1. One-dimensional case

$$\phi(a) = \begin{bmatrix} \phi_1(a) \\ \phi_2(a) \end{bmatrix} \qquad (a \in \mathbb{C})$$

$$\phi_1(a) := \frac{1+|a|^2}{1-|a|a|^2}, \quad \phi_2(a) := \frac{2a}{1-|a|a|^2}.$$

$$\phi_1(a)^2 - \big|\phi_2(a)\big|^2 = \frac{1+2|a|^2+|a|^4-2|q|^2}{\big[1-|a|^2\big]^2} =$$

$$= \frac{\big[1-|a|^2\big]^2}{\big[1-|a|^2\big]^2} = 1 \ .$$

$$\phi'(a)u = \frac{\partial}{\partial t}\Big|_{\tau=0} \phi(a+\tau u).$$

$$\phi_1'(a)u = \frac{(\overline{a}u+\overline{u}a)[1-|a|^2]-[1+|a|^2](\overline{a}u+\overline{u}a)}{\big[1-|a|^2\big]^2} = 2\frac{\overline{a}u+\overline{u}a}{\big[1-|a|^2\big]^2} \ ,$$

$$\phi_2'(a)u = \frac{2u[1-|a|^2]+2a[1+|a|^2]}{\big[1-|a|^2\big]^2} \ .$$

$$\big[\phi_1'(a)u\big]^2 - \big|\phi_2'(a)u\big|^2 = 4\frac{(\overline{a}u+\overline{u}a)^2-|u-a\overline{u}a|^2}{\big[1-|a|^2\big]^4} \ .$$

$$\begin{aligned}
(\overline{a}u+\overline{u}a)^2 - |u-a\overline{u}a|^2 &= \\
&= \underbrace{\overline{a}^2u^2}_{(1)} + \underbrace{2|a|^2|u|^2}_{(2)} + \underbrace{\overline{u}^2a^2}_{(3)} + \underbrace{|u|^2}_{(4)} - \underbrace{a\overline{u}^2a}_{(5)} - \underbrace{\overline{a}u^2\overline{a}}_{(6)} - \underbrace{|a|^4|u|^2}_{(7)} = \\
&= \underbrace{[(1)-(6)]}_{0} + \underbrace{[(3)-(5)]}_{0} + |u|^2\big(2|a|^2-1-|a|^4\big) = \\
&= -|u|^2\big[1-|a|^2\big]^2 \ .
\end{aligned}$$

$$\big[\phi_1'(a)u\big]^2 - \big|\phi_2'(a)u\big|^2 = -4\frac{|u|^2}{\big[1-|a|^2\big]^2} \ .$$

Recall that the natural Riemannian metric on the unit disc $B := \{z \in \mathbb{C} : \ |z| < 1\})$ is given by the inner products

$$\langle u, u\rangle_a^B := |g_a'(0)^{-1}u|^2 = \frac{|u|^2}{\big[1-|a|^2\big]^2} \qquad (a \in B, \ u \in \mathbb{C}).$$

Thus, in terms of the inverse $\Psi := \phi^{-1} : M \leftrightarrow B$, with

$$t := \phi_1(a), \quad x := \phi_2(a), \quad \widetilde{t} = \phi_1'(a)u, \quad \widetilde{x} = \phi_2'(a), u$$

we have $a = \Psi(t,x)$, $u = \big[\Psi'(t,x)\big](\widetilde{t},\widetilde{x})$ and

$$\widetilde{t}^2 - |\widetilde{x}|^2 = -4\big\langle \big[\Psi'(t,x)\big](\widetilde{t},\widetilde{x}), \big[\Psi'(t,x)\big](\widetilde{t},\widetilde{x})\big\rangle_{\Psi(t,x)}^B .$$

