Proc. London Math. Soc. (3) 92 (2006) 682–712
doi:10.1112/S0024611505015595

ORDERED C^*-MODULES

DAVID P. BLECHER AND WEND WERNER

1. *Introduction*

In [**34**, **35**] Weaver introduced Hilbert C^*-bimodules with involution. In the present paper we study Hilbert C^*-bimodules which have an involution and a positive cone. The reason why we became interested in this topic is that, first, ordered C^*-modules are excellent and appealing examples of *ordered operator spaces* (the latter term is defined below), and second, in order to develop a new tool to study general ordered operator spaces (the 'non-commutative Shilov boundary' of which is always an ordered C^*-module). One may also view the object of study of our paper as an apparently hitherto overlooked generalization of C^*-algebras. The ensuing theory is uncomplicated in some sense, but in other ways does possess intricate features. As an interesting by-product, it turns out that a number of properties of C^*-bimodules that, in general, are only definable using an embedding into a C^*-algebra actually turn out to be invariants of the underlying order structure. We introduce some of these invariants in §§ 2 and 3, where we principally discuss involutive structure. In § 4, we characterize the possible orderings on a C^*-bimodule X which correspond to embeddings of X as selfadjoint *ternary rings of operators* (defined below). Our characterization is in terms of selfadjoint *tripotents*, namely elements u such that $u = u^* = u^3$. In the uniform version of our theory, we will need the 'tripotent' variant of Akemann's notion (see for example [**1**; **29**, 3.11.10]) of an open projection. Hence we will also study in § 4 some of the basic properties associated with this key notion. In § 5 we characterize all the *maximal* ordered operator space orderings on X; these turn out to be automatically orderings of the type discussed above. In the final section we consider the important 'commutative' example of *involutive line bundles*. Amongst other things these will show that our results in previous sections are sharp.

We now turn to precise definitions and notation. We write X_+ for the cone of 'positive elements', that is, those with $x \geqslant 0$, in an ordered vector space. A subspace W of a vector space with involution is *selfadjoint* if $x^* \in W$ whenever $x \in W$, or in shorthand, $W^* = W$. We will say that a linear map $T : X \to Y$ between vector spaces with involution is *selfadjoint* or *$*$-linear* if $T(x^*) = T(x)^*$ for all $x \in X$. If X and Y are also ordered vector spaces then we say that T is *positive* if T is selfadjoint, and if $T(X_+) \subset Y_+$. By a *concrete ordered operator space*, we mean a selfadjoint subspace X of a C^*-algebra A, together with the positive cone $X \cap A_+$ inherited from A. If also X contains 1_A, we call X a *unital operator system*. We will not use

---

Received 30 July 2004; revised 11 July 2005.

2000 *Mathematics Subject Classification* 46L08, 47L07 (primary), 46L07, 47B60, 47L05 (secondary).

This research was supported in part by the SFB 487 Geometrische Strukturen in der Mathematik, at the Westfälische Wilhelms-Universität, supported by the Deutsche Forschungsgemeinschaft. Blecher was also supported in part by a grant from the National Science Foundation.

much from operator space theory; what we use can probably be found in any of the current books (for example [**6**]), or in most of the papers, on operator spaces. We use the term *ordered operator space* for an operator space with an involution, and with specified positive cones in $M_n(X)_{sa}$, such that there exists a completely isometric complete order embedding from X into a C*-algebra. (A complete order embedding is a completely positive map T such that T^{-1} is completely positive on $\mathrm{Ran}(T)$.) We will not need this here, but ordered operator spaces were characterized abstractly in [**37**] (see also [**38**]). We say that one ordering on X is *majorized* by another ordering if the positive cones for the first ordering are contained in the positive cones for the second ordering.

We have used $*$ above for the 'adjoint' or 'involution'; because this symbol appears so frequently in this paper we will instead write X' for the dual Banach space of a Banach space X, and for the dual operator space of an operator space X.

By a *ternary ring of operators* (or *TRO* for short) we will mean a closed subspace X of a C*-algebra B such that $XX^*X \subset X$. Sometimes we call this a TRO *in B*. The important structure on a TRO is the 'ternary product' xy^*z, which we sometimes write as $[x, y, z]$. Our interest here is in TROs within the realm of operator spaces, that is, we will always regard a TRO X as equipped with its canonical operator space structure. Thus there are canonical norms on $M_n(X)$ for each $n \in \mathbb{N}$ (which are compatible with the natural TRO structure on $M_n(X) \subset M_n(B)$). Under this assumption, an intrinsic abstract characterization of TROs may be found in [**27**] (see also [**41**]). We will not use this however. Instead we simply make a definition: a *ternary system* is an operator space X with a map $[\cdot,\cdot,\cdot] : X \times X \times X \to X$, such that there exist a TRO Z and a completely isometric surjective linear isomorphism $T : X \to Z$ satisfying $[Tx, Ty, Tz] = T([x, y, z])$ for all $x, y, z \in X$. A linear map T satisfying the last equation is called a *ternary morphism*. Even for an abstract ternary system we sometimes simply write $[x, y, z]$ as xy^*z for $x, y, z \in X$.

We write XY for the norm closure of the span of the set of products xy for $x \in X$ and $y \in Y$, assuming that such products make sense. A similar notation applies to the product of three or more sets. In contrast, in the expression $X + Y$ we are not automatically taking the norm closure. It is well known that $XX^*X = X$ for a TRO X. Also, it is clear that XX^* and X^*X are C*-algebras, which we will call the *left* and *right C*-algebras* of X respectively, and X is a (XX^*)-(X^*X)-bimodule.

The following result shows that ternary morphisms behave very similarly to $*$-homomorphisms between C*-algebras.

PROPOSITION 1.1 (See for example [**17**] or [**6**, § 8.3]). *Let $T : X \to Y$ be a ternary morphism between ternary systems. Then the following hold.*

(1) *The morphism T is completely contractive and has closed range.*

(2) *The morphism T is completely isometric if it is one-to-one.*

(3) *A linear isomorphism between ternary systems is completely isometric if and only if it is a ternary morphism.*

(4) *The quotient of a TRO X by a 'ternary ideal'* (*that is, a uniformly closed* (XX^*)-(X^*X)*-subbimodule*) *is a ternary system.*

(5) *If X is a TRO then* $\mathrm{Ker}(T)$ *is a ternary ideal, and the induced map*

$$X/\mathrm{Ker}(T) \to Y$$

is a one-to-one ternary morphism.

(6) *If X and Y are TROs, then T canonically induces a $*$-homomorphism (respectively $*$-isomorphism) $\pi : X^*X \to Y^*Y$ between the associated right C^*-algebras, via the prescription $\pi(x^*y) = T(x)^*T(y)$. If T is one-to-one (respectively one-to-one and surjective) then π is also one-to-one (respectively a $*$-isomorphism). Similar results hold for the left C^*-algebras.*

These results are mostly due to Hamana. Item (3) of the last proposition was independently proved by Z. J. Ruan in his PhD thesis, and also proved by Kirchberg. From it one easily sees that an operator space X may have at most one 'ternary product' $[\cdot,\cdot,\cdot]$ with respect to which it is a ternary system. Thus we may simply define a ternary system to be an operator space X which is linearly completely isometric to a TRO.

In this paper we will be more interested in *$*$-TROs*, by which we mean a closed selfadjoint TRO Z in a C*-algebra B. In this case, the 'left' and 'right' C*-algebras mentioned above, namely ZZ^* and Z^*Z, coincide, and equal Z^2. By a *$*$-WTRO* we mean a selfadjoint weak* closed TRO in a W*-algebra. Note that a $*$-TRO Z comes with a given positive cone Z_+, inherited from the containing C*-algebra. Analogues of parts of the last proposition are valid for selfadjoint ternary morphisms, which we also call *ternary $*$-morphisms*, between $*$-TROs. In particular, the kernel of a ternary $*$-morphism on a $*$-TRO is clearly a *ternary $*$-ideal*, by which we mean a selfadjoint ternary ideal. It is not hard, by following the usual proof of Proposition 1.1(4) (or by using Lemma 3.1(1) below), to show that the quotient of a $*$-TRO by a ternary $*$-ideal is again an involutive ternary system in a natural way which is compatible with the quotient operator space structure on Y/X. From this one easily checks the analogue of Proposition 1.1(5); that if one factors a ternary $*$-morphism on a $*$-TRO by its kernel, then one obtains a one-to-one ternary $*$-morphism on the quotient ternary system. A ternary $*$-ideal J in an involutive ternary system Z will be called a *C*-ideal* if J is ternary $*$-isomorphic to a C*-algebra.

An *ordered C*-module* is a C*-bimodule Y over a C*-algebra A with a given involution and positive cone, such that Y (with its canonical ternary product $x\langle y,z\rangle$) is 'ternary order isomorphic' to a $*$-TRO. Thus ordered C*-modules are essentially the same thing as $*$-TROs in a C*-algebra B, with their canonical inherited ordering from B. In the last paragraph of § 4, we will give a better characterization of ordered C*-modules amongst the 'involutive ternary systems' (defined below). Since they are essentially the same as $*$-TROs, it suffices to focus on the order properties of $*$-TROs: nearly all of our results on $*$-TROs will transfer in an obvious way to ordered C*-modules. Thus the reader will not see the term ordered C*-module much in this paper.

Ternary systems, and hence also $*$-TROs, are a particularly nice subclass of the *JB*-triples*. We will not define the latter objects, but we must stress that some of the techniques and ideas in the present paper originate in that field of study. As we said above, we establish a link between orderings on a $*$-TRO Z, and certain selfadjoint tripotents in Z or Z'' which happen to be 'central' (in a sense different from the JB*-triple sense of that word). With this in hand, some portion of our results may be viewed as variants in some sense of certain JB*- and JBW*-triple results. We have tried to indicate consistently, to the best of our knowledge, where a comparison with the JBW*-triple literature should be made. The reader is encouraged to consult the JBW*-triple literature (for example [**3**, **4**, **16**, **19**, **21**, **22**, **28**, **31**]; and also the work of W. Kaup (for example [**23**, **24**]), and C. M. Edwards and the late G. T. Rüttimann, of which we have cited some representatives which have some important points of contact with our paper). Note that if Z is a $*$-TRO then Z_{sa} is a real JB*-triple (see for example [**13**, **22**, **31**]), and one may then appeal to the methods and results of the real JB*-triple theory. However, we believe that the main results of our paper are quite new. In particular, we have not seen 'open tripotents' in our sense in the literature. In [**11**, **12**] this term is used in a sense which looks similar and is formally related to ours, but which in fact is certainly quite different. More relevant are the 'compact tripotents' of [**12**] for example; however there is no nice 'perp' relation between open and closed tripotents. These works were partially inspired by Akemann and Pedersen's paper [**2**].

We will use 'TRO techniques' throughout. There are several recent papers of Ruan, alone or with coauthors, concerning TROs (see for example [**14**, **30**]). Again any overlaps between this work and ours only concern very simple facts.

Acknowledgements. The authors are very grateful for support from the SFB 487 Geometrische Strukturen in der Mathematik, at the Westfälische Wilhelms-Universität, in turn supported by the Deutsche Forschungsgemeinschaft. The first author is also indebted to our colleagues there for their generous hospitality, in 2002 when this work began. Another impetus for this project was a Research Experiences for Undergraduates problem that the first author gave to then undergraduate Kay Kirkpatrick [**26**]. Finally, we thank T. Oikhberg and B. Russo for several comments after a talk the first author gave on this work; and M. Neal for much helpful information.

Some of the contents of this paper was summarized at the G.P.O.T.S. conference at the University of Urbana-Champaign in May 2003.

2. *Involutive C*-bimodules*

In this section we fix a C*-algebra B, which the reader may wish to take equal to $B(H)$ for a Hilbert space H. Then $M_2(B)$ is also canonically a C*-algebra. Indeed, $M_2(B) \cong B(H^{(2)})$ $*$-isomorphically if $B = B(H)$. Let A and Z be selfadjoint subspaces of B, and define

$$\mathcal{L} = \begin{bmatrix} A & Z \\ Z & A \end{bmatrix} = \left\{ \begin{bmatrix} a_1 & z_1 \\ z_2 & a_2 \end{bmatrix} \in M_2(B) : a_1, a_2 \in A,\ z_1, z_2 \in Z \right\}. \tag{1}$$

Then $\mathcal{L}$ is a selfadjoint subspace of $M_2(B)$, and it is uniformly closed if A and Z are closed in B. Similarly, if B is a W*-algebra, and if A and Z are weak* closed in B, then $\mathcal{L}$ is weak* closed in the W*-algebra $M_2(B)$.

Similar remarks apply to the selfadjoint closed subspace

$$\tilde{\mathcal{L}} = \left\{ \begin{bmatrix} a & z \\ z & a \end{bmatrix} \in \mathcal{L} : a \in A,\ z \in Z \right\} \tag{2}$$

of $\mathcal{L}$. Note also that $\tilde{\mathcal{L}}$ is canonically completely isometric to a subspace of $B \oplus^\infty B$. To see this, notice that the maps

$$\begin{bmatrix} x & y \\ y & x \end{bmatrix} \mapsto (x + y, x - y),$$

and

$$(x, y) \mapsto \begin{bmatrix} \frac{1}{2}(x + y) & \frac{1}{2}(x - y) \\ \frac{1}{2}(x - y) & \frac{1}{2}(x + y) \end{bmatrix},$$

are $*$-isomorphisms between $B \oplus^\infty B$ and the subspace

$$\left\{ \begin{bmatrix} x & y \\ y & x \end{bmatrix} : x, y \in B \right\}$$

of $M_2(B)$. In particular, it follows that

$$\left\| \begin{bmatrix} x & y \\ y & x \end{bmatrix} \right\| = \max\{\|x+y\|, \|x-y\|\}. \tag{3}$$

Henceforth, we will consider the case when $\mathcal{L}$ and $\tilde{\mathcal{L}}$ above are C*-subalgebras of $M_2(B)$. This occurs precisely when A is a C*-subalgebra of B, and Z is a closed selfadjoint subspace of B such that $AZ \subset Z$ and $Z^2 \subset A$. In this case we say that Z is an *involutive ternary A-submodule of B*. Then $\tilde{\mathcal{L}}$ above is canonically $*$-isomorphic to a $*$-subalgebra of $B \oplus^\infty B$. We note that any involutive ternary A-submodule is a $*$-TRO. Conversely, any $*$-TRO Z in B is an involutive ternary Z^2-submodule of B. If B is a W*-algebra, and if Z and A are weak* closed in B, then Z is a $*$-WTRO, and also $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are W*-subalgebras of $M_2(B)$.

