A note on morphisms for Hilbert-C* manifolds

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Abstract. This note deals with a clash of categories. For Hilbert-C* manifolds, ternary ring morphisms are the adequate morphisms for a transfer of some well-known results on finite dimensional symmetric spaces to the infinite dimensional set-up, whereas the class of adjointable mappings yield the appropriate definition of the cotangent bundle. It turns out that the intersection of both classes is very small.

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1. Basic definitions and results

1.1. Recall that a (left) Hilbert-C* module over a C*-algebra $\mathfrak{A}$ is a complex vector space E which is a left $\mathfrak{A}$-module with a sesquilinear pairing $E \times E \to \mathfrak{A}$ satisfying, for $r, s \in E$ and $a \in \mathfrak{A}$, the following conditions:

(i) $\langle ar, s\rangle = a\langle r, s\rangle$

(ii) $\langle r, s\rangle = \langle s, r\rangle^*$

(iii) $\langle s, s\rangle > 0$ for $s \neq 0$

(iv) Equipped with the norm
$$\|s\| = \sqrt{\|\langle s, s\rangle\|},$$
E is a Banach space.

Right Hilbert-C* modules and Hilbert bimodules are defined similarly (with appropriate compatibility conditions in the latter case). Whenever we want to refer to the algebra $\mathfrak{A}$ explicitly, we speak of a Hilbert-$\mathfrak{A}$ module.

1.2. The objects defined above coincide with the so called ternary rings of operators (TRO), which are intrinsically characterized in [8, 11]. On such a space E, a triple products $\{\cdot,\cdot,\cdot\}$ is given in such a way that E, up to (the obvious definition of) TRO-isomorphisms, is a subspace of a space of bounded Hilbert space operators $L(H)$, invariant under the triple product

$$\{x, y, z\} = xy^*z.$$

The relation to (left) Hilbert-C* modules is based on the equation

$$\{x, y, z\} = \langle x, y\rangle_\ell z,$$

connecting triple product to module action (under $\mathfrak{A} = EE^*$, where the latter algebra does not depend on the chosen embedding) as well as to the scalar product of a Hilbert-$\mathfrak{A}$ module. Not surprisingly, there is also an action of a C*-algebra $\mathfrak{B} = E^*E$ on the right, and the right Hilbert-C* module structure shows up in

$$\{x, y, z\} = x\langle y, z\rangle_r$$

Note also that the norm of an element $e \in E$ must coincide with $\|e\| = \|\{e, e, e\}\|^{1/3}$. If, in the following, the bilinear form $\langle\cdot,\cdot\rangle$ does not carry one of the indices ℓ, r the left module structure is addressed.

Definition 1.3. *Let M be a Banach manifold and $\mathfrak{A}$ a C*-algebra. M is said to be a (right-, left-) Hilbert-$\mathfrak{A}$ manifold if on each tangent space $T_p(M)$ there is given the structure of a Hilbert-$\mathfrak{A}$ module depending smoothly on base points.*

1.4. TRO-morphisms are those mappings that respect the product $\{\cdot,\cdot,\cdot\}$. These mappings differ in general from what is considered to be the natural choice for Hilbert-$\mathfrak{A}$ morphisms, the so called adjointable maps. The latter are in particular $\mathfrak{A}$-module morphisms, a property that would be too restrictive for what is needed below. Adjointable maps, on the other hand, yield a duality theory that resembles the commutative situation and thus seem to be more adequate for dealing with cotangent bundles, for example. We will come back to this point in the following section.

1.5. There is no method, in general, to provide an arbitrary Banach manifold with a Hilbert-C* structure. This is due to the fact that, on one hand, it is unclear, when a Banach spaces is (topologically, linearly isomorphic to) a Hilbert-C*-module, and, on the other hand, there is, on arbitrary Banach manifolds no substitute for smooth partitions of unity.

It is helpful, though, if M is homogeneous with respect to the action of a Banach Lie group. In such a situation, we can try to convert M into a homogeneous Hilbert-C* manifold, where we call a Banach manifold M a *homogeneous* Hilbert-C* manifold iff it is a Hilbert-C* manifold for which $\operatorname{Aut} M$ acts transitively. Here, $\operatorname{Aut} M$, the group of automorphisms of M, consists of all diffeomorphisms $\Phi : M \to M$ so that $d\Phi$ is (pointwise) a TRO-morphism.

