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A STEP TOWARD AN ASYMPTOTIC EXPANSION FOR THE CRAMÉR-VON MISES STATISTIC

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1. INTRODUCTION

The present note is a continuation of [1]. The notation used there will be kept here. Let us first review these notations. $U_1, \ldots, U_n$ denote independent r.v.-s uniformly distributed on $[0,1]$, and $F_n(t)$ the empirical distribution function of this sample. $\omega_n^2 = n\int_0^1 (F_n(t)-t)^2 dt$ is the Cramér-von Mises statistic, and $V_n(x) = P(\omega_n^2 < x)$ is its distribution function, while $f_n(t)$ denotes the characteristic function of ω_n^2. Let $V(x)$ be the (limiting) distribution function of the square integral of the Brownian bridge process. As a starting point in [1], the first author has given a complete asymptotic expansion for the Laplace-Stieltjes transform of $V_n(x)-V(x)$ in powers of $\frac{1}{n}$, and then tried to invert this expansion, without reaching the final goal. The last result (Theorem 3) of [1] reads as follows: *For a natural number* s *and positive number* ε, (1.1) $$V_n(x)-V(x) = \sum_{k=1}^{[\frac{s}{2}]} (\frac{1}{n})^k \psi_k(x)+O(n^{-\frac{1}{2}(s+1)+\varepsilon})+ + B_n^*(s,\varepsilon)$$

where the coefficient functions ψ_k *are completely specified by terms of expectations of certain functionals of the Wiener process and by the derivatives of* $V(x)$ *and*

(1.2) $$B_n^*(s,\varepsilon) = O(\int_{T_n} |\frac{f_n(t)}{t}|dt),$$

where

$$T_n = T_n(s,\varepsilon) = \{t: n^{\frac{1}{\varepsilon}(s+2)(s+4)} \leq |t| \leq \leq n^{-\varepsilon+\frac{1}{2}(s+1)}\}.$$

Thus, in order to prove the asymptotic expansion in question, it remained to prove that $B_n^*(s,\varepsilon) = O(n^{-(s+1)/2+\varepsilon})$. Unfortunately, we still cannot estimate B_n^* on this desirable way. All we can do now is a first step in this estimation procedure, the result of which will be another form of the remainder term in (1.1). This new form (derived in Sec. 2) lends itself for further analysis better than B_n^* of (1.2), and results from the fact that $V_n(x)$ is (exactly) entier $\frac{n}{2}$ times continuously differentiable. The latter fact, in turn, is a consequence of a recurrent formula for the n-dimensional volume of the intersection of an n-dimensional simplex and an n-dimensional ball. This geometric formula (proved in Sec.3) is of independent interest and provides a good hope to compile tables of exact distribution and percentage points for $V_n(x)$ and for the distribution functions of similar statistics. Some notes concerning this is contained in Sec.4.

2. THE OTHER FORM OF $B_n^*(s,\varepsilon)$

Denote the ordered sample by $U_1^{(n)},\ldots,U_n^{(n)}$. A simple integration gives the well-known alternative form of our statistic

$$(2.1)\qquad \omega_n^2 = \sum_{k=1}^{n} \left(U_k^{(n)} - \frac{2k-1}{n}\right)^2 + \frac{1}{12n}.$$

From here it follows that $\frac{1}{12n} \le \omega_n^2 \le \frac{n}{3}$, hence $V_n(\frac{1}{12n}) = 0$, $V_n(\frac{n}{3}) = 1$. Let S_n denote the simplex $\{(x_1,\ldots,x_n): 0 \le x_1 \le \ldots \le x_n \le 1\}$ in the unit cube of the n-dimensional real coordinate-space R^n. Put $c_n = (\frac{1}{2n}, \frac{3}{2n},\ldots,\frac{2n-1}{2n})$. Let $B_n(c_n,\rho(x))$ denote the n-dimensional ball with center c_n and radius $\rho(x)$. From (2.1) follows that

$$V_n(x) = n! \int\!\cdots\!\int_{S_n \cap B_n(c_n,\rho(x))} dx_1 \ldots dx_n,$$

where $\rho(x) = \sqrt{(x-\frac{1}{12n})^+}$. Here, for a real number y, $(y)^+ = \max(0,y)$. If $\mathrm{vol}_n[\cdot]$ stands for the n-dimensional Lebesgue measure, then this means that

