)V)k=-ww^{\#}vk+2vw^{\#}vk-vv^{\#}wk=-wk$$

because $v^{\#}v=1$, $w^{\#}v=0$, $v^{\#}w=0$ by orthonormality. Hence

$$\langle R(W, V)V, W\rangle=\langle -wk, wk\rangle_q=-1.$$

This concludes the proof of 9.6.

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GEODESICS AND OPERATOR MEANS IN THE SPACE OF POSITIVE OPERATORS

Dedicated to the memory of Domingo A. Herrero

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The set A^+ of positive invertible elements of a C^*-algebra has a natural structure of reductive homogeneous manifold with a Finsler metric. Because pairs of points can be joined by uniquely determined geodesics and geodesics are "short" curves, there is a natural notion of convexity: $C \subset A^+$ is *convex* if the geodesic segment joining $a, b \in C$ is contained in C. We show that this notion is related to the classical convexity of real and operator valued functions. Several results about convexity are proved in this paper. The expressions of these results are closely related to the operator means of Kubo and Ando, in particular to the geometric mean of Pusz and Woronowicz, and they produce several norm estimations and operator inequalities.

Introduction

Let A be a C^*-algebra with identity. In a series of papers the authors have studied geometrical aspects of several subsets of A or sets related to A: the set Q of all idempotents of A, the set Q_n of all n-tuples of idempotents of which are pairwise orthogonal and decompose the identity, the set G^s of all selfadjoint invertible elements of A, the set A^+ of all positive elements of G^s, etc. (see [4]–[12]). As a common feature, the group G of invertible elements of A operates on these sets defining on them

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homogeneous spaces or, more generally, discrete unions of such spaces. Each of these manifolds carries a natural connection with its corresponding exponential map and a natural Finsler structure, i.e. a natural norm on each tangent space. Several results relating the differential geometry with the metric structure, which are well known in Riemannian geometry, have been proved in this Finslerian context. In each case, the result depends on an operator norm inequality. Let us explain this assertion with some examples.

Example 1. Let Q be the set of all elements ε of A such that $\varepsilon^2 = 1$ and P the set of all selfadjoint elements of Q. For each $\varepsilon \in A$ its unitary part (in the polar decomposition), denoted $\pi(\varepsilon)$, belongs to P ([6], Sec. 3). The map $\pi : Q \to P$ is differentiable so that we can consider its tangent map $(TQ)_\varepsilon \to (TP)_{\pi(\varepsilon)}$. It turns out that this tangent map does not increase norms, if the norm (the Finsler structure) is given by $\|X\|_\varepsilon = \|\lambda^{-1}X\lambda\|$ for each $X \in (TQ)_\varepsilon$, where $\varepsilon = \lambda^2\rho$, λ positive and ρ unitary (see [6], Sec. 7). What is relevant for this paper is that this result depends on the inequality $\|\eta\| \le \|c\eta a + b\eta^* b\|$, for all $\eta, b \in A$, $a = (1 + b^*b)^{1/2}$ and $c = (1 + bb^*)^{1/2}$ (see [5]).

Example 2. Let G^s be the set of all selfadjoint invertible elements of A. As in the preceding example, given $a \in G^s$ its unitary part $\pi(a)$ belongs to P. A natural Finsler structure can be defined in G^s by $\|X\|_a = \|\lambda^{-1/2}X\lambda^{-1/2}\|$, if $a = \lambda\pi(a)$, with λ positive, is the polar decomposition of a and X belongs to $(TG^s)_a$. The fact that the tangent map $(T\pi)_a : (TG^s)_a \to (TP)_{\pi(a)}$ does not increase norms depends, again, on a norm inequality: $\|STS^{-1} + S^{-1}TS\| \ge 2\|T\|$, valid for every $S, T \in L(H)$, H a Hilbert space, S invertible and selfadjoint (see [8], [9]).

Example 3. Let A^+ be the set of all positive invertible elements of A. As a connected component of the space G^s considered before, A^+ carries a natural structure of Finsler manifold. The fact that the exponential map $\exp_a : (TA^+)_a \to A^+$ increases distances (see Sec. 1 below) is equivalent to Segal's inequality $\|e^{X+Y}\| \le \|e^{X/2}e^Y e^{X/2}\|$, valid for all selfadjoint X, Y (see [10]).

In this paper we continue the study of the Finsler properties of A^+. A^+ has a natural connection for which the geodesics can be explicitly determined. Moreover, because every two points $a, b \in A^+$ can be joined by a unique geodesic γ it is natural to call *convex* a subset C of A^+ such that for every $a, b \in C$ the geodesic segment joining them is contained in C. The main theme of this paper is that this geometric notion of convexity is deeply related to the classical convexity of certain operator valued functions. To be more precise we need to recall the following property of A^+: defining the length of a differentiable curve by means of the Finsler norm, it can be shown that, for every $a, b \in A^+$ the geodesic γ is the shortest curve joining them (see [9]) so that we get a complete metric on A^+ by putting $d(a, b) = \textit{length}$ of γ. The convexity of the balls $B = \{x \in A^+ : d(a, x) \le \alpha\}$ can be proved by showing that the functions $f(t) = (x^{-1/2}yx^{-1/2})^t$ are convex $(x, y \in A^+)$. Analogously, $\{x \in A^+ : \phi(x) \le 1\}$ (where ϕ is a positive linear map) is a convex subset of A^+.

The main result of the paper is that A^+ has the following property, which in Riemannian geometry is equivalent to the nonpositiveness of the sectional curvature (see [3]): for $s \in [0, 1]$ let $\rho_s : A^+ \to A^+$ be defined by $\rho_s(x) = \exp_a(s \exp_a^{-1}(x))$ for a fixed $a \in A^+$); then $d(\rho_s(x), \rho_s(y)) \le s \cdot d(x, y)$ for all x, $y \in A^+$. This convexity property depends on the norm inequality $\|\log(a^{-s/2} b^s a^{-s/2})\| \le s \|\log(a^{-1/2} b a^{-1/2})\|$, which is proved here.

All the geometric notions have expressions closely related to some operator means in the sense of Kubo-Ando [21]. Each result mentioned above can be translated to the language of operator means producing an operator inequality; for instance, Segal's inequality gives a lower bound for the relative entropy studied by J. I. Fujii and E. Kamei [15].

The paper is divided into three sections. Section 1 contains the description of the geometry of A^+. Section 2 contains the convexity results mentioned above. The relationship with the operator means is described in Sec. 3.

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1. Basic Facts and Inequalities

Let A be a C^*-algebra with identity, A^s the real subspace of selfadjoint elements of A, A^+ the set of all positive invertible elements of A and G the group of invertible elements of A. We describe briefly the differentiable structure of A^+ (for details, see [9]).

Because A^+ is an open subset of A^s, it carries a natural structure of (real) differentiable manifold and its tangent space $(TA^+)_a$ $(a \in A^+)$ can be identified to A^s. The group G acts on A^+ by $L_g a = (g^*)^{-1} a g^{-1}$ $(g \in G, a \in A^+)$. The action is transitive because every $a \in A^+$ can be expressed as $L_{a^{-1/2}} 1$. For a fixed $a \in A^+$, the map $\pi_a : G \to A^+$ defined by $\pi_a(g) = L_g a$ $(g \in G)$ is a principal fibre bundle with structural group $U_a = \{u \in G : \pi_a(u) = a\} = \{u \in G : u^* a u = a\}$. Set $W_1 = \{s \in G : as = s^* a\}$ and, in general, $W_g = g W_1$ for $g \in G$. An easy computation shows that $W_{gu} = W_g u$ for every $g \in G$, $u \in U_a$. Thus, by [20], $g \mapsto W_g$ is the distribution of horizontal spaces for a connection on the principal bundle $\pi_a : G \to A^+$ (the "canonical" connection). This connection induces a connection on the tangent bundle (TA^+), with covariant derivative for a tangent field X along the curve γ given by

$$\frac{DX}{dt} = \frac{dX}{dt} - \frac{1}{2}(\dot{\gamma}\gamma^{-1} X + X \gamma^{-1} \dot{\gamma}).$$

The corresponding exponential is

$$\exp_a X = e^{(1/2)Xa^{-1}} a e^{(1/2)a^{-1}X}$$

$$= a^{1/2} e^{a^{-1/2}Xa^{-1/2}} a^{1/2} \qquad (a \in A^+, X \in (TA^+)_a).$$

Observe that $\exp_a : (TA^+)_a \to A^+$ is a diffeomorphism, with inverse

$$x \mapsto a^{1/2} \log(a^{-1/2} x a^{-1/2}) a^{1/2}.$$

Furthermore, for a, b in A^+ the curve

$$\gamma(t) = a^{1/2} e^{t \log(a^{-1/2}ba^{-1/2})} a^{1/2}$$

$$= a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2}$$

is the unique geodesic in A^+ joining a to b.

Assume now that A is faithfully represented in a Hilbert space $(H, \langle\ ,\ \rangle)$. For each $a \in A^+$ define an inner product in H by $\langle \xi, \eta \rangle_a = \langle a\xi, \eta \rangle$ $(\xi, \eta \in H)$. On the other hand, each $X \in (TA^+)_a$ determines the sesquilinear form $B_X(\xi, \eta) = \langle X\xi, \eta \rangle$ $(\xi, \eta \in H)$. Notice that, if H_a denotes the Hilbert space $(H, \langle\ ,\ \rangle_a)$, then every $g \in G$ is an isometry $g : H_a \to H_{L_g a}$ and the norm of $B_X : H_a \times H_a \to \mathbf{C}$ is $\|a^{-1/2}Xa^{-1/2}\|$. This defines a Finsler metric by $\|X\|_a = \|a^{-1/2}Xa^{-1/2}\|$ $(X \in (TA^+)_a)$.

Proposition 1. *G acts isometrically on A^+ for the Finsler metric.*

Proof. Denote by T the linear map $(TL_g)_a : (TA^+)_a \to (TA^+)_{L_g a}$. Then $TX = Y$ means that $B_Y(g\xi, g\eta) = B_X(\xi, \eta)$ $(\xi, \eta \in H_a)$. Because $g : H_a \to H_{L_g a}$ is an isometry for each $g \in G$, $\|B_Y\| = \|B_X\|$ or, by the definition of the Finsler metric, $\|Y\|_{L_g a} = \|X\|_a$. More explicitly, $\|L_g X\|_{L_g a} = \|X\|_a$, as claimed.

We can recast the previous statement as follows: for any $X \in A^s$, $a \in A^+$ and $g \in G$

$$\|a^{-1/2}Xa^{-1/2}\| = \|((g^*)^{-1}ag^{-1})^{-1/2}(g^*)^{-1}Xg^{-1}((g^*)^{-1}ag^{-1})^{-1/2}\|.$$

Remark. This equality is well-known for a positive invertible X (see [19]), because in this case $\|a^{-1/2}Xa^{-1/2}\| = \inf\{\alpha > 0 : X \leq \alpha a\}$.

As usual, the *length* of a C^1 curve γ in A^+ is defined by

$$l(\gamma) = \int_0^1 \|\dot{\gamma}(t)\|_{\gamma(t)}\, dt$$

and the (*geodesic*) *distance* between a, b in A^+ is

$$d(a, b) = \inf\{l(\gamma) : \gamma \text{ joins } a \text{ and } b\}.$$

Observe that, by Proposition 1, $d(L_g a, L_g b) = d(a, b)$ for all $a, b \in A^+$ and $g \in G$.

Proposition 2. *Given $a, b \in A^+$*

$$d(a,b) = \|\log(a^{-1/2}ba^{-1/2})\|.$$

Proof. In [9], Theorem 6.3, it is proved that the geodesics are the shortest curves joining pairs of points. Then, given a, b in A^+, the distance $d(a, b)$ is given by $l(\gamma)$, where $\gamma(t) = a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2}$. Because each L_g $(g \in G)$ is an isometry, we get

$$\begin{aligned}\|\dot{\gamma}(t)\|_{\gamma(t)} &= \|a^{1/2}(\log(a^{-1/2}ba^{-1/2}))(a^{-1/2}ba^{-1/2})^t a^{1/2}\|_{\gamma(t)}\\ &= \|\log(a^{-1/2}ba^{-1/2})\|,\end{aligned}$$

and this finishes the proof.

The invariance of distance under the action by G can be rewritten as

$$\|\log(a^{-1/2}ba^{-1/2})\| = \|\log(((g^*)^{-1}ag^{-1})^{-1/2}(g^*)^{-1}bg^{-1}((g^*)^{-1}ag^{-1})^{-1/2})\|.$$

Remarks. 1. Given $a, b, c \in A^+$

$$\|\log(a^{-1/2}ba^{-1/2})\| \leq \|\log(a^{-1/2}ca^{-1/2})\| + \|\log(c^{-1/2}bc^{-1/2})\|,$$

by Proposition 2.

2. For each $a \in A^+$ and $\alpha > 0$ the exponential $\exp_a : (TA^+)_a \to A^+$ maps the ball $\{X \in (TA^+)_a : \|X\|_a \leq \alpha\}$ onto the (geodesic) ball $\{x \in A^+ : d(a, x) \leq \alpha\}$.

The Finsler structure of A^+ is not Riemannian. However, as proved in [10], [11] and [12], A^+ shares several properties of Riemannian manifolds of non-positive curvature. One of the properties is that the exponential $\exp_a : (TA^+)_a \to A^+$ increases distances: $d(\exp_a X, \exp_a Y) \geq \|X - Y\|_a$ (X, Y in $(TA^+)_a$). For $a = 1$, this gives

$$\|\log(e^{-X/2}e^{Y}e^{-X/2})\| \geq \|X - Y\|,$$

which can also be written, putting $x = e^X$, $y = e^Y$, as

$$\|\log(x^{-1/2}yx^{-1/2})\| \geq \|\log x - \log y\|.$$

This is, essentially Segal's inequality $\|e^{X/2}e^{Y}e^{X/2}\| \geq \|e^{X+Y}\|$; see [10].

2. Convexity

In a manifold with unique geodesics joining pairs of points, it is natural to label *convex* any subset C such that for all x, y in C the geodesic segment joining x and y is contained in C.

Theorem 1. *Given $a \in A^+$ and $\alpha > 0$ the ball $B = \{x \in A^+ : d(a, x) \leq \alpha\}$ is a convex set.*

Proof. Because G operates transitively and isometrically on A^+ we may assume $a = 1$, so that $B = \{x = e^X : X \in A^s, \|X\| \le \alpha\}$.

Consider therefore X, Y in $(TA^+)_1 = A^s$, with $\|X\| \le \alpha$, $\|Y\| \le \alpha$ and $x = e^X$, $y = e^Y$. The unique geodesic in A^+ joining x and y is $\gamma(t) = x^{1/2}(x^{-1/2}yx^{-1/2})^t x^{1/2}$. We want to see that $d(1, \gamma(t)) \le \alpha$, or equivalently, that $\|\log \gamma(t)\| \le \alpha$, for each t. The following remark will be useful: given a faithful representation of A in a Hilbert space H and $\xi \in H$ then the real function $f_\xi(t) = \langle \gamma(t)\xi, \xi \rangle$ is convex: in fact, $\dfrac{d^2}{dt^2} f_\xi(t) = \langle a^t(\log a)\eta, (\log a)\eta \rangle \ge 0$, where $a = x^{-1/2}yx^{-1/2}$, $\eta = x^{1/2}\xi$. As a consequence, $\max\{f_\xi(t) : 0 \le t \le 1\} = \max\{f_\xi(0), f_\xi(1)\}$. Fix $t_0 \in [0,1]$ and set $\gamma_0 = \gamma(t_0)$.

We choose a representation on a Hilbert space H and $\xi \in H$ with $\|\xi\| = 1$ such that $\|\log \gamma_0\| = \pm\langle (\log \gamma_0)\xi, \xi \rangle$. Supose first that $\|\log \gamma_0\| = \langle (\log \gamma_0)\xi, \xi \rangle$. Then, by the remark about, $f_\xi(t_0) \le \max\{f_\xi(0), f_\xi(1)\}$, which means that $\langle \gamma_0 \xi, \xi \rangle \le \langle x\xi, \xi \rangle \le \|x\| \le e^\alpha$, or that $\langle \gamma_0 \xi, \xi \rangle \le \langle y\xi, \xi \rangle \le \|y\| \le e^\alpha$. Thus, $\gamma_0 \le e^\alpha$ as operators; so we get $\log \gamma_0 \le \alpha$ and, then $\|\log \gamma_0\| = \langle (\log \gamma_0)\xi, \xi \rangle \le \alpha$. Thus, the theorem is proved in the case $\|\log \gamma_0\| = \langle (\log \gamma_0)\xi, \xi \rangle$.

Suppose now that $\|\log \gamma_0\| = -\langle (\log \gamma_0)\xi, \xi \rangle$. The hypothesis can be rewritten as $\|\log \gamma_0^{-1}\| = \langle (\log \gamma_0^{-1})\xi, \xi \rangle$. Observe that γ^{-1} is the geodesic in A^+ joining $x^{-1} = e^{-X}$ to $y^{-1} = e^{-Y}$, which also belongs to B. By the argument above applied to γ^{-1} we get $\|\log \gamma_0^{-1}\| \le \alpha$, so that $\|\log \gamma_0\| \le \alpha$.

Observe that, essentially, the theorem states that if $\|\log x\| \le \alpha$ and $\|\log y\| \le \alpha$ then $\|\log(x^{1/2}(x^{-1/2}yx^{-1/2})^t x^{1/2})\| \le \alpha$.

The next result gives an order relation between the geodesic segments and usual segments.

Theorem 2. *For every x, y in A^+ and $t \in [0,1]$*

$$x^{1/2}(x^{-1/2}yx^{-1/2})^t x^{1/2} \le (1-t)x + ty.$$

Proof. It suffices to prove that $(x^{-1/2}yx^{-1/2})^t \le 1 - t + tx^{-1/2}yx^{-1/2}$. For this, we will prove that, for every $a \in A^+$, $a^t \le 1 - t + ta$. Consider a faithful representation of A in a Hilbert space H, $\xi \in H$ and the real function $h_\xi(t) = \langle (1 - t + ta - a^t)\xi, \xi \rangle$. Then $\dfrac{d^2 h}{dt^2}\xi(t) = -\langle a^t(\log a)\xi, (\log a)\xi \rangle \le 0$ so that $h_\xi(t) \ge \min\{h_\xi(0), h_\xi(1)\} = 0$. This concludes the proof.

The next theorem exhibits another type of convex subsets of A^+.

Theorem 3. *Let $\phi : A \to \mathbf{C}$ be a positive linear functional, $a \in A^+$ and*

$$\Sigma = \{x \in A^+ : \phi(x) \le 1\}.$$

Then Σ is a convex subset of A^+.

Proof. Given x, y in Σ we will prove that $x^{1/2}(x^{-1/2}yx^{-1/2})^t x^{1/2}$ belongs to Σ, for

$t \in [0,1]$, or $f(t) = \phi(x^{1/2}(x^{-1/2}yx^{-1/2})^t x^{1/2}) \leq 1$ for $t \in [0,1]$. For this, we observe that f is convex, because $\frac{d^2}{dt^2}f(t) = \phi(x^{1/2}Ze^{tZ}Zx^{1/2}) \geq 0$, where $Z = \log(x^{-1/2}yx^{-1/2})$. Then, $f(t) \leq \max\{f(0), f(1)\} = \max\{\phi(x), \phi(y)\} \leq 1$ and this concludes the proof.

Remark. An analogous results holds if ϕ is not scalar valued but any positive map of C^*-algebras.

The next theorem gives another evidence that A^+ has some sort of "nonpositive curvature".

Theorem 4. *For every $a, b \in A^+$ and $s \in [0,1]$*

$$\|\log(a^{-s/2}b^s a^{-s/2})\| \leq s\|\log(a^{-1/2}ba^{-1/2})\|.$$

Proof. The inequality $\|a^{-s/2}b^s a^{-s/2}\| \leq \|a^{-1/2}ba^{-1/2}\|^s$ is an easy consequence of the identity $\|a^{-1/2}ba^{-1/2}\| = \inf\{\alpha > 0 : b \leq \alpha a\}$ and the fact that $f(x) = x^s$ is operator-monotone (see [19] and [2], respectively; the related inequality $\|a^s b^s\| \leq \|ab\|^s$ is also studied in [13], Chapter 1, [17] and [16]). In order to prove the theorem we shall use the following remarks: (1) for all $x \in A^+$, $\log\|x\| \leq \|\log x\|$; (2) for all $x \in A^+$, $\log\|x\| = \|\log x\|$ or $\log\|x^{-1}\| = \|\log x\|$. Set $x = a^{-s/2}b^s a^{-s/2}$. Suppose first that $\log\|x\| = \|\log x\|$. Then $\|\log(a^{-s/2}b^s a^{-s/2})\| = \log\|a^{-s/2}b^s a^{-s/2}\| \leq s\log\|a^{-1/2}ba^{-1/2}\|$ by the inequality mentioned above, and $\log\|a^{-1/2}ba^{-1/2}\| \leq \|\log(a^{-1/2}ba^{-1/2})\|$ finishes the proof if $\log\|x\| = \|\log x\|$. Suppose now that $\log\|x^{-1}\| = \|\log x\|$. Then x^{-1} is in the first case and, because $\|\log x\| = \|\log x^{-1}\|$, we get again $\|\log(a^{-s/2}b^s a^{-s/2})\| = \|\log(a^{s/2}b^{-s}a^{s/2})\| \leq s\|\log a^{1/2}b^{-1}a^{1/2}\| = s\|\log a^{-1/2}ba^{-1/2}\|$.

There is a clear geometric interpretation of the theorem above. In fact, in a Riemannian manifold M the sectional curvature is nonpositive if and only if

$$d(\rho_s(x), \rho_s(y)) \leq sd(x, y) \tag{$*$}$$

for all $x, y \in M$ and all $s \in [0,1]$, where $\rho_s(x) = \exp_p(s\exp_p^{-1}(x))$ and $p \in M$ is fixed (see [3], Lecture I). In our case $M = A^+$ is not Riemannian but inequality (*) holds. To prove this it suffices to consider $p = 1$ because G acts transitively and isometrically on A^+; then $\rho_s(x) = x^s$, $\rho_s(y) = y^s$, $d(\rho_s(x), \rho_s(y)) = \|\log(x^{-s/2}y^s x^{-s/2})\|$ and $d(x,y) = \|\log(x^{-1/2}yx^{-1/2})\|$. Inequality (*) for $\rho = 1$ is exactly Theorem 4.

3. Operator Means

Recall the notion of operator mean, as axiomatized by Kubo and Ando [21]. A *mean* is a binary operation m on the set of positive operators in a Hilbert space, satisfying the axioms

(I) If $a \leq c$ and $b \leq d$ then $m(a,b) \leq m(c,d)$.
(II) $cm(a,b)c \leq m(cac, cbc)$.
(III) If $a_n \downarrow a$ and $b_n \downarrow b$ then $m(a_n, b_n) \downarrow m(a,b)$.
(IV) $m(1,1) = 1$.

A few examples are the arithmetic mean $\frac{1}{2}(a+b)$, the left trivial mean $(a,b) \mapsto a$, the right trivial mean $(a,b) \mapsto b$, the parallel sum [1], and the geometric mean $a^{1/2}(a^{-1/2}ba^{-1/2})^{1/2}a^{1/2}$ first introduced by Pusz and Woronowicz [22]. (The geometric mean can be defined also for operators that are not necessarily invertible, but we do not pursue this issue here). The reader is referred to [2], [21] and [18] for an extensive study of the subject.

An interesting fact is the bijective correspondence between operator means and operator–monotone functions $f: \mathbf{R}^+ \to \mathbf{R}^+$. This theorem, due to Kubo and Ando [21] gives explicitly the correspondences (we only consider invertible elements):

$$m \mapsto f_m, \text{ with } f_m(t) = m(1,t);$$

$$f \mapsto m_f, \text{ with } m_f(a,b) = a^{1/2} f(a^{-1/2}ba^{-1/2})a^{1/2}.$$

This theorem has been extended for not necessarily positive functions which are operator–monotone by J. I. Fujii, M. Fujii and Y. Seo [14]. We only consider the notion of *relative entropy* (see [15]) which is defined, for invertible elements, by

$$s(a,b) = a^{1/2} \log(a^{-1/2}ba^{-1/2})a^{1/2}.$$

The geometrical results described above can be translated to this language of operator means. First, we observe that the geodesic $\gamma(t) = a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2}$ expresses a particular parametrization between the left trivial mean and the right trivial mean. The middle point of the geodesic is the geometric mean of Pusz and Woronowicz. The velocity vector $\dot\gamma(0)$ is exactly the relative entropy $s(a,b)$. Observe, also, that $s(a,b) = \exp_a^{-1}(b)$ for $\exp_a : (TA^+)_a \to A^+$.

The fact that $\exp_a$ increases distances is expressed by the inequality

$$\|\log a - \log b\| \le \log(a^{-1/2}ba^{-1/2})\|,$$

which implies the following estimation of $\|s(a,b)\|$:

$$\begin{aligned} \|\log a - \log b\| &\le \|\log(a^{-1/2}ba^{-1/2})\| \\ &= \|a^{-1/2}s(a,b)a^{-1/2}\| \\ &\le \|a^{-1}\| \, \|s(a,b)\|. \end{aligned}$$

Thus we get

Corollary. *For every $a, b \in A^+$*

$$\|s(a,b)\| \ge \frac{1}{\|a^{-1}\|} \|\log a - \log b\|.$$

Of course, many results of the preceding sections admit similar translations; for instance, the first remark following Proposition 2 reads, using the Finsler norms,

$$\|s(a,b)\|_a \leq \|s(a,c)\|_c + \|s(c,b)\|_b .$$

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The Geometry of the Space of Selfadjoint Invertible Elements in a C*–algebra

GUSTAVO CORACH, HORACIO PORTA AND LÁZARO RECHT

Let A be a C*-algebra with identity and G^s the set of all selfadjoint invertible elements of A. This paper is a study of the geometric properties of the manifold G^s. The action of the group G of invertible elements of A over G^s, given by $g \cdot a = (g^{-1})^* a g^{-1}$, defines Banach homogeneous spaces $G \to G^{s,a}$, where $G^{s,a}$ is the orbit of $a \in G^s$. It turns out that the $G^{s,a}$ are open and closed subsets of G^s and the principal bundles $G \to G^{s,a}$ carry natural connections. The horizontal lifting of (differentiable) curves γ in G^s are controlled by the differential equation $\dot{\Gamma} = -\frac{1}{2}\gamma\dot{\gamma}\Gamma$, which is called here *the transport equation* (an alternative approach based on multiplicative integrals is given in Section 8). Several G–bundles are studied, in particular the tangent bundle TG^s. One relevant point here is that the (left) polar decomposition $a = \nu\rho$ ($a \in G^s$, $\nu > 0$, ρ unitary) provides two structures: first it is easy to see that ρ is a reflection so that $\pi(a) = \rho$ defines a map $\pi : G^s \to P$ where P is the set of all $\rho \in A$ such that $\rho^* = \rho^{-1} = \rho$; second for a tangent vector $X \in T_a G^s$ the norm $\|X\|_a = \|\nu^{-1/2} X \nu^{-1/2}\|$ defines a Finsler structure on the bundle TG^s. This bundle carries a canonical connection determined by the transport equation, with covariant derivative defined by

$$D_X Y = X(Y) - \frac{1}{2}(Xa^{-1}Y + Ya^{-1}X)$$

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and parallel transport along a curve γ in G^s given by the transport function Γ of γ. Thus TG^s is endowed with the resulting structure of Finsler bundle with a transport connection. The exponential map of this connection is

$$\exp_a X = e^{-\frac{1}{2}a^{-1}X} \cdot a = e^{\frac{1}{2}a^{-1}X} a e^{\frac{1}{2}a^{-1}X}.$$

The restriction of the bundle TG^s to P splits as $TG^s|_P = TP \oplus N$ where the "normal bundle" N has over $\rho \in P$ the fiber

$$N_\rho = \{X \in T_\rho G^s : X\rho = \rho X\}.$$

The restriction to N of the exponential map is a diffeomorphism from N onto G^s which preserves the fibers. In Cheeger-Gromoll theory (see **[3]**) this is expressed by saying that P is a soul of G^s.