1.6. Suppose that M is a homogeneous Banach manifold with respect to a smooth Banach Lie group action, $o \in M$ is fixed, the isotropy subgroup at o is denoted by

H, and that $T_o(M)$ carries the structure of a Hilbert-$\mathfrak{A}$ module with form $\langle\cdot,\cdot\rangle_o$ and module map $x \mapsto a\cdot_o x$. Then a Hilbert-$\mathfrak{A}$ module on $T_p(M)$ for any $p = g(o) \in M$, is (well-)defined through

$$\langle x, y\rangle_p = \langle d_p g^{-1}(x), d_p g^{-1}(y)\rangle_0 \qquad \text{and} \qquad a \cdot_p x = d_o g\left(a \cdot_o d_p g^{-1}(x)\right),$$

where $x, y \in T_p(M)$ and $a \in \mathfrak{A}$, if, for all $h \in H$, $x, y \in T_o(M)$, and $a \in \mathfrak{A}$ we have

$$\langle d_o h(x), d_o h(y)\rangle_o = \langle x, y\rangle_o \qquad \text{as well as} \qquad d_o h\left(a \cdot_o d_o h^{-1}(x)\right) = x.$$

1.7. An important example for this situation is given in the following way. For a TRO E denote by U its open unit ball. Define a triple product for T_aM at $a \in U$ by

$$\{xyz\}_a = x(1 - a^*a)^{-1}y^*(1 - aa^*)^{-1}z,$$

so that

$$\langle x, y\rangle_a = (1 - aa^*)^{-1/2}x(1 - a^*a)^{-1}y^*(1 - aa^*)^{-1/2}$$

as well as

$$\gamma \cdot_a z = (1 - aa^*)^{1/2}\gamma(1 - aa^*)^{-1/2}, \qquad \gamma \in EE^*.$$

We will refer to this structure as the *canonical Hilbert-C* structure* on U. It is well known (cf. [4]) that $\operatorname{Hol} U$, the group of all biholomorphic automorphisms U, consists of mappings of the form $T \circ M_a$, where for $a \in U$,

$$M_a(x) := (1 - aa^*)^{-1/2}(x + a)(1 + a^*x)^{-1}(1 - a^*a)^{1/2},$$

and T is a (linear) isometry of E, restricted to U. It is furthermore well known that $\operatorname{Hol} U$ acts transitively on U, and that the isotropy subgroup at the point 0 consists of all the (linear, surjective) isometries of E, restricted to U. We have

$$d_x M_a(h) = (1 - aa^*)^{1/2}(1 + xa^*)^{-1}h(1 + a^*x)^{-1}(1 - a^*a)^{1/2}.$$

This equation is used to show that the Hilbert-C* structure from 1.7 is constructed according to the construction in 1.5 for a group G, somewhat smaller than $\operatorname{Hol} U$.

Definition 1.8. *Suppose X is a (closed) subspace of $L(H)$. Equip*

$$M_n(X) = \{(x_{ij}) \mid x_{ij} \in X \text{ for } i, j = 1, \ldots, n\}$$

with the norm it carries as a subspace of $M_n(L(H)) = L(H \oplus \cdots \oplus H)$, and, for a bounded operator $T : X \to X$, denote by $T^{(n)} = \mathrm{id}_{M_n(\mathbb{C})} \otimes X : M_n(X) \to M_n(X)$ the operator $(x_{ij}) \longmapsto (Tx_{ij})$. Then T is said to be completely bounded, iff

$$\|T\|_{cb} := \sup_{n\in\mathbb{N}} \|T^{(n)}\| < \infty.$$

Similarly, T is called a complete isometry iff each of the maps $T^{(n)}$ is an isometry.

