ON C_0-SEMIGROUPS OF HOLOMORHIC ISOMETRIES IN SPIN FACTORS

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Based on JB*-triple theory, we refine earlier results on the structure of strongly continuous one-parameter semigroups (C0-SGR) of holomorphic Carathéodory isometries of the unit ball in infinite dimensional spin factors, resulting in finite algebraic formulas in terms of joint boundary fixed points and Möbius charts and inverse Laplace transpormation.

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1. INTRODUCTION

This paper is intended to complete a series of our earlier works [14-18] with the aim of describing the C0-semigroups (C0-SGR) of Carathéodory isometries of the unit ball in a generic *infinite dimensional reflexive JB*-triple*. Recall [10,11,12] that JB*-triples are complex Banach spaces with holomorphically symmeric unit ball, and actually the infinite dimensional reflexive ones among them are finite ℓ^∞-direct sums of Cartan factors of types $1_{\rm refl}$ and 4 i.e. isometric copies of $\mathcal{L}(\mathbf{H},\mathbf{K})$ spaces with Hilbert spaces $\mathbf{H},\mathbf{K}$ such that $\dim(\mathbf{H})=\infty>\dim(\mathbf{K})$ resp. infinite dimensional spin factors. Recall also [5] that the Carathéodory metric $d_\mathbf{B}$ on the unit ball $\mathbf{B}$ of a Banach space $\mathbf{E}$ is the unique holomorphy invariant distance on $\mathbf{B}$ which coincides with the norm distance in first order around the origin (in the sense that $d_\mathbf{B}(\mathbf{0},\mathbf{x})=\|\mathbf{x}\|+o(\|\mathbf{x}\|)$). It is also well known [7] that $d_\mathbf{B}$-isometries are factor preserving. Hence the task of describing a C0-SGR of $d_\mathbf{B}$-isometries in our setting reduces just to the cases of Cartan factors of the types $1_{\rm refl},4$ and this is done already for the non-spin cases in [18]. Thus it only remains to focus our attention to an arbitrarily fixed spin factor associated with the complexification $\mathbf{H}=\mathbf{H}_0\oplus i\mathbf{H}_0$ of an infinite dimensional *real* Hilbert space $\mathbf{H}_0$ with scalar product $\langle\cdot|\cdot\rangle$ extended naturally as $\langle x_1+ix_2|y_1+iy_2\rangle=\big[\langle x_1|y_1\rangle+\langle x_2|y_2\rangle\big]+i\big[\langle x_1|y_2\rangle-\langle x_2|y_1\rangle\big]$ $(x_1,x_2,y_1,y_2\in\mathbf{H}_0)$

to $\mathbf{H}$ and the canonical conjugation $\overline{x+iy} = x - iy$ $(x, y \in \mathbf{H}_0)$. Henceforth we fix the notation $\mathbf{E} := \mathcal{S}(\mathbf{H}, \bar{\cdot})$ for this spin factor. Thus $\mathbf{E}$ is $\mathbf{H}$ (as complex vector space) equipped with the *spin norm*

$$\|x+iy\| = \Big[\big[\langle x\rangle^2 + \langle y\rangle^2\big] + 2\big[\langle x\rangle^2\langle y\rangle^2 - \langle x|y\rangle^2\big]^{1/2}\Big]^{1/2} \quad (x, y \in \mathbf{H}_0)$$

in terms of the standard abbreviation $\langle z\rangle^2 := \langle z|z\rangle$ giving rise to the square of the Hilbert norm on $\mathbf{H}$, and with the *JB*-triple product* [12]

$$\{xay\} = \langle x|a\rangle y + \langle y|a\rangle x - \langle x|\overline{y}\rangle = \langle x|a\rangle y + \langle y|a\rangle x - \langle y|\overline{x}\rangle\overline{a}. \tag{1.1}$$

In particular, for the open unit ball of $\mathbf{E}$ we can write

$$\mathbf{B} = \Big\{z \in \mathbf{H} : \langle z\rangle^2 < \frac{1}{2}\big(1 + \big|\langle z|\overline{z}\rangle\big|^2 < 1\Big\},$$

and the family of spin *tripotents* (idempotents of the triple product i.e. elements satisfying the identity $e = \{e\}^3 = \{eee\}$) has the form

$$\begin{aligned}&\mathrm{Tri}(\mathbf{E}) = \textstyle\bigcup_{k=0}^2 \mathbb{T}\cdot \mathrm{Tri}_k(\mathbf{E}) \quad \text{where}\\ &\mathrm{Tri}_0(\mathbf{E}) := \{0\}, \quad \mathrm{Tri}_1(\mathbf{E}) := \big\{e \in \mathbf{H}_0 : \langle e\rangle^2 = 1\big\},\\ &\mathrm{Tri}_2(\mathbf{E}) := \big\{u + iv :\ u, v \in \mathbf{H}_0,\ \langle u\rangle^2 = \langle v\rangle^2 = 1/4,\ \langle u|v\rangle = 0\big\}\end{aligned}$$

where $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ is the standard notation for the unit circle of the complex plane $\mathbb{C}$. To manipulate the triple product, we shall also use its binary versions i.e. the *linear-* resp. *quadratic* operator *representations* $L(x, a) : y \mapsto \{xay\}$ resp. $Q(x, y) : a \mapsto \{xay\}$with the usual abbreviations $L(a) = L(a, a)$ resp. $Q(x) = Q(x, x)$. It is crucial that any holomorphic $d_{\mathbf{B}}$-isometry Φ (even in a generic reflexive JB*-triple) admits a finite closed formula in terms of the triple product as the composition

$$\Phi = M_a \circ U, \quad M_a(x) = a + B(a)^{1/2}\big[1 + L(x, a)\big]^{-1}x \tag{1.2}$$

of a Kaup type *Möbius shift* M_a and a linear $\{\ldots\}$-homomorphism U [1,2,17] where $B(a) := 1 + 2L(a) + Q(a)^2$ is the *Bergman operator* associated with the triple product [12]. In spin factors in (1.2) we can write [21]

$$Uz = \kappa U_0 z = \kappa[U_0 x + iU_0 y], \quad (z = x + iy,\ x, y \in \mathbf{H}_0)$$

with a suitable constant $\kappa \in \mathbb{T}$ and a real-linear isometry U_0 of $\mathbf{H}_0$. In particular, the holomorphic automorphisms of $\mathbf{B}$, called also *Möbius transformations*, are precisely the transformations (1.2) where U is a surjective $\mathbf{E}$-isometry. In the sequel we shall rely upon Vesentini's *linear spin representation* [21] developed from Hierzebruch's finite dimensional considerations [9] resp. Harris' description [8] of holomorphic JC*-automorhisms: Given any operator matrix

$$G=\begin{bmatrix} M & B\\ C^{\mathrm T} & E\end{bmatrix}=\begin{bmatrix} M & b_1 & b_2\\ c_1^{\mathrm T} & E_{11} & E_{12}\\ c_2^{\mathrm T} & E_{21} & E_{22}\end{bmatrix},\quad \begin{array}{l} M\in\mathcal L(\mathbf H_0),\\ E_{k\ell}\in\mathbb R,\\ b_k,c_\ell\in\mathbf H_0\end{array}\tag{1.3}$$

such that[1] $G^T\mathrm{diag}\big(\mathrm{Id}_{\mathbf H_0},-\mathrm{Id}_{\mathbb R^2}\big)G=\mathrm{diag}\big(\mathrm{Id}_{\mathbf H_0},-\mathrm{Id}_{\mathbb R^2}\big)$, $\det(E)>0$ that is

$$\begin{array}{ll} M^{\mathrm T}M=1+C^{\mathrm T}C, & M^{\mathrm T}B=CE,\\ E^{\mathrm T}E=1+B^{\mathrm T}B, & E_{11}E_{22}>E_{12}E_{21},\end{array}\tag{1.4}$$

the mapping

$$\begin{aligned} \Phi_G(z) :=& F_G(z)/\varphi_G(z)\qquad \text{with}\\ F_G(z) :=& (b_1-ib_2)+2Mz+(z^{\mathrm T}z)(b_1+ib_2),\\ \varphi_G(z) :=& (E_{11}+E_{22}-iE_{12}+iE_{21})+2(c_1+ic_2)^{\mathrm T}z+\\ &+(E_{11}-E_{22}+iE_{12}+iE_{21})z^{\mathrm T}z\end{aligned}\tag{1.5}$$

is a holomorphic continuation of a Carathéodory isometry $\Phi\in\mathrm{Iso}(d_{\mathbf B})$ to some neighborhood of the closed unit ball $\overline{\mathbf B}$ and, conversely, any transformation $\Phi\in\mathrm{Iso}(d_{\mathbf B})$ can be written in the form $\Phi=\Phi_G|\mathbf B$ where the matrix G is determined up to a constant factor $\lambda\in\mathbb R\setminus\{0\}$.