Returning to the study of the fibration $\pi : G^s \to P$ we give a description of the fibers of π and of the group of all $g \in G$ that preserve the fibers. The tangent map $T\pi : TG^s \to TP$ decreases norms in the sense that $\|(T_a\pi)X\| \leq \|X\|_a$ $(X \in T_a G^s)$. This theorem is based on the inequality $\|STS^{-1} + S^{-1}TS\| \geq 2\|T\|$ valid for bounded linear operators S, T on a Hilbert space with S selfadjoint and invertible **[4]**. The main result of this paper is that given two points in the same fiber G^s_ρ there is a unique geodesic fully contained in G^s_ρ joining them, which is the shortest curve in G^s with the same endpoints. A basic tool of the proof is the above mentioned contraction property of $T\pi$.

In finite dimensional cases, Riemann metrics can be defined on TG^s and we show an example where the canonical connection is the Levi-Civita connection of such a metric. This paper is part of a series devoted to the study of the geometry of several reductive homogeneous spaces which appear naturally in Banach and C^*-algebra theories: the space of idempotents in a C^*-algebra (**[17]**, **[18]**, **[6]**), the space Q_n of n-tuples of idempotents decomposing the identity in a Banach algebra **[5]**, the space of relatively regular elements in a Banach algebra **[8]**. The subset A^+ of G^s of all positive invertible elements of A is also considered in **[7]**, where it is shown that the well-known Segal's inequality (see **[21]**) $\|e^{(X+Y)}\| \leq \|e^{(X/2)}e^Y e^{(X/2)}\|$, where X, Y are selfadjoint elements of A, is equivalent to the property that the exponential map of A^+ increases distances, a property which A^+ shares with Riemannian manifolds with nonpositive curvature. The geometry of some Hilbert homogeneous spaces has been previously studied by P. de la Harpe (**[12]**, **[13]**) and Finsler structure of some groups of operators on a Hilbert space has been studied by Atkin (**[1]**, **[2]**) who proves some results on uniqueness and minimality of geodesics. The transport equation of Q_n has been independently found by Daleckii and Kato (see **[9]**, **[14]** and also **[15]**, **[10]**); its geometric meaning, however, was first established in **[5]**. In the case $n = 2$, Q_2 can be identified with the space of all the reflections and its transport equation takes the same form as that of G^s, a phenomenon which will be studied in a forthcoming paper.

1. Preliminaries

Let A be a C*–algebra with 1 represented as an operator algebra in a Hilbert space H. Also denote by $G = G(A)$ the group of invertible elements of A and $G^s = G^s(A)$ the space of invertible selfadjoint elements of G. For each $a \in G^s$ there is a form B^a defined on H by $B^a(x,y) = \langle ax, y\rangle$. The B^a's are hermitian non–degenerate bilinear forms. The B^a–adjoint of $u \in A$ is $u^a = a^{-1}u^*a$. Hence the unitary group U^a of B^a consists of the $u \in G$ with the equivalent properties $u^{-1} = a^{-1}u^*a$ or $(u^*)^{-1}au^{-1} = a$.

In order to study the natural geometry of G^s we introduce the following action of G on G^s:

$$g \cdot a = (g^{-1})^* a g^{-1}.$$

This action fits into the following picture: consider $E = G^s \times H$ as a product bundle over G^s with fiber $E_a = H$ over $a \in G^s$. Then E is a pseudo–Riemannian bundle when each fiber E_a is provided with the form B^a.

E can also be considered as a G–bundle with the action

$$g(a,x) = (g \cdot a, gx).$$

It is clear that this action is isometric on fibers (because $B^{g\cdot a}(gx, gy) = B^a(x,y)$) and that the isotropy group of $a \in G^s$ for the action $g \cdot a$ is the unitary group U^a of the form B^a.

Using $B^{g\cdot a}(gx, gy) = B^a(x, y)$ with $g = \sigma(b)$ the geometric interpretation interpretation of σ is that $\sigma(b)$ an isometry from $E_a = (H, B^a)$ onto $E_b = (H, B^b)$.

In the sequel we denote $G^{s,a}$ the orbit $\{g \cdot a; g \in G\}$ of a.

1.1 PROPOSITION *The orbits $G^{s,a}$ are open and closed in G^s and for each $a \in G^s$, the map*

$$G \to G^{s,a}, \quad g \to g \cdot a$$

is a smooth principal bundle with group U^a.

Proof: It suffices to show that $G \to G^{s,a}$ has a smooth local section near $a \in G^s$. For $b \in G^s$ near a put $\sigma(b) = (b^{-1}a)^{1/2}$. Here $b^{-1}a$ is close to 1 and the square root has the usual meaning (see **[20]** for example). Routine calculations show that

$$\sigma(b) \cdot a = (((b^{-1}a)^{1/2})^{-1})^* a((b^{-1}a)^{1/2})^{-1} = b$$

so that σ is a local section, as needed. This completes the proof of 1.1.

It is readily seen that G^s has a functorial character in the category of C*-algebras and *-homomorphisms. In particular, using Michael's result **[16]** that $G(A) \to G(B)$ is a Serre fibration if $f : A \to B$ is a surjective *-homomorphism, Proposition 1.1 implies that $f : G^s(A) \to G^s(B)$ is onto if and only if every component of $G^s(B)$ contains some element of the image of f. This result is useless in the case when A is the algebra of all bounded linear operators on a Hilbert space H and B is the quotient of A by the ideal of all compact operators (the Calkin algebra of H) since in this case the natural projection $G^s(A) \to G^s(B)$ is onto (**[13]**, p. 197). However in general there is no way of lifting elements and the criterion above may be adequate.

We use $a = \nu\rho$ as the polar decomposition of a with $\nu = |a| = (a^2)^{1/2} > 0$ and with ρ unitary. Since $|a|$ and a commute we have

$$\rho^* = (|a|^{-1}a)^* = a|a|^{-1} = |a|^{-1}a = \rho$$

whence ρ is a selfadjoint unitary element of A, or $\rho^* = \rho^{-1} = \rho$.

2. The canonical connection

Denote by $\mathcal{U}^a$ the Lie algebra of U^a. It is clear that $\mathcal{U}^a$ is a subalgebra of the Lie algebra $\mathcal{G}$ of G and that $\mathcal{G}$ can be identified with A (since G is open in A). In this identification, $\mathcal{U}^a$ corresponds to the set of B^a–antisymmetric elements of A, *i. e.*,

$$\mathcal{U}^a = \{x \in A; a^{-1}x^*a = -x\}.$$

2.1 PROPOSITION *Let S^a denote the set of elements s of A which are B^a–symmetric, i. e., with $a^{-1}s^*a = s$. Then $A = \mathcal{U}^a \oplus S^a$ and the elements of U^a conjugate S^a into itself: if $s \in S^a$ and $g \in U^a$, then $gsg^{-1} \in S^a$.*

Proof: Only the last statement needs a proof:

$$a^{-1}(gsg^{-1})^*a = (a^{-1}(g^{-1})^*a)(a^{-1}s^*a)(a^{-1}g^*a) = gsg^{-1}.$$

2.2 PROPOSITION *For $g \in G$ define $W_g = \{gs; s \in S^a\}$. The the map $g \to W_g \subset T_gG(= A)$ is a distribution of horizontal spaces for a connection on the principal bundle $G \to G^{s,a}$.*

Proof: $(W_g)u = W_{gu}$ for $u \in U^a, g \in G$ is equivalent to $uS^au^{-1} = S^a$, which is shown in Proposition 2.1.

The connection defined by the distribution W_g is the *canonical connection* of the bundle $G \to G^{s,a}$.

2.3 PROPOSITION *If $\gamma(t)$, $u \le t \le v$ is a curve in $G^{s,a}$, a curve $\Gamma(t)$ in G is a horizontal lifting of $\gamma(t)$ if and only if $\Gamma(t)$ satisfies the "transport equation"*

$$\dot{\Gamma} = -\frac{1}{2}\gamma^{-1}\dot{\gamma}\Gamma.$$

Proof: Suppose that $\Gamma(t)$ lifts $\gamma(t)$, or $\Gamma(t)\cdot a = \gamma(t)$ or $(\Gamma^{-1}(t))^*a\Gamma^{-1}(t) = \gamma(t)$. Then $\gamma^{-1} = \Gamma a^{-1}\Gamma^*$ and by differentiation we get

$$-\gamma^{-1}\dot{\gamma}\gamma^{-1} = \dot{\Gamma}a^{-1}\Gamma^* + \Gamma a^{-1}\dot{\Gamma}^*$$

or

$$\begin{aligned}-\gamma^{-1}\dot{\gamma} &= \dot{\Gamma}a^{-1}\Gamma^*(\Gamma^{-1})^*a\Gamma^{-1} + \Gamma a^{-1}\dot{\Gamma}^*(\Gamma^{-1})^*a\Gamma^{-1}\\ &= (\dot{\Gamma} + M)\Gamma^{-1}\end{aligned}$$

where $M = \Gamma a^{-1}(\Gamma^{-1}\dot{\Gamma})^*a$. Hence the equation $\dot{\Gamma} = -(1/2)\gamma^{-1}\dot{\gamma}\Gamma$ holds if and only if $M = \dot{\Gamma}$. This in turn is equivalent to

$$\Gamma^{-1}\dot{\Gamma} = a^{-1}(\Gamma^{-1}\dot{\Gamma})^*a,$$

or $\Gamma^{-1}\dot{\Gamma} \in S^a$ or finally $\dot{\Gamma} \in W_\Gamma$. This completes the proof.

In the sequel we shall be interested only in solutions Γ of the transport equation with $\Gamma(u) = 1$. These satisfy $\Gamma(t) \cdot \gamma(u) = \gamma(t)$ for all $u \leq t \leq v$. This Γ will be refered to as the "transport function" of the path $\gamma(t)$ (*cf.* **[5]**, **[10]**, **[14]**, **[15]**, **[18]**). The transport function has the following fundamental property:

2.4 PROPOSITION *If $\gamma(t)$ is a curve in G^s with transport function $\Gamma(t)$ then for $g \in G$ the transport function of $g \cdot \gamma = (g^{-1})^*\gamma g^{-1}$ is $g\Gamma g^{-1}$.*

3. Induced Connections

Suppose $\mathcal{C}$ is a G–manifold ($G = G(A)$) and $\mathcal{C} \to G^s$ is a C^∞ G–Banach bundle, *i.e.*, G operates in a compatible C^∞ way on $\mathcal{C}$ and G^s. A connection D on $\mathcal{C}$ is a *transport connection* if parallel transport in $\mathcal{C}$ along a curve $a(t)$ is given by the transport function of $a(t)$. This means that a section $\sigma(t)$ of $\mathcal{C}$ along $a(t)$, $0 \leq t \leq 1$, is D–parallel is and only if $\sigma(t) = \Gamma(t)(\sigma(0))$ where $\Gamma(t)$ satisfies $\dot{\Gamma} = -(1/2)a^{-1}\dot{a}\Gamma$, $\Gamma(0) = 1$.

3.1 PROPOSITION *Transport connections are G–invariant.*

Proof: Use Proposition 2.4.

We define several transport connections resulting from the systematic use of the transport functions in appropriate contexts.

The bundle E

Let $E = G^s \times H$ as a G–bundle with the action $g(a,x) = (g \cdot a, gx)$ described above in Section 1 and define the connection on E by

$$\frac{Dv}{dt} = \frac{d}{dt}(\Gamma^{-1}(t)v(t))|_{t=0}$$

for any section $v(t) = (a(t), x(t))$ over $a(t)$.

3.2 PROPOSITION *D is a transport connection on E and*

$$D_X v = X(v) + \frac{1}{2}a^{-1}Xv.$$

The curvature of D at $a \in G^s$ is:

$$R(X,Y) = -\frac{1}{4}[a^{-1}X, a^{-1}Y].$$

Next define a Riemannian metric $\langle\langle\ ,\ \rangle\rangle$ on E as follows. For $a \in G^s$ let $a = \nu\rho$ be the polar decomposition of a with $\nu = |a| = (a^2)^{1/2} > 0$ and ρ unitary. We define on the fiber $E_a = H$ the metric

$$\langle\langle x, y\rangle\rangle_a = \langle \nu x, y\rangle = \langle \nu^{1/2}x, \nu^{1/2}y\rangle.$$

Define also a 1–form on G^s with values in A by setting at each $a \in G^s$:

$$S = -\frac{1}{2}a^{-1}[d\rho, \nu]$$

where again $a = \nu\rho$ is the polar decomposition of a.

3.3 PROPOSITION *For any tangent field X on G^s, and any sections x, y of E we have:*

$$X\langle\langle x, y\rangle\rangle - \langle\langle D_X x, y\rangle\rangle - \langle\langle x, D_X y\rangle\rangle = \langle\langle S(X)x, y\rangle\rangle.$$

Proof:

$$
\begin{aligned}
X\langle\!\langle x,y\rangle\!\rangle - \langle\!\langle \frac{Dx}{dt}, y\rangle\!\rangle - \langle\!\langle x, \frac{Dy}{dt}\rangle\!\rangle \\
&= \frac{d}{dt}\langle \nu x, y\rangle - \langle \nu(\dot{x} + \frac{1}{2}a^{-1}\dot{a}x), y\rangle \\
&\quad - \langle \nu x, (\dot{y} + \frac{1}{2}a^{-1}\dot{a}y)\rangle \\
&= \langle \dot{\nu}x, y\rangle + \langle \nu\dot{x}, y\rangle + \langle \nu x, \dot{y}\rangle \\
&\quad - \langle \nu\dot{x}, y\rangle - \frac{1}{2}\langle \nu a^{-1}\dot{a}x, y\rangle \\
&\quad - \langle \nu x, \dot{y}\rangle - \frac{1}{2}\langle \nu x, a^{-1}\dot{a}y\rangle \\
&= \langle \nu(\nu^{-1}\dot{\nu} - \frac{1}{2}a^{-1}\dot{a} - \frac{1}{2}\nu^{-1}\dot{a}a^{-1}\nu)x, y\rangle
\end{aligned}
$$

But

$$
\begin{aligned}
\nu^{-1}\dot{\nu} - \frac{1}{2}\nu^{-1}\rho(\dot{\rho}\nu + \rho\dot{\nu}) - \frac{1}{2}\nu^{-1}(\dot{\nu}\rho + \nu\dot{\rho})\rho \\
&= \nu^{-1}\dot{\nu} - \frac{1}{2}\nu^{-1}\rho\dot{\rho}\nu - \frac{1}{2}\nu^{-1}\dot{\nu} - \frac{1}{2}\nu^{-1}\dot{\nu} - \frac{1}{2}\dot{\rho}\rho \\
&= -\frac{1}{2}\nu^{-1}\rho\dot{\rho}\nu - \frac{1}{2}\dot{\rho}\rho \\
&= -\frac{1}{2}\nu^{-1}\rho\dot{\rho}\nu + \frac{1}{2}\rho\dot{\rho} \\
&= -\frac{1}{2}a^{-1}(\dot{\rho}\nu - \nu\dot{\rho}) = -\frac{1}{2}a^{-1}[\dot{\rho}, \nu],
\end{aligned}
$$

as claimed.

3.4 COROLLARY *Parallel transport on E preserves the metric on curves with $\rho =$ constant.*

The bundle M

We define M as the product bundle $M = G^s \times V$ where V is the space of bounded conjugate bilinear forms on H. The group G acts on V

by $g\beta(x,y) = \beta(g^{-1}x, g^{-1}y)$. If $\beta(t)$ is a curve in M on the curve $a(t)$ we define

$$\frac{D\beta}{dt} = \frac{d}{dt}\Big(\beta(u,v)\Big) - \beta(\frac{Du}{dt}, v) - \beta(u, \frac{Dv}{dt})$$

for any sections u, v in E. The right hand side has the form

$$\begin{aligned}\dot\beta(u,v) + &\beta(\dot u, v) + \beta(u, \dot v)\\ &- \beta(\dot u, v) - \frac{1}{2}\beta(a^{-1}\dot a u, v)\\ &- \beta(u, \dot v) - \frac{1}{2}\beta(u, a^{-1}\dot a v)\\ = \dot\beta(u,v) &- \frac{1}{2}\beta(a^{-1}\dot a u, v) - \frac{1}{2}\beta(u, a^{-1}\dot a v),\end{aligned}$$

which obviously depends only on the values of u, v at each point but not on their derivatives. This means that:

3.5 PROPOSITION *The connection on M is a transport connection with covariant derivative*

$$(D_X\beta)(u,v) = (X(\beta))(u,v) - \frac{1}{2}\beta(a^{-1}Xu, v) - \frac{1}{2}\beta(u, a^{-1}Xv)$$

The bundle $L = G^s \times A$

The elements b in A can be interpreted as bilinear forms by $\beta(u,v) = \langle bu, v\rangle$ and the connection on M induces a connection on $L = G^s \times A$ by

$$\langle \frac{D\sigma}{dt} u, v\rangle = \frac{D\beta}{dt}(u,v)$$

where $\beta(u,v) = \langle \sigma u, v\rangle$.

3.6 PROPOSITION *The connection on L is a transport connection with covariant derivative*

$$D_X\sigma = X(\sigma) - \frac{1}{2}(Xa^{-1}\sigma + \sigma a^{-1}X).$$

The curvature of D satisfies:

$$4R(X,Y)\sigma = \sigma[a^{-1}X, a^{-1}Y] - [Xa^{-1}, Ya^{-1}]\sigma.$$

Proof: The fact that D is a transport connection on L results from calculating for a fixed $b \in A$:

$$\begin{aligned}\frac{D}{dt}(\Gamma \cdot b) &= \frac{D}{dt}((\Gamma^{-1})^* b\Gamma^{-1}) \\ &= -(\Gamma^{-1})^*\dot{\Gamma}^*(\Gamma^{-1})^* b\Gamma^{-1} - (\Gamma^{-1})^* b\Gamma^{-1}\dot{\Gamma}\Gamma^{-1} \\ &\quad - \frac{1}{2}(\dot{a}a^{-1}(\Gamma^{-1})^* b\Gamma^{-1} + (\Gamma^{-1})^* b\Gamma^{-1}a^{-1}\dot{a}) \\ &= \frac{1}{2}\dot{a}a^{-1}(\Gamma^{-1})^* b\Gamma^{-1} + \frac{1}{2}(\Gamma^{-1})^* b\Gamma^{-1}a^{-1}\dot{a} \\ &\quad - \frac{1}{2}(\dot{a}a^{-1}(\Gamma^{-1})^* b\Gamma^{-1} + (\Gamma^{-1})^* b\Gamma^{-1}a^{-1}\dot{a}) = 0.\end{aligned}$$

3.7 PROPOSITION *The section $a \to B^a$ in $G^s \times A$ is parallel.*

Proof:

$$\frac{Da}{dt} = \dot{a} - \frac{1}{2}(\dot{a}a^{-1}a + aa^{-1}\dot{a}) = 0.$$

3.8 COROLLARY *The section $a \to (a,a)$ in L is parallel.*

Proof: Since $B^a(x,y) = \langle ax, y\rangle$, B^a corresponds to the tautological section in $G^s \times A$.

The metric $\langle\langle\ ,\ \rangle\rangle$ in E defines a Finsler structure on the bundle of bilinear forms $M = G^s \times V$, as follows. If $\beta \in M_a$ then

$$\|\beta\|_a = \sup\{|\beta(x,y)|; \langle\langle x,x\rangle\rangle_a \leq 1, \langle\langle y,y\rangle\rangle_a \leq 1\}.$$

With the interpretation of $u \in A$ as the bilinear form $\beta(x,y) = \langle ux, y\rangle$, this translates into a Finsler norm on the bundle $L = G^s \times A$ given explicitly by: for $u \in L_a = A$,

$$\|u\|_a = \|\nu^{-1/2}u\nu^{-1/2}\|$$

($\| \ \|$=ordinary operator norm calculated from $\langle \ , \ \rangle$).

Notice that if $a = \nu\rho = \nu^{-1/2} \cdot \rho \quad (\nu > 0,\ \rho = \text{unitary})$ then the map

$$u \to \nu^{-1/2} \cdot u, \qquad L_\rho \to L_a$$

is an isometry for the norms $\| \ \|_\rho \,(= \| \ \|), \quad \| \ \|_a$. In the sequel length of curves and related concepts refer to this metric through the usual definition

$$\mathrm{Length}(\gamma) = \int \|\dot\gamma(\mathrm{t})\|_{\gamma(\mathrm{t})}\, \mathrm{dt}.$$

The tangent bundle TG^s

The set G^s is open in the real subspace A^s of symmetric elements of A. Hence $TG^s = G^s \times A^s$ is a subbundle of $L = G^s \times A$. Since the covariant derivative in L defined by 3.6 produces symmetric results from symmetric data, we can restrict this connection to TG^s. This is the *canonical connection* on G^s, with covariant derivative defined by

$$D_X Y = X(Y) - \frac{1}{2}(Xa^{-1}Y + Ya^{-1}X)$$

and parallel transport along a curve $a(t)$ in G^s given by the transport function $\Gamma(t)$ of $a(t)$ *acting* on tangent vectors by $\Gamma(t) \cdot X = (\Gamma(t)^{-1})^* X \Gamma(t)^{-1}$. Since the term $Xa^{-1}Y + Ya^{-1}X$ in D_XY is symmetric in X and Y, the connection in TG^s is a symmetric connection. Similarly, the curvature of TG^s is given by

$$4R(X,Y)Z = Z[a^{-1}X, a^{-1}Y] - [Xa^{-1}, Ya^{-1}]Z.$$

The Finsler structure of $L = G^s \times A$ can be restricted to TG^s. In the sequel we will always consider TG^s as endowed with the resulting structure of Finsler bundle with a transport connection.

Finally we briefly describe the exponential mapping of this connection. Direct computation shows that given $a \in G^s$ and $X \in T_aG^s$, the curve

$\gamma(t) = e^{t\tilde{X}} \cdot a$, where $\tilde{X} = -(1/2)a^{-1}X$, is the geodesic with $\gamma(0) = a$, $\dot{\gamma}(0) = X$. Therefore the exponential mapping is

$$\exp_a X = e^{-a^{-1}X/2} \cdot a.$$

This can also be written as $\exp_a X = a^{1/2}e^{a^{-1/2}Xa^{-1/2}}a^{1/2}$.

4. The structure of G^s

Let $P \subset G^s$ be the set of orthogonal reflections of A, i.e., $\rho \in P$ if and only if $\rho^* = \rho = \rho^{-1}$. We define a fibration $\pi : G^s \to P$ by setting $\pi(a) = \rho$ where $a = \nu\rho$ is the polar decomposition of a. As noticed in the preliminaries section, ρ is a selfadjoint unitary, hence an element of P.

Given $\rho \in P$ we write each $u \in A$ as a 2×2 matrix

$$u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

where $u_{11} = pup$, $u_{12} = pu(1-p)$, $u_{21} = (1-p)up$, $u_{22} = (1-p)u(1-p)$, for $p = (\rho+1)/2$ the associated symmetric projection. This decomposes the algebra as $A = A_0 \oplus A_1$ where A_0 consists of the diagonal elements

$$u = \begin{pmatrix} u_{11} & 0 \\ 0 & u_{22} \end{pmatrix}$$

and A_1 consists of the codiagonal elements

$$u = \begin{pmatrix} 0 & u_{12} \\ u_{21} & 0 \end{pmatrix}.$$

Equivalently, $A_0 = \{u; u\rho = \rho u\}$, $A_1 = \{u; u\rho = -\rho u\}$. We say that degree($u$) $= 0$ for $u \in A_0$ and degree(u) $= 1$ for $u \in A_1$. Then $A = A_0 \oplus A_1$ is a $\mathbf{Z}_2$–graded algebra.

4.1 PROPOSITION *Denote by G^s_ρ the fibers $\pi^{-1}(\rho)$ of $\pi : G^s \to P$.*

a) $G^s_\rho = \{a \in G^s \cap A_0; a\rho > 0\} = \{\nu\rho; \nu > 0, \nu\rho = \rho\nu\}$.

b) *The group of all $g \in G$ that preserve the fiber G^s_ρ, i.e., $g \cdot a \in G^s_\rho$ for each $a \in G^s_\rho$ is $G \cap A_0$.*

Proof of a): $a \in G^s \cap A_0$ and $a\rho > 0$ imply $a = (a\rho)\rho$ is the polar decompostion of a.

Proof of b): Let $g \in G$ commute with ρ. Then for any $a = \nu\rho \in G^s_\rho$ we have $g \cdot a = (g^{-1})^* \nu\rho g^{-1}$. Then $g \cdot a$ is in A_0 (as a product of degree zero elements) and it is symmetric. Also $(g \cdot a)\rho = (g^{-1})^* \nu g^{-1} > 0$ so that by a) we get $g \cdot a \in G^s_\rho$. Conversely, assume that $g \in G$ acts on G^s_ρ. Then for each $\nu > 0$ with $\nu\rho = \rho\nu$, there exists $\nu' > 0$ with $\nu'\rho = \rho\nu'$ and $g \cdot (\nu\rho) = \nu'\rho$. Decomposing $g^{-1} = h_0 + h_1$ with $h_0 \in A_0$ and $h_1 \in A_1$ we get

$$\begin{aligned} \nu'\rho = g \cdot (\nu\rho) &= (h_0^* + h_1^*)\nu\rho(h_0 + h_1) \\ &= (h_0^* + h_1^*)\nu(h_0 - h_1)\rho, \end{aligned}$$

so that after cancelling ρ and comparing terms of the same degree we get

$$h_0^*\nu h_0 - h_1^*\nu h_1 = \nu' \qquad h_0^*\nu h_1 - h_1^*\nu h_0 = 0.$$

Taking $\nu = 1$ it follows that $h_0^* h_0 = \nu' + h_1^* h_1 > 0$ and h_0 is invertible. But the equality $h_0^*\nu h_1 = h_1^*\nu h_0$ can not hold for all $\nu > 0$ commuting with ρ unless $h_1 = 0$. In fact consider the example

$$\nu = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

and write

$$h_0 = \begin{pmatrix} h_{11} & 0 \\ 0 & h_{22} \end{pmatrix} \quad h_1 = \begin{pmatrix} 0 & h_{12} \\ h_{21} & 0 \end{pmatrix}.$$

Then from $h_0^*\nu h_1 = h_1^*\nu h_0$ we get

$$h_{11}^*\alpha h_{12} = h_{21}^*\beta h_{22}$$

and since we can take $\alpha, \beta > 0$ arbitrary real numbers, we get $h_{11}^* h_{12} = 0$ and $h_{21}^* h_{22} = 0$. Cancelling h_{11}^* and h_{22} we conclude that $h_{12} = 0$, $h_{21} = 0$ and therefore $h = 0$. This means that g^{-1} (whence g) has degree 0 and the proof is complete.

The restriction to P of the bundle TG^s splits as a sum $TG^s|_P = TP \oplus N$ where the "normal" bundle N is defined by $N_\rho = \{x \in T_\rho G^s; x\rho = \rho x\}$.

4.2 THEOREM *Let $\Xi : N \to G^s$ be the restriction to N of the exponential mapping of G^s, so that $\Xi(\rho, X) = e^{-\rho X/2} \cdot \rho$. Then Ξ is a diffeomorphism satisfying $\Xi(N_\rho) = G^s_\rho$.*

Proof: The inverse of Ξ is given at $a = \nu\rho$ by $\Xi^{-1}(a) = (\rho, \rho \ln \nu)$.

We close this section with the remark that geodesics in a fiber with given endpoints are unique. This follows from the fact that positive elements have unique symmetric logarithms. In fact, if $x \in G^s_\rho$ and $H = H_+ \oplus H_-$ with $H_\pm = \{x; \rho x = \pm x\}$, then

$$a = \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix}$$

can be written in a unique way as $a = e^{\tilde{X}} \cdot \rho$ where

$$\tilde{X} = \begin{pmatrix} -\frac{1}{2}X_+ & 0 \\ 0 & \frac{1}{2}X_- \end{pmatrix},$$

and $X_\pm$ symmetric. So there is a unique geodesic joining ρ with a. For arbitrary $b, a \in G^s_\rho$, operate first with a convenient $g \in G \cap A_0$ to reduce to the case $b = \rho$.

