On a class of Hilbert-C*-manifolds

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Abstract. This paper deals with a special class of 'non-compact' hermitian infinite dimensional symmetric spaces, generically denoted by U. We calculate their invariant connection very explicitly and use the concept of a Hilbert-C*-manifold so that the Banach manifold in question is of the form $\operatorname{Aut} U/H$, where $\operatorname{Aut} U$ is the automorphism group of the Hilbert-C*-manifold. We finally use results previously obtained with D.Blecher to characterize causal structure on U that comes from interpreting the elements of U as bounded Hilbert space operators.

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1. Introduction

This note has two objectives, an explicit calculation, for all vector fields, of the invariant connection on a certain type of infinite dimensional symmetric space and, using results from [1], to characterize those invariant cone fields on a similar kind of spaces that can be thought of as the result of some kind of 'quantization'. Both questions are related since the invariance of the cone fields is intimately connected to the behavior of parallel transport along geodesics.

Invariant connections for finite dimensional symmetric spaces have long been known to exist and to be unique. One has to be a little bit more careful in the infinite dimensional, Banach manifold setting since there the existence of a sufficient amount of smooth functions no longer can be proven. The type of calculations we are interested in here have been carried out under similar circumstances in [2].

Important for our approach is to use an invariant Hilbert-C*-structure on the fibers of the tangent bundle. We show that the symmetric space we are dealing with can be defined in terms of the automorphism group of this structure.

The theory of invariant cone fields on finite dimensional spaces is very well developed. A comprehensive account is [5]. We will see in the last section that the infinite dimensional theory might behave slightly different.

2. Hilbert-C*-manifolds

2.1. Recall that a (left) Hilbert-C*-module over a C*-algebra $\mathfrak{A}$ is a complex vector space E which is a left $\mathfrak{A}$-module with a sesquilinear pairing $E \times E \to \mathfrak{A}$ satisfying, for $r, s \in E$ and $a \in \mathfrak{A}$, the following requirements:

(i) $\langle ar, s\rangle = a\langle r, s\rangle$

(ii) $\langle r, s\rangle = \langle s, r\rangle^*$

(iii) $\langle s, s\rangle > 0$ for $s \neq 0$

(iv) Equipped with the norm
$$\|s\| = \sqrt{\|\langle s, s\rangle\|},$$
E is a Banach space.

Right Hilbert-C*-modules are defined similarly. Whenever we want to refer to the algebra $\mathfrak{A}$ explicitly, we speak of a Hilbert-$\mathfrak{A}$-module.

2.2. The objects defined above coincide with the so called ternary rings of operators (TRO), which are intrinsically characterized in [9, 11]. On such a space E, a triple product $\{\cdot, \cdot, \cdot\}$ is given in such a way that E, up to (the obvious definition of) TRO-isomorphisms, is a subspace of a space of bounded Hilbert space operators $L(H)$, invariant under the triple product
$$\{x, y, z\} = xy^*z.$$
The relation to (left) Hilbert-C*-modules is based on the equation
$$\{x, y, z\} = \langle x, y\rangle z,$$
connecting triple product to module action (under $\mathfrak{A} = EE^*$, where the latter algebra does not depend on the chosen embedding) as well as scalar product of a Hilbert-$\mathfrak{A}$-module. Note that in particular the norm of an element $e \in E$ must coincide with $\|e\| = \|\{e, e, e\}\|^{1/3}$.

2.3. TRO-morphisms will be those that respect the product $\{\cdot, \cdot, \cdot\}$. In the language of a Hilbert-$\mathfrak{A}$-module, a TRO-morphism is an $\mathfrak{A}$-module map preserving the form $\langle \cdot, \cdot\rangle$. This is where both categories become different since Hilbert-$\mathfrak{A}$-morphisms are usually supposed to be adjointable with respect to the pairing $\langle \cdot, \cdot\rangle$.

Definition 2.4. *Let M be a Banach manifold and $\mathfrak{A}$ a C*-algebra. M is said to be a (right-, left-) Hilbert-$\mathfrak{A}$-manifold if on each tangent space $T_p(M)$ there is given the structure of a Hilbert-$\mathfrak{A}$-module depending smoothly on base points.*

Definition 2.5. *Let M be a Hilbert-C*-manifold. The group of automorphisms, $\operatorname{Aut} M$ consists of all diffeomorphisms $\Phi : M \to M$ so that $d\Phi$ is (pointwise) a TRO-morphism.*

2.6. It is not clear under which circumstances a Banach manifold can be given the structure of a Hilbert-C*-manifold. This is so because, first, no characterization seems to be known of Banach spaces that are (topologically linear) isomorphic to Hilbert-C*-modules, and, second, because in general, there is no smooth partition of the unit that would permit the step from local to global.