2. MAIN RESULTS

Henceforth let $\mathbf\Phi:=[\Phi^t : t\in\mathbb R_+]$ denote an arbitrarily fixed C0-SGR in $\mathrm{Iso}(d_{\mathbf B})$. According to the the C0-property, we have

$$\Phi^0=\mathrm{Id}_{\mathbf B},\quad \Phi^{t+s}=\Phi^t\circ\Phi^s\ \ (t,s\ge 0),\quad t\mapsto\Phi^t(z)\text{ is continuous for any } z\in\mathbf B.$$

We shall write Φ' for the *infinitesimal generator* of $\mathbf\Phi$ that is

$$\Phi'(z):=\frac{d}{dt}\Big|_{t=0}^{+}\Phi^t(z),\quad \mathrm{dom}(\Phi'):=\big\{z\in\mathbf B:\lim_{t\to 0+}t^{-1}\big[\Phi^t(z)-z\big]\text{ exists}\big\}.$$

[1] As usually, we write $\cdot^{\mathrm T}$ for transposition and identify G with the operator $\Big[(x,\xi_1,\xi_2)^{\mathrm T}\mapsto\big(Mx+\xi_1b_1+\xi_2b_2,\langle x|c_1\rangle+\xi_1E_{11}+\xi_2E_{12},\langle x|c_2\rangle+\xi_1E_{21}+\xi_2E_{22}\big)^{\mathrm T}\Big]$ acting on columns.

In [21, Section] E. Vesentini determined the infinitesimal generator of a C0-SGR $[G^t : t \in \mathbb{R}_+]$ realizing a (1.5)-representation of $\mathbf{\Phi}$ under the hypothesis of differentiable 0-orbit $t \mapsto \Phi^t(0)$, and he outlined a method how to retrieve the terms G^t in terms of the infinitesimal generator G' without achieving closed formulas. Since the only ambiguity in the representation (1.5) is the equivalence $\Phi_G = \Phi_H \iff H = \pm G$, it is a harmless task to establish a C0-SGR $[G^t : t \in \mathbb{R}_+]$ in $\mathcal{L}(\mathbf{H}_0 \oplus \mathbb{R}^2)$ such that $\Phi^t = \Phi_{G^t}$ $(t \in \mathbb{R}_+)$. Hence Vesentini's considerations on C0-SGR of holomorphic spin isometries by means the linear representation (1.5) require no further adjustment arguments in contrast with his works on Cartan factors of type $1_{\rm refl}$. Also the technical assumption $0 \in \mathrm{dom}(\Phi')$ is harmless from the view point of finding closed formulas (observed in [19] already): $\mathrm{dom}(\Phi')$ is a dense subset in $\mathbf{B}$ and hence, by taking any point $a \in \mathrm{dom}(\Phi')$, we can pass to the C0-SGR $[\widetilde{\Phi}^t : t \in \mathbb{R}_+]$, $\widetilde{\Phi}^t := M_{-a} \circ \Phi^t \circ M_a$ with $0 \in \mathrm{dom}(\widetilde{\Phi}') = M_{-a}\big(\mathrm{dom}(\Phi')\big)$ since $M_a^{-1} = M_{-a}$.

We present a JB*-theoretical approach based on our previous works [17,18] to the structure of $\mathbf{\Phi}$ with the following improvements of earlier results.

THEOREM 2.1. (i) *There exists a Möbius transformation* Θ *defined on some neighborhood of the closed unit ball* $\overline{\mathbf{B}}$ *such that the infinitesimal generator of the C0-SGR* $\mathbf{\Psi} = [\Psi^t : t \in \mathbb{R}_+]$ *with* $\Psi^t = \Theta \circ \Phi^t \circ \Theta^{-1} \in \mathrm{Iso}(d_{\mathbf{B}})$ *is of Kaup's type, i.e.*

$$\Psi'(z) = a - \{zaz\} + U'z \qquad \big(z \in \mathbf{B} \cap \mathbf{J}\big)$$

where $a \in \mathbf{E}$, U' *is the infinitesimal generator of some C0-SGR* $[U^t : t \in \mathbb{R}_+]$ *of linear* $\mathbf{E}$*-isometries,* $\mathbf{J} = \mathrm{dom}(U')$ *is a dense subtriple of* $\mathbf{E}$ *with respect to the triple product* $\{\ldots\}$. *Furthermore there exits a tripotent* $e \in \mathrm{Tri}_k(\mathbf{E}) \cap \mathbf{J}$ *for some* $k \in \{0, 1, 2\}$ *such that*

$$a - \{eae\} + U'e = 0.$$

(ii) *There exists a C0-SGR* $[H^t : t \in \mathbb{R}_+]$ *of* $\mathbf{H}_0 \oplus \mathbb{R} \oplus \mathbb{R}$ *type real operator matrices*[2] *providing a Hierzebruch-Vesentini representation* (1.5) $\Psi^t = \Phi_{H^t}$ $(t \in \mathbb{R}_+)$ *for the C0-SGR* $\mathbf{\Psi}$ *above whose infinitesimal generator has the form*

$$(2.2) \qquad H' = \begin{bmatrix} M & b_1 & b_2 \\ b_1^{\rm T} & 0 & -\varepsilon \\ b_2^{\rm T} & \varepsilon & 0 \end{bmatrix} \quad \text{with} \quad \begin{array}{l} M = -M^{\rm T} = U'\big|\mathbf{H}_0 \cap \mathrm{dom}(U'), \\ \varepsilon \in \mathbb{R},\ b_1 = \overline{a} + a,\ ib_2 = \overline{a} - a. \end{array}$$

[2] That is $H^t = \big[H^t_{ij}\big]_{i,j=1}^3$ with $H^t_{11} \in \mathcal{L}(\mathbf{H}_0)$, $H^t_{1j} \in \mathcal{L}(\mathbb{R}, \mathbf{H_0}) \simeq \mathbf{H}_0$, $H^t_{i,1} \in \mathcal{L}(\mathbf{H_0}, \mathbb{R}) \simeq \mathbf{H}_0$ and $H^t_{1+k,1+\ell} \in \mathcal{L}(\mathbb{R}) \simeq \mathbb{R}$ $(k, \ell = 1, 2)$. We identify $\mathbf{H}_0 \oplus \mathbb{R} \oplus \mathbb{R}$ with $\mathbf{H}_0 \oplus \mathbb{R}^2$ as usually.

In terms of H', the infinitesimal generator of $\mathbf{\Psi}$ has the form

$$\Psi'(x) = \left(\frac{1}{2}b_1 - \frac{i}{2}b_2\right) + Mx - \langle x|b_1 - ib_2\rangle x + \langle x|\overline{x}\rangle \left(\frac{1}{2}b_1 + \frac{i}{2}b_2\right).$$

REMARK 2.3. As an immediate consequence, the C0-SGR $\mathbf{\Psi}$ consists of *linear* $\mathbf{E}$*-isometries* if we have $e = 0$ above. That is if the members of $\mathbf{\Phi}$ admit a common fixed point within the open unit ball $\mathbf{B}$ (namely $\Theta^{-1}(0) \in \mathrm{Fix}(\mathbf{\Phi})$) then $\mathbf{\Phi}$ is Möbius equivalent to C0-SGR of linear E-isometries, thus $\Phi^t = \Theta \circ U^t \circ \Theta^{-1}$ for some C0-SGR $[U^t : t \in \mathbb{R}_+]$ of $\mathbf{E}$-isometries. This was known by Vesentini [19] already. In contrast, as we mentioned in [17], it is still an open problem whether every C0-SGR of holomorphic non-surjective Carathéodory isometries of the unit ball in a Banach space leaving the origin fixed consists of linear maps.

The ultimate goal of our series of our papers [14–18] is establishing algebraically closed formulas for C0-SGR of holomorphic Carathéodory isometries in JB*-triple in terms of the underlying triple product. If we rely upon linear represenations, this task involves naturally the application of the Bounded Perturbation Theorem [6] leading to nested convolutions.