$$(2.2)\qquad V_n(x) = n!\,\mathrm{vol}_n[S_n \cap B_n(c_n,\rho(x))].$$

$c_n \in \mathrm{interior}(S_n)$, and the distance of c_n from two faces of S_n is equally $\frac{1}{2n}$, and from the other $n-1$ faces, equally $\frac{1}{\sqrt{2}n}$. Thus, for $\frac{1}{12n} < x < \frac{n+3}{12n^2}$, $V_n(x) = n!\,\mathrm{vol}_n[B_n(c_n,\rho(x))]$, and, if $\frac{n+3}{12n^2} < x \le \frac{n+6}{12n^2}$, then, to get $V_n(x)/n!$, we have to subtract the volume of two ball-hats from the volume of the ball. For larger x the situation becomes extremly complicated, this is why we do not know exactly $V_n(x)$. If $x \geq \frac{n}{3}$, then $S_n \subseteq B_n(c_n, \rho(x))$, and $\mathrm{vol}_n[S_n] = \frac{1}{n!}$. At the present stage we are interested only in smoothness properties of $V_n(x)$. It is quite clear that $V_n(x)$ is piecewise analytic on the whole line. We can have trouble only at $x = \frac{n+3}{12n^2}$, $\frac{n+6}{12n^2}$, and at all further x's, for which the ball knocks against the $(n-2)$-, $(n-3)$-, ..., 1-dimensional boundary of the simplex. However, in all these exceptional points the function behaves quite well, since, from (2.2) and the Corollary to Lemma 10 in Section 3 it follows

LEMMA 9. *$V_n(x)$ is everywhere $[\frac{n}{2}]$ times continuously differentiable.*

A look at the exact results in case of $n=1,2,3$ shows that this result is sharp.

Now put $\delta=\delta(s,\varepsilon)=\frac{1}{\varepsilon}(s+2)(s+4)$, and let $a=a(s,\varepsilon)$ be the smallest integer for which $\frac{1}{\delta}(\frac{s+1}{2}-\varepsilon)+1 \leq a$. Let n be larger than $2a$, and set $V_n^{(k)}(x)=\frac{d^k}{dx^k} V_n(x)$. Then, integrating by parts $(a-1)$ times,

$$f_n(t) = \int_{1/12n}^{n/3} e^{itx} V_n^{(1)}(x)dx =$$

$$= \frac{1}{(it)^{a-1}} \int_{1/12n}^{n/3} e^{itx} V_n^{(a)}(x)dx,$$

whence

$$|f_n(t)| \leq \frac{1}{|t|^{a-1}} \int_{1/12n}^{n/3} |V_n^{(a)}(x)|dx = \frac{1}{|t|^{a-1}} Q_n.$$

Thus

$$\int_{T_n} \left|\frac{f_n(t)}{t}\right| dt \le Q_n \int_{n^\delta}^{\infty} \frac{dt}{t^a} = \frac{2}{a-1} Q_n \frac{1}{n^{\delta(a-1)}} \le$$
$$\le \frac{2}{a-1} Q_n n^{-\frac{1}{2}(s+1)+\varepsilon},$$

that is

$$B_n^* = B_n^*(s,\varepsilon) = Q_n O(n^{-\frac{1}{2}(s+1)+\varepsilon}).$$

In this way we have the following variant of Theorem 3 in [1]:

THEOREM 3'. *For any natural number* s *and positive number* ε

$$V_n(x)-V(x) = \sum_{k=1}^{[s/2]} \left(\frac{1}{n}\right)^k \psi_k(x) +$$
$$+ O(n^{-\frac{1}{2}(s+1)+\varepsilon})\{1+\int_{1/12n}^{n/3} |V_n^{(a)}(x)|dx\},$$

where a *depends only on* s *and* ε.

Thus, to prove the Conjecture in [1], i.e. the complete asymptotic expansion in question, it would be enough to show that, for an arbitrary (but fixed) natural number a, the sequence $Q_n = \int_{1/12n}^{n/3} |V_n^{(a)}(x)|dx$ is bounded. The first reasonable such a is $a=49$. The boundedness of the corresponding Q_n would prove a one-term expansion, and, in particular, the $\frac{1}{n}$ rate of convergence. Quite certainly, there is no difference between 49 and a general a if we want to solve the problem, and this estimation question still seems to be not quite easy.