We will use group actions instead. Suppose M is a homogeneous space with respect to a smooth Banach Lie group action. Fix a base point $o \in M$, and denote the isotropy subgroup at o by H. Suppose that $T_o(M)$ carries the structure of a Hilbert-$\mathfrak{A}$-module with form $\langle \cdot, \cdot \rangle_o$ and module map $x \mapsto a \cdot_o x$. If for all $h \in H$, $x, y \in T_o(M)$, and $a \in \mathfrak{A}$ we have

$$\langle d_o h(x), d_o h(y) \rangle_o = \langle x, y \rangle_o \qquad \text{as well as} \qquad d_o h \left(a \cdot_o d_o h^{-1}(x) \right) = x$$

then a Hilbert-$\mathfrak{A}$-module on $T_p(M)$ for any $p = g(o) \in M$, is defined through

$$\langle x, y \rangle_p = \langle d_p g^{-1}(x), d_p g^{-1}(y) \rangle \qquad \text{and} \qquad a \cdot_p x = d_o g \left(a \cdot_o d_p g^{-1}(x) \right),$$

where $x, y \in T_p(M)$ and $a \in \mathfrak{A}$. As an illustration, if

$$\langle x, y \rangle_o \cdot_o z = \{x, y, z\}_o,$$

then both of the above conditions can be equivalently combined into

$$d_o h \{x, y, z\}_o = \{d_o h(x), d_o h(y), d_o h(z)\}_o \qquad \text{for alle } h \in H,\ x, y, z \in T_o(M),$$

so that for the TRO-structure at a

$$\{x, y, z\}_a = d_o g \{d_o g^{-1}(x), d_o g^{-1}(y), d_o g^{-1}(z)\}_o.$$

2.7. Our example here is the following. Fix a TRO E and denote by U its open unit ball. If we follow the path laid out above, we find the following invariant Hilbert-C*-structure on U. Define a triple product for $T_a M$ at $a \in U$ by

$$\{xyz\}_a = x(1 - a^*a)^{-1} y^* (1 - aa^*)^{-1} z,$$

so that

$$\langle x, y \rangle_a = (1 - aa^*)^{-1/2} x (1 - a^*a)^{-1} y^* (1 - aa^*)^{-1/2}$$

as well as

$$\gamma \cdot_a z = (1 - aa^*)^{1/2} \gamma (1 - aa^*)^{-1/2}, \qquad \gamma \in EE^*.$$

We will refer to this structure as the *canonical Hilbert-C*-structure* on U.

2.8. This definition is motivated in the following way. As shown in [4], $\operatorname{Hol} U$, the group of all biholomorphic automorphisms U, consists of mappings of the form $T \circ M_a$, where for any $a \in U$,

$$M_a(x) := (1 - aa^*)^{-1/2} (x + a)(1 + a^*x)^{-1} (1 - a^*a)^{1/2},$$

and T is a (linear) isometry of E, restricted to U. Then $\mathrm{Hol}\, U$ acts transitively on U, and the group of (linear) isometries is the isotropy subgroup at the point 0. For later use, we include here the fact that

$$d_x M_a(h) = (1-aa^*)^{1/2}(1+xa^*)^{-1}h(1+a^*x)^{-1}(1-a^*a)^{1/2}$$

as well as

$$\begin{aligned} &d_x^2 M_a(h_1,h_2) = \\ &\quad = -(1-aa^*)^{1/2}(1+xa^*)^{-1}h_2a^*(1+xa^*)h_1(1+a^*x)^{-1}(1-a^*a)^{1/2} \\ &\qquad -(1-aa^*)^{1/2}(1+xa^*)^{-1}h_1(1+a^*x)^{-1}a^*h_2(1+a^*x)^{-1}(1-a^*a)^{1/2}. \end{aligned}$$

Definition 2.9. *Suppose X is a (closed) subspace of $L(H)$. Equip*

$$M_n(X) = \{(x_{ij}) \mid x_{ij} \in X \text{ for } i,j = 1,\dots,n\}$$

with the norm it carries as a subspace of $M_n(L(H)) = L(H\oplus\cdots\oplus H)$, and, for a bounded operator $T : X \to X$, denote by $T^{(n)} = \mathrm{id}_{M_n(\mathbb{C})} \otimes X : M_n(X) \to M_n(X)$ the operator $(x_{ij}) \longmapsto (Tx_{ij})$. Then T is said to be completely bounded, iff

$$\|T\|_{cb} := \sup_{n\in\mathbb{N}} \|T^{(n)}\| < \infty.$$

Similarly, T is called a complete isometry iff each of the maps $T^{(n)}$ is an isometry.