THEOREM 2.4. (i) *In the non-linear cases $e \neq 0$ of Theorem 2.1, the C0-SGR $[H^t : t \in \mathbb{R}_+]$ is linearly equivalent to a C0-SGR $[G^t : t \in \mathbb{R}_+]$ of operator matrices of the type $\mathbb{R}^k \oplus \mathbf{H}_k \oplus \mathbb{R}^2$ where $\mathbf{H}_1 = \mathbf{H}_0 \ominus [\mathbb{R}\, e]$ if $e \in \mathrm{Tri}_1(\mathbf{E})$ resp. $\mathbf{H}_2 = \mathbf{H}_0 \ominus [(\mathbb{R}\,\mathrm{Re}(e)) \oplus (\mathbb{R}\,\mathrm{Im}(e))]$ if $e \in \mathrm{Tri}_2(\mathbf{E})$ whose generator is lower triangular perturbed with a unique non-zero superdiagonal entry. The only possibly unbounded entry of G' is located in the diagonal with the value $\mathrm{Proj}_{\mathbf{H}_k} U' \big| \mathbf{H}_k$, the remaining entries are simple algebraic expressions of the tripotent e and a parameter $\varepsilon \in \mathbb{R}$.*

(ii) *All the entries G^t_{ij} of G^t are convolution polynomials formed by $[t \mapsto U^t]$, the solution $[t \mapsto u(t)]$ of a Volterra equation $u = u * w + w$ with scalars or 2×2 matrices, exponential and trigonometric functions with coefficients in terms of the triple product, entries of U', the fixed point tripotent e and the parameter ε, respectively. The Laplace transforms $\mathcal{L}\{G^t_{ij}\}(s)$ are all rational fractions of the Laplace transforms $\mathcal{L}\{U^t\}(s)$ and $s \mapsto \mathcal{L}\{u(t)\}(s)$ with operator coefficients in Jordan triple expessions of U' and e.*

In Theorems 6.4 resp 6.9 we furnish all the details being sufficient to construct explicit finite formulas for the matrices G^t.

COROLLARY 2.5. (i) *There exists a C0-SGR* $[\widetilde{G}^t : t \in \mathbb{R}]$ *of the form* (1.3) *providing a Hierzebruch-Vesentini representation* $\Phi^t = \Phi_{\widetilde{G}^t}$ *whose entries are convolution polynomials of the C0-SGR* $[U^t : t \in \mathbb{R}_+]$*, the solution* u *of the governing scalar of* 2×2*-matrix Volterra equation and special functions with paramers in Jordan terms of the generator.*

(ii) *The C0-SGR* $[\Phi^t : t \in \mathbb{R}_+]$ *admits a group dilation in the sense that there is a strongly continuous one-parameter group* $[\widehat{\Phi}^t : t \in \mathbb{R}]$ *of a Carathéodory isometries of the unit ball of a spin factor* $\widehat{\mathbf{E}}$ *containing* $\mathbf{E}$ *as a subtriple such that* $\Phi^t = \widehat{\Phi}^2 \big| \mathbf{B}$ $(t \in \mathbb{R}_0)$.

The content of Theorem 2.1 is covered by the results of Section 4. A simple version of Theorem 2.4 with an infinitesimal generator of the form (2.2) is available immediately from Proposition (3.6) with a governing 2×2-matrix function. Theorem 3.10 improves Vesentini's approaches with infinite dimensional Riccati equations considerably. The complete version along with Corollary 2.5(i) is covered by Sections 5-6 giving a deeper geometrical insight to the structure, in particular we find an essential reduction to a scalar valued governing function in the case if some extreme point is a common fixed point of the holomorphic extensions $\overline{\Phi^t}$ to the closed unit ball.

As for numerical aspects: the linear vector field (2.2) is a bounded perturbation of the diagonal and hence its integration can be done with a Dyson-Phillips series [6] of convolution polynomials with infinite dimensional rank 4 operator matrices but consisting of 2^n monomials in the n-th summand. In contrast, our governing functions are obtained by means of Dyson-Phillips series of at most 2×2-matrices.

Corollary 2.5(ii) is a direct consequence of the fact of a theorem due to Deddens [4] stating in particular that any C0-SGR of Hilbert space isometries can be emebedded into a C0-group of sujective isometries of a larger Hilbert space. The argument is the same as in [16 Sect.5].

3. TRIANGULAR SYSTEMS WITH PERTURBATUION

Let $\mathbf{E}_1, \ldots, \mathbf{E}_n$ denote Banach spaces and let $\big[T(t) : t \in \mathbb{R}_+\big]$ be a C0-SGR of lower triangular $n \times n$ type operator matrices with entries

$$T_{ij}(t) \in \mathcal{L}(\mathbf{E}_i, \mathbf{E}_j) = \big\{\text{bded. lin. op.-s } \mathbf{E}_j \to \mathbf{E}_i\big\}, \quad T_{ij}(t) = 0 \ (i < j)$$

such that its infinitesimal generator can be written in the matrix form

$$T' = \begin{bmatrix} U_1' & 0 & 0 & \dots & 0 & 0 \\ B_{21} & U_2' & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{n-1,1} & B_{n-1,2} & B_{n-1,3} & \dots & U_{n-1}' & 0 \\ B_{n,1} & B_{n,2} & B_{n,3} & \dots & B_{n,n-1} & U_n' \end{bmatrix}$$

where each diagonal entry U_i' is the generator of a C0-SGR $\big[U_i(t) : t \in \mathbb{R}_+\big]$ of bounded linear $\mathbf{E}_i$-operators and the subdiagonal entries B_{ij} $(i > j)$ are bounded linear operators $\mathbf{E}_j \to \mathbf{E}_i$.

Recall [16, Lemma 3.8] that in the case $n = 2$ we can write

$$T(t) = \begin{bmatrix} U_1(t) & 0 \\ [U_2(t)B_{21}] * [U_1(t)] & U_2(t) \end{bmatrix} \qquad (t \in \mathbb{R}_+)$$

in terms of the *convolution*

$$f(t) * g(t) := \int_{r=0}^{t} f(t-r) g(r)\, dr.$$

By induction on n, we infer that

$$T(t) = \mathrm{diag}\big(U_1(t), \dots, U_n(t)\big) + \mathrm{subdiag}\big(T(t)\big)$$

with the entries

$$(3.1)\quad T_{ij}(t) = \sum_{(i_0,i_1,\dots,i_k)\in\mathcal{I}_{ij}} [U_{i_0}(t)B_{i_0,i_1}]*[U_{i_1}(t)B_{i_1,i_2}]*\dots*[U_{i_{k-1}}(t)B_{i_{k-1},i_k}]*[U_{i_k}(t)]$$

where $\mathcal{I}_{ij}$ denotes the family

$$\mathcal{I}_{ij} := \bigcup_{k=1}^{i-j} \Big\{(i_0, \dots, i_k) : i = i_0 > i_1 > \dots > i_{k-1} > i_k = j\Big\}$$

of all decreasing index paths (of various lengths k) between i and j.

Notice that, in general, the convolution of two strongly continuous bounded linear operator valued functions $f : \mathbb{R}_+ \to \mathcal{L}(\mathbf{F}_2, \mathbf{F}_3)$, $g : \mathbb{R}_+ \to \mathcal{L}(\mathbf{F}_1, \mathbf{F}_2)$ (i.e. we have $g(t) : \mathbf{F}_1 \to \mathbf{F}_2$ resp. $f(t) : \mathbf{F}_2 \to \mathbf{F}_3$ $(t \in \mathbb{R}_+)$ and the maps $t \mapsto g(t)x$ resp. $t \mapsto f(t)y$ are continuous for any choice of $x \in \mathbf{F}_2$ resp. $y \in \mathbf{F}_2$) between Banach spaces $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$ is well-defined and also strongly continuous

[6, Appendix]. The operation $*$ is always associative even for operator valued functions, but we have no commutativity in general. In the sequel, without danger of confusion, we write $f(t) * g(t)$ instead of the theoretically more rigorous form $\{f * g\}(t)$ if the terms f, g are expressions with variable symbol t. E.g. we have $[U_2(t)B_{21}] * [U_1(t)] = \big[\mathbf{E}_1 \ni x \mapsto \int_{s=0}^t U_2(t-s)B_{21}U_1(t)x\big]$.