2. Case of $\mathcal{L}(H)$ or TRO

$$\phi(a) = \begin{bmatrix} \phi_1(a) \\ \phi_2(a) \end{bmatrix} \qquad \big(a \in \mathcal{L}(H)\big)$$

$$\phi_1(a) := (I + a^*a)(I - a^*a)^{-1} \ , \quad \phi_2(a) := 2a(I - a^*a)^{-1} \ .$$

$$\begin{aligned}
&\phi_1(a)^2 - \phi_2(a)^*\phi_2(a) = \\
&\qquad = (I - a^*a)^{-1}(I + a^*a)^2(I - a^*a)^{-1} - (I - a^*a)^{-1}(4a^*a)(I - a^*a)^{-1} = \\
&\qquad = (I - a^*a)^{-1}\big[I + 2a^*a + (a^*a)^2 - 4a^*a\big](I - a^*a)^{-1} = \\
&\qquad = (I - a^*a)^{-1}\big[I - 2a^*a + (a^*a)^2\big](I - a^*a)^{-1} = \\
&\qquad = (I - a^*a)^{-1}\big[I - 2a^*a + (a^*a)^2\big](I - a^*a)^{-1} = \\
&\qquad = (I - a^*a)^{-1}(I - a^*a)^2(I - a^*a)^{-1} = \\
&\qquad = I \ .
\end{aligned}$$

$$\begin{aligned}
\phi_1'(a)x &= (a^*x + x^*a)(I - a^*a)^{-1} + (I + a^*a)(I - a^*a)^{-1}(a^*x + x^*a)(I - a^*a)^{-1} = \\
&= (I - a^*a)^{-1}\big[(I - a^*a)(a^*x + x^*a) + (I + a^*a)(a^*x + x^*a)\big] = \\
&= (I - a^*a)^{-1}(a^*x + x^*a)(I - a^*a)^{-1} \ . \\
\phi_2'(a)x &= 2x(I - a^*a)^{-1} + 2\,\underbrace{a(I - a^*a)^{-1}}_{(I - aa^*)^{-1}a}(a^*x + x^*a)(I - a^*a)^{-1} = \\
&= 2(I - aa^*)^{-1}\big[(I - a^*a)x + a(a^*x + x^*a)\big](I - a^*a)^{-1} = \\
&= 2(I - aa^*)^{-1}[x + ax^*a](1 - a^*a)^{-1}
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\mathrm{range}(\phi) = M := \{(t, x) \in \mathcal{L}(H) \times \mathcal{L}(H) : \ t = t^* \geq I, \ t^2 - x^*x = I\} \ , \\
&\phi : B := \{a : \ a^*a < I\} \leftrightarrow M := \{(t, x) : \ t = t^* \geq I, \ t^2 - x^*x = I\}, \\
&\Psi := \phi^{-1} : (t, x) \mapsto x(I + t)^{-1}
\end{aligned}$$

Aim. Find a convenient indefinite inner product $\langle(\bar{t}_1, \bar{x}_1), (\bar{t}_2, \bar{x}_2)\rangle_{(t,x)}$ on $\mathcal{L}(H) \times \mathcal{L}(H)$ such that $\langle(\bar{t}, \bar{x}), (\bar{t}, \bar{x})\rangle^M_{(t,x)} := \langle(\bar{t}, \bar{x}), (\bar{t}, \bar{x})\rangle_{(t,x)}$ for $(t, x) \in M$ and $(\bar{t}, \bar{x}) \in T_{(t,x)}M$ with the inner product

$$\langle(\bar{t}, \bar{x}), (\bar{t}, \bar{x})\rangle^M_{(t,x)} := -4\big\langle[\Psi'(t, x)](\bar{t}, \bar{x}), [\Psi'(t, x)](\bar{t}, \bar{x})\big\rangle^B_{\Psi(t,x)} \ \text{ for } (t, x) \in M, \ (\bar{t}, \bar{x}) \in T_{(t,x)}M$$

with the natural Riemannian inner product on the unit ball $B := \{a : \ a^*a < I\}$,

$$\begin{aligned}
\langle u, v\rangle^B_a &:= \big[[g_a'(0)]^{-1}u\big]^*\big[[g_a'(0)]^{-1}v\big] = \\
&= (I - a^*a)^{-1/2}u^*(I - aa^*)^{-1}v(I - a^*a)^{-1/2}.
\end{aligned}$$