We have the following

Theorem 1.9 ([3, 8]). *Let E be a TRO. Then $M_n(E)$ carries a distinguished TRO-structure, and the group of TRO-automorphisms of E coincides with with the group of complete isometries.*

Theorem 1.10 ([10]). *Let U be the open unit ball of a TRO E, equipped with the canonical Hilbert-C* structure, and suppose $M_n(E)$ carries the standard TRO-structure for each $n \in \mathbb{N}$. Then A diffeomorphism $\Phi : U \to U$ is a Hilbert-C* automorphism iff $\Phi = T \circ M_a$, where $a \in U$, and T is the restriction of a linear and completely isometric mapping of E to U.*

1.11. Let M again be a Hilbert-C* manifold. Then the tangent space at each point m of M carries an essentially unique norm, naturally connected to the TRO-structure by $\|x\|_m = \|\{x, x, x\}\|^{1/3}$. Since TROs embed completely isometrically into some space of bounded Hilbert space operators, this norm naturally extends to the spaces $M_n(T_m M)$. We will call a Banach manifold M an *operator Finsler manifold* in case each tangent space carries the structure of an operator space depending smoothly on base points. In this sense, any Hilbert-C* manifold carries an operator Finsler structure in a natural way. If, as before, U is the open unit ball of a fixed TRO $(E, \{\cdot, \cdot, \cdot\}, \|\cdot\|)$, furnished with its invariant Hilbert-C* structure then

Theorem 1.12 ([10]). *If the open unit ball of a TRO is equipped with the natural invariant operator Finsler structure its automorphism groups coincides with the automorphism group of the underlying homogeneous Hilbert-C* manifold.*

1.13. The reader should note that the above results support the statement that the most adequate morphisms for Hilbert-C* manifolds are those mappings whose derivatives are pointwise TRO-morphisms. That structure of this type is very natural in the treatment of symmetric spaces has been known for a long time (see [7] for example). Another result which illustrates this point is the following one, taken from [10].

Theorem 1.14. *On the Hilbert-C* manifold U there exists exactly one invariant connection whose Christoffel symbol at a is given by*

$$\Gamma_a(x, y) = y(1 - a^*a)^{-1/2}a^*(1 - aa^*)^{-1/2}x + \\ + x(1 - a^*a)^{-1/2}a^*(1 - aa^*)^{-1/2}y = 2\{x, M'_a(0)a, y\}^s_a,$$

where $\{x, y, z\}^s = \frac{1}{2}(\{z, y, x\} + \{x, y, z\})$ denotes the (symmetric) Jordan triple product on E.

1.15. It is important here that the definition of a connection for Banach manifolds cannot be, due to the scarcity of smooth functions, the usual one. The right definition seems to be [5, 1.5.1 Definition]:

Definition 1.16. *Let M be a manifold, modeled over the Banach space E, and denote the space of bounded bilinear mappings $E \times E \to E$ by $L^2(E, E)$. Then M is said to possess a connection iff there is an atlas $\mathcal{U}$ for M so that for each $U \in \mathcal{U}$*

there is a smooth mapping $\Gamma : U \to L^2(E, E)$*, called the* Christoffel symbol *of the connection on* U*, which under a change of coordinates* Φ *transforms according to*

$$\Gamma(\Phi' X, \Phi' Y) = \Phi''(X, Y) + \Phi'\Gamma(X, Y).$$

The covariant derivative of a vector field Y *in the direction of the vector field* X *is, locally, defined to be the principal part of*

$$\nabla_X Y = dX(Y) - \Gamma(X, Y),$$

where, in a chart, the principal part of $(u, X) \in U \times T(U)$ *is* X*.*

2. Ternary morphisms, adjointable mappings and the cotangent bundle

Our basic references for the structure of TROs and Hilbert-C* modules are [1], [6], [9].

2.1. Let E be a TRO which we often will suppose to be embedded into some space $L(H)$ of bounded Hilbert space operators. The *linking algebra* of E is

$$\mathcal{L}(E) = \begin{pmatrix} EE^* & E \\ E^* & E^*E \end{pmatrix}$$

Under the module action of EE^* and E^*E on E and E^*, respectively, $\mathcal{L}(E)$ carries the structure of a C*-algebra, which is independent of the chosen embedding. Furthermore, the TRO-structure of E is the canonical TRO-structure of $\mathcal{L}(E)$ restricted to E. We will also write $\mathcal{L}_0(E)$ for the subspace of $\mathcal{L}(E)$ which on the main diagonal contains finite linear combinations of elements of the form xy^* and x^*y, respectively.