We have the following

Theorem 2.10 ([3, 9]). *Let E be a TRO. Then $M_n(E)$ carries a distinguished TRO-structure, and the group of TRO-automorphisms of E coincides with with the group of complete isometries.*

2.11. The Hilbert-C*-structure from 2.7 is in fact constructed according to the construction in 2.6 for a group G, somewhat smaller than $\mathrm{Hol}\, U$.

Theorem *Let U be the open unit ball of a TRO E, equipped with the canonical Hilbert-C*-structure, and suppose $M_n(E)$ carries the standard TRO-structure for each $n \in \mathbb{N}$. Then a diffeomorphism $\Phi : U \to U$ is a Hilbert-C*-automorphism iff $\Phi = T \circ M_a$, where $a \in U$, and T is the restriction of a linear and completely isometric mapping of E to U. Furthermore, the canonical Hilbert-C*-structure on U is uniquely $\mathrm{Aut}\, U$-invariant, if $E = T_o U$ carries the given Hilbert-C*-structure.*

Proof: That each map of the form $T \circ M_a$ is in $\mathrm{Aut}\, U$ follows from the construction of the Hilbert-C*-structure on U as well as from Theorem 2.10. Since the derivative of any element $\Phi \in \mathrm{Aut}\, D$ is complex linear by definition, $\mathrm{Aut}\, D \subseteq \mathrm{Hol}\, D$, and hence $\Phi = T \circ M_a$ for some $a \in U$ and an isometry T. Because the linear map T fixes the origin, dT must be a TRO-automorphism, and the result follows, by another application of Theorem 2.10. □

3. The invariant connection

3.1. The definition of a connection for Banach manifolds cannot be, due to the scarcity of smooth functions, the usual one. We follow [6, 1.5.1 Definition].

Definition 3.2. *Let M be a manifold, modeled over the Banach space E, and denote the space of bounded bilinear mappings $E \times E \to E$ by $L^2(E,E)$. Then M is said to possess a connection iff there is an atlas $\mathcal{U}$ for M so that for each $U \in \mathcal{U}$ there is a smooth mapping $\Gamma : U \to L^2(E,E)$, called the* Christoffel symbol *of the connection on U, which under a change of coordinates Φ transforms according to*

$$\Gamma(\Phi' X, \Phi' Y) = \Phi''(X,Y) + \Phi'\Gamma(X,Y).$$

The covariant derivative of a vector field Y in the direction of the vector field X is, locally, defined to be the principal part of

$$\nabla_X Y = dX(Y) - \Gamma(X,Y),$$

where, in a chart, the principal part of $(u,X) \in U \times T(U)$ is X.

The reader should note that this definition is equivalent to specifying a smooth vector subbundle H of TTM with the property that H_p is for each $p \in TM$ closed and complementary to the tangent space $\ker(d_p\pi)$ of the fiber $E_{\pi(p)}$ through p. It is not, however, equivalent to the requirement that ∇ be $C^\infty(M)$-linear in its first variable and a derivation w.r.t. the action of $C^\infty(M)$ on vector fields, although connection as defined above do have this property.

3.3. We can keep the notion of invariance of a connection under the smooth action of a (Banach) Lie group, however. In fact, if such a group G acts on M then, for each $g \in G$ a connection $g^*\nabla$ is defined by letting

$$g^*\nabla_X Y = \nabla_{g^*X} g^*Y, \qquad g^*X(gm) = d_m g X(m).$$

Christoffel symbols then transform as in the definition above,

$$\Gamma_{g(m)}(g'(m)X(m), g'(m)Y(m)) = g''(m)(X(m),Y(m)) + g'(m)\Gamma_m(X(m),Y(m)),$$

and we call ∇ invariant under the action of G whenever $g^*\nabla = \nabla$ for all $g \in G$.