5. Projecting on the base

The basic fact of this section is the following.

5.1 THEOREM *The tangent map $T\pi : TG^s \to TP$ decreases norms.*

Proof: We want to prove that

$$\|T_a\, \pi X\| \le \|X\|_a$$

for all $a \in G^s$. Let $a(t)$ be a curve in G^s and $X = \dot{a}(t)$. Let $\rho(t) = \pi(a(t))$ and let $\Gamma(t)$ be the transport function of $\rho(t)$. Finally define $a_1(t) = \Gamma(t) \cdot a(0)$. Since $\pi(a(t)) = \pi(a_1(t))$ ($\Gamma(t)$ is unitary) we get that $X_2 = \dot{a}(0) - \dot{a}_1(0)$ is tangent to the fiber $\pi^{-1}(\rho(0))$. Next calculate at $t = 0$:

$$X_1 = \dot{a}_1 = \frac{d}{dt}(\Gamma(t) \cdot a(0)) = \frac{1}{2}(-\rho\dot{\rho}a + a\rho\dot{\rho}).$$

Writing at $t = 0$ the polar decomposition $a = \nu\rho = \rho\nu$ we get

$$X_1 = \frac{1}{2}(-\rho\dot{\rho}\rho\nu + \nu\rho\rho\dot{\rho}) = \frac{1}{2}(\dot{\rho}\nu + \nu\dot{\rho}).$$

Then calculate

$$\begin{aligned} \|X\|_a &= \|\nu^{-\frac{1}{2}} X \nu^{-\frac{1}{2}}\| \\ &= \|\nu^{-\frac{1}{2}} X_1 \nu^{-\frac{1}{2}} + \nu^{-\frac{1}{2}} X_2 \nu^{-\frac{1}{2}}\| \\ &= \|\frac{1}{2}(\nu^{-\frac{1}{2}}\dot{\rho}\nu^{\frac{1}{2}} + \nu^{\frac{1}{2}}\dot{\rho}\nu^{-\frac{1}{2}}) + \nu^{-\frac{1}{2}} X_2 \nu^{-\frac{1}{2}}\|. \end{aligned}$$

Recall the inequality ([4]):

$$\|STS^{-1} + S^{-1}TS\| \geq 2\|T\|$$

valid for any symmetric invertible operator S and any operator T. This reduces the proof of the theorem to the inequality

$$\|\nu^{-\frac{1}{2}} X \nu^{-\frac{1}{2}}\| \geq \|\nu^{-\frac{1}{2}} X_1 \nu^{-\frac{1}{2}}\|.$$

But

$$\nu^{-\frac{1}{2}} X \nu^{-\frac{1}{2}} = \nu^{-\frac{1}{2}} X_1 \nu^{-\frac{1}{2}} + \nu^{-\frac{1}{2}} X_2 \nu^{-\frac{1}{2}}$$

is the decompostion of $\nu^{-\frac{1}{2}} X \nu^{-\frac{1}{2}}$ in degree 1 and degree 0 components determined by $\rho(0)$. This is clear because $\rho\dot{\rho} = -\dot{\rho}\rho$ and X_2 is tangent to $G^S_{\rho(0)}$. Therefore if we write

$$\nu^{-\frac{1}{2}} X \nu^{-\frac{1}{2}} = \begin{pmatrix} \alpha & \beta^* \\ \beta & \gamma \end{pmatrix}$$

$$\nu^{-\frac{1}{2}} X_1 \nu^{-\frac{1}{2}} = \begin{pmatrix} 0 & \beta^* \\ \beta & 0 \end{pmatrix}$$

$$\nu^{-\frac{1}{2}} X_2 \nu^{-\frac{1}{2}} = \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix}$$

then clearly

$$\|\nu^{-\frac{1}{2}} X \nu^{-\frac{1}{2}}\| \geq \|\beta\| = \|\nu^{-\frac{1}{2}} X_1 \nu^{-\frac{1}{2}}\|.$$

5.2 THEOREM *A geodesic of length less than π contained in P is the shortest curve in G^s joining its endpoints.*

Proof: Let γ be the geodesic in P joining ρ_0 and ρ_1 and let δ be any other curve joining ρ_0 and ρ_1. Then $\delta_1 = \pi(\delta)$ is contained in P and according to Theorem 5.1, the length of δ_1 does not exceed the length of δ. Then observing that the Finsler metric of G^s restricted to P is given by ordinary operator norm, a direct application of **[18]** gives the desired minimality and uniqueness.

6. Geodesics in a fiber

Suppose $a(t)$, $0 \leq t \leq 1$ is a curve in G^s with $\pi(a(0)) = a(1)$.

Denote $\rho(t) = \pi(a(t))$, $\nu(t) = a(t)\rho(t)$, and $\Gamma(t)$ the transport function of $\rho(t)$. Next define $\sigma(t) = \Gamma^{-1}(t)a(t)\Gamma(t)$. Since $\Gamma(t)$ is unitary, the polar decomposition of σ is

$$\sigma = (\Gamma^{-1}\nu\Gamma)(\Gamma^{-1}\rho\Gamma)\ ,$$

or $\pi(\sigma) = \Gamma^{-1}\rho\Gamma = \rho(0)$ for each t. This means that σ is a curve in $G^s_{\rho(0)}$.

Observe that σ has the same endpoints as a because

$$\sigma(0) = \Gamma^{-1}(0)a(0)\Gamma(0) = a(0)$$

and by the hypothesis $\pi(a(0)) = a(1)$ we have $\rho(1) = a(1)$ and therefore $\sigma(1) = \Gamma^{-1}(1)a(1)\Gamma(1) = \Gamma^{-1}(1)\rho(1)\Gamma(1) = \rho(0) = \rho(1) = a(1)$.

We claim that

$$(\P) \qquad \|\dot{\sigma}\|_\sigma \leq \|\dot{a}\|_a\ .$$

First (use $\rho\dot\rho = -\dot\rho\rho$, $a = \nu\rho$, etc.):

$$\begin{aligned}\dot\sigma &= -\Gamma^{-1}(-\frac{1}{2}\rho\dot\rho)a\Gamma + \Gamma^{-1}a(-\frac{1}{2}\rho\dot\rho)\Gamma + \Gamma^{-1}\dot a\Gamma \\ &= \Gamma^{-1}(\frac{1}{2}(\rho\dot\rho a - a\rho\dot\rho) + \dot a)\Gamma \\ &= \Gamma^{-1}\frac{\rho\dot\nu + \dot\nu\rho}{2}\Gamma\end{aligned}$$

and therefore

$$\begin{aligned}\|\dot\sigma\|_\sigma &= \|(\Gamma^{-1}\nu^{-1/2}\Gamma)\dot\sigma(\Gamma^{-1}\nu^{-1/2}\Gamma)\| \\ &= \|\Gamma^{-1}\nu^{-1/2}\frac{\rho\dot\nu + \dot\nu\rho}{2}\nu^{-1/2}\Gamma\| \\ &= \frac{1}{2}\|\nu^{-1/2}(\rho\dot\nu + \dot\nu\rho)\nu^{-1/2}\|\ .\end{aligned}$$

On the other hand, $a = \nu\rho = \rho\nu$ gives

$$\dot a = \frac{1}{2}(\rho\dot\nu + \dot\nu\rho) + \frac{1}{2}(\dot\rho\nu + \nu\dot\rho)$$

and then

$$\|\dot a\|_a = \frac{1}{2}\|\nu^{-1/2}(\rho\dot\nu + \dot\nu\rho)\nu^{-1/2} + \nu^{-1/2}(\dot\rho\nu + \nu\dot\rho)\nu^{-1/2}\|$$

But in the matrix decomposition at each $\rho(t)$

$$\begin{aligned}\nu^{-1/2}(\rho\dot\nu + \dot\nu\rho)\nu^{-1/2} &= \begin{pmatrix}\alpha & 0\\ 0 & \gamma\end{pmatrix}\\ \nu^{-1/2}(\dot\rho\nu + \nu\dot\rho)\nu^{-1/2} &= \begin{pmatrix}0 & \beta^*\\ \beta & 0\end{pmatrix}\end{aligned}$$

(because the former commutes with ρ and the latter anticommutes with ρ). Hence

$$\left\|\begin{pmatrix}\alpha & \beta^*\\ \beta & \gamma\end{pmatrix}\right\| \geq \left\|\begin{pmatrix}\alpha & 0\\ 0 & \gamma\end{pmatrix}\right\|$$

implies $\|\dot a\|_a \geq \|\dot\sigma\|_\sigma$. This is inequality (¶) and the claim is proved.

This inequality shows that:

6.1 PROPOSITION. *For any curve joining $a \in G^s$ with $\pi(a)$, there is a shorter curve in the fiber $G^s_{\pi(a)}$ with the same endpoints.*

The following technical result is needed in the proof of Theorem 6.3:

6.2 LEMMA. *Let p be a rank 1 orthogonal projection in the Hilbert space H, $a : H \to H$ positive definite, $X : H \to H$ selfadjoint. Then*

$$\|pa^{1/2}Xa^{1/2}p\| \leq \|pap\| \, \|X\| \, .$$

Proof: Decompose $H = \mathbf{C}e \oplus H_1$ where $\|e\| = 1$, $p(e) = e$, and $H_1 = \ker(p)$. Then we have matrix representations

$$a^{1/2} = \begin{pmatrix} A & B^* \\ B & C \end{pmatrix}$$

$$X = \begin{pmatrix} \xi & \eta^* \\ \eta & \theta \end{pmatrix}$$

where A, ξ are scalars, $B \in H_1$ and $B^* : H_1 \to \mathbf{C}$ is the functional $B^*(h) = \langle h, B \rangle$, and θ, C are operators in H_1. Define also a bilinear map $F : H \times H \to \mathbf{C}$ by $F(u,v) = \langle Xu, v \rangle$. Then calculating we find that the (1,1) entry W_{11} of $W = a^{1/2}Xa^{1/2}$ is $F(Ae + B, Ae + B)$. Then

$$\|W_{11}\| \leq \|F\| \, \|Ae + B\|^2 = \|X\| \, \|Ae + B\|^2 = \|X\|(A^2 + |B|^2) \, .$$

But

$$a = (a^{1/2})^2 = \begin{pmatrix} A^2 + B^*B & AB^* + B^*C \\ BA + CB & BB^* + C^2 \end{pmatrix}$$

and so

$$\|W_{11}\| \leq \|X\| \, \|a_{11}\| \, ,$$

as claimed.

6.3 THEOREM. *The unique geodesic in G^s_ρ joining two points $a, b \in G^s_\rho$ is the shortest curve in G^s joining a and b.*

Proof: We consider first the case where $b = \rho$. Let $\omega(t)$, $0 \le t \le 1$ be a curve joining ρ and a, and $\gamma(t) = e^{t\widetilde{X}} \cdot \rho$, $0 \le t \le 1$, the geodesic in G^s_ρ joining the same endpoints where $X = \dot{\gamma}(0) \in T_\rho G^s_\rho$ and $\widetilde{X} = -\frac{1}{2}\rho X$. We will show that

$$\text{Length}(\omega) \ge \text{Length}(\gamma) .$$

By 6.1 we may assume that ω is fully contained in G^s_ρ. We handle first the case $\rho = 1$.

By changing the representation if necessary, we can find $e \in H$ with $Xe = \lambda e$, $\|e\| = 1$ and $|\lambda| = \|X\|$. Next, we decompose H as $H = \mathbf{C}e \oplus \mathbf{C}e^{\perp}$ and therefore we can obtain by compression to $\mathbf{C}e$ two curves γ_{11} and ω_{11} defined as the (1,1) entries of the matrices of γ and ω in the decomposition $H = \mathbf{C}e \oplus \mathbf{C}e$. By 6.2 we have $\text{Length}(\omega_{11}) \le \text{Length}(\omega)$. Also, $\gamma_{11}(t) = (e^{t\widetilde{X}} \cdot \rho) = e^{t\lambda}$ and

$$\|\dot{\gamma}_{11}\|_{\gamma_{11}} = |e^{t\lambda}\lambda|_{\gamma_{11}} = |e^{-t\lambda/2}e^{t\lambda}e^{-t\lambda/2}\lambda| = |\lambda|$$

so that

$$\text{Length}(\gamma_{11}) = |\lambda| = \|X\| = \text{Length}(\gamma) .$$

Since $\omega_{11}(t) > 0$ we can calculate

$$\begin{aligned} \text{Length}(\omega_{11}) &= \int_0^1 \|\dot{\omega}_{11}(t)\|_{\omega_{11}(t)} dt \\ &\int_0^1 |\omega_{11}^{-1/2}(t)\dot{\omega}_{11}(t)\omega_{11}^{-1/2}(t)| dt \\ &= \int_0^1 |\dot{\omega}_{11}(t)/\omega_{11}(t)| dt \ge |\log \omega_{11}(t)|_0^1| = |\lambda| \end{aligned}$$

since $\omega_{11}(1) = \gamma_{11}(1) = e^{\lambda}$, $\omega_{11}(0) = \gamma_{11}(0) = 1$. This shows that γ is minimal in the case $\rho = 1$.

Consider next an arbitrary ρ and decompose $H = H_+ \oplus H_-$ where $H_\pm = \{x\,;\ \rho x = \pm x\}$. Then

$$X = \begin{pmatrix} X_+ & 0 \\ 0 & X_- \end{pmatrix}, \quad \widetilde{X} = \begin{pmatrix} -\frac{1}{2}X_+ & 0 \\ 0 & +\frac{1}{2}X_- \end{pmatrix}$$

and

$$\gamma(t) = e^{t\widetilde{X}} \cdot \rho = \begin{pmatrix} e^{tX_+} & 0 \\ 0 & -e^{-tX_-} \end{pmatrix}.$$

Similarly,

$$\omega(t) = \begin{pmatrix} \omega_+(t) & 0 \\ 0 & \omega_-(t) \end{pmatrix}.$$

But,

$$\|X\| = \|X_+\| \quad \text{or} \quad \|X\| = \|X_-\|$$

and

$$\|\dot{\omega}(t)\|_{\omega(t)} \geq \|\dot{\omega}_\pm(t)\|_{\omega_\pm(t)}$$

so that by choosing the half where X keeps its norm we are (up to sign) in the case $\rho = 1$, and the proof is complete.

To complete the proof, operate with an element of $G \cap A_0$ to reduce the general case to $b = \rho$.

7. An example

We consider now the algebra A of linear endomorphisms of the Hilbert space $\mathbf{C}^2$ with the standard inner product. Then $G = GL(2, \mathbf{C})$ and G^s has three connected components determined by signature. Denote G_1^s the component consisting of the positive definite elements of A. The level manifolds $M_h = \{a; \det(a) = h\}$ of the determinant function $\det : G_1^s \to \mathbf{R}^+$ form a smooth foliation with three dimensional leaves. Also the rays $N_a = \{ra\,;\ r > 0\}$ with $a \in M_1$, form a one dimensional foliation and $\{M_h\}$ is transversal to $\{N_a\}$. The leaves M_h are the orbits of the action $g \,.\, a = (g^{-1})^* a g^{-1}$ of the subgroup $SL(2, \mathbf{C}) \subset GL(2, \mathbf{C})$ and the leaves N_a are the orbits of the center $\{z1\,;\ z \neq 0\}$ of $GL(2, \mathbf{C})$.

Since a curve through $a(0) = 1$ with $\det(a(t)) = 1$ satisfies $\mathrm{tr}(\dot{a}(0)) = 0$, by translation we have $\mathrm{tr}(a^{-1}\dot{a}) = 0$ for all curves in M_h. Then the solution Γ of the transport equation $\dot\Gamma = -\frac{1}{2}a^{-1}\dot a\Gamma$ is contained in $SL(2,\mathbf{C})$. Therefore the canonical connection on TG_1^s preserves the leaves M_h (in the sense that D_XY is tangent to M_h whenever both X and Y are), and these leaves are totally geodesic.

Introduce a Riemannian metric on G_1^s by $(X,Y)_a = \mathrm{tr}(a^{-1}Xa^{-1}Y)$ for $X,Y \in T_aG_1^s$. Writing

$$(X,Y)_a = \mathrm{tr}((a^{-1/2}Xa^{-1/2})(a^{-1/2}Ya^{-1/2}))$$

shows immediately that $(X,Y)_a$ is positive definite. The foliations $\{M_h\}$ and $\{N_a\}$ are orthogonal for $(\ ,\ )$.

7.1 PROPOSITION. *The canonical connection in TG_1^s is the Levi-Civita connection of the Riemann metric tr$(a^{-1}Xa^{-1}Y)$ and $GL(2,\mathbf{C})$ acts isometrically on G_1^s.*

Proof: We already observed that the canonical connection is symmetric. Using 3.6 one verifies that, for X,Y,Z tangent fields, it holds that

$$Z(X,Y) = (D_ZX,Y) + (X,D_ZY)$$

and this completes the proof.

The tangent space T_1M_1 to det $= 1$ at $a = 1$ is the space of symmetric 2×2 matrices with trace zero. Using

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

we can write the arbitrary element

$$X = \begin{pmatrix} y & z+ix \\ z-ix & -y \end{pmatrix}$$

in T_1M_1 as

(†) $$X = -i(xI + yJ + zK)$$

(x, y, z are *real*). Further, each $g \in SU(2)$ has the form

$$g = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1$$

and writing $\alpha = s + ui$, $\beta = v + wi$ we can expand g as

$$g = s + uI + vJ + wK \, .$$

The condition $|\alpha|^2 + |\beta|^2 = s^2 + u^2 + v^2 + w^2 = 1$ implies

$$g^{-1} = s - uI - vJ - wK = g^*$$

and therefore

$$g \cdot X = gXg^{-1} \, .$$

This shows that the action of $SU(2)$ on T_1M_1 corresponds to the action by inner automorphism of quaternions g with $|g| = 1$ on the 3-space of purely imaginary quaternions. Then with elements of $SU(2)$ we can obtain any rotation of $\mathbf{R}^3$ identified to T_1M_1 through $X \to (x, y, z)$ as in (†). In particular any plane in T_1M_1 can be mapped onto any other plane.

Observe next that $SU(2)$ operates isometrically on M_1 and leaves 1 fixed. Hence the action of $SU(2)$ leaves sectional curvature $K(X, Y) = (R(X, Y)Y, X)$ invariant. This shows that the sectional curvature in TM_1 is the same for all planes in TM_1. Then operating with $g \in SL(2, \mathbf{C})$ we conclude the M_1 has constant sectional curvature. For any pairs $X, Y \in T_1M_1$, we can calculate

$$4(R(X,Y)Y, X) = \mathrm{tr}((XY)^2) - \mathrm{tr}(X^2Y^2)$$

so that taking $X = \begin{pmatrix} \sqrt{2}/2 & 0 \\ 0 & -\sqrt{2}/2 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & \sqrt{2}/2 \\ \sqrt{2}/2 & 0 \end{pmatrix}$ we can verify that $(X, X) = (Y, Y) = 1$, $(X, Y) = 0$ and therefore the sectional curvature of M_1 is

$$\frac{1}{4}(\mathrm{tr}(XY)^2 - \mathrm{tr}(X^2Y^2)) = -\frac{1}{4} \, .$$

More generally (with the same proof!):

7.2 PROPOSITION. *The submanifolds $M_h \subset G_1^s$ defined for each $h > 0$ by* det $= h$ *have constant sectional curvature* $-1/4\sqrt{h}$.

8. Appendix

There is an alternative way of obtaining the transport function of γ in terms of multiplicative integrals (see **[19]**, **[11]**, **[22]**). Consider a curve $\gamma(t)$, $u \le t \le v$ in G^s. Assuming $\gamma(t)$ continuous we can find a partition $\Pi = \{u = t_0 \le t_1 \le \cdots \le t_n = v\}$ with $\gamma(t_i)$ and $\gamma(t_{i+1})$ close for all i. Next define

$$P_\Pi = \Big(\gamma(t_n)^{-1}\gamma(t_{n-1})\Big)^{1/2} \cdots \Big(\gamma(t_2)^{-1}\gamma(t_1)\Big)^{1/2}\Big(\gamma(t_1)^{-1}\gamma(t_0)\Big)^{1/2}$$

which makes sense because $\gamma(t_{i+1})^{-1}\gamma(t_i)$ is close to 1 for all i. Since

$$\Big(\gamma(t_{i+1})^{-1}\gamma(t_i)\Big)^{1/2} \cdot \gamma(t_i) = \gamma(t_{i+1})$$

(proof of Proposition 1.1) we get $P_\Pi \cdot \gamma(u) = \gamma(v)$. Taking limits on the partition (assume that the curve is smooth) we can define the multiplicative integral

$$P(v,u) = \lim_{\Pi} P_\Pi$$

and then

$$P(v,u) \cdot \gamma(u) = \gamma(v).$$

From the definition of P we see also that for $u \le w \le v$:

$$P(w,v)P(v,u) = P(w,u)$$

or

$$P(w,v) = P(w,u)P(v,u)^{-1} = P(w)P(v)^{-1}$$

where we abbreviate $P(t) = P(t,u)$ with u the left endpoint.

8.1 PROPOSTION *Given a smooth curve $\gamma(t)$, $u \leq t \leq v$ in G^s, the horizontal lifting $\Gamma(t)$ of $\gamma(t)$ with initial condition $\Gamma(u) = 1$ is given by $\Gamma(t) = P(t,u)$.*

Proof: We will see that $P(t,u)$ satisfies the transport equation $\dot{\Gamma} = -(1/2)\gamma^{-1}\dot{\gamma}\Gamma$. For that approximate the curve $\gamma(t)$ by a piecewise linear curve $\tau(t)$ joining $\gamma(t_0), \gamma(t_1), \cdots, \gamma(t_n)$ so that between t_i and t_{i+1} we have $\tau(t) = \gamma(t_i) + s(\gamma(t_{i+1} - \gamma(t_i)$ where $s = (t - t_i)/(t_{i+1} - t_i)$. Abbreviate $a = \gamma(t_i)$, $b = \gamma(t_{i+1})$. Then

$$\begin{aligned} \tau &= a + s(b-a) = a(1 + sa^{-1}(b-a)) \\ \dot{\tau} &= \dot{s}(b-a) \end{aligned}$$

so that letting $c = a^{-1}(b-a)$ we can write

$$\begin{aligned} \tau &= a(1+sc) \\ \tau^{-1}(b-a) &= (1+sc)^{-1}c \end{aligned}$$

and

$$\tau^{-1}\dot{\tau} = \dot{s}(1+sc)^{-1}c.$$

Then the function $T_i(t) = (1+sc)^{-1/2}$ satisfies $T_i^2(t) = (1+sc)^{-1}$ and

$$\dot{T}_i T_i + T_i \dot{T}_i = -(1+sc)^{-1}\dot{s}c(1+sc)^{-1}$$

so

$$\dot{T}_i T^{-1} + T_i \dot{T}_i T_i^{-2} = -(1+sc)^{-1}\dot{s}c = -\tau^{-1}\dot{\tau}.$$

Therefore

$$\dot{T}_i T_i^{-1} = -\frac{1}{2}\tau^{-1}\dot{\tau} - \frac{1}{2}[T_i, \dot{T}_i]T_i^{-2}.$$

Now at $t = t_i$ we have $T_i = 1$ and then $[T_i, \dot{T}_i]T_i^{-2} = 0$ there. Hence if a and b are close then:

$$\dot{T}_i T_i^{-1} = -\frac{1}{2}\tau^{-1}\dot{\tau} - K$$

with K small. Define now for $t_i \leq t \leq t_{i+1}$ the function

$$T_\Pi(t) = T_i(t)T_{i-1}(t_i)T_{i-2}(t_{i-1})\dots T_0(t_1).$$

Taking limits on the partition Π we get the function

$$T_1 = \lim_\Pi T_\Pi$$

and the identities

$$\gamma = \lim_\Pi \tau, \qquad 0 = \lim_\Pi K.$$

Hence T_1 satisfies

$$\dot{T}_1 T_1^{-1} = -\frac{1}{2}\gamma^{-1}\dot{\gamma}.$$

But $T_1 = P$. In fact, let us calculate:

$$\begin{aligned} T_i(t_{i+1}) &= (1+c)^{-1/2} \\ &= (1 + a^{-1}(b-a))^{-1/2} \\ &= (1 + a^{-1}b - 1)^{-1/2} \\ &= (a^{-1}b)^{-1/2} = (b^{-1}a)^{1/2}. \end{aligned}$$

Then

$$\begin{aligned} T_\Pi(t_n) &= T_{n-1}(t_n)T_{n-2}(t_{n-1})\dots \\ &= \Big(\gamma(t_n)^{-1}\gamma(t_{n-1})\Big)^{-1/2}\Big(\gamma(t_{n-1})^{-1}\gamma(t_{n-2})\Big)^{-1/2}\dots \end{aligned}$$

and therefore $T_1 = \lim T_\Pi = P$ as claimed.

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Jacobi Fields on Space of Positive Operators

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ABSTRACT

Let A be a C^*-algebra with 1 and denote by A^+ the set of invertible positive elements of A with its canonical connection and Finsler structure (see [2]). Then a Jacobi field $J(t)$ along a geodesic in A^+ with initial conditions $J(0)=0$ or $DJ/dt|_{t=0}$ has increasing Finsler norm for $t \geq 0$.

Let A be a C^*-algebra with 1, and denote by A^+ the set of positive invertible elements of A. We use G to denote the group of invertible elements of A. Notice that G operates on the left on A^+ by the rule $L_g a = (g^*)^{-1} a g^{-1}$ $(g \in G,\ a \in A^+)$. This action allows us to introduce a natural reductive homogenous space structure in the sense of [6]. For details see [2].

The corresponding connection—which is preserved by the group action —is given by the covariant derivative

$$\frac{DX}{dt} = \frac{dX}{dt} - \tfrac{1}{2}\left(\dot{\gamma}\gamma^{-1}X + X\gamma^{-1}\dot{\gamma}\right),$$

where X is a tangent field on A^+ along the curve γ. The exponential is given by

$$\exp_a X = e^{\frac{1}{2}Xa^{-1}} a e^{\frac{1}{2}a^{-1}X}, \qquad a \in A^+, \quad X \in TA_a^+.$$

For each $a \in A^+$ we identify TA_a^+ with $A^s = \{X \in A : X^* = X\}$. The curvature tensor for this connection is

$$R(X,Y)Z = \tfrac{1}{4}(Z[a^{-1}X, a^{-1}Y] - [Xa^{-1}, Ya^{-1}]Z)$$

for $X, Y, Z \in TA_a^+$.

The manifold A^+ has also a natural Finsler structure given by $\|X\|_a = \|a^{-1/2}Xa^{-1/2}\|$ for $X \in TA_a^+$. The group G operates by isometries for this Finsler metric.

Recall that a *Jacobi field* along a geodesic $\gamma(t)$ is a field $J(t)$ such that

$$\frac{D^2J}{dt^2} + R(J,V)V = 0, \tag{1}$$

where $V(t) = \dot\gamma(t)$. We will say that a Jacobi field is of *type* 1 [respectively, of *type* 2] if

$$\left.\frac{DJ}{dt}\right|_{t=0} = 0$$

[respectively, $J(0) = 0$].

THEOREM. *If J is a Jacobi field of type* 1 *or type* 2, *then $\|J(t)\|_{\gamma(t)}$ is an increasing function of $t \geqslant 0$.*

Proof. Notice first that by the invariance of the connection and the metric under the action of G we may assume that $\gamma(t) = e^{tX}$ is a geodesic starting at $1 \in A$, where $X = X^* \in A$. Then for the field $K(t) = e^{-\frac{1}{2}tX}J(t)e^{-\frac{1}{2}tX}$ the differential equation (1) changes into

$$4\ddot K = KX^2 + X^2K - 2XKX, \tag{2}$$

where the dots indicate the ordinary derivative with respect to t. The field J is of type 1 or 2 according to whether $\dot K(0) = 0$ or $K(0) = 0$. The theorem

then reduces to showing that

$$\|K(s)\| \leqslant \|K(t)\| \qquad \text{for} \quad 0 \leqslant s < t,$$

where the norm is the ordinary norm in the C^*-algebra A.