There exist abstract characterizations of involutive ternary A-bimodules. These bimodules are in fact very closely related to the topic of $*$-automorphisms θ of a C*-algebra M of *period* 2, that is, $\theta^2 = \mathrm{Id}$. For such an automorphism θ, let N be the fixed point algebra $\{a \in M : \theta(a) = a\}$, and let $W = \{x \in M : \theta(x) = -x\}$. Clearly W is an involutive ternary N-subbimodule of M. Conversely, if Z is an involutive ternary A-subbimodule in B, then the C*-algebra $\tilde{\mathcal{L}}$ in Equation (2) has an obvious period 2 automorphism whose fixed point algebra is isomorphic to A; and Z is appropriately isomorphic to the set of matrices $x \in \tilde{\mathcal{L}}$ with $\theta(x) = -x$. Indeed if $Z \cap A = (0)$, then one can show that $M = Z + A$ is a C*-algebra with an obvious period 2 automorphism $z + a \mapsto a - z$ (see Corollary 2.2(1) below), and then the space W mentioned above equals Z.

The following abstract characterization of involutive ternary A-bimodules, while conceptually significant, will not be technically important in the present paper, so we merely take a couple of paragraphs to sketch the details.

Definition 2.1. An *involutive C*-bimodule* over a C*-algebra A is a bimodule X over A, such that X has an involution satisfying $(ax)^* = x^* a^*$, and such that X is both a right and a left C*-module over A, with the left module inner product $[\cdot|\cdot]$ being related to the right module inner product by the formula $[x|y] = \langle x^*|y^*\rangle$, and such that two inner products are also compatible in the sense that $x\langle y|z\rangle = [x|y]z$ for all $x, y, z \in X$.

We say that such a bimodule X is a *commutative involutive C*-bimodule* if in addition $\langle y|z\rangle = [z, y]$ for all $y, z \in X$.

Since an involutive C*-bimodule is a C*-bimodule, it has a canonical operator space structure (see for example [**6**, § 8.2]).

It is clear that any involutive ternary A-submodule of a C*-algebra B is an involutive C*-bimodule in the above sense. To see the converse, we note that the proof of Theorem 12 in [**34**] may be easily adapted to show that any involutive C*-bimodule over a C*-algebra A is isomorphic, via a complete isometry which is also a ternary $*$-isomorphism and an A-A-bimodule map, to an involutive ternary A-submodule of a C*-algebra C. See also our later Corollary 3.2.

We will not use this, but involutive C*-bimodules may be equivalently defined to be a certain subclass of the Hilbert A-bimodules with involution recently developed by Weaver. Namely, it is the subclass of bimodules satisfying the natural extra condition $(x,y)z = x(y,z)$ in the language of Weaver's definition on the second page of [**34**], and which are also Hilbert C*-modules in the usual sense with regard to both of the natural sesquilinear A-valued inner products.

There are three important spaces canonically associated to a $*$-TRO, or involutive ternary A-submodule, Z in a C*-algebra B. The first is the $*$-subalgebra $A+Z$ of B, which, we will see shortly, is a C*-algebra. The case where $A = Z^2$ will feature particularly often; indeed Z^2+Z will be a C*-algebra which will play a role similar to that of the 'linking C*-algebra' in Morita equivalence theory. The second space we call the *center* of Z (following [**34**, **32**, **33**]): this is defined to be

$$\mathcal{Z}(Z) = \{z \in Z : az = za \text{ for all } a \in Z^2\}.$$

It is easy to see that $\mathcal{Z}(Z)$ is a $*$-TRO in B too. One can show that $z \in \mathcal{Z}(Z)$ if and only if $az = za$ for all $a \in A$, but since we will not use this fact we will not prove it here. We will study $\mathcal{Z}(Z)$ in more detail later in this section. The third space $J(Z)$ is defined to be $Z \cap Z^2$; this is a C*-algebra. Clearly we have

$$J(Z) \subset Z \subset Z + A.$$

These three auxiliary spaces will play a significant role for us. Note that $\mathcal{Z}(Z)$ is an invariant of the involutive ternary structure on Z. That is, a ternary $*$-isomorphism $\psi : E \to F$ between two $*$-TROs restricts to a ternary $*$-isomorphism from $\mathcal{Z}(E)$ onto $\mathcal{Z}(F)$. This may be easily seen using Proposition 1.1(6). On the other hand, Z^2+Z and $J(Z)$ are not invariants of the involutive ternary structure on Z. However, we shall see later, in Corollary 4.3, the interesting fact that Z^2+Z and $J(Z)$ *are* invariants of the order structure of Z. For this reason we defer much discussion of Z^2+Z and $J(Z)$ to § 4 which is devoted to order structure.

COROLLARY 2.2. *Let Z be an involutive ternary A-submodule of B. Then*
(1) *$A+Z$ is a closed subspace of B, and indeed is a C*-subalgebra of B;*
(2) *if, further, B is a W*-algebra, and if A and Z are weak* closed in B, then $A+Z$ is a weak* closed subspace of B, indeed is a W*-subalgebra of B;*
(3) *$J(Z) = Z \cap A$;*
(4) *$J(Z)$ is an ideal in both A and $A+Z$;*
(5) *$J(Z)$ is a complete M-ideal (in the sense of [**15**]) of Z. It is also an A-subbimodule of Z, and a ternary $*$-ideal, in fact a C*-ideal, in Z.*

Proof. (1) As remarked above, in this case the set $\tilde{\mathcal{L}}$ in Equation (2) is a C*-algebra. The map π from $\tilde{\mathcal{L}}$ to B taking the matrix written in Equation (2) to $a+z$, is a $*$-homomorphism. Therefore its range is a closed C*-subalgebra of B.

(2) Assuming the hypotheses here, we deduce, as remarked above, that the set $\tilde{\mathcal{L}}$ is a W*-algebra. It is also easy to see that π in the proof of (1) is weak* continuous. By basic von Neumann algebra theory we deduce that $\mathrm{Ran}(\pi)$ is weak* closed.

(3) Suppose that $z \in Z \cap A$. If (e_t) is an increasing approximate identity for Z^2, then by the basic theory of C*-modules we have $z = \lim_t e_t z \in Z^2 A \subset Z^2$. Thus $Z \cap A \subset Z \cap Z^2$, and so these two sets are equal.

(4) If $c \in Z \cap Z^2$ and $z \in Z$, then $zc \in Z^2 \cap Z^3 \subset Z^2 \cap Z = J(Z)$; and similarly $ac \in Z \cap Z^2$ for $a \in A$.

(5) The space $J(Z)$ is a closed A-A-submodule of the C*-bimodule Z. However, the complete M-ideals in a C*-bimodule are exactly the closed A-A-submodules (see for example [**6**, 8.5.20]). $\square$

We now return to the center $\mathcal{Z}(E)$ of a $*$-TRO E in a C*-algebra B.

LEMMA 2.3. *Let E be a $*$-TRO in a C*-algebra B. Then:*
(1) *$\mathcal{Z}(E)$ is a $*$-TRO in B;*
(2) *if $v \in \mathcal{Z}(E)$ then $vx = xv$ for all $x \in E$;*
(3) *if $v, w \in \mathcal{Z}(E)$ then vw is in the center C of E^2; also $C\mathcal{Z}(E) \subset \mathcal{Z}(E)$ and $\mathcal{Z}(E)C \subset \mathcal{Z}(E)$;*
(4) *$\mathcal{Z}(E)$ is a commutative involutive C*-bimodule over C.*

Proof. Part (1) is immediate. For (2), let $u, v, w \in \mathcal{Z}(E)$ and $x \in E$. Then

$$uvwx = u(wx)v = (wx)(uv) = (xu)wv = x(wv)u = xwvu.$$

In particular, $(uv)^2 = (vu)^2$. Setting $v = u^*$, and using the unicity of square roots, gives $uu^* = u^*u$. By the polarization identity we obtain $uv = vu$ for all $u, v \in \mathcal{Z}(E)$. Using this together with the last centered equation we obtain

$$uvwx = xwvu = xuvw.$$

Since $\mathcal{Z}(E)^3 = \mathcal{Z}(E)$, we obtain (2).

Items (3) and (4) are now easy to check directly (see also [**33**]). $\square$

LEMMA 2.4. *Let Z be a $*$-WTRO in a W*-algebra B. Let J be the weak* closure of Z^2 in B. We have:*
(1) *$Z + J$ equals the weak* closure of $Z + Z^2$ in B;*
(2) *the multiplier algebra $M(Z^2)$ is $*$-isomorphic to J;*
(3) *$M(Z + Z^2) \cong Z + J$ $*$-isomorphically;*
(4) *$J(Z)$ is a W*-subalgebra of J, and, in particular, is weak* closed in Z too;*
(5) *$Z \cong J(Z) \oplus^\infty J(Z)^\perp$, where $J(Z)^\perp = \{z \in Z : zJ(Z) = 0\}$.*

Proof. Clearly Z and J are both contained in the weak* closure of $Z + Z^2$, and hence so is their sum. Note that Z is an involutive J-subbimodule of B. Hence by Corollary 2.2(2), $Z + J$ is weak* closed. Hence $Z + J$ contains the weak* closure of $Z + Z^2$. This yields (1).

We now consider (2) (see [**14**, Appendix]). By Corollary 2.2(2), $Z + J$ is a W*-algebra, and it may therefore be represented faithfully and normally as a von Neumann algebra in $B(H)$ say. We claim that in this representation, Z^2 acts nondegenerately on H. For if η is a unit vector in the orthocomplement of Z^2H, and if $\psi = \langle \cdot \eta, \eta \rangle$, then ψ is a normal state on $Z + J$. Now Z^2 and $Z + Z^2$ have a common increasing approximate identity (e_t), and $e_t \to I_H$ is weak*. Hence $\langle \eta, \eta \rangle = \lim \langle e_t \eta, \eta \rangle = 0$, and so $\eta = 0$. Thus indeed Z^2, and also $Z + Z^2$, act nondegenerately on H. Thus we may identify $M(Z^2)$ and $M(Z+Z^2)$ with $*$-subalgebras of $B(H)$. Indeed we may identify them with $*$-subalgebras of the double commutants of Z^2 and $Z + Z^2$ respectively in $B(H)$. Thus, by the double commutant theorem and (1), we may identify them with $*$-subalgebras of J and $Z + J$ respectively.

Conversely, by routine weak* density arguments $JZ \subset Z$ since Z is weak* closed. Thus $JZ^2 \subset Z^2$, and similarly $Z^2J \subset Z^2$. Thus $J \subset M(Z^2)$, and so $J = M(Z^2)$.

Similarly $(Z+J)(Z+Z^2) \subset Z+Z^2$ and $(Z+Z^2)(Z+J) \subset Z+Z^2$, so that $Z+J = M(Z+Z^2)$. This proves (2) and (3).

For (4), simply note that $J(Z) = Z \cap J$ by Corollary 2.2(3).

Finally, let e be the identity of the W*-algebra $J(Z)$, and let 1 be the identity of $J = M(Z^2)$. If $z \in Z$ then $z = ez + (1-e)z$. We have $ez \in J(Z)$ (since the latter is an ideal), and $(1-e)z \in J(Z)^{\perp}$. This proves (5). □

3. *Involutive ternary systems*

We will use the term *involutive ternary system* for the *abstract* version of a $*$-TRO. Namely, an involutive ternary system is a ternary system X possessing an involution $*$, such that X is completely isometrically isomorphic, via a selfadjoint (that is, '$*$-linear') ternary morphism, to a $*$-TRO. We will give a useful characterization of these spaces in the next lemma. The appropriate morphisms between involutive ternary systems are of course the *ternary $*$-morphisms*, namely the selfadjoint ternary morphisms.

LEMMA 3.1. *Let X be a ternary system possessing an involution $*$. Then:*

(1) (cf. [**22**, *Proposition* 2.3]) *X is an involutive ternary system if and only if $[x,y,z]^* = [z^*,y^*,x^*]$ for all $x,y,z \in X$, and if and only if $\|[x_{ij}^*]\| = \|[x_{ji}]\|$ for all $n \in \mathbb{N}$ and $[x_{ij}] \in M_n(X)$;*

(2) *if X is an involutive ternary system possessing a predual Banach space then X is isomorphic to a $*$-WTRO via a weak* homeomorphic completely isometric ternary $*$-isomorphism.*

Proof. (1) One direction of the first equivalence is obvious. For the other, fix a completely isometric ternary morphism $\Phi_0 : X \to B(H)$. If the hypothesized identity holds, it is straightforward to check that

$$\Phi(x) := \begin{pmatrix} 0 & \Phi_0(x) \\ \Phi_0(x^*)^* & 0 \end{pmatrix}$$

is a completely isometric ternary $*$-morphism from X onto a $*$-TRO inside $M_2(B(H))$.

For the second equivalence, again one direction is obvious. For the other, suppose that X is a TRO in $B(H)$, and that $\tau : X \to X$ is a conjugate linear map satisfying $\|[\tau(x_{ij})]\|_n = \|[x_{ji}]\|$ for all matrices. The map $\theta(x) = \tau(x)^*$ is then a linear complete isometry from X onto $\{x^* \in B(H) : x \in X\}$. Thus by Proposition 1.1(3), θ is a ternary morphism. Thus $\tau([x,y,z]) = [\tau(z), \tau(y), \tau(x)]$ for all x, y and z.

(2) If, further, X has a predual, then by the characterization of dual TROs by Zettl and Effros, Ozawa and Ruan (see [**41**, § 4] and [**14**], or [**7**] for a recent alternative proof), there is a weak* homeomorphic completely isometric ternary morphism Φ_0 from X onto a weak* closed TRO inside $B(H)$. Defining Φ as in the proof of (1) yields the desired isomorphism. □

In view of Lemma 3.1(2), we define a *dual involutive ternary system* to be an involutive ternary system with a predual Banach space. By the lemma, this is the abstract version of a $*$-WTRO.

Lemma 3.1(1) immediately gives another proof of a result we mentioned after Definition 2.1.

COROLLARY 3.2. *Any involutive C^*-bimodule* (*in the sense of Definition* 2.1) *is an involutive ternary system.*

Next we discuss the second dual of a $*$-TRO. If X is an involutive ternary system, then so is X'' in a canonical way. That is, there exists one and only one way to extend the involution on X to a weak* continuous involution on X'', and with this extended involution X'' is an involutive ternary system. The 'existence' here is easy: if X is represented as a $*$-TRO in a C*-algebra B, then $X^{\perp\perp}$ is easily seen by standard arguments to be a $*$-WTRO in B''. The 'uniqueness' follows by routine weak* density considerations.