3.4. If M is the Hilbert-C*-manifold U that was defined in the last section, there can be at most one connection which is invariant under the action of the group of biholomorphic self-maps of U. This is because the difference of two of them is the difference of their Christoffel symbols which have to vanish at each point according to the way they transform under the reflections $\sigma_a = M_a \sigma_o M_{-a}$, $\sigma_o(x) = -x$. To find one, we let

$$\Gamma_o(x,y) = 0.$$

Then, for any g leaving the origin fixed, $g'' = 0$ and so Γ_0 remains zero when transformed under g. Using the transformation rule for Christoffel symbols we find

Theorem 3.5. *On the Hilbert-C*-manifold U there exists exactly one invariant connection whose Christoffel symbol at a is given by*

$$\begin{aligned}\Gamma_a(x,y) = y(1-a^*a)^{-1/2}a^*(1-aa^*)^{-1/2}x+ \\ + x(1-a^*a)^{-1/2}a^*(1-aa^*)^{-1/2}y = 2\{x, M_a'(0)a, y\}_a^s,\end{aligned}$$

where $\{x,y,z\}^s = \frac{1}{2}(\{z,y,x\}+\{x,y,z\})$ denotes the (symmetric) Jordan triple product on E.

3.6. The reader should observe that the form $\{\cdot,\cdot,\cdot\}$ defined above is parallel for the invariant connection, i.e. for all vector fields X, Y, Z and W we have

$$\nabla_W\{XYZ\} = \{\nabla_W XYZ\} + \{X\nabla_W YZ\} + \{XY\nabla_W Z\}.$$

This follows either from direct calculation (using the Jacobi identity for $\{\cdot,\cdot,\cdot\}$), or from the fact that the covariant derivative of $\{\cdot,\cdot,\cdot\}$ is invariant under reflections. This condition shows that ∇ behaves like the Levi-Civitá connection with respect to the Hilbert-C*-structure on M. Under what conditions such a connection exists under more general circumstances, is investigated in [10].

4. Causality

4.1. It is customary in a number of physical theories to pass from a Lorentzian to a Riemannian manifold ('Wick Rotation'). This is due to the difficulties one is faced with in a truly Lorentzian situation and which disappear in the Riemannian set-up. One peculiarity of Lorentzian manifolds can still be modeled in the Riemannian situation: Light cones in the fibers of the tangent bundle which specify those pairs of points on M that may interact with each other and are thus intimately connected to the notion of causality.

Definition 4.2. *A cone field on a manifold M is, for each $m \in M$, an assignment of a cone $C_m \subseteq T_m(M)$.*

Here a cone is always supposed to be pointed but not necessarily to be generating.

4.3. Our interest here is with cone fields that are invariant under the action of a group G, i.e. that $C_{gm} = d_m g(C_m)$ for all $m \in M$ and $g \in G$. In our case, G will be a certain subgroup of $\operatorname{Aut} U$. Though it might be tempting to see invariance under $\operatorname{Hol} U$ (or a large subgroup) as a substitute for diffeomorphism invariance in the complex case, the main motivation here comes from the property that the connection defined in the previous section has the property that for an invariant cone field (C_m) parallel transport along a curve γ from $\gamma(a)$ to $\gamma(b)$ ends in $C_{\gamma(b)}$ if it began in $C_{\gamma(a)}$. This is well known in the finite dimensional situation (see [7, 8]) and, in this regard, not much is changing in passing to infinite dimensions.

4.4. In the following we will suppose that 'space' is represented by a certain set of self-adjoint operators, and that on this set there is defined an invariant cone field. The question we would like to answer is this: How can such fields be characterized, which come from interpreting the points of D as bounded Hilbert space operators?

In the following, denote by E a fixed (abstract) ternary ring of operators, and by U its unit ball.

4.5. On top of the causal structure of U we need 'selfadjointness', which for us will be the existence of a '*real form*' for U, compatible with the (almost) complex structure. We do this by requiring that E carries an involutory real automorphism '*' so that $(ix)^* = -ix^*$ for all $x \in E$. Since we will be studying TRO-embeddings into $L(H)$ that respect the real form, it will be necessary to impose the additional condition that, for all $x, y, z \in E$, $\{xyz\}^* = \{z^* y^* x^*\}$. (Note that then each $x \in E$ has a unique decomposition into real and imaginary part, with some norm estimates.) A ternary ring of operators E that meets all these conditions, will be called a *-ternary ring of operators. We will suppose in the following that E is a space of this kind.

4.6. The 'space manifold' here will be the open unit ball U_{sa} of the selfadjoint part of E. U_{sa} is itself a symmetric space. If G_{sa} consists of all elements in $\operatorname{Aut} U$ which leave U_{sa} invariant, then $U_{sa} = G_{sa}/H_{sa}$, where H_{sa} comprises the TRO-automorphisms that are *-selfadjoint. In fact, it follows from $\{x, y, z\}^* = \{z^*, y^*, x^*\}$ for all $x, y, z \in E$ (and an expansion into power series) that $M_a(x)^* = M_{a^*}(x^*)$ for all $x \in U$ and so $M_a \in G_{sa}$ iff $a^* = a$. A TRO-automorphism T is in H_{sa} iff $T(x^*)^* = T(x)$ for all $x \in U$, and so

$$G_{sa} = \{T \circ M_a \mid T \in H_{sa}, a \in U_{sa}\},$$

as well as $U_{sa} = G_{sa}/H_{sa}$.