Let $(t,x)\in M$ and $(\bar t,\overline{x})\in T_{(t,x)}M$ be fixed and write

$$a := \Psi(t,x) = x(I+t)^{-1},$$

$$u := \big[\Psi'(t,x)\big](\bar t,\overline{x}) = \frac{\partial}{\partial\tau}\Big|_{\tau=0}(x+\tau\overline{x})(I+t+\tau\bar t)^{-1} = \\ = \big[\overline{x} - x(I+t)^{-1}\bar t\big](I+t)^{-1}.$$

Then we have $t = t^* = [I+x^*x]^{1/2} \geq I$ with $a^* = (I+t)^{-1}x^*$ and

$$a^*a = (I+t)^{-1}x^*x(I+t)^{-1} = (I+t)^{-1}(t^2-I)(I+t)^{-1} = \\ = (t-I)(I+t)^{-1}\ ,$$

$$I - a^*a = (t+I)(I+t)^{-1} - (t-I)(I+t)^{-1} = \\ = 2(I+t)^{-1}.$$

Therefore

$$\langle u,u\rangle_a^B = (I-a^*a)^{-1/2}u^*(I-aa^*)^{-1}u(I-a^*a)^{-1/2} = \\ = \frac{1}{2}(I+t)^{1/2}\big[\overline{x}^* - \bar t(I+t)^{-1}x^*\big](I-aa^*)^{-1}\big[\overline{x} - x(I+t)^{-1}\bar t\big](I+t)^{1/2} = \\ = \frac{1}{2}(I+t)^{1/2}[\overline{x}^* - \bar t a^*](I-aa^*)^{-1}[\overline{x} - a\bar t](I+t)^{1/2}\ .$$

Since $a^*(I-aa^*)^{-1} = (I-a^*a)^{-1}a^*$ and $(I-aa^*)(I-aa^*)^{-1} = I$, we get

$$a^*(I-aa^*)^{-1} = (I-a^*a)^{-1}a^* = \frac{1}{2}(I+t)(I+t)^{-1}x^* = \frac{1}{2}x^*\ ,$$

$$(I-aa^*)^{-1} = I + a\big[a^*(I-aa^*)^{-1}\big] = I + \frac{1}{2}x(I+t)^{-1}x^*\ .$$

Hence

$$[\overline{x}^* - \bar t a^*](I-aa^*)^{-1}[\overline{x} - a\bar t] = \\ = \overline{x}^*\underbrace{(I-aa^*)^{-1}}_{I+x(I+t)x^*/2}\overline{x} - \bar t\underbrace{a^*(I-aa^*)^{-1}}_{x^*/2}\overline{x} - \overline{x}^*\underbrace{\big[a^*(I-aa^*)^{-1}\big]^*}_{x/2}\bar t + \bar t\big[\underbrace{a^*(I-a^*a)^{-1}}_{x^*/2}\big]\underbrace{a}_{x(I+t)^{-1}}\bar t = \\ = \overline{x}^*\overline{x} + \frac{1}{2}\Big[\overline{x}^*x(1+t)^{-1}x^*\overline{x} - \bar t x^*\overline{x} - \overline{x}^*x\bar t + \bar t\big[\underbrace{x^*x}_{t^2-I=(t-I)(I+t)}(I+t)^{-1}\big]\bar t\Big] = \\ = \overline{x}^*\overline{x} - \bar t^2 + \frac{1}{2}\Big[\overline{x}^*x(1+t)^{-1}x^*\overline{x} - \bar t x^*\overline{x} - \overline{x}^*x\bar t + \bar t(t-I)\bar t\Big]\ .$$