PROPOSITION 3.3. *Let Z be a $*$-TRO in a C^*-algebra A, and set $E = Z''$, which we may identify with a subspace of the W^*-algebra A''. Then we have:*

(1) *$(Z^2)^{\perp\perp}$ in A'' equals the weak* closure N of E^2, and may also be identified with $M(E^2)$, the multiplier algebra of E^2;*
(2) *$E + N$ is weak* closed in A'';*
(3) *$J(Z)^{\perp\perp} = J(E)$ and thus $J(Z)$ is weak* dense in $J(E)$;*
(4) *$J(Z)_+$ is weak* dense in $J(E)_+$.*

Proof. (1) Let $J = (Z^2)^{\perp\perp}$. Clearly $J \subset N$. Conversely, by routine weak* density considerations, $ZE \subset J$, and so $E^2 \subset J$, giving $N = J$. Also by routine weak* density considerations, since E is an involutive ternary J-subbimodule of A'', $J = M(E^2)$ by Lemma 2.4(2).

(2) This follows from Corollary 2.2(2) (or from [**18**, I.1.14]).
(3) This follows from the above and [**18**, I.1.14].
(4) This follows from (3) and the Kaplansky density theorem. □

LEMMA 3.4. *Let Z be a $*$-TRO.*

(1) *If Z has a predual then the selfadjoint M-summands in Z are exactly the weak* closed ternary $*$-ideals, and these are exactly the subspaces pZ for a* (*necessarily unique*) *central projection p in the multiplier algebra $M(Z^2)$ such that $pz = zp$ for all $z \in Z$.*

(2) *The selfadjoint M-ideals in Z are exactly the ternary $*$-ideals, and these are exactly the subspaces of the form $Z_p = \{z \in Z : pz = z\}$ for a projection in the second dual of Z^2 such that $pz = zp$ for all $z \in Z$. In fact p may be chosen to have the additional property of being an open projection* (*see* [**29**, 3.11.10] *for the definition of this*) *in the center of the second dual of Z^2, and with this qualification the correspondence $p \mapsto Z_p$ is bijective.*

Proof (sketch). This mostly follows from the facts that

(a) the M-ideals in a TRO Z are the ternary ideals (they are also the complete M-ideals in the sense of Effros and Ruan [**15**]), and
(b) the M-summands in a weakly closed TRO are the weak* closed ternary ideals.

For example, see [**3**] and [**6**, 8.5.20].

(1) If Z has a predual then we may assume that Z is a $*$-WTRO, by Lemma 3.1(2). If N is an M-summand in Z, then by [**6**, Lemma 8.5.16], $N = P(Z)$ for an

adjointable projection P on Z, and by 8.5.13, 8.5.3, and the last line of 8.5.11, in [**6**], we see that $N = pZ$ for a unique projection $p \in M(Z^2)$. Since N is selfadjoint, $pZ = Zp$, so that $pz = pzp = zp$ for all $z \in Z$. It follows from these last equalities that p is in the center of $M(Z^2)$.

(2) It is easy to see that any space of the form Z_p above is a ternary $*$-ideal. Conversely, if E is a selfadjoint M-ideal, then by (1), $E^{\perp\perp} = pZ''$, for a central projection in $M((Z'')^2)$, or equivalently by Proposition 3.3(1), in the second dual of Z^2. Thus $E = E^{\perp\perp} \cap Z = Z_p$. For the uniqueness, suppose that p and r are two open projections with the property that if $z \in Z$ then $pz = z$ if and only if $rz = z$. If $a \in Z^2$ and if $pa = a$ then $paz = az$ for all $z \in Z$. Thus $raz = az$ for all $z \in Z$, so that $ra = a$. Let a_t be an increasing net in Z^2 converging in the weak* topology to p. Then $pa_t = a_t$, so that $ra_t = a_t$. In the limit $rp = p$. By a similar argument $pr = r$. Taking adjoints, we see that $r = p$. □

4. *Orderings on ternary systems*

In this section, we develop the basic theory of ordered ternary systems.

Lemma 4.1. *If Z is a $*$-TRO, then $J(Z) = \mathrm{Span}(Z_+)$, and $J(Z)_+ = Z_+$. Also, $J(M_n(Z)) = M_n(J(Z))$, and $M_n(Z)_+ = M_n(J(Z))_+$, for every $n \in \mathbb{N}$.*

Proof. Clearly $J(Z)_+ \subset Z_+$. If $z \in Z_+$ then since $z^2 \in Z^2$ we must have $z \in Z^2$ (square roots in a C*-algebra remain in the C*-algebra). Thus $Z_+ \subset Z \cap Z^2 = J(Z)$, and so $J(Z)_+ = Z_+$. That $J(Z) = \mathrm{Span}(Z_+)$ follows since $J(Z)$ is spanned by $J(Z)_+ = Z_+$. Finally,

$$J(M_n(Z)) = M_n(Z) \cap M_n(Z)^2 = M_n(Z) \cap M_n(Z^2) = M_n(J(Z)),$$

and so $M_n(Z)_+ = J(M_n(Z))_+ = M_n(J(Z))_+$. □

The following is clear from Lemmas 2.4(5) and 4.1.

Corollary 4.2. *If E is a $*$-WTRO then E is ternary $*$-isomorphic, via a complete order isomorphism, to $M \oplus^\infty Z$, where M is a W*-algebra, and Z is a trivially ordered $*$-WTRO.*

Corollary 4.3. *Let Z be a $*$-TRO, and let $\theta : Z \to B$ be a positive ternary $*$-morphism into a C*-algebra B.*

(1) *The map θ is completely positive.*

(2) *The map θ restricts to a $*$-homomorphism from $J(Z)$ into B. Indeed, if $\pi : Z^2 \to B$ is the $*$-homomorphism canonically associated with θ (see Proposition 1.1(6)), then $\theta = \pi$ on $J(Z)$.*

(3) *If $A = Z + Z^2$, then θ is the restriction to Z of a $*$-homomorphism from A into B. Hence θ may be extended further to a unital $*$-homomorphism from a unitization of A into a unitization of B.*

(4) *If θ is an order embedding then it is a complete order embedding. In this case, if W is the $*$-TRO $\theta(Z)$, then θ restricts to a $*$-isomorphism of $J(Z)$ onto $J(W)$, and θ is the restriction of a $*$-isomorphism between $Z + Z^2$ and $W + W^2$.*

Proof. (2) Consider the restriction of θ to $J(Z)$. We claim that a positive ternary $*$-morphism ψ between C*-algebras is a $*$-homomorphism. To see this note that by going to the second dual we may assume that the C*-algebras are unital. Then if $\psi(1) = v$, it is clear that v is a positive partial isometry. That is, v is a projection. Also $v\psi(\cdot)$ is a $*$-homomorphism, and $\psi = v^2\psi(\cdot)$ in this case. Since $v^2 = v$, this proves the claim. Using the claim, for $a, b \in J(Z)$ we have $\pi(a^*b) = \theta(a)^*\theta(b) = \theta(a^*b)$. Thus $\theta = \pi$ on $J(Z)$.

(3) Let $W = \theta(Z)$. Define $\tilde{\theta} : Z + Z^2 \to W + W^2$ by

$$\tilde{\theta}(x + yz) = \theta(x) + \theta(y)\theta(z),$$

if $x, y, z \in Z$. Namely $\tilde{\theta}$ is the unique linear extension of both θ and the $*$-homomorphism $Z^2 \to W^2$ associated with θ. It is easy to check using (2) that $\tilde{\theta}$ is a well-defined $*$-homomorphism on $Z + Z^2$. Since $Z + Z^2$ is closed, $\tilde{\theta}$ is contractive and will extend to a unital $*$-homomorphism between the unitizations.

(1) This follows from (3).

(4) Since θ is an order embedding, it maps Z_+ onto W_+. By Lemma 4.1 we see that θ maps $J(Z) = \mathrm{Span}(Z_+)$ onto $J(W) = \mathrm{Span}(W_+)$. By (2), θ is a $*$-homomorphism. To see the isomorphism between $Z + Z^2$ and $W + W^2$, use the proof of (3), and the same construction applied to θ^{-1}. □

DEFINITION 4.4. A *naturally ordered ternary system* is a ternary system which is also an ordered operator space such that X is completely order isomorphic via a ternary $*$-isomorphism to a $*$-TRO Y, where Y is given its relative cones inherited from its containing C*-algebra. The associated ordering, positive cone in X, and positive cones in $M_n(X)$ will, respectively, be referred to as a *natural ordering*, a *natural cone*, and *natural matrix cones* on X. Similarly, we refer to *natural dual orderings* and *natural dual cones* on a dual ternary system; this corresponds to the ordering or cone pulled back from a $*$-WTRO via a weak* homeomorphic ternary $*$-isomorphism.

Note that for every involutive ternary system, the trivial ordering is a natural ordering. Indeed given a $*$-TRO Y in a C*-algebra B, one may replace Y with the isomorphic subspace of $M_2(B)$ consisting of matrices of the form

$$\begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix} \quad \text{where } y \in Y.$$

If X and Y are $*$-TROs (respectively $*$-WTROs) in two C*-algebras (respectively W*-algebras) A and B, then $X \oplus^\infty Y$ is a $*$-TRO (respectively $*$-WTRO) in $A \oplus^\infty B$. It is easy to see that $\mathcal{Z}(X \oplus^\infty Y) = \mathcal{Z}(X) \oplus^\infty \mathcal{Z}(Y)$. More generally, we have the following result.

PROPOSITION 4.5. *The 'L^∞-direct sum' $\bigoplus_i^\infty X_i$ of a family of involutive ternary systems (respectively, naturally ordered ternary systems, dual involutive ternary systems, naturally dual ordered ternary systems) is again an involutive ternary system (respectively, a naturally ordered ternary system, dual involutive ternary system, naturally dual ordered ternary system). Moreover,*

$$\mathcal{Z}\left(\bigoplus_i^\infty X_i\right) = \bigoplus_i^\infty \mathcal{Z}(X_i).$$

This is the usual 'L^∞'-direct sum, with the obvious involution (and positive cones, respectively).

The following definition is a basic JBW*-triple construct (connected with the important notion of 'Peirce decompositions', etc.).

DEFINITION 4.6. Let u be a selfadjoint tripotent (that is, $u = u^* = uuu$) in $\mathcal{Z}(E)$, where E is an involutive ternary system. We define $J(u) = uEu$, also known as the Peirce 2-space. By the *canonical product* on $J(u)$ we mean the product $x \cdot y = xuy$ for $x, y \in J(u)$. It is well known, and easy to check, that with this product, and with the usual involution, $J(u)$ is a C*-algebra with identity u. The positive cone in this C*-algebra will be written as $\mathfrak{c}_u$.

It is also easy to see that $J(u)$ is a ternary $*$-ideal in E; and that if E is a $*$-WTRO then $J(u)$ is also weak* closed. The following result gives some alternative descriptions of $\mathfrak{c}_u$.

LEMMA 4.7. *If E is a $*$-TRO, and if u is a selfadjoint tripotent in $\mathcal{Z}(E)$, then*

$$\mathfrak{c}_u = \{eue^* : e \in E\} = \{x \in E : ux \in (E^2)_+,\, x = uxu\}.$$

Proof. Clearly $\mathfrak{c}_u = \{eue^* : e \in E\}$, which in turn is contained in the right-hand set. Conversely, if $ux \in (E^2)_+$ then there is a net whose terms are of the form $\sum_k e_k e_k^*$, converging to ux. Multiplying the net by u, we see that if $u^2x = x$ then x is in the closed convex hull of $\{eue^* : e \in E\}$. However, the latter set is the positive cone of $J(u)$ with its canonical C*-algebra structure, and hence is closed and convex. □

Before characterizing natural cones on $*$-TROs, we will have to tackle the $*$-WTRO case. We will use some JBW*-triple techniques, but present them in a way to make our proofs self-contained.

LEMMA 4.8. *Suppose that J is a weak* closed ternary $*$-ideal J of a $*$-WTRO E. Suppose further that J is ternary $*$-isomorphic to a W*-algebra $\mathfrak{M}$ via a mapping $\Phi : \mathfrak{M} \to J$. If 1 denotes the unit of $\mathfrak{M}$, then $u = \Phi(1)$ is a selfadjoint tripotent in $\mathcal{Z}(E)$, and $J = J(u)$.*

Conversely, if u is a selfadjoint tripotent in $\mathcal{Z}(E)$, then the ternary $*$-ideal $J(u)$ of E is ternary $*$-isomorphic to a W-algebra, via an isomorphism taking u to 1.*

Proof. Clearly, $u = \Phi(1)$ is a selfadjoint tripotent. Since Φ is a ternary $*$-morphism, we have $z = uuz = uzu = zuu$ for all $z \in J$. It follows that $J \subseteq uEu$. On the other hand, $uEu = u^3Eu \subseteq u^2E \subseteq J$, so that $J = uEu = uuE$. Similarly $J = Euu$. Thus the maps $e \mapsto uue$ and $e \mapsto ueu$ are norm-one projections from E onto J. But J is an M-summand by Lemma 3.4(1), and the projections onto such subspaces are unique (see for example [**18**, Proposition I.1.2]). Hence, $ueu = uue = euu$ for all $e \in E$. Thus

$$ue = u^3e = u^2eu = eu^3 = eu,$$

and hence $u \in \mathcal{Z}(E)$.

It is, on the other hand, well known (and in any case an easy exercise to check) that any weak* closed subspace of the form $J(u)$ is ternary $*$-isomorphic to the W*-algebra uE. The induced product on $J(u)$ is just its canonical product mentioned above the lemma. $\square$

We now link natural dual orderings with 'central tripotents'.

THEOREM 4.9. *Let E be an involutive ternary system. A given cone E_+ in E is a natural dual cone if and only if for some selfadjoint tripotent $u \in \mathcal{Z}(E)$,*

$$E_+ = \{eue^* : e \in E\}.$$

Also, given cones $M_n(E)_+ \subset M_n(E)$ for all $n \in \mathbb{N}$ are natural dual matrix cones if and only if for some selfadjoint tripotent $u \in \mathcal{Z}(E)$,

$$M_n(E)_+ = \left\{ \left[\sum_k e_{ik} u e_{jk}^* \right] : [e_{ij}] \in M_n(E) \right\}, \quad \textit{for } n \in \mathbb{N}\,.$$

Proof. ($\Leftarrow$) Fix a selfadjoint tripotent $u \in \mathcal{Z}(E)$ with $E_+ = \{eue^* : e \in E\}$. We may assume that E is a $*$-WTRO in a W*-algebra (but will not care about the induced order from this W*-algebra). Note that $eue^* = eu^5e^* = (ueu)u(ue^*u)$, since $u \in \mathcal{Z}(E)$. It then follows that $E_+ = \mathfrak{c}_u$, in the notation of Lemma 4.7, and regarding $J(u)$ as a W*-algebra as in the proof of Lemma 4.8. A similar formula clearly holds for $M_n(E)_+$, for all $n \in \mathbb{N}$. Fix a $*$-isomorphism Φ_1 of $J(u)$ onto a W*-subalgebra of some space $B(H_1)$. Then Φ_1 is a ternary $*$-isomorphism which is a complete order isomorphism onto its image. Now $J(u)$ is an M-summand of E as we observed in the proof of Lemma 4.8. Indeed $J(u) = \{uux : x \in E\}$. Let $J(u)^\perp$ be the 'complementary M-summand' $\{x - uux : x \in E\}$. This is a ternary $*$-ideal of E too, and in fact is the orthocomplement of $J(u)$ in the C*-module sense too [**36**]. The map $x \mapsto (u^2x, x - uux)$ is a completely isometric ternary $*$-isomorphism from E onto $J(u) \oplus^\infty J(u)^\perp$ (see Proposition 4.5). It is also easy to check that this map is a weak* homeomorphism, and is a complete order isomorphism when $J(u)^\perp$ is endowed with the trivial order. Let $\Phi_2 : J(u)^\perp \to B(H_2)$ be a weak* continuous ternary $*$-isomorphism onto a trivially ordered $*$-WTRO in $B(H_2)$ (see the proof of Lemma 3.1). Then $\Phi = \Phi_1 \oplus \Phi_2$ is a weak* continuous ternary $*$-isomorphism from E onto a $*$-WTRO, which also is a complete order isomorphism onto its image.