Consider first the case where the self-adjoint element $X \in A$ has the form

$$X = \sum_{i=1}^{n} \lambda_i p_i \tag{3}$$

with $\lambda_1, \lambda_2, \ldots, \lambda_n$ real numbers and $p_1, p_2, \ldots, p_n$ self-adjoint idempotent elements of A that are pairwise "disjoint," i.e., $p_i p_j = 0$ for $i \neq j$, and with $p_1 + p_2 + \cdots + p_n = 1$.

Let $s > 0$ be given. Represent A faithfully in a Hilbert space $\mathscr{H}$ in such a way that for a unit vector $x \in \mathscr{H}$ we have $\|K(s)\| = |\langle K(s)x, x\rangle|$. Also decompose $x \in \mathscr{H}$ as $x = \sum_{i=1}^n \xi_i x_i$, where x_i is a unit vector in the range of p_i and the ξ_i are scalars. Then $\sum_{i=1}^n \xi_i^2 = \|x\|^2 = 1$. Define also the matrix $k(t) = (k_{ij}(t))$ by $k_{ij}(t) = \langle K(t)x_i, x_j\rangle$ for all t.

Assume now that $t > s$ is fixed. Then

$$\begin{aligned}\|K(s)\| &= |\langle K(s)x, x\rangle| = \left|\sum k_{ij}(s)\xi_i\bar\xi_j\right| \\ &= |\langle k(s)\xi, \xi\rangle| \leqslant \|k(s)\|,\end{aligned}$$

where we write $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$. Suppose that we know that

$$\|k(s)\| \leqslant \|k(t)\|. \tag{4}$$

Choose $\eta = (\eta_1, \eta_2, \ldots, \eta_n) \in \mathbf{C}^n$ of norm 1 such that $\|k(t)\| = |\langle k(t)\eta, \eta\rangle|$. Setting $y = \sum_1^n \eta_i x_i \in \mathscr{H}$, we conclude that $\|y\| = 1$, and from $\langle k(t)\eta, \eta\rangle = \langle K(t)y, y\rangle$ it follows that

$$\begin{aligned}\|K(s)\| &\leqslant \|k(s)\| \leqslant \|k(t)\| = |\langle k(t)\eta, \eta\rangle| = |\langle K(t)y, y\rangle| \\ &\leqslant \|K(t)\|.\end{aligned}$$

Thus the theorem follows for X of the form (3) when we know (4).

To prove (4) we proceed as follows. The differential equation (2) reads

$$\ddot{k}_{ij}(t) = \delta_{ij}^2 k_{ij}(t), \tag{2 bis}$$

where $\delta_{ij} = (\lambda_i - \lambda_j)/2$. If the field J is of type 1, then the solution of (2 bis) has the form $k_{ij}(t) = C_{ij}(t)k_{ij}(0)$, where $C_{ij}(t) = \cosh(\delta_{ij}t)$, so that $k(t) = C(t) \circ k(0)$, where $\circ$ denotes the Schur product of matrices (see [4]). If the field J is of type 2, the solution of (2 bis) has the form $k_{ij}(t) = S_{ij}(t)\dot{k}_{ij}(0)$, where

$$S_{ij}(t) = \begin{cases} \dfrac{1}{\delta_{ij}} \sinh(\delta_{ij}t) & \text{for } \lambda_i \neq \lambda_j, \\ t & \text{for } \lambda_i = \lambda_j, \end{cases}$$

and then $k(t) = S(t) \circ \dot{k}(0)$.

Write $k(t) = \Xi(t,s) \circ k(s)$ with

$$\Xi(t,s)_{ij} = \begin{cases} \dfrac{C_{ij}(t)}{C_{ij}(s)} & \text{for type 1,} \\ \dfrac{S_{ij}(t)}{S_{ij}(s)} & \text{for type 2,} \end{cases}$$

and define the linear transformation $\phi : M_n(\mathbf{C}) \to M_n(\mathbf{C})$ [where $M_n(\mathbf{C})$ is the space of $n \times n$ complex matrices] by $\phi(m) = \Psi(t,s) \circ m$, where $\Psi(t,s)_{ij} = [\Xi(t,s)_{ij}]^{-1}$. It suffices to prove that the norm of ϕ is $\leqslant 1$, because then $\phi(k(t)) = k(s)$ implies (4). Abbreviate

$$f(\lambda) = \frac{\cosh(s\lambda)}{\cosh(t\lambda)}$$

for type 1, and

$$f(\lambda) = \begin{cases} \dfrac{\sinh(s\lambda)}{\sinh(t\lambda)} & \text{for } \lambda \neq 0, \\ \dfrac{s}{t} & \text{for } \lambda = 0 \end{cases}$$

for type 2. Then $\Psi(t,s)_{ij} = f(\frac{1}{2}\lambda_i - \frac{1}{2}\lambda_j)$.

In both cases f is the Fourier transform of a positive function (see [5, p. 31]). Then by Bochner's theorem (see[1]) we can conclude that f is a

positive definite function, or equivalently that $\Psi(t,s)$ *is a positive semidefinite matrix*: $\Psi(t,s) \geqslant 0$.

On the other hand, a theorem of Davis [4, 7] for matrices $U, V \in M_n(\mathbf{C})$ says that

$$\|U \circ V\| \leqslant \max_i \sqrt{d_{ii}} \max_k \sqrt{e_{kk}} \|V\|,$$

where d_{ij} are the entries of the matrix $(U^*U)^{1/2}$ and e_{ij} the entries of the matrix $(UU^*)^{1/2}$. Because the matrix $\Psi(t,s)$ is positive, we conclude that $d_{ii} = e_{ii} = \Psi(t,s)_{ii}$. Therefore $\|\phi\| \leqslant \max \Psi(t,s)_{ii} = 1$ by definition of $\Psi(t,s)$. This proves the inequality (4) (and hence the theorem) for X of the form (3).

In the general case—when X is an arbitrary self-adjoint element of A—the spectral theorem allows us to approximate X (in operator norm) by elements of the form (3). From the well-posedness of the problem (2) we conclude that $(t, X) \to K(t)$ is norm continuous, and the inequality $\|K(s)\| \leqslant \|K(t)\|$ for $0 \leqslant s < t$ for arbitrary X follows from the same inequality for X of the form (3). This concludes the proof of the theorem.

REMARK. For fixed $a \in A^+$ and $X \in TA_a^+$ the derivative of the exponential

$$(T \exp_a)_X : T(TA_a^+)_X \to TA_{\exp_a X}$$

increases norms (see [3]). This can be used to obtain a different proof of the theorem for fields of type 2.

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A GEOMETRIC INTERPRETATION OF SEGAL'S INEQUALITY $\|e^{X+Y}\| \leq \|e^{X/2}e^{Y}e^{X/2}\|$

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ABSTRACT. It is shown that the exponential mapping of the manifold of positive elements of a C^*-algebra (provided with its natural connection) increases distances (when measured in the natural Finsler structure). The proof relies on Segal's inequality $\|e^{X+Y}\| \leq \|e^{X/2}e^{Y}e^{X/2}\|$, valid for all symmetric X, Y in any C^*-algebra. In turn, this geometric inequality implies Segal's' inequality.

Let A be a C^*-algebra with 1 and denote by A^+ the set of positive invertible elements of A. Then A^+ is an open subset of A^s, the real Banach space of symmetric elements in A, and therefore, the tangent space T_aA^+ to the manifold A^+ at $a \in A^+$ can be identified to A^s. The manifold A^+ carries a natural Finsler metric (see [1]) defined by $\|X\|_a = \|a^{-1/2}Xa^{-1/2}\|$ for $X \in TA^+_a$. This norm corresponds to the following interpretation: assume A is faithfully represented in a Hilbert space $(L, \langle\ ,\ \rangle)$, and for each $a \in A^+$, define an inner product in L by $\langle \xi, \eta\rangle_a = \langle a\xi, \eta\rangle$. On the other hand, each $X \in TA^+_a$ determines the bilinear form $B(\xi, \eta) = \langle X\xi, \eta\rangle$ on L. Then the Finsler norm $\|X\|_a$ coincides with the norm of the bilinear form B in the Hilbert space $(L, \langle\ ,\ \rangle_a)$.

The group G of invertible elements of A acts on A^+ by $\mathscr{M}_g a = (g^*)^{-1}ag^{-1}$, $(g \in G,\ a \in A^+)$ making A^+ into a reductive homogeneous space (see [2]) with the natural connection given by

$$D_XY = X(Y) - \tfrac{1}{2}(Xa^{-1}Y + Ya^{-1}X),$$

where $X(Y)$ denotes the derivative of the field Y in the direction X in the Banach space A^s. In this connection, the geodesic γ with $\gamma(0) = a$ and $\dot\gamma(0) = X$ has the form $\gamma(t) = e^{tXa^{-1}/2}ae^{ta^{-1}X/2}$.

Further, for each $g \in G$ and $a \in A^+$, the map g is an isometry from the Hilbert space $(L, \langle\ ,\ \rangle_a)$ onto $(L, \langle\ ,\ \rangle_{\mathscr{M}_g a})$ and consequently the tangent map $(T\mathscr{M}_g)_a\colon TA^+_a \to TA^+_{\mathscr{M}_g a}$ is an isometry for the Finsler metric.

The geometry of A^+ in this setting was studied in [1] where, in particular, the following result is proved [1, Theorem 6.3]: *the distance $d(a, b)$ in the Finsler metric defined by*

$$d(a, b) = \inf\{\text{length}(\gamma);\ \gamma \text{ joins } a \text{ to } b\},$$

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is given by $d(a, b) =$ *length of the unique geodesic in* A^+ *joining* a *to* b, *i.e.,* $d(a, b) = \|X\|_a$ *where* $b = e^{Xa^{-1}/2} a e^{a^{-1}X/2}$.

Notice that the Finsler structure in A^+ does not come from a Riemannian metric. However, A^+ shares with Riemannian manifolds of nonpositive curvature the following metric property, which is the main result of this note.

Theorem 1. *For each* $a \in A^+$, *the exponential map* $\exp_a \colon TA_a^+ \to A^+$ *increases distances in the sense that*

$$(*) \qquad d(\exp_a X, \exp_a Y) \geq \|X - Y\|_a$$

for all X, $Y \in TA_a^+$.

Proof. Since G acts isometrically, it suffices to prove the inequality for $a = 1$. Set $x = \exp_1 X = e^X$, $y = \exp_1 Y = e^Y$. The geodesic that joins x to y in time 1 has the formula

$$\gamma(t) = e^{Zx^{-1}t/2} x e^{x^{-1}Zt/2},$$

where $b = \gamma(1) = e^{Zx^{-1}/2} x e^{x^{-1}Z/2}$. The inequality we are after is

$$\|X - Y\| \leq \|Z\|_x = \|x^{-1/2} Z x^{-1/2}\|$$

or

$$\|\log x - \log y\| \leq \|x^{-1/2} Z x^{-1/2}\|.$$

But

$$\begin{aligned} x^{-1/2} y x^{-1/2} &= x^{-1/2}(e^{Zx^{-1}/2} x e^{x^{-1}Z/2}) x^{-1/2} \\ &= e^{(x^{-1/2}Zx^{-1/2})/2} e^{(x^{-1/2}Zx^{-1/2})/2} = e^{x^{-1/2}Zx^{-1/2}}. \end{aligned}$$

Then $x^{-1/2} Z x^{-1/2} = \log(x^{-1/2} y x^{-1/2})$ so we must prove $\|\log x - \log y\| \leq \|\log(x^{-1/2} y x^{-1/2})\|$ or, changing x into x^{-1},

$$\|\log x + \log y\| \leq \|\log(x^{1/2} y x^{1/2})\|.$$

Replacing x, y by kx, ky with k a large positive number allows us to assume without loss of generality that $\log x \geq 0$ and $\log y \geq 0$. Then, the last inequality is equivalent to

$$\|e^{\log x + \log y}\| \leq \|x^{1/2} y x^{1/2}\|.$$

But this is equivalent to Segal's inequality (see [3, Theorem X.57, p. 260, vol. II], or [4])

$$(**) \qquad \|e^{X+Y}\| \leq \|e^{X/2} e^Y e^{X/2}\|$$

and this concludes the proof of Theorem 1. Obviously all steps in the proof can be reversed, so that $(**)$ implies $(*)$.

As an application of Theorem 1, consider a C^*-algebra A with a distinguished family $p_1, p_2, \ldots, p_n$ of selfadjoint orthogonal projections satisfying $p_i p_j = 0$ if $i \neq j$ and $p_1 + p_2 + \cdots + p_n = 1$. Let $B \subset A$ be the C^*-subalgebra of elements of A that commute with all p_i and $H \subset A$ be the Banach subspace of elements $h \in A$ satisfying $p_i h p_i = 0$ for each i. Let also $E = \{e^h \colon h = h^* \in H\}$.

Theorem 2. *For each $b > 0$ in B, the distance (in the Finsler metric) from b to the submanifold $E \subset A^+$ is attained at $1 \in E$.*

Proof. Set $X = \log b$. By definition $X = X_1 + \cdots + X_n$, where $X_i = p_i X p_i$. Since $\|X\| = \max \|X_i\|$, we can assume that $\|X\| = \|X_1\|$, and accordingly, we choose a faithful representation of A in a Hilbert space L with the additional property that, setting $L = L_1 \oplus \cdots \oplus L_n$ with $L_i = p_i(L)$, the subspace L_1 contains a "norming eigenvector" for X_1, i.e., a unit vector $\xi \in L_1$ with $X_1 \xi = \pm \|X_1\| \xi$. Let Y be an arbitrary selfadjoint element of H. Then, by the definition of H, $Y\xi \in L_2 \oplus \cdots \oplus L_n$ and therefore $X\xi = X_1 \xi$ is perpendicular to $Y\xi$. As a consequence we have

$$d(b, 1) = \|X\| = \|X\xi\| \leq \|X\xi - Y\xi\| \leq \|X - Y\|.$$

Then using Theorem 1, we conclude that $d(b, 1) \leq d(b, e^Y)$ and we are done.

Remark. Notice that the tangent map to $\exp$ also increases norms. In fact it suffices to show this property for $a = 1$. For that we estimate

$$\left\| \frac{e^{Y+tZ} - e^Y}{t} \right\|_{e^Y} = \frac{1}{|t|} \|e^{-Y/2} e^{Y+tZ} e^{-Y/2} - I\|$$

using Segal's inequality $\|e^{-Y/2} e^{Y+tZ} e^{-Y/2}\| \geq \|e^{tZ}\|$. Assume that $t > 0$ and that $\max \operatorname{Spec}(Z) = \|Z\|$. Then $\|e^{tZ}\| = e^{t\|Z\|} \geq 1$. Hence in this case

$$\frac{1}{|t|} \|e^{-Y/2} e^{Y+tZ} e^{-Y/2} - I\| = \frac{1}{t} (\|e^{tZ}\| - 1) \geq \frac{1}{t} (e^{t\|Z\|} - 1) \geq \|Z\|.$$

Then

$$\lim_{t \to 0+} \left\| \frac{e^{Y+tZ} - e^Y}{t} \right\|_{e^Y} \geq \|Z\|.$$

For other Z's, change Z into $-Z$.

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An Operator Inequality*

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ABSTRACT

For Hilbert space operators, with S invertible hermitian, it is proved that $\|STS^{-1}+S^{-1}TS\| \geqslant 2\|T\|$.

For a Hilbert space H consider the set G^s of bounded linear hermitian invertible operators in H and the subset $P \subset G^s$ of unitary reflections (i.e., operators with $R^* = R = R^{-1}$). If we write $A \in G^s$ as $A = NR$ with N positive and R unitary (the polar decomposition of A), then $R \in P$, and $A \to R$ defines a map $\pi : G^s \to P$. The sets G^s and P are smooth submanifolds of the C^*-algebra of bounded linear operators in H, and $\pi : G^s \to P$ is a smooth fibration. Furthermore, we introduce on G^s a natural Finsler

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structure by assigning to a tangent vector $X \in T_A G^s$ the norm $\|X\|_A = \|N^{1/2} X N^{1/2}\|$ (operator norm). In [2] we prove that the tangent map $T_A \pi : T_A G^s \to T_{\pi(A)} P$ decreases norms, together with some geometric consequences similar to those shown in [3]. The essential step in obtaining this result is the following operator inequality, whose proof is the objective of this note:

THEOREM. *Let S, T be bounded linear operators in Hilbert space, with S invertible hermitian or invertible skew-symmetric. Then*

$$\|STS^{-1} + S^{-1}TS\| \geqslant 2\|T\|. \qquad (*)$$

Proof. Changing S into iS allows us to consider only the case of S symmetric: $S^* = S$. Decompose S using its spectral measure $S = \int \lambda \, dE_\lambda$. Then $Q = \int |\lambda| \, dE_\lambda$ and $R = \int (\lambda / |\lambda|) \, dE_\lambda$ give the polar decomposition $S = QR$ of S, with $Q > 0$ and R unitary and symmetric (so that $R = R^* = R^{-1}$). Hence $S = QR = RQ$, $S^{-1} = Q^{-1}R = RQ^{-1}$, and therefore

$$\|STS^{-1} + S^{-1}TS\| = \|RQTQ^{-1}R + RQ^{-1}TQR\|$$

$$= \|QTQ^{-1} + Q^{-1}TQ\|.$$

Replacing S by Q, we may assume $S > 0$.

Next consider

$$S_1 = \int h(\lambda) \, dE_\lambda = h(S),$$

where $h(\lambda)$ is a function of the form $h(\lambda) = k/n$ for $k/n \leqslant \lambda < (k+1)/n$ and $k = 0, 1, 2, \ldots$. Then the spectrum of S_1 is finite, and $\|S - S_1\|$ and $\|S^{-1} - S_1^{-1}\|$ are small for n large. This is clear because if spectrum$(S) \subset [m, M]$, then $|h(\lambda) - \lambda| \leqslant 1/n$ and $|1/h(\lambda) - 1/\lambda| \leqslant 1/m(mn-1)$ for all $\lambda \in$ spectrum(S) and all $n > 1/m$.

Hence it suffices to prove $(*)$ under the additional hypothesis that the spectrum of S is a finite set of positive real number $\lambda_1, \lambda_2, \ldots, \lambda_k$. In this case there exists a family of orthogonal projections $\{P_F\}$ with the properties

(i) $P_F S = S P_F$,
(ii) rank$(P_F) < \infty$,
(iii) $P_F \to 1$ strongly, i.e., $\|P_F x - x\| \to 0$ for all x in the Hilbert space when F grows.

The existence of such a family $\{P_F\}$ follows simply by taking an orthonormal basis in each eigenspace $\{x;\ Sx = \lambda_j x\}$ of S, and then using as P_F the orthogonal projection on the subspace generated by a finite set F of elements of the union of these bases. For any operator A we have

$$\sup |\langle P_F A P_F x, y \rangle| = |\langle Ax, y \rangle|$$

where the sup is taken over all F for each x, y with $|x| \leqslant 1$, $|y| \leqslant 1$; hence

$$\sup_F \|P_F A P_F\| = \|A\|.$$

Since S, S^{-1} leave invariant the range of each P_F, this identity allows us to consider the special case where the Hilbert space is $\mathbf{C}^n$, and the operators are $n \times n$ matrices. Define a linear map on matrices

$$\Phi : M_n(\mathbf{C}) \to M_n(\mathbf{C})$$

by $\Phi(T) = STS^{-1} + S^{-1}TS$, where S stands now for a positive definite matrix.

We may also assume that S is diagonal by a unitary change of basis in $\mathbf{C}^n$. Denote by $s_1, s_2, \ldots, s_n$ the diagonal elements of S, so that $s_j > 0$ for all $j = 1, 2, \ldots, n$. Then $\Phi(T) = Z \cdot T$, where $Z \cdot T$ denotes the Schur product $(Z \cdot T)_{ij} = Z_{ij} T_{ij}$ of T and the fixed matrix Z with entries

$$Z_{ij} = \frac{s_i}{s_j} + \frac{s_j}{s_i}.$$

The map $\Phi : M_n(\mathbf{C}) \to M_n(\mathbf{C})$ is invertible with inverse $\Phi^{-1}(T) = W \cdot T$, where

$$W_{ij} = \frac{1}{Z_{ij}} = \frac{s_i s_j}{s_i^2 + s_j^2}.$$

Notice that $W_{ii} = \frac{1}{2}$. We claim that W is a positive definite matrix. In fact we can write $W = SMS$, where M is the matrix with $M_{ij} = 1/(s_i^2 + s_j^2)$. But the identity

$$\det\left(\frac{1}{x_i + x_j}\right) = \frac{\prod_{i<j}(x_i - x_j)^2}{\prod_{i,j}(x_i + x_j)},$$

valid for $x_i > 0$, $i = 1, 2, \ldots$ (see [1, Solution I] or [5, Absch. 7, Aufg. 3]), shows that M is positive definite, whence $W = SMS$ is also positive definite.

Recall the inequality (due to Davis [4] and Walter [6])

$$\|B \cdot C\| \leqslant \max_i \sqrt{d_{ii}} \max_j \sqrt{e_{jj}} \|C\|$$

where $\| \ \|$ denotes operator norm, B and C are $n \times n$ matrices, and d_{ij} $[e_{ij}]$ are the entries of the positive square root $(B^*B)^{1/2}$ $[(BB^*)^{1/2}]$.

In our case, we take $B = B^* = W$ and $C = T$. As proved above, $W = (B^*B)^{1/2} = (BB^*)^{1/2}$ and so $d_{ii} = e_{jj} = \frac{1}{2}$. Thus

$$\|W \cdot T\| \leqslant \tfrac{1}{2}\|T\|,$$

or $\|\Phi^{-1}(T)\| \leqslant \frac{1}{2}\|T\|$, which can be written as $2\|T\| \leqslant \|\Phi(T)\|$, an equivalent form of $(*)$.

Remark 1. If $T^* = T$ or $T^* = -T$, then the inequality $(*)$ implies $\|STS^{-1}\| \geqslant \|T\|$.

Remark 2. The companion operator

$$\Psi(T) = STS^{-1} - S^{-1}TS$$

has a geometric meaning in the context of the fibration $G^s \to P$ mentioned above. However, the norm of $\Psi(T)$ is in general unrelated to the norm of

$\Phi(T)$. For example, when $TS = ST$ we have $\Psi(T) = 0$, $\Phi(T) = 2T$. For 2×2 matrices $\|\Psi(T)\| \geqslant \|\Phi(T)\|$, where T is hermitian. Finally, if

$$S = \begin{pmatrix} k^2 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} \frac{1}{4} & \frac{-k}{k^2+1} & \frac{-k^2}{k^4+1} \\ \frac{-k}{k^2+1} & \frac{1}{4} & \frac{-k}{k^2+1} \\ \frac{-k^2}{k^4+1} & \frac{-k}{k^2+1} & \frac{1}{4} \end{pmatrix},$$

then

$$\Psi(T) = \begin{pmatrix} 0 & -\frac{k^2-1}{k^2+1} & -\frac{k^4-1}{k^4+1} \\ \frac{k^2-1}{k^2+1} & 0 & -\frac{k^2-1}{k^2+1} \\ \frac{k^4-1}{k^4+1} & \frac{k^2-1}{k^2+1} & 0 \end{pmatrix}, \quad \Phi(T) = \begin{pmatrix} \frac{1}{2} & -1 & -1 \\ -1 & \frac{1}{2} & -1 \\ -1 & -1 & \frac{1}{2} \end{pmatrix},$$

and therefore

$$\|\Psi(T)\| = \left[2\left(\frac{k^2-1}{k^2+1}\right)^2 + \left(\frac{k^4-1}{k^4+1}\right)^2\right]^{1/2}$$

and

$$\|\Phi(T)\| = \tfrac{3}{2}.$$

Taking k large, we get $\|\Psi(T)\|$ near

$$\lim_{k\to\infty} \|\Psi(T)\| = \sqrt{3},$$

and therefore $\|\Psi(T)\| > \|\Phi(T)\|$.

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The Shuffle Algorithm and Jordan Blocks

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ABSTRACT

A shuffle is the horizontal interchange of a pair of blocks of the same size in a matrix. A general algorithm using row reduction and shuffles was first introduced by Luenberger, and then used by Anstreicher and Rothblum to give an algorithm to compute generalized nullspaces. We present a new, concise proof of this shuffle algorithm, and show how the shuffle algorithm can be used in deriving the Jordan blocks for a square matrix with known eigenvalues.

INTRODUCTION

A general shuffle algorithm was first introduced by D. G. Luenberger [3], and ingeniously used by K. M. Anstreicher and U. G. Rothblum [1] to give an algorithm to compute the generalized nullspace of a square matrix and the Drazin inverse. A new, concise proof of the main theorem of [1] is presented here. The shuffle algorithm is then used to find bases for the generalized eigenspaces of a square matrix T whose eigenvalues are known. Finally we describe a simple algorithm to compute an invertible matrix P such that PTP^{-1} is in Jordan form.

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DIFFERENTIAL GEOMETRY OF SPACES OF RELATIVELY REGULAR OPERATORS

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Given an idempotent r of a Banach algebra A we study the space

$$\mathcal{S} = \mathcal{S}_r = \{(a,b) \in A \times A : ar = a,\ rb = b,\ ba = r\},$$

and in particular the fiber bundle induced by the action of the group of units G of A, and the associated bundle $\theta : \mathcal{S} \to Q = \{q \in A : q^2 = q\}$ defined by $\theta(a,b) = ab$. When A is a C^*-algebra and $r \in Q$ is symmetric, we also study a real-analytic retraction of $\mathcal{S}$ onto $\mathcal{R} = \{(a,b) \in \mathcal{S} : b = a^*\}$ related to the polar decomposition of a reflection.

INTRODUCTION

Let F be a Banach space and Q the space of projections of F, i.e. Q consists of all linear bounded operators $q \in L(F)$ such that $q^2 = q$. As in the finite dimensional case, one can consider the canonical vector bundle $\xi = \{(q,v) \in Q \times F : v \in \operatorname{im} q\}$ over Q (here we use im q for the range of q). On the other hand if E is a Banach space which is isomorphic to some im q, $q \in Q$, one defines

$$\mathcal{S} = \mathcal{S}(E,F) = \{(i,j) \in L(E,F) \times L(F,E) : ji = 1_E\}\ .$$

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There is a natural map $\theta : \mathcal{S} \to Q$ given by $\theta(i,j) = ij$, whose image Q' is open and closed in Q. The map $\theta : \mathcal{S} \to Q'$ is a principal fiber bundle with group $H = GL(E)$.

Observe that $\xi|Q' \to Q'$ is the vector bundle $\xi|Q' = \mathcal{S} \times_H E$, associated to θ and the natural representation of the group H on E.

This paper deals with the topological and geometric structure of $\mathcal{S}$. Each $(i,j) \in \mathcal{S}$ defines a decomposition of F into two direct summands, one of which is isomorphic to E. This justifies the name of *space of decompositions* we have chosen for $\mathcal{S}$.