Theorem. *For $(\bar t,\overline{x})\in T_{(t,x)}M$ with $\bar t t + t\bar t = \overline{x}^*x + x^*\overline{x}$ we conclude*

$$\langle(\bar t,\overline{x}),(\bar t,\overline{x})\rangle_{(t,x)}^M = -4\langle u,u\rangle_a^B = \\ = (I+t)^{1/2}\Big\{2\big[\bar t^2 - \overline{x}^*\overline{x}\big] - \Big[\overline{x}^*x(I+t)^{-1}x^*\overline{x} - \bar t x^*\overline{x} - \overline{x}^*x\bar t + \bar t(t-I)\bar t\Big]\Big\}(I+t)^{1/2}.$$

Remark. In the commutative case $H = \mathbb{C} \equiv \mathcal{L}(H)$ we have
$-4\langle(\bar t,\overline{x}),(\bar t,\overline{x})\rangle_{(t,x)}^M = \langle[\Psi'(t,x)](\bar t,\overline{x}),[\Psi'(t,x)](\bar t,\overline{x})\rangle_{\Psi(t,x)} = \bar t^2 - |\overline{x}|^2.$

3a. Spectral integral for $D\big(\varphi(a),\psi(a)\big)$ in $\mathcal{L}(H)$

Let $a\in\mathcal{L}(H)$ be fixed and let

$$\Omega := \mathrm{Sp}(a^*a)\setminus\{0\} = \mathrm{Sp}(aa^*)\setminus\{0\}\ ,\quad \alpha := \mathrm{Id}_\Omega : \Omega\ni\omega\mapsto\omega\ .$$

It s well-known that, there is a positive Borel measure μ on Ω and there are surjective partial isometries $Q_1,Q_2 : H\to L^2(\Omega,\mu)$ such that, by writing M_φ for the multiplication operator

$$M_\varphi(f) := \varphi f \quad (f\in L^2(\Omega,\mu)\ ,$$

we have

$$a = Q_1^* M_\alpha Q_2\ .$$

For any couple of Borelian sets $S,T\subset\Omega$, define the projection $P_{S\times T}:\mathcal{L}(H)\to\mathcal{L}(H)$ by

$$P_{S\times T}X := Q_1^*M_{1_S}Q_1 X Q_2^* M_{1_T}Q_2 \qquad (X\in\mathcal{L}(H))$$

where 1_Z stands for the indicator function of the set Z (with $1_Z(\omega)=1$ for $\omega\in Z$ and $1_Z(\omega)=0$ for $\omega\in\Omega\setminus Z$). Furthermore, let $P_{\{0\}\times\{0\}}X := (I-Q_1^*Q_1)X(I-Q_2^*Q_2)X$ and

$$P_{S\times\{0\}}X := Q_1^*M_{1_S}Q_1X(I-Q_1^*Q_1)\ ,\quad P_{\{0\}\times T}X := (I-Q_1^*Q_1)XQ_2^*M_{1_T}Q_2\ .$$

Proposition. *Let $\varphi,\psi:\Omega\cup\{0\}\to\mathbb{C}$ be a couple of continuous functions such that $\varphi(0)=\psi(0)=0$. Then*

$$D\big(\varphi(a),\psi(a)\big) = \int_{(\lambda_1,\lambda_2)\in[\Omega\cup\{0\}]^2}\frac12\big[\varphi(\lambda_1)\overline{\psi(\lambda_1)}+\varphi(\lambda_2)\overline{\psi(\lambda_2)}\big]\,P\big(d(\lambda_1,\lambda_2)\big)$$

with integration in Cauchy sense.

Remark. Given a bounded finitely additive vector valued measure $V:\mathcal{A}\to B$ where $\mathcal{A}$ is an algebra of subetes of a set $\mathcal{S}$ and B is any Banach space, a function $f:\mathcal{S}\to\mathbb{C}$ is said to be V-integrable in Cauchy sense if it can be uniformly approximated by a sequence of V-simple functions (V-measurable functions with finite range). For a V-simple function $g:\mathcal{S}\to\mathbb{C}$, using any representation of the form $g:=\sum_{k=1}^N\lambda_k 1_{A_k}$, the vector $\int g(\omega)\,V(d\omega) := \sum_{k=1}^N\lambda_k V(A_k)$ is unambigously defined. If f is the uniform limit of V-simple functions then the definition $\int g(\omega)\,V(d\omega) := \lim_{n\to\infty}\int g_n(\omega)\,V(d\omega)$ is unambigous with any uniformly convergent sequence $g_n\to g$ of V-simple functions. In the Proposition we take $\mathcal{S}:=[\Omega\cup\{0\}]^2$, $\mathcal{A}:=\big\{\text{finite disjoint unions from }\big\{S\times T:\ S,T\ \text{Borel}\subset\Omega\cup\{0\}\big\}$.