($\Rightarrow$) Suppose that E is a $*$-WTRO in $B(H)$. By Lemma 4.1, the cone of E is the cone of the W*-algebra $J(E)$. By Corollary 2.2(5) and Lemma 2.4(4), $J(E)$ is a weak* closed ternary $*$-ideal of E. So if u is the unit of $J(E)$ then, by Lemma 4.8, $u \in \mathcal{Z}(E)$, and $J(E) = J(u)$. If $e \in E$ then $eue^* = (euu)u(uue^*)$; and also $euu, uue^* \in J(E)$ since the latter is a ternary $*$-ideal. Hence

$$E_+ = J(E)_+ = \{eue^* : e \in J(E)\} = \{eue^* : e \in E\}.$$

Similar arguments apply to $M_n(E)_+$, which equals $M_n(J(E))_+$ by Lemma 4.1. $\square$

An important remark is that by Lemma 4.1 and Corollary 4.3(4), for example, a natural cone on an involutive ternary system Z completely determines the associated cone on $M_n(Z)$. Nonetheless, it is convenient to have the above explicit description of these cones.

Most of the ideas in the next two results will be familiar to those versed in the JB*-triple literature, for example [**4**, § 3; **10**; **13**; **22**]. By a selfadjoint *unitary* in a

$*$-TRO Z we mean a selfadjoint tripotent such that u^2 is an identity element for Z^2. Since we will need to include the trivial $*$-TRO $Z = \{0\}$ in the results below, we will say that 0 is a unitary in this case.

LEMMA 4.10. *Let E be a $*$-WTRO.*

(1) *The $*$-WTRO $\mathcal{Z}(E)$ possesses a selfadjoint unitary u such that $\mathcal{Z}(E) \subseteq J(u)$. Moreover, when $J(u)$ is equipped with its canonical product, $\mathcal{Z}(E)$ is a commutative unital W*-subalgebra of $J(u)$.*

(2) *An element $v \in \mathcal{Z}(E)$ is a selfadjoint tripotent if and only if it is such in $J(u)$, where u is as in* (1).

(3) *If v is a selfadjoint tripotent in $\mathcal{Z}(E)$, then there exists a selfadjoint unitary w in $\mathcal{Z}(E)$ such that $vwv = v$.*

Proof. (1) We observed earlier that $\mathcal{Z}(E)$ is a $*$-TRO in E, and indeed may be viewed as a $*$-WTRO in a W*-algebra M. The unit ball of $\mathcal{Z}(E)_{sa}$ is weak* closed in the weak* closed subspace M_{sa} of M. By the Krein–Milman theorem, the unit ball of $\mathcal{Z}(E)_{sa}$ contains an extreme point u. A standard Urysohn lemma argument shows that $A = \mathcal{Z}(E)^2$ has an identity $1_A = u^2$ (one shows that every character χ on A satisfies $\chi(u^2) = 1$). Thus if $x \in \mathcal{Z}(E)$ then $x = uux \in J(u)$, and so $\mathcal{Z}(E) \subseteq J(u)$ as desired. Since $\mathcal{Z}(E)$ is a $*$-TRO, it is clear that $\mathcal{Z}(E)$ is closed under the canonical product of $J(u)$. Now it is clear that $\mathcal{Z}(E)$ is a commutative W*-subalgebra of $J(u)$, and the unit u lies in this subalgebra.

(2) This is clear since u is unitary, and $vuvuv = u^2vvv = vvv$ if $v \in \mathcal{Z}(E)$.

(3) Let v be a selfadjoint tripotent in $\mathcal{Z}(E)$, and let u be as in (1). Then $w = v + u - vuv$ is the desired unitary. □

PROPOSITION 4.11. *Let E be a dual involutive ternary system.*

(1) *There is a bijective correspondence between natural dual orderings on E and selfadjoint tripotents of $\mathcal{Z}(E)$. This correspondence respects the ordering of cones by inclusion on one hand, and the ordering of selfadjoint tripotents given by $u_1 \leqslant u_2$ if and only if $u_1u_2u_1 = u_1$.*

(2) *If $v \in \mathcal{Z}(E)$ is a selfadjoint tripotent, then the following are equivalent:*

  (a) *v is an extreme point of* $\mathrm{Ball}(\mathcal{Z}(E)_{sa})$*;*

  (b) *v is a selfadjoint unitary of $\mathcal{Z}(E)$;*

  (c) *v is maximal with respect to the order defined in* (1)*;*

  (d) *$\mathcal{Z}(E) \subseteq J(v)$;*

  (e) *$J(v)$ contains every selfadjoint unitary of $\mathcal{Z}(E)$.*

(3) *The maximal natural dual orderings on E are in one-to-one correspondence with the selfadjoint unitaries of $\mathcal{Z}(E)$.*

(4) *Any natural dual ordering on E is majorized by a maximal natural dual ordering.*

Proof. (1) It is an easy exercise (or see the papers cited above) that $\leqslant$ is a partial ordering. Write $\mathfrak{c}_1$ and $\mathfrak{c}_2$ for the corresponding natural dual cones (see Theorem 4.9). If $\mathfrak{c}_1 \subset \mathfrak{c}_2$, then by Lemma 4.7 we have $u_1u_2 \geqslant 0$, and $u_1^2u_2^2 = u_1^2 = u_1^4$. Putting these together we have $u_1u_2 = u_1^2$, so that $u_1 \leqslant u_2$. Conversely, if $u_1 \leqslant u_2$ then $eu_1e^* = eu_1u_2u_1e^* \in \mathfrak{c}_2$. Hence $\mathfrak{c}_1 \subset \mathfrak{c}_2$. Now (1) follows readily from Theorem 4.9.

(2) These are well known, but also quite clear. For the equivalence of (a) and (b) see, for example, the hints in the proof of Lemma 4.10(1). To show that (b) implies (c) suppose that $v' \in \mathcal{Z}(E)$ is a selfadjoint tripotent with $vv'v = v$. By assumption, $vv'v = v'$. So $v = v'$. That (c) implies (b) follows from Lemma 4.10(3). Clearly, (b) $\Leftrightarrow$ (d) $\Rightarrow$ (e), and the converse is routine.

(3) This follows from (1) and (2).

(4) This follows from (1), (3) and Lemma 4.10(3). □

One may deduce structural facts about orderings on a $*$-WTRO E from the algebraic structure of the partially ordered set $\mathcal{S}(E)$ of selfadjoint tripotents in $Z(E)$ (see for example [**2**, **4**, **10**]). In fact $\mathcal{S}(E)$ is not a lattice: the 'sup' in general does not make sense without further hypotheses. However 'infs' are beautifully behaved. Indeed in our situation, it is rather clear that for any family of natural dual cones $(\mathcal{C}_i)$ on E, there is a natural dual cone which is the largest natural dual cone on E contained in every $\mathcal{C}_i$. In fact this cone is just $\bigcap_i \mathcal{C}_i$. To see that this indeed is a natural dual cone, we fix for each i a ternary order isomorphism T_i from $(E, \mathcal{C}_i)$ onto a $*$-WTRO E_i. Then define $T(x) = (T_i(x))_i$; this is a ternary order isomorphism from $\big(E, \bigcap_i \mathcal{C}_i\big)$ into $\bigoplus_i^\infty E_i$.

By Proposition 4.11, every family of selfadjoint tripotents in $\mathcal{S}(E)$, has a greatest lower bound in $\mathcal{S}(E)$. In fact there is a tidy explicit formula. If u and v are selfadjoint tripotents in $\mathcal{S}(E)$, then the greatest lower bound of $\{u, v\}$ in $\mathcal{S}(E)$ is

$$u \wedge v = \tfrac{1}{2}(uvu + vuv). \tag{4}$$

We leave the proof of this as a simple algebraic exercise (see also [**2**]). Clearly $u \in \mathcal{S}(E)$ if and only if $-u \in \mathcal{S}(E)$; however, the reader should be warned that the natural ordering $\leqslant$ on tripotents (see (1) of the previous result) behaves a little unexpectedly: $u \leqslant v$ if and only if $-u \leqslant -v$.

It is worth stating separately the fact that if E is a $*$-WTRO, and if u is the selfadjoint tripotent in $\mathcal{Z}(E)$ corresponding (as in the last proposition) to the given order on E, then $J(E)$, and the usual product on $J(E)$ (recall that we pointed out in § 2 that this is a C*-algebra), may be recaptured, in terms of the ternary structure and the tripotent u, as precisely the canonical product on $J(u)$. This observation is rather trivial (see the proof of Theorem 4.9), but is useful when E is not given as a $*$-WTRO, but instead as an abstract involutive ternary system.

Corollary 4.12. *The given cone on a $*$-WTRO E is maximal among all the natural dual cones on E if and only if E is ternary order isomorphic to $M \oplus^\infty F$, where M is a W*-algebra and F is a $*$-WTRO for which $\mathcal{Z}(F) = \{0\}$.*

Proof. ($\Rightarrow$) By Corollary 4.2, $E \cong J(E) \oplus^\infty J(E)^\perp$. Take u to be the associated selfadjoint unitary in $\mathcal{Z}(E)$ (see Proposition 4.11(4)). Note that

$$\mathcal{Z}(J(u)^\perp) \subset \mathcal{Z}(E) \cap J(u)^\perp = (0),$$

by part (2)(d) in Proposition 4.11.

($\Leftarrow$) In this case $\mathcal{Z}(E) = \mathcal{Z}(M) \subset J(E) = M$ (we are using the fact that $J(F) = 0$ if $\mathcal{Z}(F) = (0)$). Note that $(1, 0)$ is a selfadjoint central unitary with $J(u) = M$. Now appeal to Proposition 4.11. □

Remarks. (1) The last result shows that dual involutive ternary systems have a quite simple canonical form. Such canonical forms are quite common in the

JBW*-triple literature. However, as we discuss at the end of § 5, $*$-TROs F with $\mathcal{Z}(F)=\{0\}$ are not necessarily uncomplicated, unless F is 'Type I'. At the present time, a characterization of general $*$-TROs F with $\mathcal{Z}(F)=\{0\}$ seems quite unattainable.

(2) We shall see later in Proposition 5.5 that the set of maximal natural dual cones on E (characterized in the last two results above) coincides with the set of maximal ordered operator space cones on E.

We now pass to $*$-TROs. We recall, from the discussion above Proposition 3.3, that if Z is a $*$-TRO in a C*-algebra B, then $E = Z^{\perp\perp}$ is a $*$-WTRO in B''. One may deduce from Proposition 3.3(3) that there is a smallest weak* closed cone on $E = Z''$ which contains the cone given on Z, namely the weak* closure of this cone, and this is a natural cone for Z''. We call this the *canonical ordering* or *canonical second dual cone* on Z''. However, there may in general be many other (bigger) natural cones on Z'' which induce the same cone $\mathfrak{c}$ on Z (see the examples towards the end of § 6).

PROPOSITION 4.13. *Let Z be a $*$-TRO in a C*-algebra A, and let E be the $*$-WTRO Z'' in the W*-algebra A''. Suppose that $T : Z \to B$ is a $*$-linear map into a C*-algebra B.*

(1) *The map T is completely positive if and only if T'' is completely positive on E.*

(2) *If T is a ternary $*$-morphism, then so is T''. In this case, T is a complete order embedding if and only if T'' is a complete order embedding.*

Proof. (1) By Proposition 3.3, $J(E)_+ = (J(Z)'')_+$. If

$$\eta \in (Z'')_+ = J(E)_+ = (J(Z)'')_+,$$

then by a variant of Kaplansky's density theorem, there exists a net (x_t) in $J(Z)_+$ converging to η in the weak* topology. Then $T''(x_t) = T(x_t) \to T''(\eta)$. But $T(x_t) \geqslant 0$, and so $T''(\eta) \geqslant 0$. Thus T'' is a positive map. Since $T_n : M_n(Z) \to M_n(B)$ satisfies $(T_n)'' = (T'')_n$, we deduce that T'' is completely positive. The other direction follows by simply restricting T'' to Z.

(2) The first assertion follows by routine weak* density arguments. The second follows from (1) applied to T and T^{-1}. □

DEFINITION 4.14. Let Z be an involutive ternary system, and suppose that u is a selfadjoint tripotent in $\mathcal{Z}(Z'')$. If $\mathfrak{c}_u$ is the (natural) cone defined in Definition 4.6, we define $\mathfrak{d}_u$ to be the cone $\mathfrak{c}_u \cap Z$ in Z. We also write $J_u(Z)$ for the span of $\mathfrak{d}_u$ in Z, and $\mathfrak{c}'_u$ for the weak* closure of $\mathfrak{d}_u$ in Z''.

We say that a tripotent u is *open* if when we consider $J(u)$ as a W*-algebra in the canonical way (see Lemma 4.8), then u is the weak* limit in Z'' of an increasing net in $J(u)_+ \cap Z$. If Z is an involutive ternary system, then we say that an open tripotent is *central* if $u \in \mathcal{Z}(Z'')$.

We remark that this use of the word 'central' is certainly different from the usage in the JB*-triple literature (see for example [**4**]). Later we will give several alternative characterizations of open central tripotents, and list some of their basic properties. We note that a projection in the second dual of a C*-algebra is open in the usual sense if and only if it is an open tripotent in the sense above.

