Let H be a Hilbert space, $E := \mathcal{L}(H)$. We consider the mapping $\phi : \mathrm{Ball}(E) \leftrightarrow M := \{[\begin{smallmatrix} t \\ x \end{smallmatrix}] \in E_+ \times E : \ t^2 - x^* x = 1, \ t \geq 1\}$,

$$\phi(u) := \begin{bmatrix} (1 + a^* a)(1 - a^* a)^{-1} \\ 2a(1 - a^* a)^{-1} \end{bmatrix} \quad \text{with} \quad \phi^{-1} \begin{bmatrix} t \\ x \end{bmatrix} = x/(1+t) \ .$$

Remark. In the 1-dimensional commutative case $E = \mathbb{C}$, for the Möbius transformation $M_b(u) := (u + b)/(1 - \bar{b}u)$ we have

$$\phi^\# M_b \begin{bmatrix} t \\ x \end{bmatrix} = \phi \circ M_b \circ \phi^{-1} \begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} \frac{1+|b|^2}{1-|b|^2} t + \frac{2b}{1-|b|^2} x \\ \frac{2b}{1-|b|^2} t + \frac{1+|b|^2}{1-|b|^2} x \end{bmatrix} .$$

Notice that $M_{\tanh(a)}$ is the exponential of the vector field $[a - \overline{a} u^2] \frac{\partial}{\partial u}$. Hence

$$\phi^\# M_{\tanh(a)} \begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} \cosh(2a) t + \sinh(2a) x \\ \sinh(2a) t + \cosh(2a) x \end{bmatrix} \qquad \text{for} \qquad \begin{bmatrix} t \\ x \end{bmatrix} \in M \ .$$

In particular $\phi^\# M_b$ is the restriction of a linear mapping of $\mathbb{C}^2$ to M.

We are going to study the analogues in the setting of $E = \mathcal{L}(H)$.

Notation. For any $a \in \mathcal{L}(H)$ and $b \in \mathrm{Ball}(\mathcal{L}(H))$ we write

$$A_a(u) := \big[a - u a^* u\big] \frac{\partial}{\partial u}, \qquad M_b(u) := (1 - b b^*)^{-1/2} (u + b)(1 + b^* u)^{-1} (1 - b^* b)^{1/2} \ .$$

It is well-known that, for $b = [\exp(A_a)]0$, we have $M_b(u) = [\exp(A_a)]u \ \ (\|u\| < 1)$.
Recall that, for fixed $a \in \mathcal{L}(H)$ and $u \in \mathrm{Ball}\mathcal{L}(H)$, the function $u_\tau := [\exp(\tau A_a)]u$ ($\tau \in \mathbb{R}$) is the solution of the initial value problem $\frac{d}{d\tau} u_\tau = A_a(u_\tau)$, $u_0 = u$.

Proposition. *Let $a \in \mathcal{L}(H)$ and $[\begin{smallmatrix} t \\ x \end{smallmatrix}] \in M$. Then*

$$\phi^\# A_a \begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} a^* x + x^* a \\ a(1+t) + x(1+t)^{-1} x^* a \end{bmatrix} .$$

Proof. Let us write

$$u := \phi^{-1} \begin{bmatrix} t \\ x \end{bmatrix} = x(1+t)^{-1}, \quad \overline{u} := A_a(u), \quad G_\tau := \exp(\tau A_a) \ .$$

Then we have

$$\begin{aligned} \phi^\# A_a \begin{bmatrix} t \\ x \end{bmatrix} &= \frac{d}{d\tau}\Big|_{\tau=0} \phi \circ G_\tau \circ \phi^{-1} \begin{bmatrix} t \\ x \end{bmatrix} = \frac{d}{d\tau}\Big|_{\tau=0} \phi \circ G_\tau(u) = \\ &= \phi'(G_0(u)) \frac{d}{d\tau}\Big|_{\tau=0} G_\tau(u) = \phi'(u) A_a(u) = \\ &= \phi'(u) \overline{u}. \end{aligned}$$

We calculate both $\phi'(u)$ and $\overline{u}$ in terms of t, x. For the first component of ϕ,

$$\phi_1(u) = (1+u^*u)(1-u^*u)^{-1} = [(1-u^*u)+2u^*u](1-u^*u)^{-1} = \\ = 1 + u^*\phi_2(u) \ .$$

Hence and since $\phi_1(u) = x$,

$$\phi_1'(u)\overline{u} = \frac{d}{d\tau}\Big|_{\tau=0}(u+\tau\overline{u})^*\phi_2(u+\tau\overline{u}) = \overline{u}^*\phi_2(u) + u^*\phi_2'(u)\overline{u} = \\ = \overline{u}^*x + u^*\phi_2'(u)\overline{u} \ .$$