We will equip $\mathfrak{L}(E)$ with $EE^* \oplus E^*E$-valued forms, which are obtained from extending

$$\left\langle \begin{pmatrix} \langle e_1, \widehat{e}_1\rangle_\ell & g_1 \\ g_2^* & \langle e_2, \widehat{e}_2\rangle_r \end{pmatrix}, \begin{pmatrix} \langle f_1, \widehat{f}_1\rangle_\ell & h_1 \\ h_2^* & \langle f_2, \widehat{f}_2\rangle_r \end{pmatrix} \right\rangle_\ell =$$
$$= (\langle e_1, \widehat{e}_1\rangle_\ell \langle \widehat{f}_1, f_1\rangle_\ell + \langle g_1, h_1\rangle_\ell, \langle e_2, \widehat{e}_2\rangle_r \langle \widehat{f}_2, f_2\rangle_r + \langle g_2, h_2\rangle_r)$$

as well as

$$\left\langle \begin{pmatrix} \langle e_1, \widehat{e}_1\rangle_\ell & g_1 \\ g_2^* & \langle e_2, \widehat{e}_2\rangle_r \end{pmatrix}, \begin{pmatrix} \langle f_1, \widehat{f}_1\rangle_\ell & h_1 \\ h_2^* & \langle f_2, \widehat{f}_2\rangle_r \end{pmatrix} \right\rangle_r =$$
$$= (\langle \widehat{e}_1, e_1\rangle_\ell \langle f_1, \widehat{f}_1\rangle_\ell + \langle g_2, h_2\rangle_\ell, \langle \widehat{e}_2, e_2\rangle_r \langle f_2, \widehat{f}_2\rangle_r + \langle g_1, h_1\rangle_r)$$

linearly and forming limits. It is easily seen that this form provides $\mathfrak{L}(E)$ with the structure of an Hilbert-$EE^* \oplus E^*E$ bimodule. We will call this structure the *canonical* Hilbert bimodule structure on $\mathfrak{L}(E)$. Note that the left form restricts on E^* to $\langle\cdot,\cdot\rangle_r$ and vice versa. We will consider E^* equipped with the Hilbert-C* structure which is obtained in this way.

2.2. Let E, F be TROs and $\Phi : E \to F$ a TRO morphism. Important in the sequel is the following construction. Observe first that for any $x_1, \dots, x_n, y_1, \dots y_n \in E$ we have

$$\begin{aligned}\left\|\sum_{k=1}^n \Phi(x_k)\Phi(y_k)^*\right\| &= \sup\left\{\left\|\sum_{k=1}^n \Phi(x_k)\Phi(y_k)^* f\right\| \mid f \in \Phi(E), \|f\| \le 1\right\}\\ &= \sup\left\{\left\|\sum_{k=1}^n \Phi(x_k)\Phi(y_k)^*\Phi(e)\right\| \mid e \in E, \|e\| \le 1\right\}\\ &\le \sup\left\{\left\|\sum_{k=1}^n x_k y_k^* e\right\| \mid e \in E, \|e\| \le 1\right\} = \left\|\sum_{k=1}^n x_k y_k^*\right\|.\end{aligned}$$

In this estimate, the following facts have been used: In any TRO, the map

$$\theta : EE^* \to L(E), \qquad \sum_{k=1}^n x_k y_k^* \longmapsto \left(e \mapsto \sum_{k=1}^n x_k y_k^* e\right)$$

is an isometry (this we used three times), the image of Φ is a (closed) sub-TRO of F and, finally, Φ is a quotient map onto its image.

2.3. The above inequality shows that

$$\Phi_{11}\left(\sum_{k=1}^n x_k y_k^*\right) := \sum_{k=1}^n \Phi(x_k)\Phi(y_k)^*$$

is a well defined contraction which extends to a mapping $EE^* \to FF^*$, which we continue to denote by Φ_{11}. In quite the same way, we define a mapping $\Phi_{22} : E^*E \to F^*F$. Denote by $\overline{E}, \overline{F}$ the complex vector spaces conjugate to E and F, respectively, and let $\overline{\Phi} : \overline{E} \to \overline{F}$, $\overline{\Phi}(\overline{e}) = \overline{\Phi(e)}$. If $E \subseteq L(H)$ then $\overline{E}$ can be identified with the space $E^* = \{e^* \mid e \in E\}$. Put

$$\mathcal{L}(\Phi) : \mathcal{L}(E) \to \mathcal{L}(F), \qquad \mathcal{L}(\Phi) = \begin{pmatrix} \Phi_{11} & \Phi \\ \overline{\Phi} & \Phi_{22} \end{pmatrix}.$$

Then $\mathcal{L}(\Phi)$ is a C*-morphisms whose restriction to (the canonical copy of) E coincides with Φ.