3. THE RECURRENT FORMULA

Let $\mathrm{dist}_k(\cdot,\cdot)$ denote the Euclidean distance in R^k. A function $S:R^m \to \{\text{subsets of } R^n\}$ is said to be upper semicontinuous if for any $\varepsilon > 0$ there exist a $\delta > 0$ such that for $x,y\in R^m$ we have $S(y)\subseteq\{v\in R^n:\mathrm{dist}_n(v,S(x)) < \varepsilon\}$ whenever $\mathrm{dist}_m(x,y) < \delta$. Also, it is said to be concave if for each $0 \le \alpha \le 1$ and $x,y\in R^m$, $\alpha S(x)+(1-\alpha)S(y)\subseteq S(\alpha x+(1-\alpha)y)$.

Let $o = (0,\ldots,0)$ denote the origin of R^n and we write simply B^n for the closed unit ball in R^n centered in o. For $c\in R^n$ and a positive number ρ

$$B_n(c,\rho) = c+\rho B^n$$

is then the ball with center c and radius ρ. For a real ξ, $f'(\xi)$ denotes the first derivative of a function f and for $t,u\in R^n$, $\langle t,u\rangle$ stands for their inner product.

The following result (with $n+1$ replaced by the number of the $(n-1)$-dimensional faces) holds true if T is any (not necessarily bounded) convex polyhedron, but for our purpose a simplex suffices.

LEMMA 10. *If T is any simplex in R^n, and $\Lambda_n(\rho)=\mathrm{vol}_n[T\cap B_n(c,\rho)]$, then one can find simplices $T_1,\ldots,T_{n+1}$ in R^{n-1} and constants $\beta_n,\alpha_1,\ldots,\alpha_{n+1}$ so that with the functions $\Lambda_{n-1,i}(\rho)=\mathrm{vol}_{n-1}[T_i\cap\rho B^{n-1}]$ we have*

$$\Lambda_n(\rho) = \rho^n[\beta^n-\sum_{i=1}^{n+1}\alpha_i\int_0^\rho\frac{1}{\xi^{n+1}}\Lambda_{n-1,i}(\sqrt{(\xi^2-\alpha_i^2)^+})d\xi]. \tag{3.1}$$

Here the value of $\Lambda_{n-1,i}(\rho)$ equals to the $(n-1)$-dimensional volume of the intersection between the i-th face of T and the $(n-1)$-dimensional ball of radius ρ centered at the projection c_i of the center c of the original n-dimensional ball on the supporting $(n-1)$-dimensional hyperplane of the i-th face. Furthermore,

$$\alpha_i = \begin{cases} \mathrm{dist}_n(c,c_i), & \text{if } c_i-c \text{ is a non-negative multiple of } u_i \\ -\mathrm{dist}_n(c,c_i), & \text{if } c_i-c \text{ is a non-negative multiple of } u_i, \end{cases}$$

where u_i is the normal vector of the i-th face of T, pointing outward from T. At last,

$$\beta_n = \begin{cases} 0, & \text{if } c \notin T \\ \frac{1}{n}\tau_n & \text{if } c \in T, \end{cases}$$

where τ_n is the n-dimensional spatial angle of the cone formed by the rays issued from c, having an intersection with T of positive length. In particular, if c is an inner point of T then $\beta_n = \mathrm{vol}_n B^n = (\sqrt{\pi})^n / \Gamma(\frac{n}{2}+1)$.

PROOF. Without any loss of generality we suppose that the center of the ball is the origin, i.e. $c=o$, $B_n(c,\rho)=\rho B^n$.