We can express $\phi_2'(u)\overline{u}$ in algebraic terms of $u, \overline{u}$ as follows:

$$\phi_2'(u)\overline{u} = \frac{d}{d\tau}\Big|_{\tau=0}\phi_2(u+\tau\overline{u}) = \frac{d}{d\tau}\Big|_{\tau=0}2(u+\overline{u})[1+(u+\overline{u})^*(u+\overline{u})]^{-1} = \\ = 2\overline{u}[1-u^*u]^{-1} + 2u[1-u^*u]^{-1}(\overline{u}^*u + u^*\overline{u})[1-u^*u]^{-1} \ .$$

Since $u = \phi^{-1}\left[\begin{smallmatrix} t \\ x \end{smallmatrix}\right] = x(1+t)^{-1}$ and since $x^*x = t^2 - 1 = (t-1)(1+t)$, here we have

$$\begin{aligned} 1 - u^*u &= 1 - (1+t)^{-1}x^*x(1+t)^{-1} = \\ &= 1 - (1+t)^{-1}(t-1) = (1+t)^{-1}[(1+t)-(t-1)] = \\ &= 2(1+t)^{-1} \ , \\ [1-u^*u]^{-1} &= \frac{1}{2}(1+t) \ , \\ u[1-u^*u]^{-1} &= x(1+t)^{-1}\frac{1}{2}(1+t) = \frac{1}{2}x \ . \end{aligned}$$

Hence we conclude

$$\phi_2'(u)\overline{u} = 2\overline{u}\frac{1}{2}(1+t) + 2\cdot\frac{1}{2}x[\overline{u}^*x(1+t)^{-1} + (1+t)^{-1}x^*\overline{u}]\frac{1}{2}(1+t) = \\ = \overline{u}(1+t) + \frac{1}{2}x\overline{u}^*x + \frac{1}{2}x(1+t)^{-1}x^*\overline{u}(1+t) \ .$$

We can express $\overline{u}$ in terms of t, x as

$$\overline{u} = A_a(u) = a - ua^*u = a - x(1+t)^{-1}a^*x(1+t)^{-1} \ .$$

Thus

$$\begin{aligned} \phi_2'(u)\overline{u} = {} & \overbrace{a(1+t)}^{(1)}\overbrace{-x(1+t)^{-1}a^*x}^{(2)} + \\ & \overbrace{+\frac{1}{2}xa^*x}^{(3)}\overbrace{-\frac{1}{2}x(1+t)^{-1}x^*a(1+t)^{-1}x^*x}^{(4)} + \\ & \overbrace{+\frac{1}{2}x(1+t)^{-1}x^*a(1+t)}^{(5)}\overbrace{-\frac{1}{2}x(1+t)^{-1}x^*x(1+t)^{-1}a^*x}^{(6)} \ . \end{aligned}$$

The sum (2)+(3)+(6) vanishes because $x^*x = (1+t)(t-1)$ and hence

$$(2)+(3)+(6) = x\left[-(1+t)^{-1} + \frac{1}{2} - \frac{1}{2}(t-1)(1+t)^{-1}\right]a^*x =$$

$$= x(1+t)^{-1}\left[-1 + \frac{1}{2} + \frac{1}{2}t - \frac{1}{2}t + \frac{1}{2}\right]a^*x = 0 \ .$$

The sum (4)+(5) can also be simplified as

$$(4)+(5) = \frac{1}{2}x(1+t)^{-1}x^*a\big[-(1+t)^{-1}\overbrace{x^*x}^{(1+t)(t-1)} + (1+t)\big] =$$

$$= \frac{1}{2}x(1+t)^{-1}x^*a\left[-(t-1)+(1+t)\right] =$$

$$= x(1+t)^{-1}x^*a \ .$$

Summing up $(1)+\cdots+(6)$, we get

$$\phi_2'(u)\overline{u} = a(1+t) + x(1+t)^{-1}x^*a \ ,$$

$$\phi_1'(u)\overline{u} = \overline{u}^*x + u^*\phi_2'(u)\overline{u} =$$

$$= a^*x - (1+t)^{-1}x^*a(1+t)^{-1} +$$

$$+ (1+t)^{-1}x^*\big[a(1+t) + x(1+t)^{-1}x^*a\ big\big] =$$

$$= \overbrace{a^*x}^{(1)}\overbrace{-(1+t)^{-1}x^*a(1+t)^{-1}x^*x}^{(2)} +$$

$$= \overbrace{+(1+t)^{-1}x^*a(1+t)}^{(3)}\overbrace{+(1+t)^{-1}x^*x(1+t)^{-1}x^*a}^{(4)} \ .$$