4.7. In order to comply with the requirement that causality be invariant under parallel transport we have to impose the condition that the field of cones we fix in TU_{sa} must be invariant under the action of G_{sa}. We consider smooth embeddings $\Phi : U \to L(H)$ which respect

- the Hilbert-C*-structure
- the complex structure as well as the (canonical) real forms
- the action of the automorphism groups.

And we want to know: What characterizes the G_{sa} invariant cone fields that are pulled back to U via Φ? Whenever a cone field meets these properties we will call it *natural*.

4.8. Since a natural cone field is supposed to be invariant under the action of G_{sa}, we may restrict our attention to cones in $T_o U = E$. Furthermore, any cone in E that gives rise to a G_{sa} invariant field of cones has to be invariant under the action of H_{sa}. It can also be shown that under the above assumptions made,

$d\Phi$ has to respect the ternary structure of each tangent space T_pU. The question we were asking thus becomes: What properties must an H_{sa}-invariant cone in E_{sa} possess so that it is of the form $\Psi^{-1}(L(H)_+)$ for a *-ternary monomorphism $\Psi : E \to L(H)$? For the sake of simplicity, we restrict our attention to the case where E is a dual Banach space. Then

Theorem 4.9 ([1]). *Let E be a *-ternary ring of operators, which is a dual Banach space. Define the center of E by*

$$Z(E) = \{e \in E \mid exy = xye \text{ for all } x, y \in E\}$$

and call $u \in E$ tripotent whenever $\{u, u, u\} = u$. Then a cone $C \subseteq E_{sa}$ is natural iff there is a central, selfadjoint tripotent element $u \in E$ so that

$$C = C_u := \{eue^* \mid e \in E\}\,.$$

Definition 4.10. *If E is a TRO with real form *, and $u \in E_{sa}$, then u is called* rigid *iff $\Phi(u) = u$ for all *-selfadjoint TRO-automorphisms Φ of E.*

The following result is now easy to prove.

Theorem 4.11. *The only cones C_u in E that give rise to a G_{sa}-invariant causal structure on U_{sa}, coming from an embedding of E into some space $L(H)$, are those for which the central, selfadjoint element u is rigid.*

Note that the existence of a rigid selfadjoint central tripotent u is impossible in finite dimensions. In fact, $Z(E)$ is a weak*-closed commutative von Neumann algebra (see [1, Lemma 4.10]) in which the tripotents are projections, which is invariant under any TRO-automorphism Φ of E, and for which $\Phi|_{Z(E)}$ is a C*-automorphism whose properties essential here are best understood by means of the underlying homeomorphism of the spectrum of $Z(E)$.

References

[1] D. Blecher and W. Werner. Ordered C^*-modules. *Proc. London Math. Soc. 92* (2006), 682–712.

[2] G. Corach, H. Porta, and L. Recht. The geometry of spaces of projections in C*-algebras. *Adv. Math. 101 (1993)*, 59–77.

[3] M. Hamana. Triple envelopes and Shilov boundaries of operator spaces. *Math. J. Toyama Univ. 22 (1999)*, 77–93.

[4] L.A. Harris. Bounded symmetric domains in infinite dimensional spaces, in *Infinite Dimensional Holomorphy* (T.L. Hayden and T.J. Su Suffridge, eds.), Lect. Notes Math. 364, 13–40, 1974,

[5] J. Hilgert and G. Olafsson. Causal symmetric spaces. Academic Press, 1997.

[6] W. Klingenberg. Riemannian Geometry. Second Edition, Walter de Gruyter, 1995.

[7] O. Loos. Symmetric Spaces I. Benjamin, 1969.

[8] D. Mittenhuber and K.-H. Neeb On the exponential function of an ordered manifold with affine connection. *Math. Z. 218 (1995)*, 1–23.

[9] M. Neal and B. Russo Operator space characterizations of C*-algebras and ternary rings. *Pac. J. Math. 209 (2003)*, 339–364.

[10] W. Werner. Hilbert-C*-manifolds with Levi-Civitá connection. *In preparation.*

[11] H. Zettl. A characterization of ternary rings of operators. *Adv. in Math. 48 (1983)*, 117–143.

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