Let $u \in R^n$ be a unit vector and $K \subseteq R^n$ be a compact convex set. Assume that the two supporting hyperplanes of K which are ortogonal to u have intersection figures with K of less than $n-1$ dimension. Then it is easily seen that the function $f(\xi)=\mathrm{vol}_n[K \cap \{t \in R^n : \langle t,u \rangle \leq \xi\}]$ is differentiable and, for all ξ, $f'(\xi) =$

$= \mathrm{vol}_{n-1}[K\cap\{t:\langle t,u\rangle = \xi\}]$. From here it follows directly that if $u_1,\dots,u_m \in R^n$ are unit vectors with $u_j \neq \pm u_k$ $(j\neq k)$ such that the intersection figures of K with those of its supporting hyperplanes lying orthogonally to some of the u_i's $(i=1,\dots,m)$ are of less then $n-1$ dimensions, then for the function $F:R^m \to R$ defined by

$$(3.2) \qquad F(\xi_1,\dots,\xi_m) = \mathrm{vol}_n[K\cap A(\xi_1)\cap\dots\cap A(\xi_n)],$$

where $A(\xi_i)=\{t:\langle t,u_i\rangle \leq \xi_i\}\subset R^n$ $(i=1,\dots,m)$, we have

$$(3.3) \qquad \begin{aligned} \frac{\partial F}{\partial \xi_i} &= \mathrm{vol}_{n-1}[K\cap A(\xi_1)\cap\dots\cap A(\xi_{i-1})\cap \\ &\cap\{t:\langle t,u_i\rangle=\xi_i\}\cap A(\xi_{i+1})\cap\dots\cap A(\xi_n)] \quad (i=1,\dots,m). \end{aligned}$$

Now we claim that the partial derivatives $\frac{\partial F}{\partial \xi_i}$ in (3.3) are continuous on the whole R^m. Indeed, with the notations $D_j(\xi_1,\dots,\xi_m)=A(\xi_j)$, $E_j(\xi_1,\dots,\xi_m)=$ $=\{t:\langle t,u_j\rangle=\xi_j\}$ and $K(\xi_1,\dots,\xi_m)=K$, the function $\frac{\partial F}{\partial \xi_i}$ can be considered by (3.3) as the $(n-1)$-dimensional Lebesgue measure of the intersection of the set-valued concave and upper semicontinuous functions $D_1(\cdot),\dots$ $\dots,D_{i-1}(\cdot)$, $D_{i+1},\dots,D_m(\cdot),E_i(\cdot)$ and $K(\cdot)$. Thus $\frac{\partial F}{\partial \xi_i}$ is the vol_{n-1} measure of some compact convex R^n-subset valued concave and upper semicontinuous function on R^m. Hence, by the well-known Brunn-Minkowski theorem (e.g. in [2]) one can write

$$\begin{aligned} \frac{\partial F}{\partial \xi_i} &= \mathrm{vol}_{n-1}[K(\xi_1,\dots,\xi_m)\cap E_i(\xi_1,\dots,\xi_m)\cap \\ &\bigcap_{j=1}^{m} D_j(\xi_1,\dots,\xi_m)] = \end{aligned}$$

$$= \begin{cases} G_i^{n-1}(\xi_1,\dots,\xi_m), & \text{if } (\xi_1,\dots,\xi_m)\in\Omega_i, \\ 0 & \text{elsewhere in } R^m, \end{cases}$$

where $\Omega_i=\{x\in R^m: K(x)\cap E_i(x)\cap \bigcap_{j=1}^{m} D_j(x)\neq\phi\}$ and G_i is some suitable continuous concave function on the compact convex domain Ω_i. Now our assumption on the vectors u_i clearly ensures that G_i vanishes on the boundary of Ω_i which, in turn, shows the continuity of the functions $\frac{\partial F}{\partial \xi_i}$. Consequently, the function F in (3.2) is totally differentiable at any point of R^m.

In what follows, let $K=B^n$, $m=n+1$, and let the vectors $u_1,\dots,u_n$ be the normal vectors of the faces of the given simplex $T\in R^n$, directed outward from T. Denote by T_i^* the face of T orthogonal to u_i, and by E_i^* the supporting hyperplane of T_i^*. Let o_i^* be the (orthogonal) projection of o (the origin of R^n) to E_i^*, and define α_i by $\alpha_i=\langle u_i,o_i^*\rangle$ $(i=1,\dots,n+1)$. By (3.3) we have for $\rho > 0$ that

$$\frac{d}{d\rho}\left(\frac{\Lambda_n(\rho)}{\rho^n}\right) = \frac{d}{d\rho}\left(\frac{1}{\rho^n}\operatorname{vol}_n[T\cap\rho B^n]\right) =$$