Using again the identity $x^*x = (1+t)(t-1)$, here we can write

$$(2)+(3) = (1+t)^{-1}x^*a\big[-(1+t)^{-1}(1+t)(t-1) + (1+t)\big] =$$

$$= 2(1+t)^{-1}x^*a \ ,$$

$$(4) = (1+t)^{-1}(1+t)(t-1)(1+t)^{-1}x^*a = (t-1)(1+t)^{-1}x^*a =$$

$$= -\big[(1+t)-2t\big](1+t)^{-1}x^*a = -x^*a + 2t(1+t)^{-1}x^*a \ .$$

Therefore

$$\phi_1'(u)\overline{u} = [(1)+(4)] + F(2)+(3)] =$$

$$= a^*x - x^*a + 2t(1+t)^{-1}x^*a + 2(1+t)^{-1}x^*a =$$

$$= a^*x + (1+t)^{-1}\big[-(1+t) + 2t + 2\big]x^*a =$$

$$= a^*x + x^*a$$

which completes the proof. Qu.e.d.

Corollary. *If x is a normal operator and $\left[\begin{smallmatrix} t \\ x \end{smallmatrix}\right] \in M$ then $\{x, x^*, t\}$ commute and $x^*x = (t-1)(1+t)$ whence*

$$\phi^{\#}A_a\begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} a^*x + x^*a \\ a^*t + ta^* \end{bmatrix} \ .$$

The case of the real Jordan algebra of self-adjoint operators

Let henceforth only $E = \mathcal{A}(H) := \{a \in \mathcal{L}(H) : \ a = a^*\}$. In this case

$$\phi^\# A_a \begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} ax + xa \\ at + ta \end{bmatrix} = \begin{bmatrix} 0 & L_a + R_a \\ L_a + R_a & 0 \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix}$$

with the left and right multiplication operators $L_a : \mathcal{L}(H) \ni z \mapsto az$ respectivaly $R_a : \mathcal{L}(H) \ni z \mapsto za$. Since L_a and R_a commute, $\exp\big(\phi^\# A_a\big)$ has a rather simple explicite expression. Straightforward calculations with the power series $\sum_{n=0}^\infty n!^{-1} \left[\begin{smallmatrix} 0 & L_a+R_a \\ L_a+R_a & 0 \end{smallmatrix}\right]^n \left[\begin{smallmatrix} t \\ x \end{smallmatrix}\right]$, yield the following.

Theorem. *Given any self-adjoint operator* $a \in \mathcal{A}(H)$, *the Möbius transformation* $M_{\tanh(a)}$ *maps* $\mathrm{Ball}(\mathcal{A}(H))$ *onto itself,* ϕ *is a bijection between* $\mathrm{Ball}(\mathcal{A}(H))$ *and* $M \cap \mathcal{A}(H)^2 = \big\{ \left[\begin{smallmatrix} t \\ x \end{smallmatrix}\right] : \ t, x \in \mathcal{A}, \ t^2 - x^2 = 1, \ t \geq 1 \big\}$ *and*

$$\phi^\# M_{\tanh(a)} = \phi \circ [\exp A_a] \circ \phi^{-1} \begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} \cosh(L_a + R_a)t + \sinh(L_a + R_a)x \\ \sinh(L_a + R_a)t + \cosh(L_a + R_a)x \end{bmatrix} .$$

For the terms $\cosh(L_a + R_a)z$, $\sinh(L_a + R_a)z$ $(z = t, x)$ *here we have*

$$\cosh(L_a + R_a)z = \frac{1}{2}[\exp(a)]z[\exp(a)] + \frac{1}{2}[\exp(-a)]z[\exp(-a)] \ ,$$
$$\cosh(L_a + R_a)z = \frac{1}{2}[\exp(a)]z[\exp(a)] - \frac{1}{2}[\exp(-a)]z[\exp(-a)] \ .$$

Proof. For any $a, u \in \mathcal{A}(H)$, we have $A_a(u) = a - \{uau\} \in \mathcal{A}(H)$ along with $\phi_1(u) = (1 + u^2)(1 - u^2)^{-1} \in\in \mathcal{A}(H)$ and $\phi_2(u) = 2u(1 - u^2)^{-1} \in\in \mathcal{A}(H)$. Therefore the vector field A_a is tangent to $\mathcal{A}(H)$ moreover complete in $\mathcal{A}(H) \cap \mathrm{Ball}(\mathcal{L}(H) = \mathrm{Ball}(\mathcal{A}(H)$ and the vector field $\phi^\# A_a$ is complete in $\mathcal{A}(H) \cap M$. Since $\exp(A_a) = M_{\tanh(a)}$, it follows

$$\phi^\# M_{\tanh(a)} = \exp \phi^\# A_a = \exp \begin{bmatrix} 0 & L_a + R_a \\ L_a + R_a & 0 \end{bmatrix} \Big| M \ .$$