Lemma 4.15. *Let Z be an involutive ternary system, and let $E = Z''$, also an involutive ternary system in the canonical way. Let u be a selfadjoint tripotent in $\mathcal{Z}(E)$.*

(1) *We have $\mathfrak{c}'_u \subset \mathfrak{c}_u$.*

(2) *The cone $\mathfrak{d}_u$ is a natural cone on Z. That is, there is a surjective ternary order $*$-isomorphism ψ from Z with cone $\mathfrak{d}_u$ to W, for a $*$-TRO W. Also ψ'' is a ternary order $*$-isomorphism from Z'' with cone $\mathfrak{c}'_u$, onto W'' with its canonical second dual cone.*

(3) *We have $\mathfrak{c}'_u = \mathfrak{c}_v$ for a selfadjoint tripotent $v \in \mathcal{Z}(E)$ with $v \leqslant u$.*

(4) *The algebra $J_u(Z)$ is a C^*-subalgebra of $J(u)$, the latter regarded as a C^*-algebra in the canonical way (see Lemma 4.8). Also, $\mathfrak{d}_u$ is the positive cone of this C^*-algebra $J_u(Z)$.*

(5) *If Z is a $*$-TRO in a C^*-algebra B say (we do not care about the ordering induced on Z by B), then $u\mathfrak{d}_u \subset \mathfrak{d}_u^2 \subset J_u(Z)^2 \subset Z^2$. Thus $uJ_u(Z) \subset J_u(Z)^2$.*

Proof. (1) By Theorem 4.9 and Lemma 4.7, E may be regarded as a $*$-WTRO whose positive cone is $\mathfrak{c}_u$, and this is weak* closed in E. Then (1) is clear.

(2) Continuing with the argument in (1), regarding Z as a $*$-TRO inside E, we deduce that $\mathfrak{d}_u$ is just the inherited cone. Thus it is a natural cone. By Proposition 3.3(3) and Kaplansky's density theorem, $\mathfrak{c}'_u$ coincides with the canonical second dual cone induced from $\mathfrak{d}_u$.

(3) This follows from the last part of (2) and Theorem 4.9 and Proposition 4.11.

(4) This follows from the above, and Lemma 4.1.

(5) By (4), any element x of $\mathfrak{d}_u$ may be written as yuy for some $y \in \mathfrak{d}_u$. Thus $ux = u^2yy = yy \in \mathfrak{d}_u^2 \in J_u(Z)^2$. Then (5) is clear. ☐

Theorem 4.16. *Let Z be an involutive ternary system.*

(1) *The natural cones on Z are precisely the cones $\mathfrak{d}_u = \mathfrak{c}_u \cap Z$ (notation as in Lemma 4.7), for u a selfadjoint open tripotent in $\mathcal{Z}(Z'')$. A similar statement holds for the natural matrix cones on Z.*

(2) *There is a bijective correspondence between natural orderings on Z, and selfadjoint open tripotents in $\mathcal{Z}(Z'')$. This correspondence respects the ordering of the cones by inclusion on one hand, and the ordering of selfadjoint tripotents given by $u_1 \leqslant u_2$ if and only if $u_1u_2u_1 = u_1$.*

Proof. (1) If Z is a $*$-TRO, then so is $E = Z''$. Applying Theorem 4.9 to the canonical ordering on E yields a selfadjoint tripotent u with $J(u) = J(E)$, and $E_+ = \mathfrak{c}_u$. That u, the identity of the W*-algebra $J(E)$, is open follows from Proposition 3.3(3). Thus Z_+ has to be of the announced form. A similar argument applies to $M_n(Z)_+$.

The converse follows from Lemma 4.15(2).

(2) Suppose that u and v are two selfadjoint open tripotents in $\mathcal{Z}(E'')$. Set $A = \{eue^* : e \in E''\} \cap E$, and let B be the matching set for v. Suppose first that $A \subset B$. By definition, u is a weak* limit of a net in A, and hence in B. Since $\mathfrak{c}_v$ is weak* closed (being the cone of a W*-algebra), we see that u is in $\mathfrak{c}_v$. It follows as in the proof of Proposition 4.11(1), that $u \leqslant v$. Conversely, if $u \leqslant v$, then the second part of Proposition 4.11(1) shows that $A \subset B$. ☐

We now give some equivalent, and often more useful, characterizations of selfadjoint open central tripotents. But first we will need one or two more definitions and facts. Suppose that J is a C*-ideal in an involutive ternary system Z, with $\psi : B \to J$ the surjective triple $*$-isomorphism, where B is a C*-algebra. Suppose that Z is a $*$-TRO in a C*-algebra A, and that $E = Z''$ is regarded as a $*$-WTRO in the W*-algebra $M = A''$. Then $\psi'' : B'' \to J^{\perp\perp}$ is a surjective triple $*$-isomorphism onto a weak* closed triple ideal in $E = Z''$. Let $\psi''(1) = u$; by Lemma 4.8 this is a selfadjoint tripotent in $\mathcal{Z}(E)$ and $J^{\perp\perp} = J(u)$. We call u a *support tripotent* for J. If (e_t) is an increasing approximate identity for B, then it is well known that $e_t \to 1_{B''}$ weak*. Thus $\psi(e_t) \to u$ weak*. We deduce that u is an open tripotent in the sense above. Also J, with the product induced from that of B via ψ, is a C*-subalgebra of $J(u)$, the latter with its canonical product (see Corollary 4.2). Clearly $J = J^{\perp\perp} \cap Z = J(u) \cap Z$. We will see in Theorem 4.17 that u^2 is open in the usual C*-algebraic sense (from which it is easy to see that u^2 is the projection in Lemma 3.4(2)). Finally, we remark that the induced natural cone $\mathfrak{d}_u$ on Z (see Definition 4.14), equals $\psi(B_+)$, the canonical positive cone for J. To see this note that

$$\psi(B_+) = \psi''(B''_+ \cap B) = \psi''(B''_+) \cap J = \mathfrak{c}_u \cap (J(u) \cap Z) = \mathfrak{d}_u.$$

The following theorem is reminiscent of [**11**, Lemma 3.5], but in fact only seems to be formally related to that result. Indeed examples of the type in § 6 show that there can exist central selfadjoint tripotents in Z'' which are open in the sense of [**11**, **12**], but which are not related to our notion of open tripotent (all selfadjoint unitaries in the second dual are open in their sense, for example).

THEOREM 4.17. *Let Z be a $*$-TRO, and let $E = Z''$. Let u be a selfadjoint tripotent in $\mathcal{Z}(E)$. The following are equivalent:*

(i) *u is an open tripotent;*

(ii) *$u \in \mathfrak{c}'_u$ (notation as in Definition 4.14);*

(iii) *there is a net (x_t) in Z converging weak* to u, satisfying $ux_t \geqslant 0$, $u^2 x_t = x_t$ for all t, and (ux_t) is an increasing net;*

(iv) *if $J_u(Z)$ is the span of the cone $\mathfrak{d}_u$ (see Lemma 4.15) in Z, then*

$$\overline{J_u(Z)}^{w*} = J(u);$$

(v) *u is a support tripotent for a C*-ideal in Z;*

(vi) *u^2 is an open projection (in the usual sense) in $(Z^2)''$, and $u(J(u) \cap Z) \subset Z^2$;*

(vii) *$\mathfrak{c}'_u = \mathfrak{c}_u$ (notation as in Definition 4.14); that is, $\mathfrak{c}_u$ is the 'canonical second dual cone' induced by $\mathfrak{d}_u$.*

Proof. Clearly (i) is equivalent to (iii), and (iii) implies (ii), and (vii) implies (ii). We now show that (ii) implies (vii), (i) and (iv). Using Lemma 4.15, and the first part of the proof of Theorem 4.16, we have $\mathfrak{c}'_u = \mathfrak{c}_v$ for a selfadjoint open central tripotent v. Thus if (ii) holds then $u \in \mathfrak{c}_v$, so that $\mathfrak{c}_u \subset \mathfrak{c}_v = \mathfrak{c}'_u$. But by Lemma 4.15 again, $\mathfrak{c}'_u \subset \mathfrak{c}_u$. So $\mathfrak{c}_u = \mathfrak{c}'_u = \mathfrak{c}_v$. Hence $u = v$ is open. By Proposition 3.3(3) and Lemma 4.1, $\overline{J_u(Z)}^{w*} = J(u)$. Thus we have verified (vii), (i) and (iv).

(iv) $\Rightarrow$ (i) By Lemma 4.15(4), we find that $J_u(Z)$ is a C*-subalgebra of $J(u)$. Then (i) follows by Kaplansky's density theorem.

(iv) $\Rightarrow$ (vi) Given (iv), note that $J(u) \cap Z = J_u(Z)$. Hence by Lemma 4.15(5), $u(J(u) \cap Z) = uJ_u(Z) \subset J_u(Z)^2 = (J(u) \cap Z)^2 \subset Z^2$. Since (iv) implies (i), u^2 is a weak* limit of an increasing net in $u(J(u) \cap Z) \subset Z^2$. Thus u^2 is open.

(v) $\Rightarrow$ (i) We saw this in the discussion above Theorem 4.17.

(vi) $\Rightarrow$ (v) Let $J = J(u) \cap Z$; this is a ternary $*$-ideal in Z, and if (vi) holds then it is a C*-ideal in Z. Indeed J is a C*-subalgebra of the W*-algebra $J(u)$, the latter with its canonical product. By the discussion above Theorem 4.17, if (e_t) is an increasing approximate identity for this C*-subalgebra, then $e_t \to v$ weak* in $J(u)$, where v is a support tripotent for J. Clearly $vux = x$ for all $x \in J$, and hence by weak* density we have $vuv = v$. Also by the discussion above Theorem 4.17, $J = J(v) \cap Z$. Now v and v^2 are open (since (v) implies (i) and (vi)), and Lemma 3.4(2) gives $u^2 = v^2$. Thus $vuv = u = v$. □

Parts of the last result are no doubt true for general open tripotents. We now turn to another useful way of looking at open tripotents. (See also Closing Remark (2) at the end of this paper.)

Proposition 4.18. *Let Z be a $*$-TRO, and let $A = Z + Z^2$.*

(1) *Suppose that u is a selfadjoint open tripotent in $\mathcal{Z}(Z'')$. Then $p = \frac{1}{2}(u^2 + u)$ and $q = \frac{1}{2}(u^2 - u)$ are open projections in the center of A''. Moreover, $u = p - q$ and $pq = 0$.*

(2) *Suppose that $Z \cap Z^2 = (0)$. (This may always be ensured, by replacing Z by a ternary $*$-isomorphic $*$-TRO, but note that this replacement will usually change A, and the positive cone of Z.) Let $\theta : A \to A$ be the period 2 $*$-automorphism $\theta(z + a) = a - z$ for $a \in Z^2$ and $z \in Z$. Suppose that p is an open projection in the center of A'', such that $pq = 0$, where $q = \theta''(p)$. Then $u = p - q$ is a selfadjoint open tripotent in $\mathcal{Z}(Z'')$, and q is also an open projection in the center of A''.*

Proof. (1) It is clear that p and q are projections in the center of A'', and that $u = p - q$ and $pq = 0$. To see that p and q are open, let x_t be the net in Theorem 4.17(iii). Then $ux_t \in Z^2$ by Lemma 4.15(5). Thus $\frac{1}{2}(ux_t + x_t)$ is a net in A converging weak* to p, and also $p\big(\frac{1}{2}(ux_t + x_t)\big) = \frac{1}{2}(ux_t + x_t)$. From basic facts about Akemann's open projections it follows that p is one of these. The result for q is proved similarly.

(2) Clearly u is a tripotent in $\mathcal{Z}(A'')$. Also θ'' is a period 2 $*$-automorphism of A''. Suppose that (a_t) is an increasing net in A_+ converging weak* to p, with $pa_t = a_t$. Then $b_t = \theta(a_t)$ is a net in A converging weak* to $\theta''(p)$, and $\theta''(p)b_t = b_t$. Also $\theta(a_t - b_t) = b_t - a_t$. Consequently, $y_t = a_t - b_t$ is a net in Z converging weak* to u. So $u \in \mathcal{Z}(Z'')$. Clearly $u^2 y_t = (p + \theta''(p))(a_t - b_t) = a_t - b_t = y_t$. Moreover, uy_s is the weak* limit of

$$(a_t - \theta(a_t))(a_s - \theta(a_s)) = a_t a_s + \theta(a_t a_s),$$

since $\theta(a_t)a_s = \theta(a_t)\theta''(p)pa_s = 0$. Thus $uy_s = a_s + \theta(a_s) \geqslant 0$. Thus Theorem 4.17(ii) or (iii) implies that $u = p - q$ is a selfadjoint open tripotent. The last assertion follows since θ'' is a weak* continuous $*$-automorphism. □

The intersection of natural cones on a $*$-TRO is again a natural cone, as may be seen by the argument sketched above Equation (4). Hence every family of selfadjoint open tripotents has a greatest lower bound (or 'inf') amongst the selfadjoint open tripotents. The following fact is a little deeper, and should be important in future developments. It is the 'tripotent version' of Akemann's result that the inf of two (in this case central) open projections is open.

COROLLARY 4.19. *Let Z be a $*$-TRO, and let u and v be two selfadjoint open tripotents in $\mathcal{Z}(Z'')$. Then the greatest tripotent $u \wedge v$ in $\mathcal{Z}(Z'')$ majorized by u and v is open, and is given by Equation* (4). *Also, $\mathfrak{d}_u \cap \mathfrak{d}_v = \mathfrak{d}_{u\wedge v}$, in the notation of Lemma* 4.7.

Proof. The last assertion is clear:

$$\mathfrak{d}_u \cap \mathfrak{d}_v = \mathfrak{c}_u \cap \mathfrak{c}_v \cap Z = \mathfrak{c}_{u\wedge v} \cap Z = \mathfrak{d}_{u\wedge v}.$$

To see that $u \wedge v$ (as given by Equation (4)) is open, note first that by easy algebra,

$$\tfrac{1}{2}\big((u\wedge v)^2 + u\wedge v\big) = \tfrac{1}{2}(u^2+u)\cdot\tfrac{1}{2}(v^2+v).$$

The latter is a product of two commuting open projections (by Proposition 4.18(1)), and hence is open. Similarly, $\frac{1}{2}((u\wedge v)^2 - u\wedge v)$ is open. Now the result is easy to see using Proposition 4.18(2). $\square$

REMARK. As in the JBW*-triple case (see for example [**10**]), a family of selfadjoint open central tripotents which is bounded above by a selfadjoint open central tripotent, has a 'sup'. Also, the sum of two 'orthogonal' selfadjoint open tripotents is clearly an open tripotent. Otherwise the 'sup' of tripotents does not make sense in general.

The following is an 'ordered variant' of a very useful result due to Youngson [**40**]. We will not use this result in our paper, but it seemed worth including in view of the importance of Youngson's original result.