Proof. By definition, $\varphi(a):=Q_1^*M_\varphi Q_2$ and $\psi(a):=Q_1^*M_\psi Q_2$. Let φ_n be the simple function assuming the value $\varphi(k/n)$ on the set $I_{k,n}:=[(k-1)/n,k/n)\cap\Omega$ (for $n=1,2,\ldots$ with $k=1,\ldots,n\cdot\mathrm{entier}(\|a\|)+1$). We define $\psi_n(a)$ analogously. Then

$$D\big(\varphi_n(a),\psi_n(a)\big) = \sum_{k,\ell}\frac12\big[\varphi(k/n)\overline{\psi(k/n)}+\varphi(\ell/n)\overline{\psi(\ell/n)}\big]\,P\big(I_{n,k}\times I_{n,\ell}\big)\ .$$

For $n\to\infty$ we have $\varphi_n\to\varphi$, $\psi_n\to\psi$ uniformly, implying $\varphi_n(a)\to\varphi(a)$, $\psi_n(a)\to\psi(a)$ and $D\big(\varphi_n(a),\psi_n(a)\big)\to D\big(\varphi(a),\psi(a)\big)$ in $\mathcal{L}\big(\mathcal{L}(H)\big)$. Hence the statement is immediate. $\square$

3b. Spectral integral for $D\big(\varphi(a),\psi(a)\big)$ in $\mathcal{H}_3(\mathbb{O})$

Given $a \in \mathcal{H}_3(\mathbb{O})$, there is an orthogonal frame (in Jordan sense) $G_0 := \{g_1, g_2, g_3\}$ and there are scalars $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$ such that

$$a = \sum_{\ell=1}^{3} \lambda_\ell g_\ell, \quad D\big(\varphi(a),\psi(a)\big) = \sum_{\ell=1}^{3} \varphi(\lambda_\ell)\overline{\psi(\lambda_\ell)} D\big(g_\ell, g_\ell\big) \ .$$

The frame G_0 can be extended to a covering grid

$$G = \big\{ g_j : \ j = 1, \dots 27 \big\}$$

of linearly independent tripotents such that

$$D\big(g_j, g_j\big) g_k = \langle \gamma_j, \gamma_k \rangle g_k \quad (j,k = 1, \dots, 27)$$

where $\big\{\gamma_j : \ j = 1, \dots 27\big\} \subset \mathbb{R}^2 7$ is the 1-part of the 3-graded root system E_6. Actually, after a suitable change of indices we may assume that

$$\begin{aligned} G = G_0 \cup G_{\{1,2\}} \cup G_{\{2,3\}} \cup G_{\{1,3\}} \ \text{ with } \ & G_{\{1,2\}} := \{g_j : \ 4 \leq j \leq 11\}, \\ & G_{\{2,3\}} := \{g_j : \ 12 \leq j \leq 19\}, \\ & G_{\{1,3\}} := \{g_j : \ 20 \leq j \leq 27\} \end{aligned}$$

and. for $\ell = 1,2,3,$ we have

$$\begin{aligned} &\big\{ g \in G : \ D\big(g_\ell, g_\ell\big) g = g \big\} = \{g_\ell\} \ , \\ &\big\{ g \in G : \ D\big(g_\ell, g_\ell\big) g = (1/2) g \big\} = \textstyle\bigcup_{m \in \{1,2,3\} \setminus \{\ell\}} G_{\{\ell,m\}} \ , \\ &\big\{ g \in G : \ D\big(g_\ell, g_\ell\big) g = 0 \big\} = \big[G_0 \setminus \{g_\ell\}\big] \cup G_{\{1,2,3\} \setminus \{\ell\}} \ . \end{aligned}$$