Here we have

$$\exp \begin{bmatrix} 0 & L_a + R_a \\ L_a + R_a & 0 \end{bmatrix} = \sum_{n=0}^\infty \frac{1}{n!} \begin{bmatrix} 0 & L_a + R_a \\ L_a + R_a & 0 \end{bmatrix}^n =$$
$$= \sum_{k=0}^\infty \frac{1}{(2k)!} \begin{bmatrix} 0 & L_a + R_a \\ L_a + R_a & 0 \end{bmatrix}^{2k} + \sum_{k=0}^\infty \frac{1}{(2k+1)!} \begin{bmatrix} 0 & L_a + R_a \\ L_a + R_a & 0 \end{bmatrix}^{2k+1} =$$
$$= \sum_{k=0}^\infty \frac{1}{(2k)!} \begin{bmatrix} (L_a + R_a)^{2k} & 0 \\ 0 & (L_a + R_a)^{2k} \end{bmatrix} + \sum_{k=0}^\infty \frac{1}{(2k+1)!} \begin{bmatrix} 0 & (L_a + R_a)^{2k+1} \\ (L_a + R_a)^{2k+1} & 0 \end{bmatrix} =$$
$$= \begin{bmatrix} \cosh(L_a + R_a) & 0 \\ 0 & \cosh(L_a + R_a) \end{bmatrix} + \begin{bmatrix} 0 & \sinh(L_a + R_a) \\ \sinh(L_a + R_a)^{2k+1} & 0 \end{bmatrix} .$$

In general, the operations L_x and R_x commute $(L_x(R_y u) = x(uy) = (xu)y = L_x(R_y u)$. Therefore

$$\begin{aligned}\cosh(L_a + R_a) &= \frac{1}{2}\exp(L_a + R_a) + \frac{1}{2}\exp(-L_a - R_a) = \\ &= \frac{1}{2}\exp(L_a)\exp(R_a) + \frac{1}{2}\exp(-L_a)\exp(-R_a) = \\ &= \frac{1}{2}L_{\exp(a)}R_{\exp(a)} + \frac{1}{2}L_{\exp(a)}R_{\exp(a)} :\ z \mapsto \frac{1}{2}e^a z e^a + \frac{1}{2}e^{-a} z e^{-a}.\end{aligned}$$

Similarly $\sinh(L_a + R_a) :\ z \mapsto \frac{1}{2}e^a z e^a - \frac{1}{2}e^{-a} z e^{-a}$ which completes the proof. Qu.e.d.

Case of JB*-triples

Throughout this section let E be a JB*-triple with the triple product $\{uvw\}$ and structure derivations $D(a,b) : u \mapsto \{abu\}$. We define the mapping $\phi : \mathrm{Ball}(E) \to \mathcal{L}(E) \times E$ as

$$\phi(u) := \begin{bmatrix} \phi_1(u) \\ \phi_2(u) \end{bmatrix} \qquad \text{where} \qquad \begin{array}{l} \phi_2(u) := 2\big[1 - D(u,u)\big]^{-1} u \ , \\ \phi_1(u) := 1 + D\big(\phi_1(u), u\big) \ . \end{array}$$

We shall also write $M := \mathrm{range}(\phi) = \big\{\phi(u) : \ u \in \mathrm{Ball}(E)\big\}$.

Lemma. *On M we have* $\phi^{-1} \left[\begin{smallmatrix} t \\ x \end{smallmatrix}\right] = (1+t)^{-1} x$.

Proof. Let $u \in \mathrm{Ball}(E)$. We have to see that $u = (1+t)^{-1}x$ that is $(1+t)u = x$ with

$$x := 2\big[1 - D(u,u)\big]^{-1} u \ , \qquad t := 1 + D(x,u) \ .$$

Recall that, using the customary notation $u^{2k+1} := D(u,u)^k u$ $(k = 0, 1, \ldots)$, we have the identity $\{u^{2k+1} u^{2\ell+1} u^{2m+1}\} = u^{2(k+\ell+m)+1}$. Therefore

$$x = 2\big[1 - D(u,u)\big]^{-1} u = 2\sum_{k=0}^{\infty} D(u,u)^k = 2\sum_{k=0}^{\infty} u^{2k+1} \ ,$$

$$(1+t)u = \big[2 + D(x,u)\big]u = \Big[2 + 2\sum_{k=0}^{\infty} D\big(u^{2k+1}, u\big)\Big]u = 2u + 2\sum_{k=0}^{\infty} D\big(u^{2k+1}, u\big)u\Big] =$$

$$= 2u^1 + 2\sum_{k=0}^{\infty} u^{2k+3} = 2\sum_{k=0}^{\infty} u^{2k+1} = x \ . \quad \text{Qu.e.d.}$$