2.4. The morphisms for Hilbert-C* modules prevailing in the literature are the adjointable maps, i.e. maps $\Phi : E \to F$ between Hilbert-$\mathfrak{A}$ modules E and F such that there is an adjoint $\Phi^* : F \to E$ so that $\langle\Phi(e), f\rangle = \langle e, \Phi^*(f)\rangle$ for all $e \in E$ and $f \in F$. Adjointable maps are $\mathfrak{A}$-module mappings. Had we based our approach to Hilbert-C* manifolds on these mappings then the action of the isotropy group in the previous section would have dropped out of this category. The definition of the cotangent bundle, on the other hand, depends on duality, and, as we will see below, adjointable maps seem to be a much better choice in this case.

2.5. If E is a Hilbert-module then we say that E has a *real form* iff there is an involutory and antilinear mapping $e \mapsto e^*$ on E such that $\{a,b,c\}^* = \{c^*, b^*, a^*\}$ for all $a,b,c \in E$ (cf. [2]). If $\Phi : E \to L(H)$ is a TRO-embedding, then $\overline{\Phi}(a) = \Phi(a^*)^*$ and

$$e \longmapsto \begin{pmatrix} 0 & \Phi(e) \\ \overline{\Phi}(e) & 0 \end{pmatrix}$$

is a TRO-embedding of E into $L(H \oplus H)$ which respects the real form. If we apply this to the linking algebra of E, the involution on E then coincides with the one of $\mathfrak{L}(E)$ restricted to (the canonical copy of) E. In this way, we obtain an anti-isomorphism between the C*-algebras EE^* and E^*E, denoted by S and given by

$$S\left(\sum_i \widehat{e}_i e_i^*\right) = \sum_i e_i \widehat{e}_i^*.$$

Equivalently, for all $x, y, z \in E$,

$$zS(\langle x,y\rangle_\ell) = (\langle x,y\rangle_\ell z^*)^* = z\langle y^*, x^*\rangle_r.$$

2.6. It should be noted, though, that existence of a real form is a rather restrictive condition. For example, if $(H, \langle\cdot,\cdot\rangle)$ is a (complex) Hilbert space which we equip with the structure

$$\{h_1, h_2, h_3\} = \langle h_1, h_2\rangle h_3$$

then H possesses a real form only in case its dimension equals one. In fact, for any pair of norm-one elements x, y, self-adjoint under the involution $*$, we have

$$y = (\langle x,x\rangle y)^* = \langle y,x\rangle x,$$

whence the real space of self-adjoint elements is one-dimensional as must consequently be H itself. (This also follows from the fact that $HH^* = \mathbb{C}$ and H^*H consists of the compact operators on H.)

Lemma 2.7. *Let E and F be Hilbert-$\mathfrak{A}$ bimodules. Suppose that F has a real form and that $\Phi : E \to F$ has an adjoint $\Phi^* : F \to E$. Then all of $\overline{\Phi} : E^* \to F^*$, Φ_{11} and Φ_{22} are adjointable, and $\overline{\Phi}^* = \overline{\Phi^*}$. If the linking algebras carry their canonical Hilbert module structures, then $\mathcal{L}(\Phi)$ has the adjoint*

$$\mathcal{L}(\Phi)^* = \begin{pmatrix} \Phi_{11} & \Phi^* \\ \overline{\Phi}^* & \Phi_{22} \end{pmatrix}.$$