THEOREM 4.20. *Let P be a completely positive completely contractive idempotent map on a $*$-TRO Z. Then $P(Z)$ is an involutive ternary system, and $P(Z)\cap Z_+ = P(Z_+)$ is a natural cone for this system. A similar assertion holds for the matrix cones. Thus $P(Z)$ with these matrix cones is completely isometrically completely order isomorphic to a $*$-TRO.*

Proof. Since P is positive, $P(Z_+) \subset P(Z)\cap Z_+$. Hence

$$P(Z)\cap Z_+ = P(P(Z)\cap Z_+) \subset P(Z_+).$$

So $P(Z_+) = P(Z)\cap Z_+$.

By Youngson's theorem [**40**], $P(Z)$ is a ternary system with new ternary product $[Px, Py, Pz] = P(P(x)P(y)P(z))$. Since $[Px, Py, Pz]^* = [Pz^*, Py^*, Px^*]$, it follows from Lemma 3.1 that $P(Z)$ is an involutive ternary system.

Since $P(Z_+) \subset Z_+$, we have, by Lemma 4.1, $P(J(Z)) \subset J(Z)$. Hence P restricts to a completely positive completely contractive idempotent map on the C*-algebra $J(Z)$. By a slight variation of a well-known result of Choi and Effros (use [**9**, Theorem 3.1] in conjunction with [**8**, Lemma 3.9]), $P(J(Z))$ is a C*-algebra with respect to the new product $P(ab)$, for $a, b \in P(J(Z))$, and the map $x \mapsto P(x)$ from $J(Z)$ into the C*-algebra $P(J(Z))$ is completely positive. On the other hand, since $J(Z)$ is a ternary $*$-ideal in Z, if $P(z)$ or $P(x)$ is in $P(J(Z)) \subset J(Z)$, then $P(P(x)P(y)P(z)) \subset P(J(Z))$. We deduce that $P(J(Z))$ is a ternary $*$-ideal in $P(Z)$. Another part of the Choi and Effros result states that $P(P(x)P(y)) = P(xP(y)) = P(P(x)y)$ for $x, y \in J(Z)$. This translates to the assertion that the identity map is a ternary $*$-morphism from $P(J(Z))$ to $P(J(Z))$

with its C*-algebra product. Hence $P(J(Z))$ is a C*-ideal in $P(Z)$. Thus by facts given in the last section, the positive cone $\mathfrak{c}$ in $P(J(Z))$ (coming from the fact above that $P(J(Z))$ is a C*-algebra in a new product) is a natural cone for $P(Z)$. Note that a representative element of $\mathfrak{c}$ is $P(a^*a)$, for $a \in P(J(Z))$, which is certainly contained in $P(Z_+)$. Conversely, $P(Z_+) = P(J(Z)_+) \subset \mathfrak{c}$, by the observation above about the map $x \mapsto P(x)$ being completely positive. Thus $P(Z_+) = \mathfrak{c}$. We leave the 'matrix cones' version as an exercise. $\square$

An *ordered C*-module* is an involutive C*-bimodule Y over a C*-algebra A (in the sense of Definition 2.1), with a given positive cone, such that Y (with its canonical ternary product $x\langle y, z\rangle$) is 'ternary order isomorphic' to a $*$-TRO. Note that every involutive C*-bimodule Y is canonically an involutive ternary system (see Corollary 3.2), and hence the notions of selfadjoint open tripotents, etc. make sense. Thus by Theorem 4.16, the ordered C*-modules are exactly the involutive C*-bimodules Y, with a positive cone of the form $\{e\langle u|e^*\rangle : e \in Y''\} \cap Y$, for a (selfadjoint, as always) open tripotent u in $\mathcal{Z}(Y'')$.

5. *Maximally ordered and unorderable $*$-TROs*

Definition 5.1. We say that a $*$-TRO Z is *maximally ordered* if its given cone is maximal amongst the ordered operator space cones on Z (see Definition 4.4). This is equivalent to saying that every completely positive complete isometry from Z into a C*-algebra is a complete order injection.

We say that a $*$-TRO Z is *unorderable* if the only ordered operator space cone on Z is the trivial one.

Lemma 5.2. *If J is a ternary $*$-ideal in a $*$-TRO Z, then the involutive ternary system Z/J (see the end of § 1) possesses a natural ordering for which the canonical quotient ternary $*$-morphism $Z \to Z/J$ is completely positive.*

Proof. We consider Z'' with its canonical second dual ordering. Now $J^{\perp\perp}$ is a weak* closed ternary $*$-ideal in Z'', and hence equals qZ'' for a central projection q (see Lemma 3.4). If $p = 1 - q$ then $(Z/J)'' \cong Z''/J^{\perp\perp} \cong pZ''$. We may thus identify Z/J as a $*$-TRO inside the $*$-WTRO pZ''. This endows Z/J with natural matrix cones. Let $q_J : Z \to Z/J$ be the quotient ternary $*$-morphism. It is easy to see that if $z \in Z_+$ then $pz \geqslant 0$ in Z'', so that $q_J(z)$ is in the cone just defined in Z/J. A similar argument applies to matrices, so that q_J is completely positive. $\square$

Henceforth, whenever we refer to 'the natural ordering on Z/J', it will be the one considered in the last lemma.

Lemma 5.3. *Let Z be an ordered ternary system. Then the given ordering on Z is majorized by a natural ordering for Z. In particular, it follows that a maximal ordered operator space cone on Z is necessarily a natural cone on Z.*

Proof. If $T : Z \to B$ is a completely positive complete isometry into a C*-algebra B, let W be the $*$-TRO generated by $T(Z)$ in B. By Hamana's theorem (see [**17**] or [**6**, Theorem 8.3.9]), there exists a canonical surjective ternary morphism $\theta : W \to Z$ with $\theta \circ T = I_Z$. It is easy to see that θ is a ternary

$*$-morphism. If $N = \operatorname{Ker}\theta$, consider the quotient W/N, with the natural ordering discussed in Lemma 5.2. If q_N is the completely positive quotient ternary $*$-morphism $W \to W/N$, then we obtain a surjective ternary $*$-isomorphism $\rho : Z \to W/N$ with $\rho = q_N \circ T$. Note that ρ is completely positive. Also ρ induces a natural ordering on Z, namely the one pulled back from W/N. The result is now clear. $\square$

THEOREM 5.4. *Suppose Z is a ternary system which is also an ordered operator space. Then Z has a maximal ordered operator space cone containing the given one, and this cone is natural.*

Proof. By the previous lemma we may assume that the given order on Z is a natural order. By Theorem 4.16, this order corresponds to a selfadjoint open central tripotent u_0. We consider the set $\mathcal{S}$ of all selfadjoint open central tripotents in $E = Z''$ majorizing u_0, with the usual ordering of tripotents (see Proposition 4.11(1)). By Theorem 4.16 and Lemma 5.3, we will have the required result if we can show that $\mathcal{S}$ has a maximal element. We use Zorn's lemma. Suppose that (w_t) is an increasing net in $\mathcal{S}$, and let u be a weak* limit point of the net. We may assume that $w_t \to u$ weak* (by replacing the net by a subnet). If $r \geqslant s \geqslant t$ then $w_r w_s w_t = w_r w_s^3 w_t = w_s^2 w_t = w_t$. Taking the limit over r gives $u w_s w_t = w_t$. Then taking the limit over s gives $u u w_t = w_t$. Hence u is a selfadjoint tripotent. It is also easy to see that u majorizes the net, and hence also u_0. Finally, to see that u is open, we use Theorem 4.17(ii). Since each w_t is in $\mathfrak{c}'_{w_t} \subset \mathfrak{c}'_u$, we have $u \in \mathfrak{c}'_u$. $\square$

PROPOSITION 5.5. *Let E be a $*$-WTRO.*

(1) *If J is a C*-ideal in E, then $\bar{J}^{w*}$ is a weak* closed ternary $*$-ideal in E which is ternary $*$-isomorphic to a W*-algebra.*

(2) *Every ordered operator space cone on E is contained in a natural dual cone.*

(3) *The maximal ordered operator space cones on E coincide with the maximal natural dual cones on E.*

Proof. (1) It is easy to see that $\bar{J}^{w*}$ is a weak* closed ternary $*$-ideal in E. Suppose that A is a C*-algebra, and $\varphi : A \to J$ is a ternary $*$-isomorphism. Let (e_t) be an increasing approximate identity for A. We may assume that $\varphi(e_t) \to u \in \bar{J}^{w*}$ in the weak* topology. Clearly u is selfadjoint, and

$$u\varphi(a)\varphi(b) = \lim \varphi(e_t ab) = \varphi(ab) \quad \text{for all } a, b \in A.$$

Taking $a = e_t$, and taking the limit, yields $u^2 y = y$ for all $y \in J$. Hence $u^2 y = y$ for all $y \in \bar{J}^{w*}$. Thus u is a tripotent in $\bar{J}^{w*}$. Clearly the above argument also gives $\varphi(a)\varphi(b)u = \varphi(ab)$ for all $a, b \in A$, so that u is in the center of $\bar{J}^{w*}$. It follows that $u\bar{J}^{w*}$ is a W*-subalgebra of the W*-algebra $\overline{E^2}^{w*}$. Thus by Lemma 4.8 we deduce that $u \in \mathcal{Z}(E)$, and that $\bar{J}^{w*} = J(u)$.

(2) Any ordered operator space cone on E is contained in a natural cone, by Lemma 5.3. Suppose that $\mathfrak{C}$ is a natural cone on E; thus $(E, \mathfrak{C})$ is ternary order isomorphic to a $*$-TRO Z. Then E has a ternary $*$-ideal J which is ternary $*$-isomorphic to $J(Z)$. Thus by (1), E contains a weak* closed ternary $*$-ideal J' containing J, which is ternary $*$-isomorphic to a W*-algebra. By Lemma 4.8, $J' = J(w)$ for a selfadjoint tripotent w in $\mathcal{Z}(E)$. Also $\mathfrak{C} \subset \mathfrak{c}_w$, since J is a C*-subalgebra

of $J(w)$. The result is completed by an appeal to Proposition 4.11(4). The result for the matrix cones is proved similarly.

(3) This follows from (2). □

The following result follows immediately from the above and Corollary 4.12.

COROLLARY 5.6. *A $*$-WTRO E is maximally ordered if and only if E is completely order isomorphic via a ternary $*$-isomorphism to $M \oplus^\infty F$, where M is a W*-algebra and F is a $*$-WTRO which is unorderable. Also, E is unorderable if and only if $\mathcal{Z}(E) = \{0\}$.*

COROLLARY 5.7. *Let Z be an involutive ternary system. Then Z is unorderable if and only if Z contains no non-zero C*-ideals, and if and only if there are no non-zero selfadjoint open tripotents in $\mathcal{Z}(Z'')$.*

COROLLARY 5.8. *If Z is a $*$-TRO, and if $\mathcal{Z}(Z'') = \{0\}$, then Z is unorderable; and also $\mathcal{Z}(Z) = \{0\}$.*

We do not have a tidy characterization of general maximally ordered $*$-TROs. In fact the situation here seems quite complicated (for example, see the examples in § 6). We give a couple of partial results.

COROLLARY 5.9. *Let Z be a $*$-TRO such that Z'' with its canonical ordering is maximally ordered (or equivalently such that the selfadjoint open central tripotent corresponding to the given ordering is unitary). Then:*

(1) Z is maximally ordered;

(2) the involutive ternary system $Z/J(Z)$ is unorderable.

In particular, a $*$-TRO Z is maximally ordered if Z'' is a C*-algebra.

Proof. (1) If $T : Z \to B$ is a completely positive complete isometry into a C*-algebra then T'' is completely positive by Proposition 4.13. By hypothesis, T'' is a complete order embedding, and therefore so is T, by restriction.

(2) If $Z/J(Z)$ had a non-trivial natural ordering, then so would

$$(Z/J(Z))'' \cong Z''/J(Z)^{\perp\perp}.$$

However, since $J(Z)^{\perp\perp} = J(u)$ for a selfadjoint unitary u in $\mathcal{Z}(Z'')$, it follows that $Z''/J(Z)^{\perp\perp} \cong \{e - ueu : e \in Z''\}$. Hence the latter involutive ternary system has a non-trivial natural dual ordering, and hence has a non-zero selfadjoint tripotent $w \in \mathcal{Z}(Z'')$ (by Theorem 4.9). Since $w+u \geqslant u$, this contradicts Proposition 4.11(2).

Finally, note that C*-algebras are maximally ordered, as one can see, for example, by combining (1) with Lemma 5.3 (or see for example [**39**]). □

REMARKS. (1) The converse of Corollary 5.9(1) is false, as we show in an example towards the end of § 6. That is, Z may be maximally ordered without Z'' (with its canonical cone) being maximally ordered.

(2) It is not true in general that for a maximally ordered $*$-TRO Z, $Z/J(Z)$ is necessarily unorderable. Counterexamples may be found amongst the examples considered towards the end of § 6, together with the fact (proved above Proposition 6.3) that the quotient of a commutative involutive C*-bimodule (in the sense of

Definition 2.1) by a ternary $*$-ideal is again a commutative involutive C*-bimodule. However we do have the following result.

PROPOSITION 5.10. *If Z is a $*$-TRO such that $Z/J(Z)$ is unorderable, then Z is maximally ordered.*

Proof. Let v be the open tripotent corresponding to the given ordering on Z. Let $E = Z''$ as usual. Then $J(E) = J(v) = J(Z)^{\perp\perp}$ (see Proposition 3.3). If Z is not maximally ordered, then there exists an open tripotent $u \geqslant v$ with $u \neq v$. We identify the $*$-TRO $\{x - vxv : x \in E\}$ with $(Z/J(Z))''$ via the canonical isomorphisms

$$\{x - vxv : x \in E\} \cong E/J(v) \cong (Z/J(Z))''.$$

Then $u - v$ is a non-zero central tripotent in this $*$-TRO; if we can show that it is also open, we will have the desired contradiction, by Corollary 5.7. If (x_t) is as in Theorem 4.17(iii), then it is easy to check using Theorem 4.17(iv) that $x_t + J(Z) = x_t + J_v(Z)$, which in $(Z/J(Z))''$ may be identified with $x_t - vx_tv$, converges in the weak* topology of $(Z/J(Z))''$ to $u - v$. We conclude with an appeal to Theorem 4.17(ii). Note that $(u - v)(x_t - vx_tv) = (u - v)x_t \geqslant 0$, since $ux_t \geqslant 0$, and $(u - v)x_t = (u - v)^2ux_t$ and $(u - v)^2 \leqslant u^2$. □

We now turn to a variant of the topological boundary of an open set. Let A be a C*-algebra, and let p and q be respectively an open and a closed central projection in A''. We say that *q is contained in the boundary of p* if $pq = 0$ and if $rp \neq 0$ whenever r is an open central projection in A'' such that $rq \neq 0$.