$$= \frac{d}{d\rho}\left(\operatorname{vol}_n[\tfrac{1}{\rho}T\cap B^n]\right) =$$

$$= \frac{d}{d\rho}F\left(\frac{\alpha_1}{\rho},\dots,\frac{\alpha_{n+1}}{\rho}\right) =$$

$$= -\sum_{i=1}^{n+1}\frac{\alpha_i}{\rho^2}\operatorname{vol}_{n-1}[B^n\cap\{t:\langle t,u_i\rangle=\frac{\alpha_i}{\rho}\}\cap$$

$$\cap\bigcap_{j=1}^{n+1}\{t:\langle t,u_j\rangle\le\frac{\alpha_j}{\rho}\}] =$$

$$= -\sum_{i=1}^{n+1}\frac{\alpha_i}{\rho^{n+1}}\operatorname{vol}_{n-1}[\rho B^n\cap\{t:\langle t,u_i\rangle = \alpha_i\}\cap$$

$$\bigcap \bigcap_{j=1}^{n+1} \{t : \langle t, u_j \rangle \le \alpha_j\}] = -\sum_{i=1}^{n+1} \frac{\alpha_i}{\rho^{n+1}} \mathrm{vol}_{n-1}[T_i^* \cap \rho B^n].$$

Observe now that $T_i^* \cap \rho B^n$ here can be written in the form $T_i^* \cap (E_i^* \cap \rho B^n)$ where $E_i^* \cap \rho B^n$ is a ball of center o_i^* and radius $\sqrt{\rho^2 - \alpha_i^2}$ in the $(n-1)$-dimensional affin subspace E_i^* if $\rho \ge |\alpha_i| = \mathrm{dist}_n(o, o_i^*)$, and is Φ for $0 \le \rho < |\alpha_i|$. Hence, considering an isometry $H_i : E_i \to R^{n-1}$, with $H_i(o_i^*) = o \in R^{n-1}$ $(i=1,\dots,n+1)$ (otherwise arbitrary), for the choice $T_i = H_i(T_i^*)$ we have

$$\mathrm{vol}_{n-1}[T_i^* \cap \rho B^n] = \mathrm{vol}_{n-1}[H_i(T_i^* \cap \rho B^n)] =$$

$$= \mathrm{vol}_{n-1}[T_i \cap \sqrt{(\rho^2 - \alpha_i^2)^+} B^{n-1}] \qquad (i=1,2,\dots,n+1).$$

Hence, the case $c=o$ of formula (3.1) follows with

$$\beta_n = \lim_{\rho \searrow 0} (\Lambda_n(\rho)/\rho^n) = \lim_{\rho \searrow 0} \mathrm{vol}_n[T \cap \rho B^n],$$

i.e.,

$$\beta_n = \begin{cases} 0, & \text{if } o \notin T, \\ \frac{1}{n} \tau_n, & \text{if } o \in T, \end{cases}$$

where τ_n is the n-dimensional spatial angle of the cone $(0,\infty) \times T$. But then the lemma also follows in the stated generality.

Viewing now the volume as a function of the radius we immediately have the following

COROLLARY. *$\Lambda_n(\rho)$ is $d(n) = [\frac{n}{2}]$ times continuously differentiable.*

PROOF. For $n=1$, $d(n)=0$, i.e. the claim is only the continuity of $\Lambda_1(\rho)$, and this is true. Suppose that for some $k \geq 2$ the assertion holds for $k-1$. This means that the functions $\Lambda_{k-1,i}^{(d(k-1))}$ $(i=1,\dots,k+1)$, the derivatives of order $d(k-1)$ of $\Lambda_{k-1,i}$, exist and continuous. It follows then from (3.1) that Λ_k is $d(k-1)+1$ $(\geq d(k))$ times continuously differentiable over the set $(0,\infty)\setminus\{\alpha_1,\dots,\alpha_{k+1}\}$. For $\xi \leq \alpha_i$, $\Lambda_{k-1,i}(\sqrt{(\xi^2-\alpha_i^2)^+}) \equiv 0$, and to prove that Λ_k is $d(k)$ times continuously differentiable also in the points $\alpha_1,\dots,\alpha_{k+1}$, it is enough to show that

$$\text{(3.4)} \qquad \lim_{\xi\searrow\alpha_i} \Lambda_{k-1,i}^{(d(k)-1)}(\sqrt{\xi^2-\alpha_i^2}) = 0 \qquad (i=1,\dots,k+1).$$