Proof: Using the real form of F we have, for all $y \in E$ and $x, z \in F$

$$x\langle \overline{\Phi}(y^*), z\rangle_r = (\langle z^*, \Phi(y)\rangle_\ell x^*)^* = (\langle \Phi^*(z^*), y\rangle_\ell x^*)^* = \\ = x\langle y^*, \Phi^*(z^*)^*\rangle_r = x\langle y^*, \overline{\Phi^*}(z)\rangle_r$$

and, consequently, that $\overline{\Phi}$ has the adjoint $\overline{\Phi}^* = \overline{\Phi^*}$. (Recall that E^* carries the right module structure.) With this fact kept in mind, we find

$$\left\langle \Phi_{11}\left(\sum_i e_i\widehat{e}_i^*\right), \sum_j f_j\widehat{f}_j^*\right\rangle = \sum_{i,j}\Phi(e_i)\overline{\Phi}(\widehat{e}_i^*)\widehat{f}_j f_j^* = \sum_{i,j} e_i\overline{\Phi^*}(\widehat{e}_i^*)\overline{\Phi}^*(\widehat{f}_j^*)^* f_j^* =$$

$$= \sum_{i,j} e_i\widehat{e}_i^*\Phi(\widehat{f}_j)\Phi(f_j)^* = \left\langle \sum_i e_i\widehat{e}_i^*, \Phi_{11}\left(\sum_j f_j\widehat{f}_j^*\right)\right\rangle$$

and so Φ_{11} is adjointable, as well. A similar calculation proves that also Φ_{22} is adjointable, from which the statement on $\mathcal{L}(\Phi)$ follows. □

Proposition 2.8. *Let E and F be Hilbert-$\mathfrak{A}$ bimodules. Suppose further, that F has a real form. Then the mapping $\Phi : E \to F$ is adjointable iff $\mathfrak{L}(\Phi)$ is adjointable for the canonical Hilbert module structure on the linking algebras of E and F, respectively. Furthermore, there exist central projections $p_{1,2}$ in $\mathcal{M}\mathfrak{A}$, the multiplier algebra of $\mathfrak{A}$ such that $a_{1,2} \in \mathfrak{A}$ such that*

$$\langle\Phi(x), \Phi(y)\rangle_\ell = \langle x, y\rangle_\ell p_1 \qquad \textit{and} \qquad \langle\Phi(x), \Phi(y)\rangle_r = p_2\langle x, y\rangle_r$$

for all $x, y \in E$

Proof: The first statement is a direct consequence of the above Lemma. The second follows from the fact that a map $x \mapsto px$, where p is in the multiplier algebra of $\mathfrak{A}$ is a C*-morphism iff p is a central projection. □

2.9. Suppose that E is a Hilbert-$\mathfrak{A}$ module and that $\Phi : E \to \mathfrak{A}$ is adjointable. If $\mathfrak{A}$ has a unit, then, for all $e \in E$,

$$\Phi(e) = \langle\Phi(e), 1\rangle = \langle e, \Phi^*(1)\rangle$$

It follows that Φ is adjointable iff there is an element $z \in \mathfrak{A}$ such that $\Phi(e) = \langle e, z\rangle$ for all $e \in \mathfrak{A}$. (Note that this statement remains true also when $\mathfrak{A}$ does not have a unit. The element z is then in $\mathcal{M}(\mathfrak{A})$.) Thus, concerning duality, Hilbert-$\mathfrak{A}$ modules behave like Hilbert spaces, so that the cotangent bundle of a Hilbert-C* manifold behaves like its classical counterpart when the underlying morphisms are adjointable mappings. This is not so for TRO morphism, as will be shown below. The difference, between TRO-morphisms and adjointable maps can be measured by the size of the intersection between these classes of mappings:

Corollary 2.10. *Let E be a Hilbert-$\mathfrak{A}$ bimodule. Then $\varphi : E \to \mathfrak{A}$ is adjointable and a TRO-morphism iff there is an element $w \in E$ such that $ww* \in \mathfrak{A}$ is a central projection and*

$$\varphi(e) = \langle e, w\rangle$$

for all $e \in E$.

Proof: By the above results, there are elements $a \in E$ and $b \in \mathcal{M}(\mathfrak{A})$ so that $b = b^* = b^2$ is central, and

$$b\langle x, y\rangle = \langle \varphi(x), \varphi(y)\rangle = \langle a, x\rangle\langle y, a\rangle = \langle \{a, x, y\}, a\rangle$$

for all $x, y \in E$. Using (two-sided, approximate) units in $EE^* = E^*E = \mathfrak{A}$ we find that $aa^* = b$, and the result follows. □

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