PROPOSITION 5.11. *Let Z be a $*$-TRO, let u be a selfadjoint open tripotent in $\mathcal{Z}(Z'')$, and define p and q as in Proposition 4.18(1). Write 1 for the identity of $\mathcal{Z}(Z'')^2$ (which exists by, for example, the proof of Lemma 4.10(1)). Suppose that $1-u^2$ is contained in the boundary of both p and q. Then u is a maximal selfadjoint open tripotent in $\mathcal{Z}(Z'')$. Hence the corresponding cone $\mathfrak{d}_u$ is a maximal cone for Z.*

Proof. Suppose that $v \geqslant u$, but that $v \neq u$. Then $u^2 \neq v^2$. Hence either $\frac{1}{2}(v^2 + v) \neq \frac{1}{2}(u^2 + u)$ or $\frac{1}{2}(v^2 - v) \neq \frac{1}{2}(u^2 - u)$. Assume the former (the other case is similar). Thus $\frac{1}{2}(v^2 + v)(1 - u^2) \neq 0$. By hypothesis, $\frac{1}{2}(v^2 + v) \cdot \frac{1}{2}(u^2 - u) \neq 0$. However $(v^2 + v)(u^2 - u) = u^2 + u - u - u^2 = 0$, a contradiction. □

REMARK. We imagine that a modification of the ideas in the last corollary yields a *characterization* of maximal selfadjoint open tripotents, and therefore also of maximal cones. Indeed in the commutative case the 'boundary' hypothesis in Proposition 5.11 is necessary and sufficient (see the later Corollary 6.8). In the general case this converse requires further investigation.

EXAMPLE. If M is a finite-dimensional W*-algebra, it is easy to see that the cones on M may be characterized as follows. If the center of M is n-dimensional, the span of n minimal central projections $\{p_1, \ldots, p_n\}$, then there are 3^n possible natural cones on M, namely the product of the natural cone of M, with each of the 3^n selfadjoint tripotents $(\alpha_1 p_1) \oplus (\alpha_2 p_2) \oplus \ldots \oplus (\alpha_n p_n)$, where $\alpha_k \in \{-1, 0, 1\}$.

There are 2^n maximal ordered operator space cones on X, namely the cones above, but with $\alpha_k \in \{-1, 1\}$.

EXAMPLE. If E is a finite-dimensional involutive ternary system, then as above we may write $E \cong M \oplus^\infty F$, where M is a finite-dimensional W*-algebra, and F is an unorderable finite-dimensional involutive ternary system. The possible natural cones on E are exactly those in the last example.

In fact it is not difficult to characterize the unorderable finite-dimensional involutive ternary systems. They are precisely the $*$-TROs $eN(1-e) + (1-e)Ne$, for a finite-dimensional W*-algebra N and a projection $e \in N$. There is a similar characterization valid for all 'Type I' unorderable involutive dual ternary systems. However, in general, it is not true that general (non Type I) unorderable involutive dual ternary systems are of the form $eN(1-e) + (1-e)Ne$, for a W*-algebra N and a projection $e \in N$.

We will discuss unitizations of $*$-TROs in a projected sequel paper.

6. *Examples and remarks*

We now consider a special class of involutive ternary systems, namely the $*$-TROs Z in $C(K)$ spaces. For brevity, we will omit most proofs in this section, since our main purpose here is to look at certain examples showing that some results in earlier sections are best possible. Also, some of this material is well known in this case, and the rest is not particularly difficult (full proofs were given in an earlier distributed version of the present paper). We begin by remarking that in this case the 'center' $\mathcal{Z}(Z)$ (defined in § 2) is simply Z, and the 'center' of Z'' is simply Z''. It follows from the earlier theory that Z'' is ternary $*$-isomorphic to an L^∞ space, a fact which plays a role in the results below. In fact, the $*$-TROs Z in $C(K)$ spaces may be given several abstract characterizations. For example, they are the commutative involutive C*-bimodules of Definition 2.1. They may also be viewed as being the space of sections vanishing at infinity for certain line bundles. Then there is the following characterization related to the notion of 'C_σ spaces' (see for example [**16**, **25**]). Namely, suppose that Ω is a locally compact Hausdorff space, and that $\tau : \Omega \to \Omega$ is a homeomorphism with $\tau \circ \tau = I_\Omega$. This corresponds to a period 2 automorphism of $C_0(\Omega)$. Let W be the corresponding $*$-TRO, namely

$$W = \{f \in C_0(\Omega) : f \circ \tau = -f\}. \tag{5}$$

Then W is a $*$-TRO in $C_0(\Omega)$. In fact, every $*$-TRO in a commutative C*-algebra arises in this way. Some form of this is essentially well known (see for example [**25**, **31**] and references therein).

THEOREM 6.1. *If Z is a $*$-TRO in a commutative C*-algebra, then Z is ternary $*$-isomorphic to a ternary system W of the form in* (5), *where Ω is a locally compact subspace of* $\mathrm{Ball}(Z')$ *with the weak* topology, and τ is simply 'change of sign': namely $\tau(\varphi) = -\varphi$. Indeed Ω may be taken to be* $\mathrm{ext}(\mathrm{Ball}(Z'_{sa}))$.

Proof (sketch). The set Z'_{sa}, of selfadjoint functionals in Z', is closed in Z' in the weak* topology. Set $\Omega = \mathrm{ext}(\mathrm{Ball}(Z'_{sa}))$, which is a closed subset of $\mathrm{ext}(\mathrm{Ball}(Z'))$, and therefore is locally compact in the weak* topology (see [**16**]). Let

$$W = \{f \in C_0(\Omega) : f(-\psi) = -f(\psi) \text{ for all } \psi \in \Omega\}. \tag{6}$$

Then W is a space of the form (5). Define $\Phi : Z \to W$ by $\Phi(z)(\psi) = \psi(z)$, for $z \in Z$ and $\psi \in \Omega$. This is a ternary $*$-morphism by a result in [**16**], and is easily checked to be one-to-one. Hence Φ is an isometry. Let

$$B = \{f \in C_0(\Omega) : f(-\psi) = f(\psi) \text{ for all } \psi \in \Omega\},$$

a commutative C*-algebra. One may define an equivalence relation $\equiv$ on Ω, by identifying $-\psi$ and ψ. Then $S = \Omega/\equiv$ is a locally compact Hausdorff space, and $B \cong C_0(S)$ as C*-algebras. It is easy to see that W^2 strongly separates points of S (that is, given two distinct points of S, there is an element of W^2 which is 0 at the first and non-zero at the second). Thus by the Stone–Weierstrass theorem $W^2 = B \cong C_0(S)$. Also $\Phi(Z)^2$ strongly separates points of S, so that by, for example, Theorem 4.20 in [**5**] it follows that Φ is surjective. □

Henceforth in this section, we reserve the letters W and Ω for the spaces introduced in the proof of the last result. By that result, W 'is' the generic commutative involutive ternary system. A subset C of Ω will be said to be *symmetric* if $C = -C$, and *antisymmetric* if $C \cap (-C) = \emptyset$. Given a closed symmetric subset C of Ω, we will write

$$F_C = \{f \in W : f(x) = 0 \text{ for all } x \in C\}. \tag{7}$$

If S is as in the last proof, then we saw that $W^2 \cong C_0(S)$ $*$-isomorphically. Also, the open projections in $C_0(S)''$ correspond to the open sets in S, and hence to the symmetric open sets in Ω. This together with Lemma 3.4 yields the following result.

Proposition 6.2. *A subspace N of W is a ternary $*$-ideal of W if and only if there is a closed symmetric subset C of Ω such that $N = F_C$. Moreover, such a set C is uniquely determined by N.*

Remark. If C is a closed symmetric subset of Ω, define a ternary $*$-morphism $W \to \{f \in C_b(C) : f(-\psi) = -f(\psi) \text{ for all } \psi \in C\}$ by $f \mapsto f_{|C}$. The kernel of this ternary morphism is F_C, so that W/F_C is ternary $*$-isomorphic to a $*$-TRO in $C_b(C)$. This shows that the class we are investigating in this section, namely the $*$-TROs in commutative C*-algebras, is closed under quotients by ternary $*$-ideals.

Proposition 6.3. *Let $U \subset \Omega$ be an open antisymmetric subset, and let $C = \Omega \setminus (U \cup (-U))$. Then F_C is a C*-ideal in W. Conversely, if C is a closed subset C of Ω with $C = -C$, such that F_C is a C*-ideal, then there is an open set $U \subset \Omega$ with $U \cap (-U) = \emptyset$, and $U \cup (-U) = C^c$.*

Proof. Given U with the asserted properties, define $\theta : C_0(U) \to F_C$ by

$$\theta(f)(x) = \begin{cases} f(x) & \text{if } x \in U, \\ -f(-x) & \text{if } x \in (-U), \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\theta(f)$ is continuous on U, and therefore also on $-U$. In the interior of C the function $\theta(f)$ is clearly continuous. Finally, if x is in the boundary of C, and if

$\epsilon > 0$ is given then there is a compact $K \subset U$ with $|f| < \epsilon$ on $U \setminus K$. Choose an open symmetric set V containing x which does not intersect $K \cup (-K)$. On V it is clear that $|\theta(f)| < \epsilon$. Thus $\theta(f)$ is continuous at x. Indeed it is clear that $\theta(f) \in C_0(\Omega)$, so that $\theta(f) \in F_C$. Clearly θ is a one-to-one ternary $*$-morphism. Given $g \in F_C$ let f be the restriction of g to U. If $\epsilon > 0$ is given, let $K = \{w \in \Omega : |g(w)| \geqslant \epsilon\}$; this is compact and is a symmetric subset of $U \cup (-U)$. Thus $K \cap U$ is a compact subset of U, so that $f \in C_0(U)$. Clearly $\theta(f) = g$.

Conversely, suppose that B is a commutative C*-algebra, and suppose that $\theta : B \to W \subset C_0(\Omega)$ is a one-to-one ternary $*$-morphism onto $N = F_C$. Let (e_t) be an increasing approximate identity for B. Then the weak* limit v of $(\theta(e_t))$ is the support tripotent for N (see the discussion before Theorem 4.17). For $\omega \in \Omega$ let δ_ω be 'evaluation at ω'. This yields a canonical map $\Omega \to W : \omega \mapsto \delta_\omega$. Define $h(\omega) = v(\delta_\omega)$ for $\omega \in \Omega$. Clearly h is a real-valued function on Ω, and $h(\psi) = \lim_t \theta(e_t)(\psi)$ for all $\psi \in \Omega$. If $g = \theta(b)$ for $b \in B$, then we have

$$h(\psi)g(\psi)^2 = \lim_t \theta(e_t)(\psi)g(\psi)^2 = \lim_t \theta(e_t b^2)(\psi) = \theta(b^2)(\psi). \tag{8}$$

Similarly, $h^2 g = g$ for all $g \in N$. Thus $h(\psi) = 1$ or -1 for every $\psi \notin C$. Let $U = h^{-1}(1)$; then $-U = h^{-1}(-1)$, and these are disjoint subsets of $\Omega \setminus C$. If $\psi \notin C$ and $g \in N$ with $|g(\psi)| = \alpha \neq 0$, let $V = |g|^{-1}\big(\big(\frac{1}{2}\alpha, \infty\big)\big)$ be a symmetric open set in $\Omega \setminus C$, containing ψ. Now by (8), $hg^2 = \theta(b^2)$ is continuous on V, so that h is continuous on V. Hence h is continuous on $\Omega \setminus C$, and so U and $-U$ are open. □

Definition 6.4. Given an open antisymmetric set $U \subset \Omega$ as in the last proposition, define $\mathfrak{C}_U$ to be the cone in W consisting of those functions $f \in W$ with $f(x) \geqslant 0$ if $x \in U$, and $f(x) = 0$ for $x \in C = \Omega \setminus (U \cup (-U))$. (Necessarily then $f \leqslant 0$ on $-U$.)

Remark. In fact, the support tripotent v of a C*-ideal 'restricts' to a Borel function h on Ω. There is a canonical map $\rho : \mathrm{Bo}(\Omega) \to C_0(\Omega)''$, where $\mathrm{Bo}(\Omega)$ is the C*-algebra of bounded Borel measurable functions on Ω, and $\rho(h) = v$. As a corollary one may deduce that $\mathfrak{C}_U = \mathfrak{c}_v \cap W$, where $\mathfrak{c}_v$ is as in Definition 4.6.

Corollary 6.5. (1) *If U is an open antisymmetric set $U \subset \Omega$, then $\mathfrak{C}_U$ is a natural cone.*

(2) *Given a natural cone $\mathfrak{C}$ in W, then there is a unique open antisymmetric set $U \subset \Omega$ such that $\mathfrak{C} = \mathfrak{C}_U$. Indeed $U = \bigcup_{g \in \mathfrak{C}} g^{-1}((0, \infty))$.*

Proof. The first part of the proof of Proposition 6.3 shows that $\mathfrak{C}_U$ is the canonical positive cone in a C*-ideal of W. By the centered equation before Theorem 4.17, and the discussion above it, it follows that $\mathfrak{C}_U$ is a natural cone.

To see (2), first note that if $\mathfrak{C}$ is a natural cone in W, then $N = \mathrm{Span}(\mathfrak{C})$ is a C*-ideal. By Propositions 6.2 and 6.3, we obtain the associated open set U. Let θ be as in the (second part of the) last proof. If $\pi : B \to N^2$ is the canonical $*$-isomorphism associated with θ (see Proposition 1.1(6)), then $\theta(b) = v\pi(b)$ for all $b \in B$. Indeed

$$\theta(b)(\psi) = \lim_t \theta(e_t^2 b)(\psi) = \lim_t \theta(e_t)(\psi) \lim_t (\theta(e_t)\theta(b))(\psi) = v(\psi)\pi(b)(\psi).$$

Thus since $b \geqslant 0$ if and only if $\pi(b) \geqslant 0$, and since $v = 1$ on the open set U, it follows that a function g in F_C is in $\mathfrak{C}$ (that is, of the form $\theta(b)$ for a $b \in B_+$) if and only if $g \in \mathfrak{C}_U$. Also note that $x \in U$ if and only if $g(x) > 0$ for some $g \in \theta(B_+)$. To see this note that if $x \in U$ then $g(x) \geqslant 0$ for all $g \in \theta(B_+)$ by the above. If $g(x) = 0$ for all $g \in \theta(B_+)$, then $g(x) = 0$ for all $g \in \theta(B) = N$, which is impossible. Conversely, if $g(x) > 0$ for some $g \in \theta(B_+)$ then certainly $x \notin C$ since $g \in F_C$. If $x \in (-U)$ then $-x \in U$, so that $g(-x) = -g(x) \geqslant 0$, which is impossible. So $x \in U$. □

Corollary 6.6. *Suppose that we have two natural cones $\mathfrak{C}_1$ and $\mathfrak{C}_2$ on Z, or equivalently on W, and that U_1 and U_2 are the two corresponding open antisymmetric subsets of Ω. Then $U_1 \subset U_2$ if and only if $\mathfrak{C}_1 \subset \mathfrak{C}_2$. In particular, $\mathfrak{C}_{U_1} = \mathfrak{C}_{U_2}$ if and only if $U_1 = U_2$.*

One may use the last results to show that if Z is a non-trivial (that is, non-zero) commutative involutive ternary system, then Z possesses a non-trivial natural ordering. Putting this together with Theorem 5.4 we have the following.