It follows

$$D\big(\varphi(a),\psi(a)\big) g_j = \sum_{\ell=1}^{3} \varphi(\lambda_\ell)\overline{\psi(\lambda_\ell)} \big\langle \gamma_\ell, \gamma_j \big\rangle g_j \qquad (j = 1, \dots, 27).$$

Or alternatively, with the projections

$$\begin{aligned} &P_{\{\ell\}} : \mathcal{H}_3(\mathbb{O}) \to \mathbb{C} g_\ell && (1 \leq \ell \leq 3), \\ &P_{\{\ell,m\}} : \mathcal{H}_3(\mathbb{O}) \to \sum_{g \in G_{\{\ell,m\}}} \mathbb{C} g && (1 \leq \ell < m \leq 3) \end{aligned}$$

we can write

$$D\big(\varphi(a),\psi(a)\big) = \sum_{1 \leq \ell \leq m \leq 3} \Big[\frac{1}{2} \varphi(\lambda_\ell)\overline{\psi(\lambda_\ell)} + \frac{1}{2} \varphi(\lambda_m)\overline{\psi(\lambda_m)} \Big] P_{\{\ell,m\}} \ .$$

4. Jordan case

$(E,\{\ldots\})$ JB*-triple,

$$
\begin{aligned}
&D(a,b)x := \{abx\}, \qquad Q(a,b)x := \{axb\},\\
&B(a,b)x := I - 2D(a,b) + Q(a,a)Q(b,b) =\\
&\qquad = [I - D(a,b)]^2 \ ,\\
&M_a(x) := a + B(a,a)^{1/2}[I + D(x,a)]^{-1}x, \qquad M_{-a} = [M_a]^{-1},\\
&M_a'(x)v = \partial/\partial t\big|_{t=0} M_a(x+tv) =\\
&\qquad = B(a,a)^{1/2}[I + D(x,a)]^{-2}v =\\
&\qquad = B(a,a)^{1/2}B(x,-a)^{-1}v.
\end{aligned}
$$

Remark. In the commutative case $E = \mathbb{C}$, $\{xyz\} = x\overline{y}z$ we have $B(a,b)z = [1 - 2a\overline{b} + a^2\overline{b}^2]z$ and $B(a,a)z = [1 - |a|^2]^2 z$. Also $M_a(z) = (z+a)/(a - \overline{a}z)$. The identity $\phi_1(a)^2 - |\phi_2(a)|^2 = 1$ can be written as $\phi_1(a)^2 - D\big(\phi_2(a), \phi_2(a)\big) = I$. Here $\phi_2(a) = 2a[1 - |a|^2]^{-1} = 2[I - D(a,a)]^{-1}a = 2[M_a'(0)]^{-1}a = 2[(M_a)^{-1}]'(M_a(0))a$. Thus $\phi_2(a) = 2M_{-a}'(a)a$

$$
\phi(a) = \begin{bmatrix} \phi_1(a) \\ \phi_2(a) \end{bmatrix} \qquad \big(a \in E\big)
$$

$$
\mathrm{range}(\phi) = \left\{ \begin{bmatrix} t \\ x \end{bmatrix} \in \mathcal{L}(E) \times E : \ t^2 - D(x,x) = I, \ \mathrm{Sp}(t) > 0 \right\}
$$

$$
\begin{aligned}
&\phi_2(a) := 2[I - D(a,a)]^{-1}a = 2M_{-a}'(a)a \ ,\\
&\phi_1(a) := [I + D(\phi_1(a), \phi_1(a))]^{1/2} \ \stackrel{?}{=}\ B\big(\phi_2(a), -\phi_2(a)\big)^{1/4} \ .
\end{aligned}
$$

Conjecture. A natural candidate for a trailer space (as subspace of $\mathcal{L}(E) \times E$) would be the *Jordan supertriple* of $(E,\{\ldots\})$.