Next we consider a C0-SGR $[G(t) : t \in \mathbb{R}_+]$ of operator matrices

$$G(t) = \begin{bmatrix} G_{11}(t) & G_{12}(t) \\ G21(t) & G_{22}(t) \end{bmatrix}, \quad G' = \begin{bmatrix} V_1' & C \\ B & V_2' \end{bmatrix} \tag{3.2}$$

where the lower tringular part is the infinitesimal generator of a triangular C0-SGR of the type described previously: we are given a C0-SGR

$$T(t) = \big[T_{ij}(t)\big]_{i,j=1}^2 = \begin{bmatrix} V_1(t) & 0 \\ [V_2(t)B] * [V_1(t)] & V_2(t) \end{bmatrix}, \quad T' = \begin{bmatrix} V_1' & 0 \\ B & V_2' \end{bmatrix}$$

where $[V_i(t) : t \in \mathbb{R}_+]$ $(i = 1, 2)$ are C0-SGR in some Banach spaces $\mathbf{F}_1, \mathbf{F}_2$ and $B : \mathbf{F}_1 \to \mathbf{F}_2$ is a *bounded* linear operator. In applications here we only consider cases with $\dim(\mathbf{F}_2) < \infty$ and with $[V_1(t) : t \in \mathbb{R}_+]$ being lower triangular type with isometric diagonal. Since $[G(t) : t \in \mathbb{R}_+]$ is a bounded perturbation of $[T(t) : t \in \mathbb{R}_+]$, by [6 III.Cor.1.7] we have the *Volterra convolution equation*

$$G(t) = \int_{r=0}^t T(t-r) \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} G(r)\, dr + T(t). \tag{3.3}$$

Thus $G = \begin{bmatrix} 0 & V_1 C \\ 0 & V_2 * (BV_1C) \end{bmatrix} * G + T,$ that is

$$\begin{aligned} G_{11} &= (V_1C) * G_{21} + V_1, \qquad & G_{12} &= (V_1C) * G_{22}, \\ G_{21} &= w * G_{21} + V_2 * (BV_1), \qquad & G_{22} &= w * G_{22} + V_2. \end{aligned} \tag{3.4}$$

in terms of the operator valued function

$$w := [V_2 * (BV_1C)] : t \mapsto \int_{r=0}^t V_2(t-r)BV_1(r)C\, dr. \tag{3.5}$$

We shall call the solution $u : \mathbb{R}_+ \to \mathcal{L}(\mathbf{F}_2)$ of the Volterra convolution equation

$$u = w + u * w$$

with the function (3.5) the *governing function* of the system (3.2).

PROPOSITION 3.6. *In terms of the governing function u of (3.2) we have*

$$G = \begin{bmatrix} 1 & V_1C \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} V_1 & 0 \\ V_2 * (BV_1) + u * V_2 * (BV_1) & V_2 + u * V_2 \end{bmatrix}. \tag{3.7}$$

Proof. According to (3.4), for $j = 1, 2$ we have $G_{2j} = w * G_{2j} + T_{2j}$ i.e. $T_{2j} = G_{2j} - w * G_{2j}$. Therefore

$$\begin{aligned} T_{2j} + u * T_{2j} &= G_{2j} - w * G_{2j} + u * G_{2j} - u * w * G_{2j} = \\ &= G_{2j} + \big(- w + u - u * w \big) * G_{2j} = G_{2j}. \end{aligned}$$

Hence also $G_{1j} = (V_1C) * G_{2j} + T_{1j} = (V_1C) * \big[T_{2j} + u * T_{2j}\big] + T_{1j}$. It follows

$$\begin{aligned} &G_{21} = V_2 * (BV_1) + u * V_2 * (BV_1), \quad G_{22} = V_2 + u * V_2, \\ &G_{11} = (V_1C) * [V_2 * (BV_1) + u * V_2 * (BV_1)] + V_1, \; G_{12} = (V_1C) * [V_2 + u * V_2] \end{aligned}$$

whence the stated matrix convolution form is immediate.

COROLLARY 3.8. *For any $x_1 \in \mathbf{F}_1$ and $x_2 \in \mathbf{F}_2$. the functions $t \mapsto G_{ij}(t)x_j$ with $(i,j) \neq (1,1)$ are continuously differentiable.*

Proof. This is a folklore consequence of the Newton-Leibniz formula and the local uniform continuity of continuous Banach space valued functions of a real variable that we have $f * g \in \mathcal{C}^1(\mathbb{R}_+, \mathbf{K})$ whenever $\mathbf{K}, \mathbf{L}$ Banach spaces, $f \in \mathcal{C}\big(\mathbb{R}_+, \mathcal{L}(\mathbf{K}, \mathbf{L})\big)$, $g \in \mathcal{C}(\mathbb{R}_+, \mathbf{L})$ and f or g is continuously differentiable.[3] In the expressions of G_{12}, G_{21} and G_{22} every monomial involves the factor V_2. However, since $\dim(\mathbf{F}_2 < \infty$ and $V' \in \mathcal{L}(\mathbf{F}_2)$ by assumption, necessarily $V_2(t) = \exp(tV_2')\ t \in \mathbb{R}_+)$ i.e. the function V_2 is analytic. Hence the statement is immediate.

As a first relevant consequence, we can integrate the vector fields (2.2) in terms of convolution polynomials of the rotation group

$$R^t = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}, \; R' = \frac{d}{dt}\Big|_{t=0} R^t = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{3.9}$$

the C0-SGR $[U^t : t \in \mathbb{R}_+]$ with infinitesimal generator M.

[3] $\mathcal{C}^n(\mathbb{R}_+, \mathbf{K})$ is the usual notation for the family of n times continuously differentiable functions $[0, \infty) \to \mathbf{K}$.

THEOREM 3.10. *The C0-SGR* $[H^t : t \in \mathbb{R}_+]$ *of infinitesimal generator* (2.2) *has the convolution form*

$$(3.10)\quad H^t = \begin{bmatrix} 1 & U^t C \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} U^t & 0 \\ R^{\varepsilon t} * (BU^t) + u * R^{\varepsilon t} * (BU^t) & R^{\varepsilon t} + u(t) * R^{\varepsilon t} \end{bmatrix}.$$

where $B = \begin{bmatrix} b_1^{\mathrm{T}} \\ b_2^{\mathrm{T}} \end{bmatrix}$, $C = \begin{bmatrix} b_1 & b_2 \end{bmatrix}$ *and the governing function* u *is associated with*

$$w(t) = R^{\varepsilon t} * \left[BU^t B^{\mathrm{T}}\right] = \begin{bmatrix} \cos \varepsilon t & -\sin \varepsilon t \\ \sin \varepsilon t & \cos \varepsilon t \end{bmatrix} * \left[\langle b_k | U^t b_\ell \rangle\right]_{k,\ell=1}^2.$$

REMARK 3.11. In Theorem 3.10 we have $\dim(\mathbf{F}_2) \leq 2$, that is the operators $w(t)$ $(t \in \mathbb{R}_+)$ can be regarded a 2×2 or 1×1 matrices with entries depending continuously on the parameter t. Analogously as in the classical scalar case, with the (matrix valued) kernel function $K(t,s) := w(t-s)$ and with the spectral norm we can see that the (necessarily unique) solution of the equation $u = w + w * u$ is the Neumann sum $u = \sum_{n=1}^{\infty} w^{*n} = w + w * w + w * w * w + \cdots$ with locally uniform convergence because $\max_{t \in [0,\tau]} \|w^{*n}(t)\| \leq \left[\max_{t \in [0,\tau]} \|w\|\right]^n \tau^n / n!$ as it can be seen by a straightforward induction on n.

COROLLARY 3.12. *The C0-SGR* (3.2) *is a convolution polynomial of the operator functions* $V_1(\cdot), V_2(\cdot)$ *and the solution* $u : \mathbb{R}_+ \to \mathcal{L}(\mathbf{F}_2)$ *of the Volterra equation* $u = [V_2 * (BV_1 C)] + u * [V_2 * (BV_1 C)]$.

REMARK 3.13. The matrix function u above is no convolution polynomial but an infinite convolution Neumann series of V_1 and V_1. We can apply the *Laplace transform* [3]

$$\mathcal{L}\{f(t)\}(s) := \int_{t=0}^{\infty} f(t) e^{-ts} dt$$

with strongy continuous bounded operator valued functions to infer finite rational formulas for the entries $G_{ij}(t)$ in terms of convolution and the C0-SGR $[V(t) : t \in \mathbb{R}_+]$. Notice that the product rule $\mathcal{L}\{f(t) * g(t)\}(s) = \mathcal{L}\{f(t)\}(s)\, \mathcal{L}\{g(t)\}(s)$ holds even in operator context with strongly continuous outcome [6]. In particular from the relation $u = w + u * w$ we infer that

$$\mathcal{L}\{u(t)\}(s) = \left[1 - \mathcal{L}\{V_2(t)\}(s) B \mathcal{L}\{V_1(t)\}(s) C\right]^{-1} \mathcal{L}\{V_2(t)\}(s) B \mathcal{L}\{V_1(t)\}(s) C.$$

COR0LLARY 3.14. *The entries* $\mathcal{L}\{G_{ij}(t)\}(s)$ *of the Laplace transform of the C0-SGR* (3.2) *are rational fractions of the Laplace transforms* $\mathcal{L}\{V_j(t)\}(s)$.