It is readily seen that $\mathcal{S}$ is a closed analytic submanifold of $L(E,F) \times L(F,E)$. The projections define locally trivial fiber bundles with affine fibers $pr_1 : \mathcal{S} \to I$, $pr_2 : \mathcal{S} \to J$; I (resp. J) consists of all direct monomorphisms (resp. epimorphisms) from E into F (resp. from F onto E). As a consequence, $\mathcal{S}$, I and J are homotopy equivalent. The group $G = GL(F)$ of all invertible operators on F operates on $\mathcal{S}$ by $g.(i,j) = (gi, jg^{-1})$; for each $(i,j) \in \mathcal{S}$ the map $\tau = \tau_{(i,j)} : g \mapsto g.(i,j)$ defines a principal fiber bundle over the orbit $\mathcal{S}'$ of (i,j) in $\mathcal{S}$. In particular, curves in $\mathcal{S}'$ lift to G. But for C^1 curves an explicit lift can be found as the solution of a linear differential equation, as shown below. This way of lifting curves is related to the geometry of the bundle $\tau : G \to \mathcal{S}'$ in a very precise sense: a natural connection is defined on τ and the horizontal lifts of C^1 curves in $\mathcal{S}'$ are, precisely, the solutions mentioned above. Several geometric invariants of the connection are calculated. Also a natural connection is defined on the principal bundle $\theta : \mathcal{S}(E,F) \to Q'$. In this case, the horizontal lift of a C^1 curve γ in Q' with origin $\theta(i,j)$ is $(\Gamma(t)i, j\Gamma(t)^{-1})$, where Γ is the solution of the initial value problem $\dot{\Gamma} = (\dot{\gamma}\gamma - \gamma\dot{\gamma})\Gamma$, $\Gamma(0) = 1$.

If E and F are Hilbert spaces, it is natural to consider the "symmetric part" of $\mathcal{S}$, i.e. the space $\mathcal{R} = \{(i,j) \in \mathcal{S} : i^* = j\}$. The unitary group $\mathcal{U}$ of F acts on $\mathcal{R}$ and defines, for each $(i,i^*) \in \mathcal{R}$, a principal fiber bundle $\mathcal{U} \to \mathcal{R}'$ (= orbit of (i,i^*) in $\mathcal{R}$) with a connection induced by that of $\mathcal{S}'$.

Observe that $\theta(i,i^*) \in \mathcal{P} = \{p \in L(F) : p^2 = p,\ p^* = p\}$. In [5] it is shown that the polar decomposition, which can be seen as a map $G \to \mathcal{U}$, induces a map $\pi : Q \to \mathcal{P}$; we call $p = \pi(q)$ the *polar decomposition* of q.

We conclude the paper by introducing a "polar decomposition" of pairs $(i,j) \in \mathcal{S}$, which seems to play a relevant role in the study of the geometry of $\mathcal{S}$.

The results described above are developed in the following, more general, context: given a Banach algebra A and $r \in Q = \{q \in A : q^2 = q\}$, we consider $\mathcal{S} = \mathcal{S}_r = \{(a,b) \in A \times A : ar = a,\ rb = b,\ ba = r\}$. We show that $\mathcal{S}(E,F)$ can be identified with a convenient $\mathcal{S}_r$ (see the end of §1). All the results we get for $\mathcal{S}$ can be translated to the spatial case, $\mathcal{S}(E,F)$, with no extra effort. One advantage of this presentation is that it has functorial character. Of course, in the last part of the paper, which is concerned with Hilbert space operators, we deal with C^*-algebras instead of general Banach algebras.

This paper is part of a series ([5], [6], [7], [18], [19]) devoted to several aspects of the geometry of spaces of projections of a Banach algebra A. The space $\mathcal{S}(E,F)$, when $E = A$ and $F = A^n$, has been considered in [4]. Prof. B.E. Johnson suggested some generalizations which motivated our interest in the space $\mathcal{S}$. In another context, Taylor [23] has also studied the space $\mathcal{S}$ (see, in particular, section 3.6 of his paper). Related aspects can be found in papers by Douady [10] and Koschorke [13]. Finally, we mention the connections of our work with that of Gramsch concerning relative regular elements of topological algebras. Recall that $a \in A$ is *relative regular* or *pseudo inversible* if there exists $b \in A$ such that $aba = a$ and $bab = b$ (see [1], [16], [3]). In this case ab and ba are idempotent and it is clear that the union of the spaces $\mathcal{S}_r$, when r runs through Q, coincides with the set $\mathcal{W}$ of such pairs (a,b). The differential structure of $\mathcal{W}$ has been considered by Gramsch [11] whose work has some connections with ours.

The minimality of geodesics in the fibers of $\pi : Q \to P$ in a C^*-algebra

A has been studied in [5] and [19]. We study this problem for the case of $\pi : \mathcal{S} \to \mathcal{R}$ in a forthcoming paper.

We thank Prof. B.E. Johnson for several valuable comments.

§1. ANALYTIC STRUCTURE OF $\mathcal{S}$

Let A be a Banach algebra with identity 1, G its group of units and Q the set of idempotent elements of A. Throughout r is a fixed element of Q and $\mathcal{S}$ is defined by

$$\mathcal{S} = \mathcal{S}_r = \{(a,b) \in A \times A \,:\, ar = a,\ rb = b,\ ba = r\}.$$

1.1. Theorem *$\mathcal{S}$ is a closed analytic submanifold of $A \times A$. The tangent space of $\mathcal{S}$ at $(a,b) \in \mathcal{S}$ is (isomorphic to)* $\{(X,Y) \in A \times A : Xr = X,\ rY = Y,\ bX + Ya = 0\}$.

Proof. Observe, first, that $\mathcal{S}$ is contained in the direct (=complemented) subspace M of $A \times A$ consisting of all pairs (x,y) such that $xr = x$, $ry = y$ (a complement of M is $\{(x,y) : xr = 0,\ ry = 0\}$). Then, it suffices to show that $\mathcal{S}$ is a closed analytic submanifold of M. Observe, next, that $\mathcal{S} = \phi^{-1}(r)$, where $\phi : M \to rAr$ is defined by $\phi(x,y) = yx = (ry)(xr)$; then, by the implicit function theorem, it suffices to show that $T = T_{(a,b)}\phi$ is right invertible. For this, an easy computation shows that $T(X,Y) = Ya + bX$ and $W : rAr \to M$ defined by $W(Z) = (1/2)(aZ, Zb)$ verify $TW = id$.

1.2. Proposition *For $g \in G$ the map $(a,b) \mapsto (ag^{-1}, gb)$ defines a diffeomorphism from $\mathcal{S} = \mathcal{S}_r$ onto $\mathcal{S}_{grg^{-1}}$.*

Thus, $\mathcal{S}_r$ depends, as a Banach manifold, not on r but on its conjugation class. We shall see later (5.8) that, as a principal fiber bundle over Q, $\mathcal{S}_r$ depends on a certain equivalence class of r, which is smaller than its conjugation class.

1.3. Remark Recall that every idempotent $q \in Q$ induces matrix representations for all elements of A,

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where $a_{11} = qaq,\ a_{12} = qa(1-q),\ a_{21} = (1-q)aq,\ a_{22} = (1-q)a(1-q)$. Product in A corresponds ot matrix multiplication in this representation. In this matrix decomposition induced by r, an element $(a,b) \in \mathcal{S}$ can be written as

$$\left(\begin{pmatrix} a_1 & 0 \\ a_2 & 0 \end{pmatrix}, \begin{pmatrix} b_1 & b_2 \\ 0 & 0 \end{pmatrix} \right)$$

with $b_1 a_1 + b_2 a_2 = 1$.

We end this section by observing that the sets $\mathcal{S}(E,F)$ described in the Introduction are special cases of $\mathcal{S} = \mathcal{S}_r$. In fact, let $(i_0, j_0) \in \mathcal{S}(E,F)$ be fixed and let $A = L(F),\ r = i_0 j_0 \in Q = Q(L(F))$. Then $\Upsilon : \mathcal{S}(E,F) \to \mathcal{S}_r$ defined by $\Upsilon(i,j) = (ij_0, i_0 j)$ is a diffeomorphism with inverse $\Upsilon^{-1}(a,b) = (ai_0, j_0 b)$.

§2. THE FIBRATION $G \to \mathcal{S}$

Let $\tau : G \to \mathcal{S}$ be the map defined by $\tau(g) = g.(a,b) = (ga_0, b_0 g^{-1})$, where (a_0, b_0) is a fixed element of $\mathcal{S}$. There is a neighborhood U of (a_0, b_0) in $\mathcal{S}$ such that $\sigma_0(a,b) = ab_0 + (1-ab)(1-a_0 b_0)$ is invertible for all (a,b) in U. In fact, $\sigma_0(a,b) \cdot (a_0 b + (1-a_0 b_0)(1-ab)) = 1 - (1-q)q_0(1-q)$ (where $q_0 = a_0 b_0,\ q = ab$), which is invertible for q_0 close to q. The map $\sigma_0 : U \to G$ is a cross section of τ, i.e. $\tau(\sigma_0(a,b)) = (a,b)$ for all $(a,b) \in U$. More generally, if $(a_1, b_1) = \tau(g)$ and $U_1 = g.U$ then $\sigma_1 : U_1 \to G$ defined by $\sigma_1(a,b) = g\sigma_0(g^{-1}a, bg) = g\sigma_0(g^{-1}.(a,b))$, is a cross section of τ over U_1. These remarks show that the image $\mathcal{S}'$ of τ is open in $\mathcal{S}$ and that $\tau : G \to \mathcal{S}'$ admits local cross sections. Moreover, if $K = \{g \in G : ga_0 = a_0,\ b_0 g =$

$b_0\}$ (K is the *isotropy group* of (a_0, b_0)), then $\psi : U_1 \times K \to \tau^{-1}(U_1)$, defined by $\psi((a,b),h) = \sigma_1(a,b)h$, is a diffeomorphism, with $\psi^{-1}(g) = (\tau(g), \sigma_1(\tau(g))^{-1}g)$. This proves the following result:

2.1. Theorem *The map $\pi : G \to S'$ is a principal fiber bundle with structural group $K = \{g : ga_0 = a_0,\ b_0 g = b_0\}$.*

2.2. Corollary *τ induces the exact homotopy sequence*

$$\cdots \to \pi_{i+1}(\mathcal{S}', (a_0, b_0)) \to \pi_i(K, 1) \to \pi_i(G, 1) \to \pi_i(\mathcal{S}', (a_0, b_0)) \to \dots$$

In particular, if G is contractible then $\pi_{i+1}(\mathcal{S}', (a_0, b_0))$ is isomorphic to $\pi_i(K)$.

2.3. Remarks i) In terms of the idempotent $q_0 = a_0 b_0$, K can be expressed as $\{g \in G : gq_0 = q_0 g = q_0\}$; thus, in the matrix representation determined by q_0, K consists of all invertible matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}.$$

ii) Consider the commutative diagram

$$\begin{tikzcd} & G \arrow[dl, "\tau"'] \arrow[dr, "\rho"] & \\ S \arrow[rr, "\theta"'] & & Q \end{tikzcd}$$

where $\rho(g) = gp_0 g^{-1}$ and $\theta(a, b) = ab$. Then the isotropy group of q_0 in G is $H' = \{g \in G : gq_0 = q_0 g\}$, i.e. H' consists of all invertible matrices (with respect to q_0) of the form $\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$; thus K is a normal subgroup of H' and therefore $\mathcal{S} \to Q$ is a principal fiber bundle, as will be shown in section

5 (cf. [22]).

iii) If $(a_1, b_1) = \tau(g)$ then the tangent map of τ at g, $T_g\tau : T_gG \to T_{(a_1,b_1)}\mathcal{S}' = T_{(a_1,b_1)}\mathcal{S}$ is given by $(T_g\tau)(X) = (Xa_0, -b_0g^{-1}Xg^{-1})$, $X \in T_gG \simeq A$. (As a rule we shall write tangent vectors with capital letters). In particular, for $g = 1$ we get $(T_1\tau)(X) = (Xa_0, -b_0X)$ so that the kernel of $T_1\tau$ consists of all vectors X in A such that $Xq_0 = q_0X = 0$; in other words, $\ker(T_1\tau) = \{X \in A : X = (1-q_0)X(1-q_0)\}$. In q_0–matrix form,

$$\ker(T_1\tau) = \left\{\begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}\right\},$$

which is clear because it consists of tangent vectors at

$$H_0 = \left\{\begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}\right\}.$$

iv) If we let (a_0, b_0) vary in $\mathcal{S}$ and consider the corresponding maps $\tau = \tau_{(a_0,b_0)}$, the results above show that $\mathcal{S}$ is a discrete union of homogeneous spaces $\mathcal{S}' \subset \mathcal{S}$ of G (here *discrete* means that each $\mathcal{S}'$ is open and closed in $\mathcal{S}$). This type of Banach manifolds has been considered by Raeburn [20]. By combining his results and ours it is rather easy to obtain a homotopy equivalence between $\mathcal{S}(B\widehat{\otimes}A)$ and the space of continuous maps $C(X(B), \mathcal{S}(A))$, where B is a complex commutative Banach algebra with identity and $X(B)$ is the space of maximal ideals of B [20].

v) Even if we shall not consider the functorial character of $\mathcal{S}$, it is worth mentioning that an epimorphism $f : A \to C$ of Banach algebras induces a Serre fibration $\mathcal{S}(A) \to \mathcal{S}(C)$. The proof of this fact follows the argument used in [6].

2.4. Example Let F be a Hilbert space, E a closed subspace and $p \in A = L(F)$ the orthogonal projection on E. Let $\mathcal{S} = \mathcal{S}_p$, so that the isotropy group of $(p,p) \in \mathcal{S}$ is

$$H_0 = \{g \in G = GL(F) : g\,|_E = 1_E,\ g(E^{\perp}) = E^{\perp}\}\ .$$

Thus, H_0 is isomorphic to $GL(E^{\perp})$. When the codimension of E in F is ∞, Kuiper's theorem [14] shows that G and H are contractible, so that, by 2.2, $\pi_i(\mathcal{S}) = 0$ for all $i \geq 1$.

We turn now to the study of the projections

$$pr_1 : (a,b) \mapsto a,\ pr_2(a,b) \mapsto b.$$

Denote by I the image of $\mathcal{S}$ by pr_1 and J the image of $\mathcal{S}$ by pr_2. We study pr_1, the case of the other projection being similar. Let $(a_0, b_0) \in \mathcal{S}$ and $\tau : G \to \mathcal{S}$ as before. Consider a local cross section of $pr_1 \circ \tau : G \to I$ (for instance, $s(a) = 1 + ab_0 - a_0b_0$ for $a \in I$ close to a_0, or $s_g(a) = g(1 + g^{-1}ab_0 - a_0b_0)$ for a close to ga_0). Let N be the left annihilator of $a_0 : N = \{X \in A : Xa_0 = 0\}$. Then $\psi : pr_1^{-1}(V) \to V \times N$ defined by $\psi(a,b) = (a, b\ s_g(a) - b_0)$, where V is the domain of the section s_g, is a diffeomorphism and $\psi^{-1}(a,X) = (a_1(X + b_0)s_g(a)^{-1})$. Thus $pr_1 : \mathcal{S} \to I$ is a locally trivial fiber bundle with affine (a fortiori contractible) fiber. Analogous results hold for $pr_2 : \mathcal{S} \to J$. In particular $\mathcal{S}$, I, J have the same homotopy type (in the case $\mathcal{S}(E,F)$ where E is a Banach algebra and $F = E^n$, this result has been obtained in [4]).

In certain cases, the surjectivity of the map $\tau : G \to \mathcal{S}$ is related to the so called stability problems in K-theory. For instance when B is a Banach algebra, $A = M_n(B)$ and r is the idempotent matrix with 1 in the (1,1) entry and 0 elsewhere, $\mathcal{S}$ can be identified with $\{((x_1, \ldots, x_n), (y_1, \ldots, y_n)) \in B^n \times B^n : \sum_{k=1}^{n} y_k x_k = 1\}$; K-algebraists look for sufficient conditions for the map $G \to pr_1\mathcal{S} = \{(x_1, \ldots, x_n)\}$, which assigns to every invertible matrix $\sigma \in G$ its first column, to be onto (see [15] for an excellent description of these problems). The results above show that those conditions depend only on the homotopy type of G and $\mathcal{S}$ (see [4]).

§3. THE TRANSPORT EQUATION

We use the notations of §2. From the fibration properties of $\tau : G \to \mathcal{S}'$ it follows that for every continuous curve $\gamma : [\alpha, \beta] \to \mathcal{S}'$ and every $g \in G$ such that $\tau(g) = \gamma(\alpha)$ there exists a continuous curve $\Gamma : [\alpha, \beta] \to G$ such that $\tau\Gamma = \gamma$ and $\Gamma(\alpha) = g$. We shall prove that, for C^1 curves γ, a lifting Γ can be obtained by solving a linear differential equation. Indeed, a more involved procedure yields a rectifiable continuous lifting Γ under the assumption that γ is rectifiable and continuous. See [18] for such a construction for the case of curves in Q, where multiplicative integrals are used (see Volterra [24], Schlesinger [21], Potapov [17] and Daleckii [8]).

Consider a C^1 curve $\gamma : [\alpha, \beta] \to \mathcal{S}'$ with $\gamma(\alpha) = (a_0, b_0)$ and take a partition $\Pi : \alpha = t_0 < t_1 < \cdots < t_n = t$ of $[\alpha, t]$ $(t \in [\alpha, \beta])$ such that, for each $k = 0, 1, \ldots, n-1$, $g_{k+1} = a(t_{k+1})b(t_k) + (1 - q(t_k)) \in G$, where $\gamma(t) = (a(t), b(t))$, $q(t) = a(t)b(t)$. Observe that $g_{k+1} \cdot (a(t_k), b(t_k)) = (a(t_{k+1}), b(t_{k+1}))$, so that $\tau(g_{k+1}g_k \cdots g_1) = \gamma(t_{k+1})$. Set $g_\Pi = g_n \cdots g_1$. It can be shown, under the weaker assumption that γ is rectifiable and continuous, that the limit $\Gamma(t) = \lim_{\|\Pi\| \to 0} g_\Pi$ exists for all $t \in [\alpha, \beta]$ and Γ is a rectifiable continuous curve $[\alpha, \beta] \to G$ such that $\tau\Gamma = \gamma$. We shall assume this fact without proof and derive a differential equation such that its unique solution Γ with initial value $g \in G$ with $\gamma(\alpha) = \tau(g)$, satisfies $\tau\Gamma = \gamma$.

First, observe that, for any real number h small enough, $\Gamma(t+h) - \Gamma(t) = g_{t+h}\Gamma(t) - \Gamma(t) + o(h)$, where $g_{t+h} = a(t+h)b(t) + (1 - q(t+h))(1 - q(t))$

and $o(h)/|h| \to 0$ when $h \to 0$. Then

$$\begin{aligned} g_{t+h} - 1 &= a(t+h)b(t) - q(t+h) - q(t) + q(t+h)q(t) \\ &= a(t+h)(b(t) - b(t+h)) + (q(t+h) - q(t))q(t) \end{aligned}$$

and $$\begin{aligned} \dot{\Gamma}(t) &= \lim_{h\to 0} \frac{\Gamma(t+h) - \Gamma(t)}{h} = \lim_{h \to 0} \frac{g(t+h) - 1}{h} \Gamma(t) \\ &= (-a(t)\dot{b}(t) + \dot{q}(t)q(t)\Gamma(t)\,. \end{aligned}$$

3.1. Theorem *Let $\gamma = (a, b) : [\alpha, \beta] \to S'$ be a C^1 curve. The solution of the initial value problem*

(3.1.1) $$\dot{\Gamma} = (-a\dot{b} + \dot{q}q)\Gamma, \ \Gamma(\alpha) = 1$$

is a C^1 curve $\Gamma : [\alpha, \beta] \to G$ such that

$$\tau(\Gamma(t)) = \gamma(t) \quad \textit{for all} \ \ t \in [\alpha, \beta].$$

Proof The existence, uniqueness and invertibility of the solution of 3.1.1 follow from the general theory (see e.g. [3], [9]). It suffices to show that $\tau\Gamma = \gamma$, i.e. $\Gamma(t)a_0 = a(t)$ and $b_0\Gamma(t)^{-1} = b(t)$ for all t. Since the curves $\Gamma^{-1}a$, $b\Gamma$ begin at a_0, b_0, respectively, it suffices to show that they are constant.

First, write the differential equation as

$$\dot{\Gamma} = (-a\dot{b} + \dot{a}b + a\dot{b}ab)\Gamma$$

since $ba = r$ and $ar = a$, so that $\dot{a}r = \dot{a}$.

$$\begin{aligned}\text{Then}\quad (\Gamma^{-1}a)^{\cdot} &= -\Gamma^{-1}\dot{\Gamma}\Gamma^{-1}a + \Gamma^{-1}\dot{a}\\ &= -\Gamma^{-1}(-a\dot{b} + \dot{a}b + a\dot{b}ab)\Gamma\Gamma^{-1}a + \Gamma^{-1}\dot{a}\\ &= -\Gamma^{-1}(-a\dot{b}a + \dot{a}ba + a\dot{b}aba - \dot{a})\\ &= -\Gamma^{-1}(-a\dot{b}a + \dot{a} + a\dot{b}a - \dot{a})\\ &= 0\,, \quad \text{and}\end{aligned}$$

$$\begin{aligned}(b\Gamma)^{\cdot} &= \dot{b}\Gamma + b\dot{\Gamma}\\ &= \dot{b}\Gamma + b(-a\dot{b} + \dot{a}b + a\dot{b}ab)\Gamma\\ &= (\dot{b} - r\dot{b} + b\dot{a}b + r\dot{b}ab)\Gamma\\ &= (b\dot{a}b + \dot{b}ab)\Gamma\,, \quad \text{since } rb = b\\ &= 0\end{aligned}$$

since $ba \equiv r$ so that $0 = \dot{r} = \dot{b}a + b\dot{a}$.

§4. THE CONNECTION ON $G \to \mathcal{S}'$

As usual, let (a_0, b_0) be a fixed element of $\mathcal{S}$ and $\tau : G \to \mathcal{S}'$ be the corresponding fibration. A tangent vector $X \in T_gG$ is called *vertical* if $(T_g\tau)X = 0$. We denote V_g the set of all vertical vectors at $g \in G$:

$$V_g = \{X \in T_gG : (T_g\tau)X = 0\}\,.$$

As remarked before (2.3.3) $V_g = \{X \in T_gG : Xq_0 = 0 \text{ and } q_0g^{-1}X = 0\}$, where $q_0 = a_0b_0 \in Q$. It is easy to see that $V_g = gV_1$ and $V_1 = \{X \in T_1G : X = (1-q_0)X(1-q_0)\}$. But then it is clear that $H_1 := \{X \in T_1G : (1-q_0)X(1-q_0) = 0\}$ is a direct supplement of V_1 in $A = T_1G$.

Define $H_g := gH_1$ for all $g \in G$. Then $H_g \oplus V_g = T_1G$ and $H_gh = H_{gh}$ for all $g \in G$, $h \in K = \{g \in G : gq_0 = q_0g = q_0\}$. Finally we have the operators v_g, $h_g \in L(A)$ defined by $v_g(X) = g(1-q_0)g^{-1}X(1-q_0)$, $h_g(X) = gq_0g^{-1}Xq_0 + gq_0g^{-1}X(1-q_0) + g(1-q_0)g^{-1}Xq_0$ for $X \in A = T_gG$.

It is clear that v_g and h_g are the projections of $L(A)$ determined by the decomposition $A = V_g \oplus H_g$. It is also clear that v_g and h_g depend analytically on g.

All these remarks, put together, prove that the subspace distribution $g \mapsto H_g$ defines a connection on the principal bundle $\pi : G \to \mathcal{S}'$.

The *connection form* ω is defined for $X \in T_gG$ by $\omega_g X = v_1(g^{-1}X)$. Then $\omega_g X = (1-q_0)g^{-1}X(1-q_0)$. The *2-form* $d\omega$ is easily calculated from its definition, for $X, Y \in T_gG$:

$$\begin{aligned} d\omega_g(X,Y) &= (1/2)\{X\,.\,\omega_g Y - Y\,.\,\omega_g X - \omega_g[X,Y]\} \\ &= (1/2)(1-q_0)[g^{-1}Y,\, g^{-1}X](1-q_0) \\ &= (1/2)\omega_1[g^{-1}Y,\, g^{-1}X]\,. \end{aligned}$$

Analogously, the *curvature form* Ω of the connections is

$$\begin{aligned} \Omega_g(X,Y) &= d\omega_g(h_gX,\, h_gY) \\ &= (1/2)(1-q_0)\{g^{-1}Yq_0g^{-1}X - g^{-1}Xq_0g^{-1}Y\}(1-q_0)\,, \end{aligned}$$

and obviously we get the *structure equations*

$$d\omega_g(X,Y) + (1/2)[\omega_g X,\, \omega_g Y] = \Omega_g(X,Y)\,.$$

Recall [12, p.69] that a differentiable curve $\gamma : [\alpha, \beta] \to \mathcal{S}'$ admits a unique *horizontal lift* with origin 1, i.e. a differentiable curve $\Gamma : [\alpha, \beta] \to G$ such that $\Gamma(\alpha) = 1$, $\pi(\Gamma(t)) = \gamma(t)$ and $\dot{\Gamma}(t) \in H_{\Gamma(t)}$ for all $t \in [\alpha, \beta]$. We show next that the lifting constructed in Theorem 3.1 by means of the differential equation $\dot{\Gamma} = (-a\dot{b} + \dot{q}q)\Gamma$ is exactly the horizontal lift of $\gamma = (a, b)$.

4.1. Theorem *For every C^1 curve $\gamma : [\alpha, \beta] \to \mathcal{S}'$ the horizontal lift Γ with origin g_0, for $\pi(g_0) = \gamma(\alpha)$, is the solution of the initial value problem*

$$(4.1.1) \qquad \dot{\Gamma} = (-a\dot{b} + \dot{q}q)\Gamma, \quad \Gamma(\alpha) = g_0 \,.$$

Proof Both objects, the horizontal lift and the solution of the differential equation, are unique, so it suffices to prove that the horizontal lift Γ with origin g_0 satisfies (4.1.1).

By definition, $\pi(\Gamma(t)) = \gamma(t)$, $\Gamma(\alpha) = g_0$ and $\dot{\Gamma}(t) \in H_{\Gamma(t)} = \Gamma(t)H_1$; this means that $\Gamma(t)a_0 = a(t)$, $b_0\Gamma(t)^{-1} = b(t)$ and $\Gamma(t)^{-1}\dot{\Gamma}(t) \in H_1$ for all t, so that (from now on we omit writing the variable t) $(1-q_0)\Gamma^{-1}\dot{\Gamma}(1-q_0) = 0$; but $q_0 = a_0b_0 = \Gamma^{-1}ab\Gamma$, so

$$(4.1.2) \qquad 0 = (1-q_0)\Gamma^{-1}\dot{\Gamma}(1-q_0) = \Gamma^{-1}(1-ab)\dot{\Gamma}\Gamma^{-1}(1-ab)\,.$$

On the other hand, $\dot{a} = \dot{\Gamma}a_0$, $\dot{b} = -b_0\Gamma^{-1}\dot{\Gamma}\Gamma^{-1}$ and

$$\begin{aligned}
\dot{\Gamma}\Gamma^{-1} + a\dot{b} - \dot{q}q &= \dot{\Gamma}\Gamma^{-1} + a(-b_0\Gamma^{-1}\dot{\Gamma}\Gamma^{-1}) - (\dot{a}bab + a\dot{b}ab) \\
&= \dot{\Gamma}\Gamma^{-1} + a(-(b\Gamma)\Gamma^{-1}\dot{\Gamma}\Gamma^{-1})(\dot{\Gamma}a_0b - ab_0\Gamma^{-1}\dot{\Gamma}\Gamma^{-1}ab) \\
&= \dot{\Gamma}\Gamma^{-1} - ab\dot{\Gamma}\Gamma^{-1} - \dot{\Gamma}\Gamma^{1}ab + ab\dot{\Gamma}\Gamma^{-1}ab \\
&= (1-ab)\dot{\Gamma}\Gamma^{-1}(1-ab) = 0\,, \quad \text{by (4.1.2)}.
\end{aligned}$$

This shows that $\dot{\Gamma} = (-a\dot{b} + \dot{q}q)\Gamma$, which proves the theorem.

From the transport equation, a natural connection on the tangent bundle $T\mathcal{S}$ is constructed as follows.

