Work in progress

On non-commutative Minkowski spheres

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MOTIVATION

$M := \{(t,x) \in \mathbb{R} \times \mathbb{R}^3 : \; t^2 - \|x\|^2 = 1, \; t \geq 1\}$

$M =$ [MINKOWSKI SPHERE in upper light cone]

$\mathbb{R}^3 \equiv \mathrm{Mat}(1,3,\mathbb{R}), \quad \|a\| = [aa^{\mathrm{T}}]^{1/2}$

$$\phi(a) = \begin{bmatrix} \phi_1(a) \\ \phi_2(a) \end{bmatrix} \qquad \left(a \in \mathbb{R}^3, \; \|a\| \neq 1\right)$$

$$\phi_1(a) := \frac{1 + \|a\|^2}{1 - \|a\|^2}, \quad \phi_2(a) := 2(1 - \|a\|^2 a)^{-1} a \; .$$

Observation: $\phi : \mathrm{Ball}(\mathbb{R}^3) \leftrightarrow M$ by MAPLE (simplify)

MOTIVATION

$$\phi^{-1}(t,x,y,z) = \begin{cases} \frac{r}{1+\sqrt{1+r^2}}(x,y,z) \\ \qquad \text{if} \;\; (t,x,y,z) \neq (1,0,0,0) \\ \\ (0,0,0) \\ \qquad \text{if} \;\; (t,x,y,z) = (1,0,0,0) \end{cases}$$

where $\quad r^2 := x^2 + y^2 + z^2$

ALTERNATIVE VIEW

$\mathrm{Mat}(3,3,\mathbb{R}) \times \mathbb{R}^3$ instead of $\mathbb{R} \times \mathbb{R}^3$

$$M := \{(t,x) \in \underbrace{\mathrm{Mat}(3,3,\mathbb{R})_+}_{pos.self-adj.op} \times \mathbb{R}^3 : \ t^2 - \|x\|^2 = 1, \ t \geq I\}$$

$$\phi(a) = \begin{bmatrix} \phi_1(a) \\ \phi_2(a) \end{bmatrix} \qquad \left(a \in \mathbb{R}^3, \ \|a\| \neq 1\right)$$

$$\phi_1(a) := \left(I + a^{\mathrm{T}} a\right)\left(I - a^{\mathrm{T}} a\right)^{-1},$$

$$\phi_2(a) := 2a\left(I - a^{\mathrm{T}} a\right)^{-1}.$$

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Observation

$$\begin{aligned}
\phi_1(a)^2 - \phi_2(a)^{\mathrm{T}}\phi_2(a) = \\
&= \underbrace{(I - a^{\mathrm{T}}a)^{-1}}\,(I + a^{\mathrm{T}}a)^2\,\underbrace{(I - a^{\mathrm{T}}a)^{-1}} - \\
&\quad - \underbrace{(I - a^{\mathrm{T}}a)^{-1}}\,(4a^{\mathrm{T}}a)\,\underbrace{(I - a^{\mathrm{T}}a)^{-1}} = \\
&= (I - a^{\mathrm{T}}a)^{-1}(I - a^{\mathrm{T}}a)^2(I - a^{\mathrm{T}}a)^{-1} = \\
&= I\ .
\end{aligned}$$

NON-COMMUTATIVE SCALARS

Alternative case: $\langle a|a\rangle := a^{\mathrm{T}} a \in \mathrm{Mat}(3,3,\mathbb{R})$

Hilbert C^*-modules

E vector space, $\mathcal{A}$ C^*-algebra with 1 (over $\mathbb{R}, \mathbb{C}$)

$\left\{ \begin{array}{c} \alpha x \in E \text{ LEFT} \\ x\alpha \in E \text{ RIGHT} \end{array} \right\}$ $\mathcal{A}$-module structure on E

$\langle x|y\rangle \in \mathcal{A}$ non-comm scalar prod.