4. KAUP'S TYPE VECTOR FIELDS AND FIXED POINTS

Henceforth $\mathbf{E}$ denotes an arbitrarily fixed spin factor with trailer Hilbert space $\mathbf{H} = \mathbf{H}_0 \oplus i\mathbf{H}_0$ and unit ball $\mathbf{B}$ as described in Section 1 with the triple product (1.1). We also reserve the notation $\mathbf{\Phi} = [\Phi^t : t \in \mathbb{R}_+]$ for a fixed C0-SGR in $\mathrm{Iso}(d_\mathbf{B})$ such that $0 \in \mathrm{dom}(\Phi')$ and we write $a = \Phi'(0)$. As mentioned, this can be done up to Möbius equivance, that is without loss of generality. It is also well-known (valid for all reflexive JB*-triples [17]) that the infinitesimal generator Φ' is of Kaup's type, that is

$$\Phi'(x) = a - \{xax\} + iAx \qquad \big(x \in \mathbf{J} \cap \mathbf{B}\big) \tag{4.1}$$

where $\mathbf{J}$ is a dense (complex) subtriple of $\mathbf{E}$ and iA is the infinitesimal generator of a C0-SGR $[U^t : t \in \mathbb{R}_+]$ of $\mathbf{E}$-isometries [4] with $\mathrm{dom}(iA) = \mathbf{J}$. On the other hand, Vesentini [21] proved that the generic form for $\mathbf{\Phi}$ is given by the linear representation (1.3 – 5) with a C0-SGR

$$G^t = \big[G^t_{ij}\big]^3_{i,j=1} = \begin{bmatrix} M_t & b^t_1 & b^t_2 \\ c^t_1 & 0 & -\varepsilon_t \\ c_2 & \varepsilon & 0 \end{bmatrix} = \begin{bmatrix} M_t & B_t \\ C_t & E_t \end{bmatrix}$$

where the infinitesimal generator has the form

$$G' = \begin{bmatrix} M & b_1 & b_2 \\ b_1^{\mathrm{T}} & 0 & -\varepsilon \\ b_2^{\mathrm{T}} & \varepsilon & 0 \end{bmatrix} \quad \text{where} \quad \begin{matrix} b_1 := 2\mathrm{Re}(a), b_2 := -2\mathrm{Im}(a), \\ M = \overline{M} = -M^{\mathrm{T}}, \varepsilon \in \mathbb{R} \end{matrix}. \tag{4.2}$$

This corresponds to the transcription

$$\Phi'(x) = \big(\frac{1}{2}b_1 - \frac{i}{2}b_2\big) + Mx + i\varepsilon x - \langle x|b_1 - ib_2\rangle x + \langle x|\overline{x}\rangle \big(\frac{1}{2}b_1 + \frac{i}{2}b_2\big) \tag{4.1$'$}$$

of (4.1) in terms of the real Hilbert space $\mathbf{H}_0$. Since G' is a finite rank perturbation of the operator matrix $\mathrm{diag}\big(M, \left[\begin{smallmatrix} 0 & -\varepsilon \\ \varepsilon & 0 \end{smallmatrix}\right]\big)$ with domain $\mathrm{dom}(M') \oplus \mathbb{R}^2$, by the Bounded Perturbation Theorem [6] we have also $\mathrm{dom}(G') = \mathrm{dom}(M) \oplus \mathbb{R}^2$ and hence

$$\mathrm{dom}(M) = \big\{z \in \mathbf{H} : [t \mapsto M_t z] \in \mathcal{C}^1(\mathbb{R}_+, \mathbf{H})\big\}. \tag{4.3}$$

Notice any choice above with $b_1, b_2 \in \mathbf{H}_0$, $\varepsilon \in \mathbb{R}$ and $M : \mathrm{dom}(M) \to \mathbf{E}$ being a the complex linear extension of a maximal closed antisymmetric unbounded $\mathbf{H}_0$-operator is admissible.

[4] Extendig the term $\mathbf{E}$-*hermitian* used by Kaup [KaupR] in bounded context, we may say that A is a (possibly) unbounded $\mathbf{E}$-*symmetric* operator.

REMARK 4.4. (i) Vesentini proved [21] that $\frac{d}{dt}\big|_{t=0} F_{G^t}(x)/\varphi_{G^t}(x) = \big[$right hand side of $(4.1')\big]$ under the tacitly used hypothesis that $x \in \mathrm{dom}(M)$, $\varphi_{\mathrm{Id}}(x) \neq 0$ and $\Phi'(0)$ exists.

(ii) Forerunner results are due to Hierzebruch [9] in finite dimensions and L. Harris [8] in general setting. W. Kaup [12] established first (4.1) for *uniformly* continuous groups $[\Phi^t : t \in \mathbb{R}]$ (necessarily with $\mathrm{dom}(A) = \mathbf{E}$) for the complete holomorphic vector fields of a bounded circular symmetric Banach space domain and derived the JB*-axioms from their Banach-Lie algebra.

(iii) The fact that $\mathbf{J}$ is closed under the triple product even in the setting of generic reflexive JB*-triples is shown in [17,2].

(iv) Notice also that there is a misprint related to $(1.3-5)$ in [21p.438.l.11]: "$\delta(G(X)) = 2\big(X\big|C_1 - iC_2\big)$" should stay instead of "$\delta(G(X)) = 2\big(X\big|C_1 - C_2\big)$".

PROPOSITION 4.5. *Suppose z is a common fixed point of the continuous extensions $\overline{\Phi^t}$ of the maps Φ^t onto the closed unit ball $\overline{\mathbf{B}}$. Then we have $z \in \mathbf{J}$.*

Proof. We may apply Corollary (3.8) to the C0-SGR $[G^t : t \in \mathbb{R}]$ since all the entries G'_{ij} resp. G^t_{ij} with $(i,i) \neq (1,1)$ are necessarily of rank 1. Hence we conclude that the maps $t \mapsto G^t_{i,j}$ with indices $(i,j) \neq (1,1)$ are all continuously differentiable. In particular, the the scalar function resp. vector valued functions

$$\begin{aligned} t \mapsto \varphi_{G^t}(z) &= (E^t_{11} + E^t_{22} - iE^t_{12} + iE^t_{21}) + 2(c_1 + ic_2)^{\mathrm{T}} z + \\ &\quad + (E^t_{11} - E^t_{22} + iE^t_{12} + iE^t_{21}) z^{\mathrm{T}} z, \\ t \mapsto F_{G^t}(z) - 2Mz &= (b^t_1 - ib^t_2) + (z^{\mathrm{T}} z)(b^t_1 + ib^t_2) \end{aligned}$$

with (1.5) are continuously differentiable. On the other hand, since $[G^t : t \in \mathbb{R}_+]$ is C0-SGR, for $t \searrow 0$ we have $M_t z \to z$, $b^t_j, c^t_j \to 0$ in $\mathbf{H}_0$ resp. $E^t_{1,1}, E^t_{22} \to 1$ and $E^t_{21}, E^t_{22} \to 0$ in $\mathbb{R}$. It is also well-known [17] that each map Φ^t admits a holomorphic extension to the ball of radius $\|\Phi^t(0)\|^{-1} > 1$. Therefore we have $\varphi_{G^t}(z) \neq 0$ for the denominator function of (1.5) and

$$z = \Phi^t(z) = F_{G^t}(z)/\varphi_{G^t}(z) \qquad (0 \leq t \leq \tau)$$

for some $\tau > 0$. Hence we conclude that the vector valued function

$$t \mapsto 2M_t z = \varphi_{G^t}(z) z - \big[(b^t_1 - ib^t_2) + (z^{\mathrm{T}} z)(b^t_1 + ib^t_2)\big]$$

is continuously differentiable in some right neighborhood of $0 \in \mathbb{R}_+$ and hence on the whole non-negative semiaxis [6]. The proof is complete in view of (4.3).

REMARK 4.6. A similar argument can be applied with the projective representation of reflexive a TRO (Cartan factor of type 1). In [18] we used this fact tacitly in the concluding sentence of the proof of Lemma 3.1. On the other hand, all the arguments, used in course of the proof there, are purely Jordan theoretical and rely upon only the finite rank property (i.e. every element is a finite linear combination of a Jordan orthognal family) without referring to any further specific featutes of reflexive TRO-s. Thus the following holds as well:

LEMMA 4.7. *In a reflexive JB*-triple* $\mathbf{F}$*, every point* $z \in \overline{\mathbf{B_F}}$ *of the closed unit ball can be mapped into a tripotent with a suitable Möbius shift which preserves the intersection the open unit ball with the* (*necessarily finite dimensional*) *subtriple* $\mathbf{J}_z$ *generated by* z*.*

Proof. It is well-known [11] that reflexivity in JB*-triple is nothing else than being of finite rank. Thus we have a (unique) finite decomposition of the form $z = \sum_{k=1}^r \lambda_k e_k$ where $e_1, \dots, e_r \in \mathrm{Tri}(\mathbf{F})$ are pairwise Jordan orthogonal tripotents (in the sense that $L(e_k, e_\ell) = 0$ for $k \neq \ell$) and $\lambda_1 > \cdots > \lambda_r > 0$ with $\lambda_1 = \|z\|$. If $\|z\| < 1$ i.e. $\lambda_1 < 1$ then the Möbius shift (1.2) with $a := -z$ is well-defined and takes z into the origin. It also is well-known (cf. [18]) that $\mathbf{J}_a = \sum_{k=1}^r \mathbb{C} e_k$, and $\mathbf{B_F} \cap \mathbf{J}_z = \{\sum_{k=1}^r \zeta_k e_k : |\zeta_1|, \dots, \|\zeta_k\| <\}$. Furthermore, in the case $1 = \lambda_1 = \|z\|$, the Möbius shift (1.2) with $a := -\sum_{k>1} \lambda_k e_k$ takes z into $\lambda_1 e_1 = e_1$ and maps $\mathbf{B_F} \cap \mathbf{J}_z$ onto itself.