Corollary 6.7. *Let Z be a non-trivial (that is, non-zero) commutative involutive ternary system. Then Z possesses a non-trivial maximal ordering, and this ordering is a natural one.*

Corollary 6.8. *If U is an open subset of Ω with $U \cap (-U) = \emptyset$, then the following are equivalent:*

(i) $\mathfrak{C}_U$ *(the space of functions in W that are non-negative on U) is a maximal natural cone of W;*
(ii) $\Omega \setminus (U \cup (-U)) = \mathrm{Bdy}(U)$;
(iii) $\Omega \setminus (U \cup (-U)) = \mathrm{Bdy}(-U)$;
(iv) $\mathrm{Bdy}(U) = \mathrm{Bdy}(-U)$ *and $\Omega \setminus (U \cup (-U))$ has no interior points;*
(v) *there is no larger open subset U' of Ω containing U with $U' \cap (-U') = \emptyset$.*

Any maximal cone of W is of the form in (i).

From Corollaries 6.8 and 6.6, we deduce that there is a bijection between maximal cones (which we know from an earlier result are always natural) on a commutative involutive ternary system, and the class of open antisymmetric subsets U characterized in Corollary 6.8. Indeed we may call such a subset U an 'antipodal coloring' of Ω, and then maximal cones are in one-to-one correspondence with such 'antipodal colorings'. From this correspondence it is easy to construct interesting very explicit examples of maximal cones on commutative involutive ternary systems. The example we shall consider in the remainder of this section is as follows. Let S^2 be the unit sphere, and Z the $*$-TRO $\{f \in C(S^2) : f(-x) = -f(x)\}$. This is clearly a trivially ordered $*$-TRO in $C(S^2)$. By an 'antipodal coloring of the sphere' we mean an open subset U of the sphere (called blue), which does not intersect $-U$ (called red), such that the boundary of U is the boundary of $(-U)$, and this latter set has no interior. Thinking about such colorings of the sphere it is clear that there is a rich profusion of them that are quite different topologically. One may also, if one wishes, choose the coloring so that the measure of $C = \Omega \setminus (U \cap (-U))$ is positive.

Example. In this example we show that given a natural cone $\mathfrak{c}$ on a $*$-TRO Z, there may exist many distinct maximal orderings on the second dual $E = Z''$, whose restriction to Z is $\mathfrak{c}$. This can be done even when $\mathfrak{c}$ is (0) or is a maximal ordering on Z. We will use the fact from Proposition 4.11 that any selfadjoint unitary u in $\mathcal{Z}(E)$ gives a maximal cone $\mathfrak{c}$ on E such that $z \in \mathfrak{c}$ if and only if $uz \geqslant 0$. Let

$$Z = \{f \in C(S^2) : f(-x) = -f(x)\},$$

and let H be the open upper hemisphere of S^2. Let P and Q be disjoint Borel sets which together partition H, each of which is dense in H. Let $\mathbb{T}$ be the 'equator' in S^2. Let v be the function from H to $\{-1, 1\}$ which is 1 on P and -1 on Q. Then v is a unitary in the C*-algebra $\mathrm{Bo}(H)$ of bounded Borel functions on H. Let $\theta : \mathrm{Bo}(H) \to C_0(H)''$ be the canonical unital $*$-homomorphism. There is a canonical ternary $*$-morphism $\nu : C_0(H) \to Z$ defined by

$$\nu(f)(x) = \begin{cases} f(x) & \text{if } x \in H, \\ -f(-x) & \text{if } x \in (-H), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\psi = \nu'' : C_0(H)'' \to Z''$ is a one-to-one ternary $*$-morphism. Let $\rho = \psi \circ \theta$ and $V = \rho(v)$. Then V is a selfadjoint tripotent in Z''. We may choose, as in Lemma 4.10(3), a selfadjoint unitary u in Z'' with $u \geqslant V$. As we saw in Proposition 4.11, we may endow Z'' with a maximal cone $\mathfrak{C}_u$ (which depends on P). We claim that the restriction of $\mathfrak{C}$ to Z is trivial. Indeed if $f \in Z$ with $fu \geqslant 0$, then $fuV^2 = fV \geqslant 0$. If $\omega \in H$, and if δ_ω is 'evaluation at ω' (which is a character of $C(S^2)$), then we have $(fV)(\delta_\omega) = f(\omega)V(\delta_\omega) \geqslant 0$. However $V(\delta_\omega) = 1$ if $\omega \in P$, because

$$V(\delta_\omega) = \nu''(\theta(v))(\delta_\omega) = \theta(v)(\nu'(\delta_\omega)) = \theta(v)(\delta_\omega) = v(\omega) = 1.$$

Similarly, $V(\delta_\omega) = -1$ if $\omega \in Q$. It follows that $f = 0$ on H, and so $f = 0$ on S^2.

To see that one may obtain many different cones on Z'' restricting to the trivial cone on Z, simply choose a different partition P', Q' of H. It is easy to argue that the associated cone on Z'' is necessarily different from $\mathfrak{C}_u$ above.

The existence of multiple orderings on $E = Z''$ restricting to the same maximal cone is very similar. Suppose that $\mathfrak{c} = \mathfrak{C}_U$ is a maximal cone on Z corresponding to a maximal open set U as in Corollary 6.8, and suppose that v is the associated maximal open tripotent in $\mathcal{Z}(E)$. Thus $p = v^2$ is open; let $q = 1 - p$ be the complementary projection. It is easy to see that Eq contains many linearly independent selfadjoint tripotents w with $w^2 = q$. (If E^2q was 1-dimensional it would be easy to argue that F_C^2 has codimension 1, that is, it is a maximal ideal in Z^2. Here F_C is as in Proposition 6.3. This forces $S^2 \setminus (U \cup (-U)) = \{\zeta, -\zeta\}$ for some $\zeta \in S^2$, which is absurd.) Any such w gives rise as before to a selfadjoint unitary $u \geqslant v \in E$, and hence to a maximal natural cone $\mathfrak{d}_u$ on Z (see Theorem 4.16). We claim that $\mathfrak{c}_u \cap Z = \mathfrak{C}_U$. Indeed if $x \in Z$ then $x \in \mathfrak{c}_u$ if and only if $ux \geqslant 0$. Similarly, if $x \in Z$ then $x \in \mathfrak{C}_U$ if and only if $vx \geqslant 0$ and $px = x$. The latter condition implies that $ux = uvvx = vx \geqslant 0$. Thus $\mathfrak{C}_U \subset \mathfrak{c}_u \cap Z$. For the other direction, note that if $x \in Z$ and if $ux \geqslant 0$, then as before $(vx)(\varphi) = x(\varphi) \geqslant 0$ for $\varphi \in U$. Thus $x \leqslant 0$ on $-U$. It follows from Corollary 6.8 that $x \in \mathfrak{C}_U$.

Closing remarks. (1) The results of this paper can be applied to so-called *causal structures* on infinite-dimensional manifolds M (see for example [**20**]). We will go into details elsewhere.

(2) Some time after this paper was written, M. Neal pointed out to us the following interesting facts. First, a simple calculation shows that one may characterize our 'central selfadjoint tripotents' in a $*$-TRO Z, as precisely the tripotents u for which $u = u^*$ and $uu^*z = zu^*u$ for every z in Z, and which also have the property that the involution $x \mapsto ux^*u$ on the Pierce 2 space of u coincides with the given involution on Z. Secondly, selfadjoint open tripotents u in Z may also be characterized by the property that the matrix $\frac{1}{2}(z + z^2)$, where z is the 2×2 matrix with zero main diagonal entries and a repeated u in the other two entries, is an open projection in $\mathcal{L}(Z)''$, where $\mathcal{L}(Z)$ is the linking C*-algebra of Z. Proving this pretty and useful characterization is not hard, once one is aware of the result. It may be seen as an improvement on Proposition 4.18. A variant is also true for open tripotents in general TROs.

References

1. C. Akemann, 'The general Stone–Weierstrass problem', *J. Funct. Anal.* 4 (1969) 277–294.

2. C. Akemann and G. Pedersen, 'Facial structure in operator algebra theory', *Proc. London Math. Soc.* (3) 64 (1992) 418–448.

3. T. Barton and R. Timoney, 'Weak*-continuity of Jordan triple products and its applications', *Math. Scand.* 59 (1986) 177–191.

4. M. Battaglia, 'Order theoretic type decomposition of JBW*-triples', *Quart. J. Math. Oxford* 42 (1991) 129–147.

5. D. P. Blecher, 'The Shilov boundary of an operator space, and the characterization theorems', *J. Funct. Anal.* 182 (2001) 280–343.

6. D. P. Blecher and C. Le Merdy, *Operator algebras and their modules*, London Mathematical Society Monographs 30 (Oxford University Press, 2004).

7. D. P. Blecher and B. Magajna, 'Duality and operator algebras: automatic weak* continuity', *J. Funct. Anal.* 224 (2005) 386–407.

8. M. D. Choi and E. G. Effros, 'The completely positive lifting problem for C*-algebras', *Ann. of Math.* 104 (1976) 585–609.

9. M.-D. Choi and E. G. Effros, 'Injectivity and operator spaces', *J. Funct. Anal.* 24 (1977) 156–209.

10. C. M. Edwards and G. T. Rüttimann, 'On the facial structure of the unit balls in a JBW*-triple and its predual', *J. London Math. Soc.* (2) 38 (1988) 317–332.

11. C. M. Edwards and G. T. Rüttimann, 'Inner ideals in C*-algebras', *Math. Ann.* 290 (1991) 621–628.

12. C. M. Edwards and G. T. Rüttimann, 'Compact tripotents in bi-dual JBW*-triples', *Math. Proc. Cambridge Philos. Soc.* 120 (1996) 155–173.

13. C. M. Edwards and G. T. Rüttimann, 'The facial and inner ideal structure of a real JBW*-triple', *Math. Nach.* 222 (2001) 159–184.

14. E. G. Effros, N. Ozawa and Z. J. Ruan, 'On injectivity and nuclearity for operator spaces', *Duke Math. J.* 110 (2001) 489–521.

15. E. G. Effros and Z. J. Ruan, 'Mapping spaces and liftings for operator spaces', *Proc. London Math. Soc.* (3) 69 (1994) 171–197.

16. Y. Friedman and B. Russo, 'Function representation of commutative operator ternary systems', *J. London Math. Soc.* (2) 27 (1983) 513–524.

17. M. Hamana, 'Triple envelopes and Silov boundaries of operator spaces', *Math. J. Toyama Univ.* 22 (1999) 77–93.

18. P. Harmand, D. Werner and W. Werner, *M-ideals in Banach spaces and Banach algebras*, Lecture Notes in Mathematics 1547 (Springer, Berlin, 1993).

19. L. A. Harris, 'A generalization of C*-algebras', *Proc. London Math. Soc.* (3) 42 (1981) 331–361.

20. J. Hilgert and G. Olafsson, *Causal symmetric spaces*, Perspectives in Mathematics 18 (Academic Press, San Diego, 1997).

21. G. Horn, 'Characterization of the predual and ideal structure of a JBW*-triple', *Math. Scand.* 61 (1987) 117–133.

22. J. M. Isidro, W. Kaup and A. Rodriguez-Palacios, 'On real forms of JB*-triples', *Manuscripta Math.* 86 (1995) 311–335.

23. W. Kaup, 'A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces', *Math. Z.* 138 (1983) 503–529.

24. W. KAUP, 'Contractive projections on Jordan C*-algebras and generalizations', *Math. Scand.* 54 (1984) 95–100.
25. W. KAUP, 'On JB*-triples defined by fibre bundles', *Manuscripta Math.* 87 (1995) 379–403.
26. K. KIRKPATRICK, 'The Shilov boundary and M-structure of operator spaces', Research Experiences for Undergraduates paper, Houston, 2001.
27. M. NEAL and B. RUSSO, 'Operator space characterizations of C*-algebras and ternary rings', *Pacific J. Math.* 209 (2003) 339–364.
28. M. NEAL and B. RUSSO, 'State spaces of JB*-triples', *Math. Ann.* 328 (2004) 585–624.
29. G. K. PEDERSEN, *C*-algebras and their automorphism groups* (Academic Press, New York, 1979).
30. Z.-J. RUAN, 'Type decomposition and rectangular AFD property for W^*-TRO's', *Canad. J. Math.* 56 (2004) 843–870.
31. B. RUSSO, 'Structure of JB*-triples', *Jordan algebras*, Oberwolfach, 1992 (ed. W. Kaup *et al.*; de Gruyter, Berlin, 1994) 209–280.
32. J.-L. SAUVAGEOT, 'Tangent bimodules and locality for dissipative operators on C*-algebras', *Quantum probability and applications* (ed. L. Accardi and W. von Waldenfels), Lecture Notes in Mathematics 1396 (Springer, New York, 1989) 322–338.
33. M. SKEIDE, 'Hilbert modules in quantum electro dynamics and quantum probability', *Comm. Math. Phys.* 192 (1998) 569–604.
34. N. WEAVER, 'Hilbert bimodules with involution', *Canad. Math. Bull.* 44 (2001) 355–369.
35. N. WEAVER, *Mathematical quantization* (CRC Press, Boca Raton, 2001).
36. N. E. WEGGE-OLSEN, *K-theory and C*-algebras* (Oxford University Press, 1993).
37. W. WERNER, 'Subspaces of $L(H)$ that are $*$-invariant', *J. Funct. Anal.* 193 (2002) 207–223.
38. W. WERNER, 'Multipliers on matrix ordered operator spaces and some K-groups', *J. Funct. Anal.* 206 (2004) 356–378.
39. W. WINTER, 'Covering dimension for nuclear C*-algebras', Thesis, Westfälische Wilhelms-Universität, Münster, 2001.
40. M. A. YOUNGSON, 'Completely contractive projections on C^*-algebras', *Quart. J. Math. Oxford* 34 (1983) 507–511.
41. H. ZETTL, 'A characterization of ternary rings of operators', *Adv. Math.* 48 (1983) 117–143.

David P. Blecher
Department of Mathematics
University of Houston
Houston
TX 77204-3008
USA

dblecher@math.uh.edu

Wend Werner
Mathematisches Institut
Einsteinstrasse 62
D-48149 Münster
Germany

wwerner@math.uni-muenster.de