$\langle y|x\rangle = \langle x|y\rangle^*$, $\begin{cases} \alpha\langle x|y\rangle = \langle \alpha x|y\rangle \\ \langle x|y\rangle\alpha = \langle x\alpha|y\rangle \end{cases}$

Example. $\mathrm{Mat}(1,3,\mathbb{R}) \sim \mathcal{L}(\mathbb{R}^3,\mathbb{R})$
Motivation$\to$left $\mathcal{A} = \mathrm{Mat}1,1,\mathbb{R} \equiv \mathbb{R}$,
Alternative$\to$right $\mathcal{A} = \mathrm{Mat}3,3,\mathbb{R}$, $\mathrm{Mat}(1,3,\mathbb{R}) \hookleftarrow \mathrm{Mat}(3,3,\mathbb{R})$.

TRO SETTING

Ternary **R**ing of **O**perators

H Hilbert space $\quad \mathcal{T} \subset \mathcal{L}(H) \quad$ subspace preserving xy^*z

TRO $=$ Hilbert C^*-module

$$\text{gen. constants: } \mathcal{A}_{left} = \{x^*y : x, y \in \mathcal{T}\}^{C_1^* span} \quad \mathcal{A}\mathcal{T} \subset \mathcal{T}$$
$$\mathcal{A}_{right} = \{xy^* : x, y \in \mathcal{T}\}^{C_1^* span} \quad \mathcal{T}\mathcal{A} \subset \mathcal{T}$$

Remark. Enveloping TRO: $\mathcal{L}(H)$. IT SUFFICES

$$\phi(a) = \begin{bmatrix} \phi_1(a) \\ \phi_2(a) \end{bmatrix} \qquad \big(a \in \mathrm{Ball}(\mathcal{L}(H))\big)$$
$$\phi_1(a) := (1 + a^*a)(1 - a^*a)^{-1}, \quad \phi_2(a) := 2a(1 - a^*a)^{-1}$$

TRO SETTING

Proposition. $\phi : \mathrm{Ball}\big(\mathcal{L}(H)\big) \leftrightarrow M$ *where*

$$M := \big\{[t,x] \in \mathcal{L}(H)^2 : \underbrace{t \geq 0}_{pod.def},\ t^2 - x^* x = 1\big\}.$$

moreover $\qquad \phi^{-1}[t,x] = x(1+t)^{-1}$

Corollary. *If* $\big(E, \mathcal{A}, \langle .|.\rangle\big)$ *is a right [left] Hilbert* C^**-module,*

$$\phi_1(a) := (1 + \langle a|a\rangle)(1 - \langle a|a\rangle)^{-1},$$
$$\phi_2(a) := 2a(1 - \langle a|a\rangle)^{-1}\ \big[resp.\ 2(1 - \langle a|a\rangle)^{-1} a\big]$$

then $\quad [\phi_1, \phi_2] : \mathrm{Ball}(E) \leftrightarrow \big\{[t,x] \in \mathcal{A}_0 \times E :\ t^2 - \langle x|x\rangle = 1\big\}$

GEOMETRIC ISSUE (motivation)

Example. $H := \mathbb{C}, \quad \phi(a) = \left[\frac{1+|a|^2}{1+|a|^2}, \frac{2a}{1+|a|^2}\right] \implies$

$$\left[\phi_1'(a)u\right]^2 - \left|\phi_2'(a)u\right|^2 = -4\frac{|u|^2}{\left[1-|a|^2\right]^2} .$$

Riemann(-Poincaré-Bergman) metric on $B := \mathrm{Ball}(\mathbb{C})$ is given by

$$\langle u|u\rangle_a^B := |g_a'(0)^{-1}u|^2 = \frac{|u|^2}{\left[1-|a|^2\right]^2} \qquad (a \in B,\ u \in \mathbb{C})$$

$g_a(z) := (z+a)/(1-\bar{a}z)$ Möbius transformation.