COROLLARY 4.8. *Since* $\mathbf{J} = \mathrm{dom}(\Phi')$ *ia a Jordan subtriple of* $\mathbf{E}$*, given any point* $z \in \overline{\mathbf{B}} \cap \mathbf{J}$ *in the closed unit ball of the spin factor* $\overline{\mathbf{E}}$*, there is a Möbius transformation* Θ (*actually a Möbius shift composed with a modulus* 1 *scalar multiplication*) *such that* $\Theta(z) \in \bigcup_{k=0}^2 \mathrm{Tri}_k(\mathbf{E})$ *and* $0 \in \Theta(\mathbf{J}_z) \subset \bigcap_{t \in \mathbb{R}_+} \mathrm{dom}(\Theta \circ \Phi^t \circ \Theta^{-1})$*.*

Since the holomorphic Carathéodory isometries of the unit ball of a reflexive JB*-triple are factor preserving [17] and since the analogous statement to Proposition 4.5 is trivial for finite dimensional Cartan factors, in view of [18, Lemma 3.1] we have proved the following

THEOREM 4.9. *Any C0-SGR of holomorphic Carathéodory isometries of the open unit ball in a reflexive JB*-triple is Möbius equivalent to a C0-SGR with Kaup's type infinitesimal generator whose membes admit a tripotential common fixed point when extended continuously to the closed unit ball.*

5. TRIANGULARIZATION WITH FIXED POINTS

Throughout this section we keep the previously established setting and notations. Furthermore we write $[U^t : t \in \mathbb{R}_+]$ for the C0-SGR of bounded linear $\mathbf{E}$-isometries with

$$U' = M + i\varepsilon = \big[\mathbb{C}\text{-lin. extension of } M \text{ to } \mathbf{J} = \mathrm{dom}(M) \oplus i\mathrm{dom}(M)\big] + \varepsilon \mathrm{Id}_{\mathbf{E}}.$$

According to Corollary (4.8) and in view of Remark 4.4(i) applied at the common fixed points z of the maps $\overline{\Phi^t}$ resulting in $z \in \mathrm{dom}(\Phi') = \mathbf{J} \cap \mathbf{B}$ with $\Phi'(z) = 0$, we are lead to the followig alternatives.

ALTERNATIVES 5.1. Any C0-SGR in $\mathrm{Iso}(d_{\mathbf{B}})$ is Möbius eqivalent to a C0-SGR $\mathbf{\Phi} = [\Phi^t : t \in \mathbb{R}_+]$ with Kaup's type infinitesimal generator $(4.1')$ and admitting a tripotent $e \in \mathrm{Tri}_k(\mathbf{E}) \cap \mathrm{Fix}(\mathbf{\Phi})$ $(k = 0, 1, 2)$ such that

Case (0): $e = 0$. Then $\Phi'(0) = 0$, $\Phi'(x) = Mx + i\varepsilon x \quad (x \in \mathbf{J} \cap \mathbf{B})$, that is $\Phi^t = U^t\big|\mathbf{B}$ (proved by Vesentini [21] already).

Case (1): $e \in \mathrm{Tri}_1(\mathbf{E})$ is a real extreme point of $\overline{\mathbf{B}}$, $e = \overline{e}$, $\langle e|e\rangle = 1$. Then

$$\begin{aligned} 0 = \Phi'(e) &= a + iAe - \{ea^*e\} = \\ &= \big(\frac{1}{2}b_1 - \frac{i}{2}b_2\big) + Me + i\varepsilon e - \big\langle e\big|b_1 - ib_2\big\rangle e + \big\langle e\big|e\big\rangle\big(\frac{1}{2}b_1 + \frac{i}{2}b_2\big) = \\ &= b_1 - \big\langle b_1\big|e\big\rangle e + Me + i\big(\varepsilon - \big\langle b_2\big|e\big\rangle\big)e. \end{aligned}$$

Case (2): $e \in \mathrm{Tri}_2$, is a face middle point of $\overline{\mathbf{B}}$, $e \perp \overline{e}$, $\langle e\rangle^2 = 1/2$. Then

$$\begin{aligned} 0 = \Phi'(e) &= \big(\frac{1}{2}b_1 - \frac{i}{2}b_2\big) + Me + i\varepsilon e - \big\langle e\big|b_1 - ib_2\big\rangle e = \\ &= \frac{1}{2}b_1 - \big\langle b_1\big|e\big\rangle e + Me + i\big(\varepsilon - \big\langle b_2\big|e\big\rangle\big)e - \frac{i}{2}b_2. \end{aligned}$$

Our next aim will be to find almost lower triangular linearly equivalent forms for the infinitesimal generator G' of the Hierzebruch-Vesentini representation $\Phi^t = \Phi_{G^t}$ with a C0-SGR $[G^t : t \in \mathbb{R}_+]$ of real operator matrices over $\mathbf{H}_0 \oplus \mathbb{R}^2$ by means of the refinements

$$\begin{aligned} &\mathbf{H}_0 = [\mathbb{R}e] \oplus \mathbf{H}_1 \simeq \mathbb{R} \oplus \mathbf{H}_1 \quad \text{with} \quad \mathbf{H}_1 := \mathbf{H}_0 \ominus [\mathbb{R}e], \\ &\mathbf{H}_0 \simeq \mathbb{R}^2 \oplus \mathbf{H}_2 \quad \text{with} \quad \mathbf{H}_2 := \mathbf{H}_0 \ominus \big([\mathbb{R}\,\mathrm{Re}(e)] \oplus [\mathbb{R}\,\mathrm{Im}(e)]\big) \end{aligned}$$

of the underlying spaces in Cases 1-2, respectively. We shall also write P_j $(j = 1, 2)$ for the orthogonal projections $P_{\mathbf{H}_j} : \mathbf{H}_0 \to \mathbf{H}_j$ for short.

Case (1). With the orthogonal decompositions

$$b_j := \rho_j e + x_j, \quad \rho_j = \langle b_j | e \rangle \in \mathbb{R},\ x_j \perp e \quad (j = 1, 2)$$

we have $0 = x_1 + Me + i(\varepsilon - \rho_2)e$ implying $\rho_2 = \varepsilon$ and $Me = -x_1 \in \mathbf{H}_1$. Hence, with the restricted operator $M_1 := P_1 M \big| \mathbf{H}_1 \cap \mathrm{dom}(M)$, we can write

$$G' = \begin{bmatrix} M & b_1 & b_2 \\ b_1^{\mathrm{T}} & 0 & -\varepsilon \\ b_2^{\mathrm{T}} & \varepsilon & 0 \end{bmatrix} = \begin{bmatrix} 0 & -(Me)^{\mathrm{T}} & \rho_1 & -\varepsilon \\ Me & M_1 & -Me & x_2 \\ \rho_1 & -(Me)^{\mathrm{T}} & 0 & -\varepsilon \\ -\varepsilon & x_2^{\mathrm{T}} & \varepsilon & 0 \end{bmatrix} = \begin{bmatrix} 0 & x_1^{\mathrm{T}} & \rho_1 & -\varepsilon \\ -x_1 & M_1 & x_1 & x_2 \\ \rho_1 & x_1^{\mathrm{T}} & 0 & -\varepsilon \\ -\varepsilon & x_2^{\mathrm{T}} & \varepsilon & 0 \end{bmatrix}.$$

Almost triagular linearly similar form can be obtained with the coordinatization matrices

$$\text{(5.2)} \quad S_1 := \begin{bmatrix} 1/2 & 0 & 0 & 1 \\ 0 & I_1 & 0 & 0 \\ -1/2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad S_1^{-1} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & I_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix}, \quad I_1 = \mathrm{Id}_{\mathbf{H}_1}$$

as

$$\text{(5.3)} \qquad S_1^{-1} G' S_1 = \begin{bmatrix} -\rho_1 & 0 & 0 & 0 \\ -x_1 & M_1 & x_2 & 0 \\ -\varepsilon & x_2^{\mathrm{T}} & 0 & 0 \\ 0 & x_1^{\mathrm{T}} & -\varepsilon & \rho_1 \end{bmatrix}.$$