Given a C^1 curve $(a,b) = (a(t), b(t))$ in $\mathcal{S}$ and a tangent vector field (U,V) along (a,b), the covariant derivative $\frac{D}{dt}(U,V)$ is $\frac{d}{dt}(\Gamma^{-1}U, V\Gamma)|_{t=0}$. Then $\frac{D}{dt}(U,V) = (\dot{U} - cU, \dot{V} + Vc)$, where $c = -a\dot{b} + \dot{a}b + a\dot{b}ab$. The curve (a,b) is a geodesic of this connection if and only if it verifies $\ddot{a} = c\dot{a}$ and $\ddot{b} = -\dot{b}c$. The following proposition, whose proof is straightforward, describes all geodesics through the point $(a_0, b_0) \in \mathcal{S}$.

4.2. Proposition *Given a tangent vector* $(X, Y) \in T_{(a_0,b_0)}\mathcal{S}$ *the curve* $\gamma(t) = (g(t)a_0, b_0 g(t)^{-1})$, *where* $g(t) = e^{t(X_0 - a_0 Y + a_0 Y a_0 b_0)}$, *is the geodesic through* (a_0, b_0) *with the initial velocity vector* (X, Y).

4.3. Remark The exponential map of the connection is given by

$$\operatorname{Exp}_{(a_0,b_0)}(X,Y) = e^{(Xb - aY + aYab)} \ ,$$

for $(X, T) \in T_{(a_0,b_0)}(X, Y)$.

§5. THE PRINCIPAL BUNDLE $\mathcal{S} \to Q$

In this section, we study the map $\theta : \mathcal{S} \to Q$ defined by $\theta(a, b) = ab$. We shall prove that θ is a principal fiber bundle with structure group $H = \{h \in G : hr = rh\,,\ (1 - r)h = 1 - r\}$ and define a connection on θ. We also determine the horizontal liftings, with respect to this connection, of curves in Q.

5.1. Lemma *H acts freely on $\mathcal{S}$ by* $(a, b) \,.\, h = (ah,\ h^{-1}b)$.

Proof Observe, first, that in terms of r the elements of H have the form

$$\begin{pmatrix} h_{11} & 0 \\ 0 & 1 \end{pmatrix}.$$

Suppose that $(a, b) \,.\, h = (a, b)$ for some $(a, b) \in \mathcal{S},\ \ h \in H$. This means that $ah = a$ and $h^{-1}b = b$, or, matricially,

$$\begin{pmatrix} a_1 h_{11} & 0 \\ a_2 h_{11} & 0 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ a_2 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} h_{11}^{-1} b_1 & h_{11}^{-1} b_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \\ 0 & 0 \end{pmatrix}$$

so that $a_1 b_{11} = a_1, \quad a_2 h_{11} = a_2$ and multiplying at left by b_1 and b_2, respectively, and adding we get $(b_1 a_1 + b_2 a_2) h_{11} = b_1 a_1 + b_2 a_2$, which means precisely, $h_{11} = 1$, i.e. $h = 1$.

Observe that the action of H preserves the fibers $\theta^{-1}(q)$ $(q = ab)$.

5.2. Lemma *Given* $(a,b),(a',b')$ *in* $\theta^{-1}(q)$ *there exists a unique* $h \in H$ *such that* $(a',b') = (a,b)\,.\,h$.

Proof We continue with the same matrix notation. If $(a',b') = (a,b)\,.\,h$ then $a_1 h_{11} = a'_1$ and $a_2 h_{11} = a'_2$, so that, $b_1 a_1 h_{11} = b_1 a'_1$, $b_2 a_2 h_{11} = b_2 a'_2$ and $h_{11} = (b_1 a_1 + b_2 a_2) h_{11} = b_1 a'_1 + b_2 a'_2$. Thus, h_{11} (and so h) is completely determined by (a,b) and (a',b'). For the existence, it suffices to show that

$$h = \begin{pmatrix} h_{11} & 0 \\ 0 & 1 \end{pmatrix}$$

is actually invertible. But it follows from the equalities $a_i b_k = a'_i b'_k$ that $h_{11}(b'_1 a_1 + b'_2 a_2) = (b'_1 a_1 + b'_2 a_2) h_{11} = 1$, so that h is invertible.

5.3. Lemma *The map* $\theta : \mathcal{S} \to Q$ *admits local sections; in particular, its image is open and closed in* Q.

Proof Let $q = ab = \theta(a,b) \in Q$ and $\pi_q : G \to Q$ be defined by $\pi_q(g) = gqg^{-1}$. It is well known that π_q admits analytic local sections [18], [6]. Then, there exists a neighborhood $V \subset Q$ of q and an analytic map $\sigma' : V \to G$ such that $\sigma'(s) g \sigma'(s)^{-1} = s$ for all $s \in V$. Thus, $\sigma : V \to \mathcal{S}$ defined by $\sigma(s) = \sigma'(s)\,.\,(a,b) = (\sigma'(s)a,\, b\sigma'(s)^{-1})$ is an analytic cross section of θ. This shows, also, that $\theta(\mathcal{S})$ is open in Q. But $Q \backslash \theta(\mathcal{S})$ is the union of sets of the form $\theta(\mathcal{S}_{r'})$ so that $\theta(\mathcal{S})$ is also closed.

The lemmas above imply, together, the following result

5.4. Theorem *The map* $\theta : \mathcal{S} \to Q$ *is a principal fiber bundle with structural group* H.

5.5. Remarks 1) The image of θ clearly contains the similarity orbit of r, but examples can be easily obtained in which it is strictly greater. In fact, when $A = L(F)$ the image of $\theta : \mathcal{S} \to Q$ consist of all $q \in Q$ such that $\operatorname{im} q$ is isomorphic to $\operatorname{im} r$; however, if $q \in Q$ belongs to the similarity orbit of r then $\ker q$ is isomorphic to $\ker r$ (and, of course, $\operatorname{im} q$ is isomorphic to $\operatorname{im} r$). It can be shown that $\theta(\mathcal{S}_r)$ is the equivalence class of r for the

relation $\sim$ defined on Q as follows: $q \sim q'$ if there exists w, z in A such that $wq = q'w$, $qz = zq'$, $zwq = q$ and $wzq' = q'$.
2) Observe that, in the spatial case $\mathcal{S}(E,F)$, the associated vector bundle corresponding to the action of the group $H = GL(E)$ over E coincides with the canonical vector bundle $\{(ij, \text{ im } ij) : (i,j) \in \mathcal{S}(E,F)\}$.

Next, a connection will be defined on the bundle $\theta : \mathcal{S} \to Q$. Recall that for $(a,b) \in \mathcal{S}$, the tangent space $T = T_{(a,b)}\mathcal{S}$ is

$$T = \{(X,Y) : Xr = X,\ rY = Y,\ Ya + bX = 0\}\,.$$

Observe also that the vertical vectors, i.e. the elements of T which are tangent to the fiber $\theta^{-1}(ab)$, form the subspace

$$V_{(a,b)} = \{(X,Y) \in T : Xb + aY = 0\}\,.$$

Let us define the following direct complement of $V_{(a,b)}$ in T:

$$H_{(a,b)} = \{(X,Y) \in T : bX = 0\}$$

and consider the projection $F_{(a,b)} : T \to T$

$$F_{(a,b)}(X,Y) = (abX,\ -bXb)\,.$$

It is straightforward verification that F is the projection onto $V_{(a,b)}$ with kernel $H_{(a,b)}$.
5.6. Lemma *Given $(a,b), (a',b') \in \theta^{-1}(q)$ and $h \in H$ such that $(a',b') = (a,b)\,.\,h$ the respective projections $F_{(a,b)}$, $F_{(a',b')}$ verify $F_{(a',b')}((X,Y)\,.\,h) = (F_{(a,b)}(X,Y))\,.\,h$ for all $(X,Y) \in T$.*

This equivariance property, whose proof is straightforward, shows that the distribution $(a,b) \mapsto H_{(a,b)}$ defines a connection on $\theta : \mathcal{S} \to Q$. We shall determine the horizontal liftings of C^1 curves in Q.
5.7. Theorem *Let $\gamma : [\alpha, \beta] \to Q$ be a C^1 curve with origin $\gamma(\alpha) = q = ab$,*

for some $(a,b) \in \mathcal{S}$. *Let* $\Gamma : [\alpha,\beta] \to G$ *be the unique solution of the initial value problem*

$$\dot{\Gamma} = [\dot{\gamma},\gamma]\Gamma, \qquad \Gamma(\alpha) = 1 .$$

Then $\delta(t) = \Gamma(t).(a,b) = (\Gamma(t)a,\ b\Gamma(t)^{-1})$ *is the horizontal lift of* γ *with origin* (a,b).

Proof By the uniqueness of horizontal liftings with fixed origin, it suffices to show that $\Gamma(t)ab\Gamma(t)^{-1} = \gamma(t)$ and $\dot{\delta}(t) \in H_{\delta(t)}$ for all $t \in [\alpha,\beta]$. The first property is known ([18], [6]). Observe that

$$\dot{\delta}(t) = (\dot{\Gamma}(t)a,\ -b\Gamma(t)^{-1}\dot{\Gamma}(t)\Gamma(t)^{-1})$$

and

$$H_{\delta(t)} = \{(X,Y) \in T_{\delta(t)} : b\Gamma(t)^{-1}X = 0\}.$$

We must show, then, that $b\Gamma(t)^{-1}\dot{\Gamma}(t)a = 0$. But Γ is the horizontal lift of γ to G so that

$$q\Gamma(t)^{-1}\dot{\Gamma}(t)q = 0$$

(see [6,(4.6)]). Multiplying by b on the left and by a on the right and using $rb = b$ and $ar = a$ we get the desired equality.

5.8. Remark In 1.2 we have observed that, as Banach manifolds, $\mathcal{S}_r$ and $\mathcal{S}_{grg^{-1}}$ are isomorphic for all $g \in G$. We now prove that, as principal fiber bundles, $\theta_r : \mathcal{S}_r \to Q$ and $\theta_{r'} : \mathcal{S}_{r'} \to Q$ are isomorphic if $rr' = r'$ and $r'r = r$.

In fact, using matrix representations in terms of r: we have

$$r = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ r' = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$$

for some x; the map

$$\Upsilon_{rr'} : \left(\begin{pmatrix} a_1 & 0 \\ a_2 & 0 \end{pmatrix}, \begin{pmatrix} b_1 & b_2 \\ 0 & 0 \end{pmatrix} \right)$$

$$\longmapsto \left(\begin{pmatrix} a_1 & a_1 x \\ a_2 & a_2 x \end{pmatrix}, \begin{pmatrix} b_1 & b_2 \\ 0 & 0 \end{pmatrix} \right)$$

is a diffeomorphism of $\mathcal{S}_r$ onto $\mathcal{S}_{r'}$, the isomorphism of H_r onto $H_{r'}$ having the form

$$\nu : \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} h & hx - x \\ 0 & 1 \end{pmatrix}.$$

It is easily verified that: a) the isomorphism respects also the connections, i.e. horizontal vectors on $T_{(a,b)}\mathcal{S}_r$ are sent onto horizontal vectors in $T_{(a',b')}\mathcal{S}_{r'}$, and b) $\Upsilon_{rr'}((a,b)\,.\,h) = (\Upsilon_{rr'}(a,b))\,.\,\nu(h)$ for all $(a,b) \in \mathcal{S}_r$, $h \in H_r$.

§6. THE C^*–ALGEBRA SETTING

When E and F are Hilbert spaces, the most relevant pairs "inclusion–projection" (i,j) are those in which the inclusion i is an isometry whose image is orthogonal to the kernel of the projection j.

6.1. Proposition *Let E, F be Hilbert spaces and $(i,j) \in \mathcal{S} = \mathcal{S}(E,F)$. Then i is an isometry and* $\ker j \perp$ *im* i *if and only if $j = i^*$.*

Proof Suppose that $j = i^*$, so that $i^*i = 1_E$. Then clearly i^* is an isometry and, for $y \in \ker j$ and $x \in E$, $\langle y, i(x)\rangle = \langle i^*y, x\rangle = \langle j(y), x\rangle = 0$. Conversely, let i be an isometry with im $i \perp \ker j$. For $i \in \ker j$ and $x \in E$ we have $\langle j(y), x\rangle = 0 = \langle y, i(x)\rangle = \langle i^*(y), x\rangle$, so that $j = i^*$ on $\ker j$; for $y \in (\ker j)^\perp = \operatorname{im} i$ and $x \in E$, $\langle i^*(y), x\rangle = \langle i^*(i(x')), x\rangle$ for some $x' \in E$, so $\langle i^*(y), x\rangle = \langle i(x'), i(x)\rangle = \langle x', x\rangle = \langle ji(x'), x\rangle = \langle j(y), x\rangle$ and $j = i^*$ on $(\ker j)^\perp$.

Let A be a C^*–algebra with unitary group $\mathcal{U}$ and $\mathcal{P} = \{p \in Q : p^* = p\}$. Given a fixed $r \in \mathcal{P}$ we consider the subset $\mathcal{R}$ of $\mathcal{S}$ consisting of all pairs (a,b) such that $b = a^*$, i.e. $\mathcal{R} = \{(a,a^*) : ar = a,\ a^*a = r\}$.

The following facts can be proven as above:

6.2. $\mathcal{R}$ is a closed real–analytic submanifold of $A \times A$; for $(a, a^*) \in \mathcal{R}$, $T_{(a,a^*)}\mathcal{R}$ can be identified with $\{(X, X^*) \in A \times A : Xr = X,\ a^*X + X^*a = 0\}$.

6.3. $\mathcal{U}$ acts on $\mathcal{R}$ by inner automorphisms defining, for each $(a, a^*) \in \mathcal{R}$, a principal fiber bundle $\tau : \mathcal{U} \to \mathcal{R}$ with structure group $H_0' = \{u \in \mathcal{U} : ua = a\}$.

6.4. There is a principal connection on $\tau : \mathcal{U} \to \mathcal{R}$; for $u \in \mathcal{U}$ the horizontal vectors are those $X \in A$ such that $X^* = X$ and $(1 - aa^*)uX(1 - aa^*) = 0$; the horizontal lifting of a C^1-curve $\gamma : [\alpha, \beta] \to \mathcal{R}$ with origin u_0 ($\tau(u_0) = \gamma(\alpha)$) is the solution of the initial value problem

$$\dot{\Gamma} = (-a\dot{a}^* + \dot{a}a^* + a\dot{a}^*aa^*)\Gamma,\ \Gamma(\alpha) = u_0\ .$$

6.5. The map $\theta : \mathcal{R} \to \mathcal{P}$ is a principal fiber bundle with structure group $H = \{u \in \mathcal{U} : ur = ru,\ (1 - r)u = 1 - r\}$.

6.6. There is a principal connection on $\theta : \mathcal{R} \to \mathcal{P}$ whose horizontal vectors at (a, a^*) are those (X, X^*) satisfying $a^*X = 0$; the horizontal lifting with origin (a, a^*) of a C^1 curve $\gamma : [\alpha, \beta] \to \mathcal{P}$ such that $\gamma(\alpha) = aa^*$ is given by $t \mapsto (\Gamma(t)a,\ a^*\Gamma(t)^*)$, where Γ is the unique solution of the initial value problem $\dot{\Gamma} = [\dot{\gamma}, \gamma]\Gamma,\ \Gamma(\alpha) = 1$ (Γ is unitary because $[\dot{\gamma}, \gamma]$ is antisymmetric [9]).

6.7. Remark When E is a Hilbert space and $A = L(E)$ then $\mathcal{R}$ can be thought as the space of all partial isometries with the same kernel as r; thus, the map θ is a way of fibering over $\mathcal{P}$ these partial isometries.

We proceed now to define the "polar decomposition" of a pair $(a, b) \in \mathcal{S}$. The definition of the polar decomposition of an idempotent q of A is in order [6, §2]. Given $q \in Q$, $2q - 1$ is invertible so it admits the (uniquely determined) polar decomposition $2q - 1 = \lambda^2\rho$ where $\lambda > 0$ and $\rho^* = \rho^{-1}$. It follows that $\rho^2 = 1$ and $\lambda\rho = \rho\lambda^{-1}$, so that $p = (1/2)(\rho + 1) \in \mathcal{P}$. Define $\pi(q) = p$; p is what we call the polar decomposition of q.

In order to find what should be the polar decomposition of a pair $(a,b) \in \mathcal{S}$ be, we start by considering the spatial case. Thus, as in the beginning of the section, $\mathcal{S} = \mathcal{S}(E,F)$ where E and F are Hilbert spaces and $\mathcal{R} = \mathcal{R}(E,F) = \{(k,k^*) : k^*k = 1_E\}$.

Given $(i,j) \in \mathcal{S}$ we look for a pair $(k,k^*) \in \mathcal{R}$ such that if $ij = q$ then $kk^* = \pi(q)$. Thus we have to twist (i,j) into (i',j') in order to transform $\operatorname{im} q$ into $\operatorname{im}\pi(q)$ and then convert i' in an isometry without changing $\operatorname{im}(i'j')$.

6.8. Theorem *There exists a real-analytic retraction* $\Pi : \mathcal{S} \to \mathcal{R}$ *such that the following diagram commutes*

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\Pi"] \arrow[d, "\theta"'] & \mathcal{R} \arrow[d, "\theta"] \\ \mathcal{Q} \arrow[r, "\pi"'] & \mathcal{P} \end{tikzcd}$$

where $\theta(i,j) = ij$.

Proof Write $q = ij$ and $p = \pi(q)$. Using the same notations as above, we observe that $\lambda^{-1}q = p\lambda^{-1}$: in fact, $\lambda^{-1}q = \lambda^{-1}(1/2)(\lambda^2\rho+1) = (1/2)(\lambda\rho+\lambda^{-1}) = (1/2)(\rho\lambda^{-1}+\lambda^{-1}) = (1/2)(\rho+1)\lambda^{-1} = p\lambda^{-1}$, where we have used, for the third equality, that $\lambda\rho = \rho\lambda^{-1}$.

Thus $\lambda^{-1} \in GL(F)$ satisfies $\lambda^{-1}(\operatorname{im} i) = \lambda^{-1}(\operatorname{im} q) = \operatorname{im} p$. Now $(x,y) \mapsto \langle \lambda^{-1}ix, \lambda^{-1}iy \rangle$ is an inner product in E, so there exists a positive $g \in GL(E)$ such that $\langle \lambda^{-1}ix, \lambda^{-1}iy \rangle = \langle gx, gy \rangle$ for all $x, y \in E$. This shows that $k = \lambda^{-1}ig^{-1} : E \to F$ is an isometry and then $(k,k^*) \in \mathcal{R}$. Notice that, when i is an isometry, i.e. $(i,i^*) \in \mathcal{R}$, then $\lambda = 1_F$, $g = 1_E$ and $k = i$.

Define $\Pi(i,j) = (k,k^*)$. The analyticity of Π is obviously determined by that of π. Thus, it suffices to prove that the diagram commutes, i.e. that $kk^* = p$. But, kk^* and p are both orthogonal projections with the same image, for $\operatorname{im} kk^* = \operatorname{im} k = \operatorname{im} \lambda^{-1} i g^{-1} = \operatorname{im} \lambda^{-1} i = \operatorname{im} p$. It follows then that $kk^* = p$.

In the general case we have:

6.10. Theorem *Let A be a C^*–algebra and $r \in Q$. Then there is a real-analytic retraction $\Pi : \mathcal{S}_r \to \mathcal{R}_{\pi(r)}$ such that the diagram (6.9) commutes.*

Proof In view of Remark 5.8 we can suppose that $r \in \mathcal{P}$ (replacing, if necessary, $r \in Q$ by the unique $r' \in \mathcal{P}$ such that $r'r = r^*$ and $rr' = r'$). Given $(a, b) \in \mathcal{S}$ let $q = ab$ and decompose, as usual, $2q-1 = \lambda^2 \rho$ with $\lambda > 0$ and $\rho^* = \rho = \rho^{-1}$. Observe first that $a^* \lambda^{-2} a + 1 - r$ is a positive invertible element (with inverse $b\lambda^2 b^* + 1 - r$) so that $\alpha = \lambda^{-1} a(a^* \lambda^{-2} a + 1 - r)^{-1/2}$ is a well-defined element of A. It suffices to verify (a) $\alpha^* \alpha = r$, (b) $\alpha r = \alpha$ and (c) $\alpha \alpha^* = p = \pi(q)$ and define $\Pi(a, b) = (\alpha, \alpha^*)$.

Proof of (a): $\alpha^* \alpha = \{(a^* \lambda^{-2} a + 1 - r)^{-1/2} a^* \lambda^{-1}\}\{\lambda^{-1} a(a^* \lambda^{-2} a + 1 - r)^{-1/2}\} = (a^* \lambda^{-2} a + 1 - r)^{-1/2} a^* \lambda^{-2} a(a^* \lambda^{-2} a + 1 - r)^{-1/2} = 1 - (a^* \lambda^{-2} a + 1 - r)^{-1/2}(1 - r)(a^* \lambda^{-2} a + 1 - r)^{-1/2} = 1 - (1 - r) = r$, because $(a^* \lambda^{-2} a + 1 - r)(1 - r) = 1 - r$ (see Lemma 6.11 below).

Proof of (b): $\alpha = \lambda^{-1} a(a^* \lambda^{-2} a + 1 - r)^{-1/2} = \lambda^{-1} a(a^* \lambda^{-2} a)^{-1/2}$, where the inverse square root is taken in the C^* algebra rAr, whose unit is r, and then $\alpha r = \lambda^{-1} a(a^* \lambda^{-2} a)^{-1/2} r = \lambda^{-1} a(a^* \lambda^{-2} a)^{-1/2} = \alpha$, because $ar = r$ (see Lemma 6.11 below).

Proof of (c): $\alpha \alpha^* = \lambda^{-1} a(a^* \lambda^{-2} a)^{-1} a^* \lambda^{-1} = \lambda^{-1} q \lambda^2 q^* \lambda^{-1}$; now $q = \lambda p \lambda^{-1}$ by (6.8), so that $q^* = \lambda^{-1} p \lambda$, $q \lambda^2 q^* = \lambda p \lambda^{-1} \lambda^2 \lambda^{-1} p \lambda = \lambda p \lambda$ and $\alpha \alpha^* = p$, as claimed.

Thus, the proof of Theorem 6.10 is finished, modulo the following lemma, whose proof is an easy exercise:

6.11. Lemma *Let A be a C^*–algebra, $x, c \in A$, such that $x^* = x$ and $xc = x$. Then for every continuous function f on the spectrum of x, $f(x)c = f(x)$.*

6.12. Remark Denote $\mathcal{S}\,|\,\mathcal{P} = \{(a, b) \in \mathcal{S} : ab \in \mathcal{P}\} = \theta^{-1}(\mathcal{P})$. Then $\mathcal{S}|\mathcal{P}$ can be identified with $\mathcal{R} \times_U H$, where H has the same meaning as before, and for every $u \in U$, $h \in H$ and $\alpha \mathcal{R}(\alpha, h)$ is identified with

$(\alpha u, u^{-1}h)$; the projection $\mathcal{S}|\mathcal{P} \to \mathcal{R} \times_U H$ is determined by $(a,b) \mapsto (\alpha, h)$, where h is the unique element of H such that $(a,b) = (\alpha, \alpha^*) \,.\, h$ (see 5.2).

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DIFFERENTIAL GEOMETRY OF SYSTEMS OF PROJECTIONS IN BANACH ALGEBRAS

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Let A be a Banach algebra, n a positive integer and $Q_n = \{(q_1, \ldots, q_n) \in A^n : q_i q_k = \delta_{ik} q_i,\ q_1 + \cdots + q_n = 1\}$. The differential geometry of Q_n, as a discrete union of homogeneous spaces of the group G of units of A is studied, a connection on the principal bundle $G \to Q_n$ is defined and invariants of the associated connection on the tangent bundle TQ_n are determined.

Introduction. The structure of the set Q of all idempotent elements of a Banach algebra A plays a fundamental role in several aspects of spectral theory. This work deals with the differential structure of the space

$$Q_n = \left\{ (q_1, \ldots, q_n) \in A^n : q_i q_k = \delta_{ik} q_i,\ \sum_{i=1}^{n} q_i = 1 \right\}$$

of systems of n "orthogonal" projections in A.

The manifold Q_n appears as a universal model when certain polynomial equations are considered. More precisely, if $\alpha_1, \ldots, \alpha_n$ are *different* complex numbers and $\alpha(X)$ denotes the polynomial $(X - \alpha_1) \cdots (X - \alpha_n)$, then the set $A_\alpha = \{a \in A : \alpha(a) = 0\}$ is a closed submanifold which is diffeomorphic to Q_n. Thus Q_n is the model for all simple algebraic elements of A of degree n. Moreover, Q_n plays a role in the study of arbitrary algebraic (in particular, nilpotent) elements (see **[AS]**).

Section 1 contains the description of the differential structure of Q_n and A_α as closed analytic submanifolds of A^n and A, respectively; it contains also the proof that Q_n and A_α are diffeomorphic.

Using Kaplansky's notion of SBI-rings, we recover a result of Barnes **[Ba]** concerning the surjectivity of $A_\alpha \to B_\alpha$ when B is the quotient of A by its Jacobson radical. In §2 we show that Q_n is a discrete union of homogeneous spaces of G, the group of units of A; this fact, together with a classical result of Michael **[Mi]**, shows that an epimorphism $f: A \to B$ of Banach algebras induces Serre fibrations $Q_n(A) \to Q_n(B)$ and $A_\alpha \to B_\alpha$. In §3 we obtain an explicit way of lifting differentiable curves in Q_n to G by solving a linear differential equation which we call the *transport equation*; this fact is due to Daleckii and S. G. Krein **[DK]** and T. Kato **[Ka1]** but its geometrical meaning is new. In fact, in §4 we define a connection in the principal bundle $G \to Q_n$ and show that the horizontal liftings of differentiable curves in Q_n are precisely the solutions of the transport equation.

Several invariants of the tangent bundle of Q_n are calculated in §5 (covariant derivative, curvature, geodesics, etc.). As observed by Kato **[Ka1]**, **[Ka2**, II.4**]** the lifting theorem has important applications in quantum mechanics (see **[Ga]**, **[GS]**). A remark about C^*-algebras is in order: our results extend to the case of some involution algebras, in particular to all C^*-algebras. For instance, the transport equation has a unitary solution if the curve has selfadjoint values; in a forthcoming paper the immersion of

$$P_n = \{p \in Q_n : p_i^* = p_i, \ i = 1, \dots, n\}$$

into Q_n will be studied, together with associated fibrations $Q_n \to P_n$.

Concerning the references, the reader may consult Rickart's book **[Ri]** for the literature up to 1960; the topology of the space of idempotents $Q = Q_2$ has been considered in **[PR1]**, **[Ra]**, **[Ko]**, **[Ze]**, **[Au]**, **[Gr]** and with special emphasis on the differential struture of Q in **[Ra]**, **[Gr]**, **[Ki]**, **[HK]**; for the transport equation the reader may consult **[Ka1]** and **[DK2]**; in **[PR2]** the differential geometry of $P = P_2$ is needed for the study of minimality of geodesics; see also **[CPR2]** for a related problem; finally, the case of algebraic operators on Hilbert space, the reader may consult the books **[He]** and **[AFVH]**. In particular, some problems concerning the set P_n in this context are discussed in **[CH]**. The set Q_n appears, implicitly or explicitly, in various works; we only mention **[Ja**, p. 54**]**, **[Ka2**, II.5**]** and **[DK2**, Chapter IV**]**.

1. Differential structure of systems of projections. Let A be a real or complex algebra with identity 1. Denote by $G = G(A)$ the group of units of A and by $Q = Q(A)$ the set of all idempotents of A.

Suppose that the polynomial $\alpha(X) = \prod_{i=1}^n (X - \alpha_i)$ has different roots $\alpha_1, \dots, \alpha_n$ in the field. Let $g_j(X) = \prod_{i \neq j}(X - \alpha_i)$ and $q_j(X) = g_j(X)/g_j(\alpha_j)$. Then $q_j(X)$ has degree $n-1$, $q_j(\alpha_i) = \delta_{ji}$, for $i \neq j$ $q_i(X)q_j(X) = h(X)\alpha(X)$ for some polynomial $h(X)$ and $\sum_{i=1}^n q_i(X) = 1$ (because $1 - \sum_{i=1}^n q_i(X)$ has degree $\leq n-1$ and it vanishes at n values, the α_j).