Observation. $\Psi := \phi^{-1}, \implies$

$$\widetilde{t}^2 - |\widetilde{x}|^2 = -4\left\langle \left[\Psi'(t,x)\right](\widetilde{t},\widetilde{x}) \middle| \left[\Psi'(t,x)\right](\widetilde{t},\widetilde{x}) \right\rangle_{\Psi(t,x)}^B .$$

GEOMETRIC ISSUE (TRO setting)

$\phi : B := \mathrm{Ball}\big(\mathcal{L}(H)\big) \leftrightarrow M := \{(t,x) : \ t \geq 0, \ t^2 - x^* x = 1\}, \qquad \Psi := \phi^{-1}$
$g_a(z) := (1 - aa^*)^{-1/2}(z + a)(1 - a^* z)^{-1}(1 - a^* a)^{-1/2}$ Möbius trf.

$$\begin{aligned} \langle u|v\rangle_a^B &:= \big[[g_a'(0)]^{-1}u\big]^*\big[[g_a'(0)]^{-1}v\big] = \\ &= (1 - a^*a)^{-1/2}u^*(1 - aa^*)^{-1}v(1 - a^*a)^{-1/2}. \end{aligned}$$

Theorem. *For* $(\overline{t},\overline{x}) \in T_{(t,x)}M$ *we have* $\overline{t}t + t\overline{t} = \overline{x}^* x + x^* \overline{x}$ *and*
$$\langle(\overline{t},\overline{x})|(\overline{t},\overline{x})\rangle_{(t,x)}^M := -4\Big\langle \big[\Psi'(t,x)\big](\overline{t},\overline{x})\Big|\Psi'(t,x)\big](\overline{t},\overline{x})\Big\rangle_{\Psi(t,x)}^B =$$
$$(I+t)^{1/2}\Big\{2\big[\overline{t}^2 - \overline{x}^*\overline{x}\big] - \Big[\overline{x}^* x(I+t)^{-1}x^*\overline{x} - \overline{t}x^*\overline{x} - \overline{x}^* x\overline{t} + \overline{t}(t-I)\overline{t}\Big]\Big\}(I+t)^{1/2}$$

Problem. Find "smart" *indefinite* inner products $\langle . | . \rangle_{(t,x)}$ on $\mathcal{L}(H)^2$ extending $\phi^{\#}\langle . | . \rangle_{(t,x)}$ from $T_{t,x}M$

GEOMETRIC ISSUE (further motivation)

Remark. In the 1-dimensional case $H = \mathbb{C}$, we have

1) $$\phi^{\#} g_b \begin{bmatrix} t \\ x \end{bmatrix} = \phi \circ g_b \circ \phi^{-1} \begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} \frac{1+|b|^2}{1-|b|^2} t + \frac{2b}{1-|b|^2} x \\ \frac{2b}{1-|b|^2} t + \frac{1+|b|^2}{1-|b|^2} x \end{bmatrix};$$

2) $g_{\tanh(a)}$ is the exponential of the vector field $[a - \bar{a} u^2] \frac{\partial}{\partial u}$. Hence

$$\phi^{\#} g_{\tanh(a)} \begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} \cosh(2a)t + \sinh(2a)x \\ \sinh(2a)t + \cosh(2a)x \end{bmatrix} \qquad \text{for} \quad \begin{bmatrix} t \\ x \end{bmatrix} \in M .$$

In particular $\phi^{\#} g_b$ is the restriction of a linear mapping of $\mathbb{C}^2$ to M.