REMARK 5.4. The operator M_1 is the restriction of the bounded perturbation $P_1 M P_1 = \big[1 - P_{\mathbb{R}e}\big] M \big[1 - P_{\mathbb{R}e}\big] = M - P_{\mathbb{R}e} M - M P_{\mathbb{R}e} + P_{\mathbb{R}e} M P_{\mathbb{R}e}$ of M to $\mathrm{ran}(P_1) = \mathrm{ran}\big(P_{\mathbb{R}e}\big)^{\perp}$. Since M is the infinitesimal generator of the C0-SGR $[U^t | \mathbf{H}_0 : t \in \mathbb{R}_+]$ of $\mathbf{H}$-isometries, M_1 is a possibly unbounded maximal skew symmetric closed $\mathbb{R}$-linear $\mathbf{H}_1$-operator being the infinitesimal generator of a C0-SGR $[U_1^t : t \in \mathbb{R}_+]$ of $\mathbf{H}_1$-isometries. The matrix $S_1^{-1} G' S_1$ is lower triangular if $x_2 = 0$ that is if $b_2 = 2\,\mathrm{Im}\big(\Phi'(0)\big) \in \mathbb{R}e$. This latter means that either $\mathrm{Im}\big(\Phi'(0)\big) = 0$ or one of $\pm \mathrm{Im}(b_2)/\|\mathrm{Im}\big(\Phi'(0)\big)\|$ is common fixed point of $[\overline{\Phi^t} : t \in \mathbb{R}_+]$ and it belongs to $\mathrm{ext}\big(\overline{\mathbf{B}}\big)$ in the same time.

Case (2) We may assume without loss of generality that

$$e = \frac{1}{2} u + \frac{i}{2} v, \quad u \perp v \in \mathbf{H}_0, \quad \langle u \rangle^2 = \langle v \rangle^2 = 1.$$

Since M is the complexification of a possibly unbounded real antisymmetric $\mathbf{E} = (\mathbf{H}_0 \oplus i\mathbf{H}_0)$-operator, $M = \overline{M} \subset -M^{\mathrm{T}} = -\overline{M}^* = -\overline{M^*}$ along with $\mathrm{dom}(M) = \{\overline{x} : x \in \mathrm{dom}(M)\}$, we have $u, v \in \mathrm{dom}(M)$ with

$$\langle Mu|u\rangle = \langle Mv|v\rangle = \langle Mu|v\rangle + \langle Mv|u\rangle = 0,$$
$$\langle Me|e\rangle = -\frac{i}{2}\langle Mu|v\rangle, \quad \langle Me|\overline{e}\rangle = 0.$$

Hence, using the identities $\langle b_j|u\rangle = \langle u|b_j\rangle$ resp. $\langle b_j|v\rangle = \langle v|b_j\rangle$, we get

$$0 = \langle \Phi'(e)|e\rangle = \Big\langle \frac{1}{2}b_1 - \frac{i}{2}b_2\Big|e\Big\rangle + \langle Me|e\rangle + \frac{i}{2}\varepsilon - \Big\langle e\Big|\frac{1}{2}b_1 - \frac{i}{2}b_2\Big\rangle =$$
$$= \frac{i}{2}\big[\varepsilon - \langle Mu|v\rangle - \langle b_1|v\rangle - \langle b_2|u\rangle\big],$$
$$0 = \langle \Phi'(e)|\overline{e}\rangle = \Big\langle \frac{1}{2}b_1 - \frac{i}{2}b_2\Big|\overline{e}\Big\rangle = \frac{1}{4}\big[\langle b_1|u\rangle + \langle b_2|v\rangle + i\langle b_1|v\rangle - i\langle b_2|u\rangle\big].$$

Considering the real and imaginary parts, therefore

(5.5) $\langle b_1|u\rangle = -\langle b_2|v\rangle,\ \langle b_1|v\rangle = \langle b_2|u\rangle,\ \langle Mu|v\rangle = \varepsilon - \langle b_1|v\rangle - \langle b_2|u\rangle = \varepsilon - 2\langle b_2|u\rangle.$

Thus in terms of the orthogonal decompositions resp. constant

$$b_j = \rho_j u + \sigma_j v + x_j, \quad x_j \in \{u,v\}^\perp = \mathbf{H}_2 \ \ (j = 1,2), \quad \mu := \langle Mu|v\rangle$$

we have

$$\sigma_2 = -\rho_1, \quad \sigma_1 = \rho_2, \quad \mu = \varepsilon - 2\rho_2.$$

Hence, with the notations

$$M_2 := P_2 M\big|\mathbf{H}_2 \cap \mathrm{dom}(M), \quad q_1 := P_2 Mu, \quad q_2 := P_2 Mv$$

we can write

$$G' = \begin{bmatrix} M & b_1 & b_2 \\ b_1^{\mathrm{T}} & 0 & -\varepsilon \\ b_2^{\mathrm{T}} & \varepsilon & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\mu & -q_1^{\mathrm{T}} & \rho_1 & \rho_2 \\ \mu & 0 & -q_2^{\mathrm{T}} & \rho_2 & -\rho_1 \\ q_1 & q_2 & M_2 & x_1 & x_2 \\ \rho_1 & \rho_2 & x_1^{\mathrm{T}} & 0 & -\varepsilon \\ \rho_2 & -\rho_1 & x_2^{\mathrm{T}} & \varepsilon & 0 \end{bmatrix}.$$

Observe that from the relation

$$0 = P_2\Phi'(e) = P_2\Big[\Big(\frac{1}{2}b_1 - \frac{i}{2}b_2\Big) + Me + i\varepsilon e - \langle e|b_1 - ib_2\rangle e\Big] =$$
$$= \frac{1}{2}\big[x_1 - ix_2 + PM(u+iv) + 0\big]$$

we infer also $q_1 = -x_1$ and $q_2 = x_2$. Therefore

$$G' = \begin{bmatrix} 0 & 2\rho_2-\varepsilon & x_1^{\mathrm{T}} & \rho_1 & \rho_2 \\ \varepsilon-2\rho_2 & 0 & -x_2^{\mathrm{T}} & \rho_2 & -\rho_1 \\ -x_1 & x_2 & M_2 & x_1 & x_2 \\ \rho_1 & \rho_2 & x_1^{\mathrm{T}} & 0 & -\varepsilon \\ \rho_2 & -\rho_1 & x_2^{\mathrm{T}} & \varepsilon & 0 \end{bmatrix}.$$

A convenient quasi lower triagular form can be obtained with the coordinatization matrices

$$(5.6)\quad S_2 := \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & I_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad S_2^{-1} := \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & I_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad I_2 := \mathrm{Id}_{\mathbf{H}_2}$$

as

$$(5.7)\qquad S_2^{-1} G' S_2 = \begin{bmatrix} -\rho_1 & \varepsilon-\rho_2 & 0 & 2\varepsilon & 0 \\ \rho_2-\varepsilon & -\rho_1 & 0 & 0 & 2\varepsilon \\ x_2 & -x_1 & M_2 & 0 & 0 \\ \rho_2 & \rho_1 & x_1^{\mathrm{T}} & \rho_1 & -\varepsilon-\rho_2 \\ -\rho_1 & \rho_2 & x_2^{\mathrm{T}} & \rho_2+\varepsilon & \rho_1 \end{bmatrix}.$$

REMARK 5.8. Analogously as in Remark 5.4, the operator M_2 is a possibly unbounded maximal skew symmetric closed $\mathbb{R}$-linear $\mathbf{H}_2$-operator being the infinitesimal generator of a C0-SGR $[U_2^t : t \in \mathbb{R}_+]$ of $\mathbf{H}_2$-isometries. The matrix $S_2^{-1} G' S_2$ is upper triangular if $\varepsilon = 0$. In view of (5.5), this means that $Mv - 2b_2 \perp u$.

6. INTEGRATION OF THE QUASI-TRIANGULAR SYSTEMS

With the tools developed in of Section 3, it is a voluminous but feasible work to find explicit formulas for the entries of the matrices in the Hierzebruch-Vesentini representation $[G^t : t \in \mathbb{R}_+]$ described in Section 5. This will be done below.