Let A_α denote the solution set of α, i.e., the set of all $a \in A$ with $\alpha(a) = 0$.

1.1. PROPOSITION. *Let* $a \in A(\alpha)$. *Then*

(i) $\sum_{i=1}^{n} q_i(a) = 1$;
(ii) $q_i(a)q_j(a) = 0$ *if* $i \neq j$;
(iii) $q_i(a) \in Q$, $i = 1, \ldots, n$;
(iv) $q_i(a)a = aq_i(a) = \alpha_i q_i(a)$, $i = 1, \ldots, n$.

Proof. (i) follows from $\sum_{i=1}^{n} q_i(X) = 1$ and (ii) follows from the equality $q_i(X)q_j(X) = h(X)\alpha(X)$. From (i) and (ii),

$$q_i(a) = q_i(a)\sum_{k=1}^{n} q_k(a) = \sum_{k=1}^{n} q_i(a)q_k(a) = q_i(a)^2,$$

which gives (iii). Finally from $\alpha(X) = c(X - \alpha_i)q_i(X)$ (with $c = g_i(\alpha_i) \neq 0$) it follows that $0 = \alpha(a) = c(aq_i(a) - \alpha_i q_i(a))$ and this completes the proof because $q_i(a)$ commutes with a.

Let $Q_n = Q_n(A)$ denote the set of all n-tuples of idempotents q_i of A which satisfy $q_i q_j = 0$ if $i \neq j$ and $\sum_{i=1}^{n} q_i = 1$.

1.2. PROPOSITION. *The mapping* $a \to (q_i(a), \ldots, q_n(a))$ *is a bijection from* A_α *onto* Q_n *whose inverse is* $(q_1, \ldots, q_n) \to \sum_{i=1}^{n} \alpha_i q_i$.

The proof is a straightforward application of Proposition 1.1. Thus, from a set-theoretical view point, Q_n is a universal model for the sets A_α. We shall extend this result to the differential geometry setting.

1.3. REMARK. I. Kaplansky introduced the notion of SBI-rings (SBI = suitable for building idempotents) as those rings A such that the natural mapping $Q(A) \to Q(A/R)$ is onto, where R is the Jacobson radical of A.

It is known that for a SBI-ring A, the map $Q_n(A) \to Q_n(A/R)$ is also onto for each $n = 1, 2, \ldots$ (see [**Ja**, p. 54]).

It is also known that all Banach algebras are SBI [**Ri**, p. 58]. These facts and 1.2 imply that, for every $\alpha = (\alpha_1, \ldots, \alpha_n)$ (with $\alpha_l \neq \alpha_k$), $A_\alpha \to (A/R)_\alpha$ is onto, a result due to Barnes [**Ba**, Theorem 7].

From now on, we will assume that A is a real or complex Banach algebra with identity. For n-tuples $Z = (Z_1, \ldots, Z_n)$ in A^n we use the norm $\|Z\| = \max_{1 \leq i \leq n} \|Z_i\|$. The general facts on Banach algebras and Banach manifolds needed below can be found in [**Ri**] and [**La**], respectively.

1.4. THEOREM. *Let* $a \in A_\alpha$ *be a fixed element,* $q = q(a) = (q_1(a), \dots, q_n(a)) \in Q_n$ *the corresponding system of idempotents. Set*

$$T = \{X \in A;\ q_i X q_i = 0 \textit{ for all } i = 1, \dots, n\},$$
$$S = \{Y \in A;\ q_k Y q_l = 0 \textit{ for all } k \neq l\}.$$

1.4.(i) A *is the Banach space direct sum* $A = T \oplus S$.
1.4.(ii) *For each* $Z = X + Y$, $X \in T$, $Y \in S$, *set*

$$X' = \sum_{i \neq k} q_i X q_k / (\alpha_k - \alpha_i)$$

and define

$$\phi(Z) = \exp(X')(a + Y)\exp(-X').$$

Then ϕ *is a diffeomorphism from a neighborhood* U *of* $O \in A$ *onto a neighborhood* V *of* a. *Moreover,* $\phi|_{U \cap T}$ *is a homeomorphism onto* $V \cap A_\alpha$.

Proof. It is clear that every $Z \in A$ decomposes as $X + Y$, where

$$X = \sum_{j \neq k} q_j Z q_k \in T \quad \text{and}$$

$$Y = \sum_l q_l Z q_l \in S, \quad \text{for } \sum_{l=1}^n q_l = 1 \quad \text{and}$$
$$Z = \left(\sum q_l\right) Z \left(\sum q_l\right) = \sum_{j \neq k} q_j Z q_k + \sum_l q_l Z q_l.$$

It is also clear that the decomposition is topological, for T and S are respectively defined as the images of the projections

$$Z \to \sum_{j \neq k} q_j Z q_k \quad \text{and} \quad Z \to \sum_l q_l Z q_l.$$

An easy computation shows that the derivative of ϕ at O is the identity: in fact, for $Y \in S$ $D\phi(O)Y = Y$ obviously; for $X \in T$ $D\phi(O)X = [X', a] = X'a - aX' = X$; the assertion follows from the decomposition $A = T \oplus S$.

Then, by the inverse function theorem, there exist open neighborhoods U' of O and V' of a such that ϕ maps U' diffeomorphically onto V'. Consider next $Z = X + Y$ with $\phi(Z) \in A_\alpha$. Since

$\phi(Z) = M(a+Y)M^{-1}$, then $a+Y$ is also a root of α. Then $O = \prod_i(a+Y-\alpha_i)$ and using Prop. 1.1.(iv):

$$\begin{aligned} O &= q_j \prod_i (a+Y-\alpha_i) = q_j \prod_i (\alpha_j + Y - \alpha_i) \\ &= q_j Y L \end{aligned}$$

where $L = \prod_{j\neq i}(Y-(\alpha_i - \alpha_j))$. If Y has small norm ($\|Y\| < \min\{|\alpha_i - \alpha_j|,\ i \neq j\}$ suffices) then L is invertible and therefore $q_j Y = 0$ for each j. Hence $\phi(Z) \in A_\alpha$ with Y small implies $Z \in T$. This means that (perhaps for smaller neighborhoods) ϕ is a homeomorphism from $U' \cap T$ onto $V' \cap V_\alpha$.

Considering the maps ϕ as analytic local coordinates in A, we obtain:

1.5. COROLLARY. *A_α is a closed analytic submanifold of A whose tangent space at $a \in A_\alpha$ can be identified to the Banach space T.*

1.6. REMARKS. (i) The choice of the chart ϕ may seem rather artificial; for instance, the derivative at O of $\phi_1(X+Y) = \exp(X)(a+Y)\exp(-X)$ is $X+Y \to Xa - aX + Y = [X, a] + Y$ and the equalities $q_i[X, a]q_j = (\alpha_j - \alpha_i)q_i X q_j$ $(i \neq j)$ show that $D\phi_1(O)$ maps T onto T and S onto S. Thus, ϕ_1 also provides charts for the analytic structure of A_α. However, we have chosen the map ϕ because it is the exponential map of the natural connection to be studied later (see §4). This remarks applies also to the charts chosen below for Q_n.

(ii) An obvious consequence of 1.3 is that A_α is locally arcwise connected for all α as above. For the simpler case of $\alpha(X) = X(X-1)$ this is a result of Zemanek [**Ze**, 3.2] for complex Banach algebras, which was generalized for real algebras by Aupetit [**Au**, p. 413]. However both results have been also proved in [**PR1**, 4.3] (see also 2.2(iii) below).

1.7. THEOREM. *Q_n is a closed submanifold of A^n.*

Proof. Fix $q \in Q_n$ and define $T' = \{X = (X_1, \dots, X_n) \in A^n \colon q_r X_i q_s = 0$ for $r \neq i$ and $s \neq i$ or $r = s = i$, and $q_i X_i q_k + q_i X_k q_k = 0$ for $i \neq k\}$.

The map $\theta\colon A^n \to A^n$, $\theta(Z_i, \ldots, Z_n) = (X_1, \ldots, X_n)$ defined by

$$X_1 = \sum_{i>1} q_1 Z_1 q_i + q_i Z_1 q_1,$$

$$X_2 = \left(\sum_{i>2} q_2 Z_2 q_i + q_i Z_2 q_i\right) - (q_1 Z_1 q_2 + q_2 Z_1 q_1),$$

$$\vdots$$

$$X_k = \sum_{i>k} (q_k Z_k q_i + q_i Z_k q_k) - \sum_{i<k} (q_i Z_i q_k + q_k Z_i q_i) \qquad (k \le n-1),$$

$$X_n = -\sum_{k=1}^{n-1} X_k$$

is a projection onto T' whose kernel is the set S' of all $Y = (Y_1, \ldots, Y_n) \in A^n$ with $q_r Y_i q_s = 0$ for $r = i$ and $s > i$ or $s = i$ and $r > i$.

Thus $T' \oplus S' = A^n$. For $X \in T'$ put

$$\widetilde{X} = \sum_{i \ne j} \widetilde{X}_{ij} \quad \text{where } \widetilde{X}_{ij} = \begin{cases} q_i X_j q_j & \text{if } j < i, \\ -q_i X_i q_j & \text{if } i < j. \end{cases}$$

Observe that $q_i \widetilde{X} q_i = 0$ for $i = 1, \ldots, n$.

Consider now the map $\psi\colon A^n \to A^n$ defined by

$$\psi(Z)_i = \psi(X+Y)_i = \exp(\widetilde{X})(q_i Y_i)\exp(-\widetilde{X})$$

for $X \in T'$, $Y \in S'$. Then $D\psi(O)Y = Y$ for $Y \in S'$ and, calculating,

$$(D\psi(O)X)_i = [\widetilde{X}, q_i] = X_i \quad \text{for } X \in T', \ i = 1, \ldots, n.$$

This means that $D\psi(O) =$ identity and ψ is a diffeomorphism from a neighborhood of O onto a neighborhood of q. For $Y \in S'$ such that $\|Y\| < 1$ it is easily shown that $q + Y \in Q_n$ if and only if $Y = O$. This completes the proof.

Remark. According to Proposition 1.2, the bijections connecting A_α and Q_n are given by algebraic expressions.

The next result, whose proof follows easily from the theorems above, shows that Q_n is a universal model for the sets A_α of simple algebraic elements of degree n.

1.8. Theorem. *The map $a \to (q_1(a), \ldots, q_n(a))$ is a diffeomorphism from A_α onto Q_n whose inverse is given by $(q_1, \ldots, q_n) \to$*

$\sum_{i=1}^n \alpha_i q_i$. *Consequently, for any other* $\beta = (\beta_1, \dots, \beta_n)$ *with* $\beta_i \neq \beta_j$ *the map* $a \to \sum_{i=1}^n \beta_i q_i(a)$ *is a diffeomorphism from* A_α *onto* A_β.

2. Fibrations. The group G of invertible elements of A acts on Q_n by inner automorphisms on each coordinate: if $g \in G$ and $q = (q_1, \dots, q_n) \in P_n$ then $gqg^{-1} = (gq_1g^{-1}, \dots, gq_ng^{-1}) \in Q_n$.

2.1. THEOREM. *Let* q *be a fixed element of* Q_n *and define* $\pi: G \to Q_n$ *by* $\pi(g) = gqg^{-1}$. *Then*

(i) *there exist an open neighborhood* U *of* q *in* Q_n *and a local section* $\sigma: U \to G$ *of* π;

(ii) *the orbit* $V_q = \{gqg^{-1}: g \in G\}$ *is open (and closed) in* Q_n;

(iii) $\pi: G \to V_q$ *is a principal fiber bundle with structure group* $G_0 = \{g \in G: gq_1 = q_1 g,\ i = 1, \dots, n\}$.

Therefore Q_n *is a discrete union of homogeneous spaces of* G.

Proof. Given $q' \in Q_n$ define

$$\sigma(q') = \langle q, q' \rangle = q_1' q_1 + \cdots + q_n' q_n.$$

It is clear that $\sigma(q) = 1$ and $\sigma(q) q_i = q_i' \sigma(q')$. Thus, for every q' in a neighborhood U of q, we have $\sigma(q') \in G$ and $\sigma(q') q \sigma(q')^{-1} = q'$. This proves (i) and (ii) and the rest of the statement follows from standard arguments (see [**St**, §7]).

2.2. REMARKS. (i) An invertible element g belongs to G_0 if and only if $q_k g q_l = 0$ for all $k \neq l$. Thus, the Lie algebra of G_0 can be identified to $\{X \in A: q_k X q_l = 0 \text{ for all } k \neq l\}$.

(ii) With the notations of 2.1 and 1.6 it is easy to describe trivializations of the tangent bundle TQ_n and of a suplement NQ_n of TQ_n in the trivial bundle $\varepsilon: Q_n \times A^n \to Q_n$. We call NQ_n the "normal bundle" of Q_n. Given $q \in Q_n$, let $U_q = \{q' \in Q_n: \sigma(q') \in G\}$. Then $h: U_q \times A^n \to U_q \times A^n$, defined by

$$h(q', Z) = (q', \sigma(q') Z \sigma(q')^{-1}),$$

is a diffeomorphism which trivializes simultaneously $\tau: TQ_n \to Q_n$ and a bundle $\nu: NQ_n \to Q_n$ where $(NQ_n)_q = S'$ (as in 1.6).

(iii) Given $q \in Q_n$, its connected component (in Q_n) can be described as the set $\{gqg^{-1}: g \in G^0\}$, where G^0 is the connected component of 1 in G: in fact, it suffices to replace G by G^0 in the proof of 2.1. Of course, similar statements hold for A_α. This generalizes [**Ze**, Theorem 3.3] and [**Au**].

2.3. COROLLARY. *Consider a fixed* $q \in Q_n$ *and a continuous curve* $\gamma: [0, 1] \to Q_n$ *such that* $\gamma(0) = q$. *Then, there exists a continuous curve* $\Gamma: [0, 1] \to G$ *such that* $\Gamma(0) = 1$ *and* $\pi \circ \gamma = \gamma$, *where* $\pi(g) = gqg^{-1}$.

We consider now the behaviour of the functor Q_n under epimorphisms.

Let $f: A \to B$ be a continuous homomorphism of Banach algebras which preserves the identity

Clearly f induces maps $G(f): G(A) \to G(B)$, and $f_n: Q_n(A) \to Q_n(B)$. We shall prove that f_n is a Serre fibration when f is an epimorphism **[Sp]**.

2.4. THEOREM. *Let* $f: A \to B$ *be a (continuous) epimorphism of Banach algebras. Then* $f_n: Q_n(A) \to Q_n(B)$ *is a Serre fibration. In particular,* f_n *is onto if and only if its image intersects every connected component of* $Q_n(B)$.

Proof. Replacing A and B by $C(I^m, A)$ (= algebra of all maps $I^m \to A$) and $C(I^m, B)$ respectively (where $I = [0, 1]$), it suffices to show that if $\gamma: I \to Q_n(B)$ is such that $\gamma(0) = q' = f_n(q)$ for some $q \in Q_n(A)$ there exists a curve $\tilde{\gamma}: I \to Q_n(A)$ such that $f_n \circ \tilde{\gamma} = \gamma$.

For this, we consider the commutative diagram

$$\begin{array}{ccc} G(A) & \xrightarrow{f} & G(B) \\ {\scriptstyle \pi_q}\downarrow & & \downarrow{\scriptstyle \pi_{q'}} \\ Q_n(A) & \xrightarrow[f_n]{} & Q_n(B) \end{array}$$

where $\pi_q(g) = gqg^{-1}$, $\pi_{q'}(h) = hq'h^{-1}$ $(g \in G(A),\ h \in G(B))$. By the local triviality of $\pi_{q'}$ proved in 2.1, there is a curve $\delta: I \to G(B)$ with $\delta(0) = 1$ and $\pi_{q'}\delta = \gamma$. Michael **[Mi]** proved that $f: G(A) \to G(B)$ is a Serre fibration; therefore, there is a curve $\varepsilon: I \to G(A)$ such that $\varepsilon(0) = 1$ and $f \circ \varepsilon = \delta$. To finish the proof it suffices to define $\tilde{\gamma} = \pi_q \circ \varepsilon$, which satisfies $f_n \circ \tilde{\gamma} = \gamma$.

The next theorem extends results of Raeburn **[Ra]** concerning the set $\pi_0(P(A \hat{\otimes} B))$ of all connected components of the idempotents of $A \hat{\otimes} B$, where A is supposed to be commutative.

We omit its proof and that of the proposition below because they are simple combination of Raeburn's techniques without previous results.

2.5. PROPOSITION (*cf.* [**Ra**, p. 383]). *Let A be a Banach algebra and $B_1, \ldots, B_n$ be open balls in $\mathbf{C}$ with pairwise disjoint closures, centered at $\alpha_1, \ldots, \alpha_n$, respectively. Let $U = B_1 \cup \cdots \cup B_n$ and $A_U = \{a \in A:$ the spectrum of a is contained in $U\}$. Then A_U is open in A and $f = (f_1, \ldots, f_n): A_U \to A^n$ is an analytic retraction onto Q_n, where $f_i: U \to \mathbf{C}$ is defined by $f_i(z) = \delta_{ik}$ for $z \in B_k$ and $f_n(a)$ is obtained by means of the holomorphic functional calculus.*

2.6. THEOREM (*cf.* [**Ra**, 4.5, 4.7]). *Let A and B be complex Banach algebras. Suppose that A is commutative with spectrum X. Then the Gelfand map $A \to C(X)$ induces bijections*

$$\pi_0(Q_n(A\hat{\otimes}B)) \to [X, Q_n(B)],$$
$$\{Q_n(A\hat{\otimes}B)\} \to \{Q_n(C(X, B))\}$$

where $[\ ,\ ]$ denotes homotopy classes of maps and $\{Q_n(C)\}$ is the set of orbits of the action of $G(C)$ on $Q_n(C)$.

2.7. REMARK. If A is the algebra of complex continuous functions on the 3-sphere, B is the algebra of all 2×2-matrices over $\mathbf{C}$ and $n = 2$, we reobtain the example of [**PR1**, 7.13].

3. Lifting C^1-curves. The transport equation. In this section we describe a method which leads to a lifting Γ of a curve $\gamma: [a, b] \to Q_n$, as in Corollary 2.3, valid when γ is rectifiable and continuous. For the sake of simplicity we only consider $n = 2$, the general case being similar and somewhat more involved. The reader can find the details (for $n = 2$) in [**PR1**]. Our present interest in this construction lies in that it leads to the transport equation.

Consider a continuous rectifiable curve $\gamma: [a, t] \to Q$ and a partition $\Pi: t_0 = a < t_1 < \cdots < t_n = t$ such that $\|\gamma_k - \gamma_{k+1}\| < 1$ $(k = 0, \ldots, n-1)$, where $\gamma_k = \gamma(t_k)$; then

$$\sigma_k = \gamma_k\gamma_{k-1} + (1-\gamma_k)(1-\gamma_{k-1}) \in G \qquad (k = 0, \ldots, n-1) \quad \text{and}$$

$$\sigma_k\gamma_0\sigma_1^{-1} = \gamma_1,$$
$$\sigma_2\sigma_1\gamma_0\sigma_1^{-1}\sigma_2^{-1} = \sigma_2\gamma_1\sigma_2^{-1} = \gamma_2, \ldots, \sigma_n\cdots\sigma_1\gamma_0\sigma_1^{-1}\cdots\sigma_n^{-1} = \gamma_n.$$

Thus, σ can be thought of as a "discrete" curve of units which conjugates γ_0 with γ_n. Putting $u(\Pi) = \sigma_n \cdots \sigma_1$, it can be shown [**PR1**, §5] that the limit $\Gamma(t) = \lim u(\Pi)$, when the length of the partition Π tends to zero, exists and defines a unit of the algebra. Moreover

$\Gamma\colon [a, b] \to G$ is continuous and rectifiable. If the original curve γ has a continuous derivative, then the value

$$\begin{aligned}
&(1/h)(\Gamma(t+h) - \Gamma(t) \quad \text{is, approximately,}\\
&(1/h)(\sigma_{t+h}\Gamma(t) - \Gamma(t)), \quad \text{where}\\
&\sigma_{t+h} = \gamma(t+h)\gamma(t) + (1-\gamma(t+h))(1-\gamma(t)).
\end{aligned}$$

Then,

$$\begin{aligned}
(1/h)(\Gamma(t+h) - \gamma(t)) &\cong (1/h)(\sigma_{t+h} - 1)\Gamma(t)\\
&= (1/h)(2\gamma(t+h)\gamma(t) - \gamma(t+h) - \gamma(t))\Gamma(t)\\
&= (1/h)\{\gamma(t+h)(\gamma(t) - \gamma(t+h)) + (\gamma(t+h) - \gamma(t))\gamma(t)\}\Gamma(t)
\end{aligned}$$

and

$$\begin{aligned}
\dot{\Gamma}(t) &= \lim_{h\to 0}(1/h)(\Gamma(t+h) - \Gamma(t))\\
&= \{-\gamma(t)\dot{\gamma}(t) + \dot{\gamma}(t)\gamma(t)\}\Gamma(t).
\end{aligned}$$

Thus, the lifting Γ of γ constructed by the limiting process described above satisfies the initial values problem

$$\begin{aligned}
&\dot{\Gamma} = (\dot{\gamma}\gamma - \gamma\dot{\gamma}),\\
&\Gamma(0) = 1.
\end{aligned}$$

In the general case $n > 2$ the initial value problem is

$$\begin{aligned}
&\dot{\Gamma} = \left(\sum_1^n \dot{\gamma}_k\gamma_k\right)\Gamma,\\
&\Gamma(0) = 1,
\end{aligned}$$

where $\gamma = (\gamma_1, \dots, \gamma_n)\colon [a, b] \to Q_n$ is of class C^1. Observe that $\sum_1^2 \dot{\gamma}_k\gamma_k = \dot{\gamma}_1\gamma_1 - \dot{\gamma}_1(1-\gamma_1) = \dot{\gamma}_1\gamma_1 - \gamma_1\dot{\gamma}_1$ because $\gamma_2 = 1 - \gamma_2$ and $\dot{\gamma}_1 = \dot{\gamma}_1\gamma_1 + \gamma_1\dot{\gamma}_1$ (differentiate $\gamma_1^2 = \gamma_1$).

As we said before, we shall not justify all the assertions about Γ. Instead we include the proof of the following result due to Daleckii, Krein and Kato, for the sake of completeness (see [**DK2**, IV, Theorem 1.1]).

3.1. THEOREM. *Let $\gamma\colon [a, b] \to Q_n$ be a C^1 curve. Then, the unique solution in A of the initial conditions problem*

$$\begin{aligned}
&\dot{\Gamma} = \hat{\gamma}\Gamma,\\
&\Gamma(a) = 1,
\end{aligned}$$

where $\hat{\gamma} = \sum_{k=1}^{n} \dot{\gamma}_k \gamma_k$, *satisfies*

(i) $\Gamma(t) \in G \quad (t \in [a, b])$,

(ii) $\Gamma(t)\gamma(a)\Gamma(t)^{-1} = \gamma(t) \quad (t \in [a, b])$.

Proof. Existence and uniqueness of Γ follow from general facts [**La**, p. 71]. To prove (i) consider the companion problem

$$\begin{cases} \dot{\Delta} = -\Delta\hat{\gamma}, \\ \Delta(a) = 1, \end{cases}$$

and observe that $(\Delta\Gamma)^{\cdot} = \dot{\Delta}\Gamma + \Delta\dot{\Gamma} = 0$. Then $\Delta\Gamma$ is constant on $[a, b]$ and, since $\Delta(a) = \Gamma(a) = 1$, it is $\Delta\Gamma \equiv 1$. Thus $\Gamma(t)$ is left invertible in A; moreover, $\Gamma(t)$ belongs to the connected component of the identity in the set of left invertible elements. It is easy to see that this component is completely contained in G. This proves (i).

To see (ii) we compute $(\Gamma^{-1}\gamma_k\Gamma)^{\cdot} \quad (k = 1, \dots, n)$:

$$\begin{aligned} (\Gamma^{-1}\gamma_k\Gamma)^{\cdot} &= -\Gamma^{-1}\dot{\Gamma}\Gamma^{-1}\gamma_k\Gamma + \Gamma^{-1}\dot{\gamma}_k\Gamma + \Gamma^{-1}\gamma_k\dot{\Gamma} \\ &= -\Gamma^{-1}\{\hat{\gamma}\gamma_k - \dot{\gamma}_k - \gamma_k\hat{\gamma}\}\Gamma; \end{aligned}$$

observe that $\hat{\gamma}\gamma_k = (\sum \dot{\gamma}_i\gamma_i)\gamma_k = \dot{\gamma}_k\gamma_k$, because $\gamma_i\gamma_k = 0$ for $i \neq k$, and that $\gamma_k\hat{\gamma} = \gamma_k(\sum \dot{\gamma}_i\gamma_i) = -\gamma_k(\sum \gamma_i\dot{\gamma}_i) = -\gamma_k\dot{\gamma}_k$, because $\dot{\gamma}_k = \dot{\gamma}_k\gamma_k + \gamma_k\dot{\gamma}_k$ and $\sum \dot{\gamma}_k = (\sum \gamma_k)^{\cdot} = 1^{\cdot} = 0$. Thus

$$(\gamma^{-1}\gamma_k\Gamma)^{\cdot} = -\Gamma^{-1}\{\dot{\gamma}_k\gamma_k - \dot{\gamma}_k + \gamma_k\dot{\gamma}_k\}\Gamma = 0$$

and $\Gamma^{-1}\gamma_k\Gamma$ is constantly $\gamma_k(a)$. This completes the proof of (ii).

3.2. Remark. The proof of part (i) could have been omitted because it is a general fact that the solution of $\dot{\Gamma} = \varphi\Gamma$, $\Gamma(a) = 1$, where $\varphi: [a, b] \to A$ is a continuous curve, is a curve of invertible element of A.

If A is an involutive Banach algebra, i.e. there exists a continuous antilinear mapping $x \to x^*$ such that $(xy)^* = y^*x^*$, $1^* = 1$ and $x^{**} = x \quad (x, y \in A)$, we consider the unitary group of A

$$U = \{u \in G: u^{-1} = u^*\}$$

and the selfadjoint part of Q_n

$$P_n = \{p = (p_1, \dots, p_n) \in Q_n: p_k^* = p_k \quad (k = 1, \dots, n)\}.$$

For these algebras more specific results hold. We omit the details about the differential structure of P_n.

3.3. COROLLARY. *If* $\gamma: [a, b] \to P_n$ *is a* C^1 *curve then the solution of* $\dot{\Gamma} = \hat{\gamma}\Gamma$, $\Gamma(a) = 1$, *defines a curve* $\Gamma: [a, b] \to U$ *which conjugates the curve* γ.

Proof. It suffices to show that $\Gamma(t) \in U$ for every $t \in [a, b]$. Observe first that

$$\begin{aligned}\dot{\Gamma}^* &= \left\{\left(\sum \dot{\gamma}_k\gamma_k\right)\Gamma^* = \Gamma^*\left(\sum \dot{\gamma}_k\gamma_k\right)^*\right. \\ &= \Gamma^*\left(\sum \gamma_k\dot{\gamma}_k\right) = -\Gamma^*\left(\sum \dot{\gamma}_k\gamma_k\right),\end{aligned}$$

because

$$\sum \dot{\gamma}_k\gamma_k + \gamma_k\dot{\gamma}_k = \sum \dot{\gamma}_k = \left(\sum \gamma_k\right)^{\cdot} = 1^{\cdot} = 0.$$

Thus $(\Gamma^*\Gamma)^{\cdot} = \dot{\Gamma}^*\Gamma + \Gamma^*\dot{\Gamma} = 0$ and $\Gamma^*\Gamma$ is constant. But $\Gamma(0) = \Gamma^*(0) = 1$, so $\Gamma^*\Gamma = 1$. Now, $\Gamma(t)$ is invertible for all t, by Theorem 3.1, so $\Gamma(t)^* = \Gamma(t)^{-1}$.