GEOMETRIC ISSUE (TRO setting)

Notation. $A_a(u) := \big[a - ua^*u\big]\frac{\partial}{\partial u} \quad \big(a \in \mathcal{L}(H)\big)$

For fixed $a \in \mathcal{L}(H)$ and $u \in \mathrm{Ball}\mathcal{L}(H)$,
$u_\tau := [\exp(\tau A_a)]u \ (\tau \in \mathbb{R})$ satisfies $\frac{d}{d\tau}u_\tau = A_a(u_\tau),\ u_0 = u.$

Well-known: $b = [\exp(A_a)]0 \ \Rightarrow \ g_b(u) = [\exp(A_a)]u \ \ (\|u\| < 1).$

Proposition. *Let $a \in \mathcal{L}(H)$ and $\left[\begin{smallmatrix} t \\ x \end{smallmatrix}\right] \in M$. Then*

$$\phi^{\#} A_a \begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} a^*x + x^*a \\ a(1+t) + x(1+t)^{-1}x^*a \end{bmatrix}.$$

BEYOND TRO: Self-adjoint operators

$\mathcal{A} = \mathcal{A}(H) = \{a \in \mathcal{H} : a^* = a\}$ REAL JORDAN ALGEBRA in $\mathcal{L}(H)$

$$\phi^\# A_a \begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} ax + xa \\ at + ta \end{bmatrix} = \begin{bmatrix} 0 & L_a + R_a \\ L_a + R_a & 0 \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix}$$

Theorem. $a \in \mathcal{A} \Rightarrow$ 1) $g_{\tanh(a)} : \mathrm{Ball}(\mathcal{A}) \leftrightarrow \mathrm{Ball}(\mathcal{A})$,

2) $\phi : \mathrm{Ball}(\mathcal{A}) \leftrightarrow M \cap \mathcal{A}^2 = \left\{ \left[\begin{smallmatrix} t \\ x \end{smallmatrix}\right] \in \mathcal{A}_+ \times \mathcal{A} : t^2 - x^2 = 1 \right\}$

3) $\phi^\# g_{\tanh(a)} = \phi \circ [\exp A_a] \circ \phi^{-1} \left[\begin{smallmatrix} t \\ x \end{smallmatrix}\right] = \begin{bmatrix} \cosh(L_a+R_a)t + \sinh(L_a+R_a)x \\ \sinh(L_a+R_a)t + \cosh(L_a+R_a)x \end{bmatrix}$

$$\cosh(L_a + R_a)z = \frac{1}{2}[\exp(a)]z[\exp(a)] + \frac{1}{2}[\exp(-a)]z[\exp(-a)] \ ,$$
$$\cosh(L_a + R_a)z = \frac{1}{2}[\exp(a)]z[\exp(a)] - \frac{1}{2}[\exp(-a)]z[\exp(-a)] \ .$$

BEYOND TRO: JB*-triples

Observation. $\phi_1(a) = 1 + 2a^*\phi_2(a)$ in TRO case

$\big(E, \|.\|, \{\ldots\}\big)$ JB*-triple: $\{\ldots\} : E^3 \to E$, $D(a,b) : x \mapsto \{abx\}$

1) $\{xyz\}$ symmetric bilinear in x, z, antilinear in y
2) Jordan-identity for $\{\ldots\}$
3) $\|\{aaa\}\| = \|a\|^3$
4) $\big\|\exp\big(\zeta D(a,a)\big)\big\| \leq 1$ if $\mathrm{Re}(\zeta) \leq 0$

Canonical JB*-product on $\mathcal{L}(H)$: $\{xyz\} := \frac{1}{2}xy^*z + \frac{1}{2}zy^*x$

$$\phi_2(a) := 2\big[1 - D(a,a)\big]^{-1}a, \quad \phi_1(a) := 1 + \underbrace{D\big(\phi_2(a), a\big)}_{=D\left(a,\phi_2(a)\right)} \in \mathcal{L}(E)$$

Proposition. $\phi^{-1}\begin{bmatrix} t \\ x \end{bmatrix} = (1+t)^{-1}x$ *for* $\begin{bmatrix} t \\ x \end{bmatrix} \in \mathrm{range}(\phi)$