But, by (3.1) again, $\Lambda_{k-1,i}(\tau) = c_{k-1,i}\tau^{k-1}$ with some constant $c_{k-1,i}$, if τ is small enough. Hence, to show (3.4) is the same thing as checking

$$\text{(3.5)} \qquad \lim_{\xi\searrow\alpha} \frac{d^\nu}{d\xi^\nu}(\xi^2-\alpha^2)^{\frac{k-1}{2}} = 0 \qquad (\nu=d(k)-1=\left[\tfrac{k}{2}\right]-1).$$

But

$$\frac{d^\nu}{d\xi^\nu}(\xi^2-\alpha^2)^{\frac{k-1}{2}} = \sum_{j=1}^{\left[\frac{k}{2}\right]-1} p_j(\xi)(\xi^2-\alpha^2)^{\frac{k-1}{2}-j},$$

where the p_j'-s are some polynomials, whence (3.5) follows. By induction the Corollary is proved.

Taking the simplex S_n of Section 2 as an example we see that the Corollary cannot, in general, be improved. On the other hand, we saw that we have troubles with differentiability in those ρ's only, where the ball knocks against different dimensional $(n-1,n-2,\dots,n)$ faces of the boundary of the simplex. In fact, it is easy to prove that $\Lambda_n(\rho)$ is piecewise analytic on

$(0,\infty)$.

4. COMMENTS ON $V_n(x)$

Because of (2.2) and Lemma 10, $V_n(x)$ has the following form

$$V_n(x) = n!\Lambda_n(\rho(x)) =$$

(4.1)
$$= n!\rho^n(x)\Big\{\beta_n - \sum_{i=1}^{n+1} \alpha_i \int_0^{\rho(x)} \frac{1}{\xi^{n+1}}\Lambda_{n-1,i}(\sqrt{(\xi^2-\alpha_i^2)^+})\,d\xi\Big\},$$

where $\rho(x)=\sqrt{(x-1/12n)^+}$; $\alpha_1=\alpha_2=\frac{1}{2n}$, $\alpha_3=\ldots=\alpha_{n+1}=$ $=\frac{1}{\sqrt{2n}}$; $\Lambda_{n-1,1}=\Lambda_{n-1,2}$, $\Lambda_{n-1,3}=\ldots=\Lambda_{n-1,n+1}$ and $\beta_n =$ $=\pi^{n/2}/\Gamma(\frac{n}{2}+1)$, since c_n is an inner point of S_n. It is easily seen that S_n do not have obtuse angles. It follows then, that the projections of c_n will be inner points of the $(n-1)$-dimensional faces, the projections of these projections will be inner points of the $(n-2)$--dimensional faces of the $(n-1)$-dimensional faces, etc. So, when evaluating $\Lambda_{n-1,i}(\sqrt{(\xi^2-\alpha_i^2)^+})$ by (4.1), the corresponding constants $\beta_{n-1,i}$ will again be the volume of the $(n-1)$-dimensional unit ball $i=1,\ldots,n+1$, and all the corresponding constants α_{ij} $(j=1,\ldots,n)$ will be positive, and this phenomenon is persistent with the decrease of dimension. In this sense the recursion in (4.1) is "homogeneous". But one also notes that in the second step it will not be true that two of the α's is the same and the rest is again the same (i.e. the ball reaches two faces at the same time, and, a bit later, it reaches the other faces again at the same time). This "regularity" disappears after the step, as seen starting out from three dimensions.

Much work has been done to compile tables of percentage points for ω_n^2 and similar statistics, in particular by STEPHENS. A survey and comparison can be found in KNOTT [3]. In fact, Knott's results are proved to be the most accurate. All these results, tables, are based on some kind of approximation of $V_n(x)$. *In principle*, formula (4.1) gives the possibility of the exact tabulation. Lemma 10 is also applicable for other similar statistics, e.g., for the M_n^2 statistic of Durbin and Knott. $n=20$ seems to be accessible on a computer. Unfortunately, our computer facilities here are not adequate at present to do this work.

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