3.4. REMARK. Of course many liftings of γ may exist. But Γ is the unique horizontal lifting of γ with respect to the connection we shall define in the next section. This fact completes Kato's remark [**Ka**, II.4.2, Remark 4.4]. Moreover, if our σ's, used to obtain the transport equation, are multiplied (at left or at right) by $(1 - (\gamma_k - \gamma_{k-1})^2)^{-1/2}$, where $(1-r)^{-1/2} = \sum_{m=0}^{\infty}\binom{-1/2}{m}(-r)^m$ for $\|r\| < 1$, we get a different "discrete" lifting of γ but in the limit it becomes the same continuous curve Γ. In this sense, the local solution [**Ka**, p. 102, (4.18)]

$$\Gamma_1(t) = (1 - (\gamma(t) - \gamma(0))^2)^{-1/2}(\gamma(t)\gamma(0) + (1 - \gamma(t)))(1 - \gamma(0))$$

is related to the global solution Γ.

4. The connection. Let $q \in Q_n$ be fixed and $\pi: G \to Q_n$ defined by $\pi(g) = gqg^{-1} = (gq_1g^{-1}, \dots, gq_ng^{-1})$. It is very easy to show that the derivative of π at $g \in G(T\pi)_g: (TG)_g: (TG)_g \to (TQ_n)_{\pi(g)}$ is given by

$$(T\pi)_g(X) = g[g^{-1}X, q]g^{-1} \qquad (X \in (TG)_g)$$

where $[Z, q] = ([Z, q_1], \dots, [Z, q_n])$ for all $Z \in A$.

We say that $X \in (TG)_g$ is *vertical* if $(T\pi)_g(X) = 0$ or, what is the same, if $[g^{-1}X, q] = 0$. Then, if $V_g = \{X \in (TG)_g: [g^{-1}X, q] = 0\}$, it is clear that $V_g = g \cdot V_1$ and that

$$\begin{aligned} V_1 &= \{X \in A = (TG)_g : [X, q] = 0\} \\ &= \{X \in A : q_k X q_i = 0 \text{ for all } i \neq k\} \\ &= \left\{ \sum_{i=1}^{n} q_i X q_i : X \in A \right\}. \end{aligned}$$

This shows that

$$\begin{aligned} H_1 &= \{X \in A : q_i X q_i = 0 \ (i = 1, \ldots, n)\} \\ &= \left\{ \sum_{k \neq i} q_k X q_i : X \in A \right\} \end{aligned}$$

is a supplement of V_1 in A $(= (TG)_1)$ and, in general $H_g = gH_1$ is a supplement of V_g in A $(= (TG)_g)$. Moreover, $H_g \cdot h = H_{gh}$ $(g \in G, h \in H)$. Finally, the projections $h_g : (TG)_g \to H_g$, $v_g : (TG)_g \to V_g$ given by

$$\begin{aligned} h_g(X) &= g \sum_{i \neq k} q_k g^{-1} X q_i, \\ v_g(X) &= g \sum_{i=1}^{n} q_i g^{-1} X q_i, \end{aligned}$$

verify

$$\begin{aligned} h_g(X) &= g h_1(g^{-1} X), \\ v_g(X) &= g v_1(g^{-1} X). \end{aligned}$$

Clearly the mappings $g \to h_g$ and $g \to v_g$ from G into the bounded linear operators on A are differentiable. All these facts show that $g \to H_g$ defines a connection in the principal bundle $\pi : G \to Q'_n$.

For the theory of connections we refer the reader to [**KN**]. However, we are dealing with Banach manifolds and bundles, which requires a few notational changes.

From now on by "curve" we mean a C^∞ curve.

Given a curve $\gamma : [\alpha, \beta] \to Q_n$, a *horizontal lifting* of γ is a curve $\Gamma : [\alpha, \beta] \to G$ such that $\pi\Gamma = \gamma$ and $\dot{\Gamma}(t) \in H_{\Gamma(t)}$ $(t \in [\alpha, \beta])$.

It is a general fact that, for each $g_0 \in G$ such that $\gamma(\alpha) = g_0 p g_0^{-1}$, there is a unique horizontal lifting Γ such that $\Gamma(\alpha) = g_0$. In particular, if $\gamma(\alpha) = q$ there is a unique horizontal lifting Γ such that $\Gamma(\alpha) = 1$.

4.1. THEOREM. *Given a curve $\gamma\colon [\alpha, \beta] \to Q_n$ the horizontal lifting Γ such that $\Gamma(\alpha) = 1$ is the solution of the transport equation*

$$\dot\Gamma = \hat\gamma\Gamma, \quad \textit{where } \hat\gamma = \sum_{i=1}^{n} \dot\gamma_i\gamma_i, \tag{4.2}$$

with initial condition $\Gamma(\alpha) = 1$.

Proof. We have seen that the solution Γ of (4.2) is a lifting of π, i.e. $\pi\circ\Gamma = \gamma$ (see 3.1). By the uniqueness of both objects it suffices to show that the horizontal lifting Γ with $\Gamma(\alpha) = 1$ satisfies (4.2). We recall that Γ satisfies

$$\Gamma(t)q\Gamma(t)^{-1} = \gamma(t) \qquad (t\in[\alpha, \beta]), \tag{4.3}$$

$$\dot\Gamma \in H_\Gamma = \Gamma H_1, \text{ i.e. } \dot\Gamma(t)\in\Gamma(t)H_1 \qquad (t\in[\alpha, \beta]) \tag{4.4}$$

or, what is the same

$$\Gamma^{-1}\gamma\Gamma = q \tag{4.5}$$

and

$$\Gamma^{-1}\dot\Gamma \in H_1. \tag{4.6}$$

Differentiating (4.5) we get $0 = \Gamma^{-1}(-\dot\Gamma\Gamma^{-1}\gamma + \dot\gamma + \gamma\dot\Gamma\Gamma^{-1})\Gamma$ and cancelling Γ^{-1} and Γ, we get

$$\dot\gamma = [\dot\Gamma\Gamma^{-1}, \gamma]. \tag{4.7}$$

Now, (4.6) means that $q_i\Gamma^{-1}\dot\Gamma q_1 = 0$, $(i = 1, \dots, n)$, which can also be written as

$$q\Gamma^{-1}\dot\Gamma = \Gamma^{-1}\dot\Gamma(1-q). \tag{4.8}$$

Replacing (4.5) in (4.8) we get $\Gamma^{-1}\gamma\dot\Gamma = \Gamma^{-1}\dot\Gamma - \Gamma^{-1}\dot\Gamma\Gamma^{-1}\gamma\Gamma$ which, after cancellation, gives

$$\gamma\dot\Gamma\Gamma^{-1} = \dot\Gamma\Gamma^{-1}(1-\gamma) \tag{4.9}$$

and

$$\dot\Gamma\Gamma^{-1}\gamma = (1-\gamma)\dot\Gamma\Gamma^{-1}. \tag{4.10}$$

Finally,

$$\begin{aligned}\hat\gamma\Gamma &= \left(\sum_i^n \dot\gamma_i\gamma_i\right)\Gamma \\ &= \sum_1^n [\dot\Gamma\Gamma^{-1}, \gamma_i]\gamma_i\Gamma \qquad \text{(by 4.7)} \\ &= \sum_1^n \{\dot\Gamma\Gamma^{-1}\gamma_i - \gamma_i\dot\Gamma\Gamma^{-1}\gamma_i\}\Gamma.\end{aligned}$$

This last expression coincides with $\dot{\Gamma}$ because $\gamma_i\dot{\Gamma}\Gamma^{-1} = \dot{\Gamma}\Gamma^{-1}(1-\gamma_i)$ by (4.9) and therefore $\gamma_i\dot{\Gamma}\Gamma^{-1}\gamma_i = \dot{\Gamma}\Gamma^{-1}(1-\gamma_i)\gamma_i = 0$. This proves the theorem.

4.11. REMARK. In general, if $\gamma\colon [\alpha, \beta] \to Q_n$ is a curve with origin $q' = g_0 q g_0^{-1}$ then Γ is the horizontal lifting with origin g_0 if and only if it is the solution of the problem $\dot{\Gamma} = \hat{\gamma}\Gamma$, $\Gamma(\alpha) = g_0$.

We compute next the 1-form, the 2-form and the curvature form of the connection.

We recall that the 1-form θ assigns to each $X \in (TG)_g$ the horizontal component of $g^{-1}X \in (TG)_1 = \mathscr{L}$, the Lie algebra of H. More explicitly,

$$\theta_g X = v_1(g^{-1}X) = g^{-1}v_g(X) = \sum_{i=1}^{n} q_i g^{-1} X q_i.$$

The 2-form $d\theta$ of the connection is defined by

$$d\theta(X, Y) = \tfrac{1}{2}\{X \cdot \theta Y - Y \cdot \theta X - \theta([X, Y])\},$$

where $X, Y \in (TG)_g$, $[\ ,\ ]$ denotes the Lie bracket and $Z \cdot W$ denotes the derivative of W in the direction of Z, i.e. W is extended to a vector field on a neighborhood of g and given a curve $\delta\colon (-\varepsilon, \varepsilon) \to G$ such that $\delta(0) = g$ and $\dot{\delta}(0) = Z$,

$$Z \cdot W = \frac{d}{dt_{t=0}} W(\delta(t)).$$

Although the notation is the same, the Lie bracket should not be confused with the commutator bracket of the algebra.

From the computations

$$\begin{aligned} X \cdot \theta Y &= X \cdot \left(\sum_{i=1}^{n} q_i g^{-1} Y q_i\right) \\ &= -\sum_{i=1}^{n} q_i g^{-1} X g^{-1} Y q_i + \sum_{i=1}^{n} q_i g^{-1} X \cdot Y q_i, \\ Y \cdot \theta X &= -\sum_{i=1}^{n} q_i g^{-1} Y g^{-1} X q_i + \sum_{i=1}^{n} q_i g^{-1} Y \cdot X q_i, \end{aligned}$$

and

$$\theta([X, Y]) = \sum_{i=1}^{n} q_i g^{-1}[X, Y] q_i,$$

we get

$$\begin{aligned} d\theta(X, Y) &= \frac{1}{2}\sum_{i=1}^{n} q_i[g^{-1}Y, g^{-1}X]q_i \\ &= \left(-\frac{1}{2}\right)\sum_{i=1}^{n} q_i[g^{-1}X, g^{-1}Y]q_i. \end{aligned}$$

The *horizontal differential of* θ, also called the *curvature form of the connection* is $\Omega(X, Y) = d\theta(h_gX, h_gY)$ for $[X, Y] \in (TG)_g$. Explicitly

$$\begin{aligned} \Omega(X, Y) &= \left(-\frac{1}{2}\right)\sum_{i=1}^{n} q_i[g^{-1}h_gX, g^{-1}h_gY]q_i \\ &= \left(-\frac{1}{2}\right)\sum_{i=1}^{n} q_i\left[\sum_{k\neq l} q_kg^{-1}Xq_l, \sum_{r\neq s} q_rg^{-1}Yq_s\right] q_i \\ &= \left(-\frac{1}{2}\right)\sum_{i=1}^{n} q_ig^{-1}\{X(1-q_i)g^{-1}Y - Y(1-q_i)g^{-1}X\}q_i \\ &= \left(-\frac{1}{2}\right)\sum_{i=1}^{n} q_ig^{-1}\{X\bar{q}_ig^{-1}Y - Y\bar{q}_ig^{-1}X\}q_i, \\ &\qquad \left(\text{where } \bar{q}_k = 1 - q_k = \sum_{i\neq k} q_i\right) \\ &= \left(-\frac{1}{2}\right)\sum_{i=1}^{n} q_ig^{-1}(Xg^{-1}Y - Yg^{-1}X - Xq_ig^{-1}Y + Yq_ig^{-1}X)q_i. \end{aligned}$$

The *structure equation* $\Omega(X, Y) = d\theta(X, Y) + (\frac{1}{2})[\theta X, \theta Y]$ is thus trivially satisfied.

5. Calculations on the tangent bundle, geodesics. Consider $q \in Q_n$ fixed and let $A_1 = \{X \in A : q_iXq_i = 0, \ i = 1, \dots, n\}$ (in §4 we called it H_1). It is clear that $H = \{g \in G : gq_i = q_ig, \ i = 1, \dots, n\}$ operates at left on A_1 by $h \cdot X := hXh^{-1}$.

Thus we define the associated bundle of $\pi : G \to Q_n$ with standard fibre A_1, denoted by $G \otimes A_1 \to Q_n$, where $G \otimes A_1 := G \times A_1/\sim$, $(g, X) \sim (gh, h^{-1}X)$ for $h \in H$ and the map $G \otimes A_1 \to Q_n$ is determined by $(g, X) \to \pi(g)$. It is a general fact that this vector bundle is isomorphic to the tangent bundle TQ_n, by means of $(g, X) \to (\pi(g), gXg^{-1}) \in (TQ_n)_{\pi(g)}$. Given a curve $\gamma : [\alpha, \beta] \to$

Q_n the parallel displacement of the fibre $(TQ_n)_{\gamma(\alpha)}$ along γ from α to $t \in [\alpha, \beta]$ is defined by $\tau_\alpha^t \colon (TQ_n)_{\gamma(\alpha)} \to (TQ_n)_{\gamma(t)}$, $\tau_\alpha^t(Z) = \Gamma(t)Z\Gamma(t)^{-1}$, where Γ is the horizontal lifting of γ with origin $\Gamma(\alpha) = 1$.

Given $X \in (TQ_n)_q$ and a vector field Z defined near q the *covariant derivative* $D_X Z$ is $D_X Z := X \cdot Z + [Z, \widetilde{X}]$, where

$$\widetilde{X} = \sum_{i=1}^{n} X_i q_i \quad \text{and} \quad X \cdot Z = \frac{d}{dt_{t=0}} Z(\delta(t))$$

for a curve $\delta \colon (-\varepsilon, \varepsilon) \to Q_n$ such that $\delta(0) = q$ and $\dot{\delta}(0) = X$.

5.1. Proposition. *For every curve $a \colon [\alpha, \beta] \to A^n$ the element $Da/dt = \dot{a} + [a, \hat{\gamma}]$ is well defined and has the following properties:*

(a) *if $\gamma_i a \gamma_i = 0$ for all $i = 1, \dots, n$ then $\gamma_i (Da/dt) \gamma_i = 0$ for all $i = 1, \dots, n$ (in other words, Da/dt is tangent if a is tangent).*

(b) *if $\gamma_i a \gamma_k = 0$ for all $i \neq k$ then $\gamma_i (Da/dt) \gamma_k = 0$ for all $i \neq k$ (i.e. Da/dt is normal if a is normal).*

Proof. (a) Differentiating $\gamma_i a \gamma_i = 0$ we get

$$0 = \dot{\gamma}_i a \gamma_i + \gamma_i \dot{a} \gamma_i + \gamma_i a \dot{\gamma}_i.$$

Multiplying by γ_i at right and left we have

$$\gamma_i \dot{\gamma}_i a \gamma_i + \gamma_i \dot{a} \gamma_i + \gamma_i a \dot{\gamma}_i \gamma_i = 0. \tag{5.2}$$

On the other hand

$$\begin{aligned} \gamma_i \frac{Da}{dt} \gamma_i &= \gamma_i \dot{a} \gamma_i + \gamma_i [a, \hat{\gamma}] \gamma_i \\ &= \gamma_i \dot{a} \gamma_i + \gamma_i \left(a \sum \dot{\gamma}_k \gamma_k - \sum \dot{\gamma}_k \gamma_k a \right) \gamma_i \\ &= \gamma_i \dot{a} \gamma_i + \gamma_i a \dot{\gamma}_i \gamma_i - \gamma_i \sum \dot{\gamma}_k \gamma_k a \gamma_i \end{aligned}$$

and $\gamma_i \sum_k \dot{\gamma}_k \gamma_k = \gamma_i \sum_k (1 - \gamma_k) \dot{\gamma}_k$ because $\dot{\gamma}_k = \dot{\gamma}_k \gamma_k + \gamma_k \dot{\gamma}_k$ (differentiate $\gamma_k^2 = \gamma_k$); thus

$$\gamma_i \sum_k \dot{\gamma}_k \gamma_k = \gamma_i \sum_k \dot{\gamma}_k - \gamma_i \sum \gamma_k \dot{\gamma}_k = -\gamma_i \dot{\gamma}_i,$$

because $\sum_k \dot{\gamma}_k = 0$ and $\gamma_i \gamma_k = 0$ if $i \neq k$.

This shows that

$$\gamma_i \frac{Da}{dt} \gamma_i = \gamma_i \dot{a} \gamma_i + \gamma_i a \dot{\gamma}_i \gamma_i + \gamma_i \dot{\gamma}_i a \gamma_i = 0, \qquad \text{by (4.2).}$$

The proof of (b) is similar.

This shows that for every vector field Y of Q_n along γ, the formula $Da/dt = \dot{Y} + [Y, \hat{\gamma}]$ defines another vector field of Q_n, the *covariant derivative* of Y.

The *torsion* of the connection, defined by $T(X, Y) = D_X Y - D_Y X - [X, Y]$ in general, turns out to be in our case

$$(5.3) \qquad T(X, Y) = [Y, \widetilde{X}] - [X, \widetilde{Y}],$$

where $X, Y \in (TQ_n)_g$ and $\widetilde{X} = \sum_{i=1}^n X_i q_i$, $\widetilde{Y} = \sum_{i=1}^n Y_i q_i$.

5.4. REMARK. For $n = 2$ the connection is symmetric, in the sense that its torsion is zero everywhere: in fact, for $n = 2$ we have $X_1 + X_2 = 0$, $Y_1 + Y_2 = 0$, $q_1 + q_2 = 1$, $q_i X_i = X_i(1 - q_i)$, $q_i X_j = -X_i q_j$.

These equalities, when replaced in (4.3), prove the assertion. However, for $n > 3$ this is no longer true.

The *curvature* of the connection, expressed by $R(X, Y)Z = D_X(D_Y Z) - D_Y(D_X Z) - D_{[X, Y]}Z$ for $X, Y, Z \in (TQ_n)_q$, is given, in our case, by

$$(5.5) \qquad R(X, Y)Z = \left[\sum_{i=1}^n [X_i, Y_i]q_i, Z\right]$$

or, abbreviating

$$(5.6) \qquad R(X, Y)Z = [[X, Y]^\sim, Z].$$

We study now the geodesic curves of the connection, that is, the curves $\gamma\colon [\alpha, \beta] \to Q_n$ such that $D\dot{\gamma}/dt = 0$. It is a well-known fact that this condition is equivalent to $\tau_\alpha^t(\dot{\gamma}(\alpha)) = \dot{\gamma}(t)$, $(t \in [\alpha, \beta])$. The equation defining the geodesic curves can be written as

$$(5.7) \qquad \ddot{\gamma}_k + [\dot{\gamma}_k, \hat{\gamma}] = 0, \qquad k = 1, \dots, n.$$

Using the commutation rules obtained from $\sum \gamma_i = 1$, $\gamma_i^2 = \gamma_i$ and $\gamma_i \gamma_k = 0$ for $i \neq k$, we get

(i) $\dot{\gamma}_i \gamma_i = (1 - \gamma_i)\dot{\gamma}_i$ $(i = 1, \dots, n)$;
(ii) $\dot{\gamma}_i \gamma_k + \gamma_i \dot{\gamma}_k = 0$ $(i \neq k)$;
(iii) $\sum_i^n \dot{\gamma}_k = 0$;
(iv) $\gamma_i \dot{\gamma}_i^2 = \dot{\gamma}_i^2 \gamma_i$ $(i = 1, \dots, n)$;
(v) $\gamma_i \dot{\gamma}_i \gamma_i = 0$ $(i = 1, \dots, n)$.

These equalities imply that (5.7) is equivalent to

$$(5.8) \quad \ddot{\gamma}_k + \gamma_k\left(\sum_1^n \dot{\gamma}_i^2\right) + \left(\sum_1^n \dot{\gamma}_i^2\right)\gamma_k - 2\dot{\gamma}_k^2 = 0, \qquad (k = 1, \dots, n).$$

It is easy to exhibit all the solutions of (5.8) which satisfy $\gamma(t) \in Q_n$ for all t. In fact, for $q \in Q_n$, $X \in (TQ_n)_q$, $\gamma(t) = e^{t\widetilde{X}} q e^{-t\widetilde{X}}$ $(t \in R)$, satisfies (5.8) and all the solutions of (5.8) with the additional condition $\gamma(t) \in Q_n$, have this form. The connection is also complete, in the sense that its geodesics are defined for all $t \in R$, and the exponential map of the connection is given by

$$\mathrm{Exp}_q : (TQ_n)_q \to Q_n, \quad \mathrm{Exp}_q(X) = e^{\widetilde{X}} q e^{-\widetilde{X}}.$$

Properties of minimality of length of geodesics are studied in a forthcoming paper ([**CPR2**]).

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Two C^*–Algebra Inequalities

Gustavo Corach, Horacio Porta[1] and Lázaro Recht[1]

Dedicated to Mischa Cotlar on his 75th birthday.

The purpose of this note is to prove the following inequalities

(1) $$\|\eta\| \leq \|c\eta a \pm b\eta^* b\|$$

(2) $$\|c\eta b^* \pm b\eta^* c\| \leq K\, \|c\eta a \pm b\eta^* b\|$$

where η and b are elements of a C^*–algebra $\mathcal{A}$, a and c are the positive square roots of $1 + b^*b$ and $1 + bb^*$ respectively, and $K = 2\|b\|\sqrt{1 + \|b\|^2}/(1 + 2\|b\|^2)$.

Inequality (1) has a geometric interpretation that can be briefly described as follows. Denote by Q and P the subsets of $\mathcal{A}$ formed by all projections (=idempotent elements) and all self-adjoint projections. The polar decomposition allows us to define a retraction $\pi : Q \to P$ from Q to P and inequality (1) applies to conclude that the tangent map $d\pi : TQ \to TP$ is a contraction. Inequality (2) has similar application to the study of hyperbolic unitary groups. Suppose $p \in P$ and $B(x, y) = \langle (2p - I)x, y\rangle$ (here we assume that $\mathcal{A}$ is represented in a Hilbert space). Denote by U the group of B–unitary elements of $\mathcal{A}$ and by V the subset of positive elements of U. The "bending" of V in U can be partially described by the following fact: for $Z \in TV_v$ decompose $v^{-1}Z = Z_0 + Z_1$ where Z_0 commutes with p and Z_1 anticommutes with p; then $\|Z_0\| < \|Z_1\|$. This follows from inequality (2). For details and complete proofs we refer to our forthcoming paper *The structure of projections in a C^*–algebra.*

Some abbreviations will be helpful: $r = \|b\|$, $T = ba^{-1} = c^{-1}b$, $D = b^*b(1 + b^*b)^{-1} = T^*T$, $F = bb^*(1 + bb^*)^{-1} = TT^*$. Define also real-linear maps from $\mathcal{A}$ into itself by $\Delta(\eta) = T\eta^*T$, $\Psi(\eta) = \eta - \Delta(\eta)$, $\Phi(\eta) = c\eta a - b\eta^* b$. Calculations give $\Delta^{2n}(\eta) = F^n \eta D^n$, $\Delta^{2n+1}(\eta) = F^n T\eta^* TD^n$ for $n = 0, 1, 2, \ldots$ Hence using $\|D\| = \|F\| = r^2/(1 + r^2)$, $\|T\| = r/\sqrt{1 + r^2}$ we get

$$\|\Delta^{2n}(\eta)\| \leq (r^2/(1 + r^2))^{2n}\|\eta\|$$
$$\|\Delta^{2n+1}(\eta)\| \leq (r^2/(1 + r^2))^{2n+1}\|\eta\|$$

so that $\sum_{k=0}^{\infty} \Delta^k$ converges to the inverse of $\Psi = I - \Delta$. This means that Φ is also invertible, and

$$\Phi^{-1}(\xi) = \Psi^{-1}(c^{-1}\xi a^{-1}) = \sum_{k=0}^{\infty} \Delta^k (c^{-1}\xi a^{-1})$$

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whence

$$\|\Phi^{-1}(\xi)\| \leq \Big(\sum_{k=0}^{\infty}\big(\frac{r^2}{1+r^2}\big)^k\Big)\,\|c^{-1}\|\,\|a^{-1}\|\,\|\xi\|.$$

But $\|a^{-1}\| = \|c^{-1}\| = (1+r^2)^{-1/2}$ implies

$$\Big(\sum_{k=0}^{\infty}\big(\frac{r^2}{1+r^2}\big)^k\Big)\|c^{-1}\|\,\|a^{-1}\| = \frac{1}{1-(r^2/(1+r^2))}\cdot\frac{1}{1+r^2} = 1$$

and so $\|\Phi^{-1}(\xi)\| \leq \|\xi\|$. Setting $\xi = c\eta a - b\eta^* b$ gives (1) with $-$ sign. Changing η into $i\eta$ gives the $+$ sign.

To prove (2) we will calculate:

$$\begin{aligned}&c\Phi^{-1}(\xi)b^* - b\big(\Phi^{-1}(\xi)\big)^*c\\ &\quad= \sum_{k=0}^{\infty}\big(c\Delta^k(c^{-1}\xi a^{-1})b^* - b\big(\Delta^k(c^{-1}\xi a^{-1})\big)^*c\big).\end{aligned}$$

The term of order $2n$ reads

$$\begin{aligned}&c\Delta^{2n}(c^{-1}\xi a^{-1})b^* - b\big(\Delta^{2n}(c^{-1}\xi a^{-1})\big)^*c\\ &\quad= cF^n c^{-1}\xi a^{-1}D^n b^* - bD^n a^{-1}\xi^* c^{-1}F^n c.\end{aligned}$$

The identities $cF = Fc$, $Db^* = b^*F$, $T = ba^{-1}$, and $T^* = a^{-1}b^*$ reduce this expression to $F^n\big(\xi T^* - T\xi^*\big)F^n$. Similarly for terms of odd order $2n+1$:

$$\begin{aligned}&c\Delta^{2n+1}(c^{-1}\xi a^{-1})b^* - b\big(\Delta^{2n+1}(c^{-1}\xi a^{-1})\big)^*c\\ &\quad= cF^nTa^{-1}\xi^*c^{-1}TD^nb^* - bD^nT^*c^{-1}\xi a^{-1}T^*F^nc,\end{aligned}$$

and using now also the identities $cTa^{-1} = T$, $c^{-1}Tb^* = bT^*c^{-1} = TT^* = F$, and $bD = Fb$ we obtain the simpler expression $F^n\big(T\xi^*F - F\xi T^*\big)F^n$. Thus combining the terms of order $2n$ and $2n+1$ the series has the form

$$\begin{aligned}&c\Phi^{-1}(\xi)b^* - b\big(\Phi^{-1}(\xi)\big)^*c\\ &\quad= \sum_{n=0}^{\infty}F^n\big((1-F)\xi T^* - T\xi^*(1-F)\big)F^n.\end{aligned}$$

From $\|1-F\| = \|(1+bb^*)^{-1}\| = (1+r^2)^{-1}$ and $\|T\| = r/\sqrt{1+r^2}$ we get

$$\|(1-F)\xi T^* - T\xi(1-F)\| \leq 2r(1+r^2)^{-3/2}\|\xi\|$$

and therefore

$$\begin{aligned}&\|c\Phi^{-1}(\xi)b^* - b\big(\Phi^{-1}(\xi)\big)^*c\|\\ &\quad\leq \Big(\sum_{n=0}^{\infty}\|F\|^{2n}2r(1+r^2)^{-3/2}\Big)\|\xi\|\\ &\quad= \left(\sum_{n=0}^{\infty}\Big(\frac{r^2}{1+r^2}\Big)^{2n}\cdot 2r(1+r^2)^{-3/2}\right)\|\xi\|.\end{aligned}$$

This proves (2) with $-$ signs; changing η into $i\eta$ produces the $+$ signs, so the proof is complete.

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