Case (1). We can write the matrix (5.3) in the triangular block form

$$S_1^{-1} G' S_1 = \begin{bmatrix} -\rho_1 & 0 & 0 \\ -x_1 & T'_{22} & 0 \\ 0 & T'_{32} & \rho_1 \end{bmatrix}, \quad T'_{22} = \begin{bmatrix} M_1 & x_2 \\ x_2^{\mathrm{T}} & 0 \end{bmatrix}, \quad \begin{array}{l} T'_{12} = -\begin{bmatrix} x_1^{\mathrm{T}} & \varepsilon \end{bmatrix}^{\mathrm{T}}, \\ T'_{32} = \begin{bmatrix} x_1^{\mathrm{T}} & -\varepsilon \end{bmatrix}. \end{array}$$

Notice that T'_{22} is the compression of an infinitesimal generator into a 1-codimensional subspace of $\mathbf{J}$ and therefore it is the infinitesimal generator of a C0-SGR $[T^t_{22} : t \in \mathbb{R}_+]$ of operator matrices. According to (3.1), in terms of $[T^t_{22} : t \in \mathbb{R}_+]$ we can write

$$
(6.1) \qquad S_1^{-1} G^t S_1 = \begin{bmatrix} e^{-\rho_1 t} & 0 & 0 \\ T^t_{21} & T^t_{22} & 0 \\ T^t_{31} & T^t_{32} & e^{\rho_1 t} \end{bmatrix}, \qquad \begin{aligned} &T^t_{21} = T^t_{22}\big[-x_1^{\mathrm{T}}, -\varepsilon\big]^{\mathrm{T}} * \; e^{-\rho_1 t}, \\ &T^t_{32} = e^{\rho_1 t} * \big[x_1^{\mathrm{T}}, -\varepsilon\big] T^t_{22}, \\ &T^t_{31} = e^{\rho_1 t} * \big[x_2^{\mathrm{T}}, -\varepsilon\big] T^t_{22} \begin{bmatrix} -x_1 \\ -\varepsilon \end{bmatrix} * e^{-i\rho_1 t}. \end{aligned}
$$

We can evalute the entries $\big[T^t_{22}\big]_{ij}$ $(i,j = 1,2)$ of the matrices T^t_{22} by the aid of Proposition 3.6 applied with $V_2 := [t \mapsto 1]$, $B := x_2^{\mathrm{T}}\big(\simeq [\mathbf{H}_1 \ni z \mapsto \langle z|x_2\rangle]\big)$, $C := x_2\big(\simeq [\mathbb{R} \ni \xi \mapsto \xi x_2]\big)$, and $V_1 := [t \mapsto W^t]$ where $[W^t : t \in \mathbb{R}_+]$ is the C0-SGR of $\mathbf{H}_1$-operators with the generator M_1. Let us introduce the scalar valued function $u \in \mathcal{C}(\mathbb{R})$ as the solution of the Volterra convolution equation

$$
(6.2) \quad u = w + u * w \quad \text{where} \quad w(t) := 1 * \big[x_2^{\mathrm{T}} W^t x_2\big] = \int_{r=0}^t \big\langle W^t x_2 \big| x_2 \big\rangle \, dr.
$$

It follows

$$
(6.3) \qquad \begin{aligned} &\big[T^t_{22}\big]_{11} = \big[W^t x_2\big] * \big[1 * x_2^{\mathrm{T}} W^t + u(t) * 1 * x_2^{\mathrm{T}} W^t\big] + W^t, \\ &\big[T^t_{22}\big]_{12} = \big[W^t x_2\big] * \big[1 + u(t) * 1\big], \\ &\big[T^t_{22}\big]_{21} = 1 * \big(x_2^{\mathrm{T}} W^t\big) + u(t) * \big[1 * \big(x_2^{\mathrm{T}} W^t\big)\big], \\ &\big[T^t_{22}\big]_{22} = 1 + u(t) * 1. \end{aligned}
$$

Notice that if $\varphi \in \mathcal{C}(\mathbb{R}_+)$, even in with operator functions $g \in \mathcal{C}\big(\mathbb{R}_+, \mathcal{L}(\mathbf{F}_2, \mathbf{F}_1)\big)$ we have $\big[\varphi(t)\mathrm{Id}_{\mathbf{F}_2}\big] * g(t) = g(t) * \big[\varphi(t)\mathrm{Id}_{\mathbf{F}_1}\big]$ since $\int_{r=0}^t \varphi(t-r) g(r) x \, dr = \int_{s=0}^t g(t-s)\varphi(s) x \, ds$ $(x \in \mathbf{F}_2)$. Thus we can rearrange (6.3) by shifting the scalar terms to the left in the form

$$
(6.3^{(i)}) \qquad T^t_{22} = \widetilde{u}(t) * \begin{bmatrix} [W^t x_2] * [x_2^{\mathrm{T}} W^t] & W^t x_2 \\ x_2^{\mathrm{T}} W^t & 0 \end{bmatrix} + \mathrm{diag}\big(W^t, \widetilde{u}(t)\big).
$$

in terms of the primitive of the governing function

$$
(6.2') \qquad \widetilde{u}(t) = 1 + 1 * u(t) = 1 + \int_{s=0}^t \int_{r=0}^s \big\langle W^t x_2 \big| x_2 \big\rangle \, dr \, ds.
$$

Substituting back into (6.1), we get

$$
\begin{aligned} T^t_{21} &= T^t_{22} \begin{bmatrix} -x_1 \\ -\varepsilon \end{bmatrix} * e^{-\rho_1 t} = e^{-\rho_1 t} * T^t_{22} \begin{bmatrix} -x_1 \\ -\varepsilon \end{bmatrix} = \\ &= -e^{-\rho_1 t} * \widetilde{u}(t) * \begin{bmatrix} [W^t x_2] * [x_2^{\mathrm{T}} W^t] x_1 + \varepsilon W^t x_2 \\ -x_2 W^t x_1 \end{bmatrix} - e^{-\rho_1 t} * \begin{bmatrix} W^t x_1 \\ \varepsilon \, \widetilde{u}(t) \end{bmatrix}. \end{aligned}
$$

Since each term $x_2^{\mathrm{T}} W^t x_1 = \big[\langle W^t x_1 | x_2\rangle\big]$ is a scalar (real 1×1-matrix),

$$(6.3^{(ii)})\qquad \begin{aligned} &T^t_{21} = \\ &= -\begin{bmatrix} \big[e^{-\rho_1 t} * \widetilde{u}(t) * \langle W^t x_1 | x_2\rangle + \varepsilon e^{-\rho_1 t} * \widetilde{u}(t)\big] * W^t x_2 + e^{-\rho_1 t} * W^t x_1 \\ e^{-\rho_1 t} * \widetilde{u}(t) * \langle W^t x_1 | x_2\rangle + \varepsilon\, e^{-\rho_1 t} * \widetilde{u}(t) \end{bmatrix}. \end{aligned}$$

Analogously

$$(6.3^{(iii)})\qquad \begin{aligned} T^t_{32} &= e^{\rho_1 t} * \big[x_1, -\varepsilon\big] T^t_{22} = e^{\rho_1 t} * \Big[x_1^{\mathrm{T}} W^t - \varepsilon\, \widetilde{u}(t)\Big] + \\ &= + e^{\rho_1 t} * \widetilde{u}(t) * \Big[x_1^{\mathrm{T}} [W^t x_2][x_2^{\mathrm{T}} W^t] - \varepsilon\, x_2^{\mathrm{T}} W^t, x_1^{\mathrm{T}} W^t x_2\Big] = \\ &= \Big[\big[e^{\rho_1 t} * \widetilde{u}(t) * \langle W^t x_2 | x_1\rangle - \varepsilon e^{\rho_1 t} * \widetilde{u}(t)\big] * x_2^{\mathrm{T}} W^t + e^{\rho_1 t} * x_1^{\mathrm{T}} W^t, \\ &\qquad e^{\rho_1 t} * \widetilde{u}(t) * \langle W^t x_2 | x_1\rangle - \varepsilon\, e^{\rho_1 t} * \widetilde{u}(u)\Big]. \end{aligned}$$

Furthermore

$$(6.3^{(iv)})\qquad \begin{aligned} T^t_{31} &= e^{\rho_1 t} * \big[x_2^{\mathrm{T}}, -\varepsilon\big] T^t_{22} \begin{bmatrix} -x_1 \\ -\varepsilon \end{bmatrix} * e^{-\rho_1 t} = \\ &= -e^{\rho_1 t} * e^{-\rho_1 t} * \big[x_2^{\mathrm{T}}, -\varepsilon\big] T^t_{22} \begin{bmatrix} x_1 \\ \varepsilon \end{bmatrix} = \\ &= -\frac{1}{\rho_1} \sinh(\rho_1 t) * \Big(x_2^{\mathrm{T}} \big[\widetilde{u}(t) * [W^t x_2] * [x_2^{\mathrm{T}} W^t] + W^t\big] x_1 + \\ &\qquad + \varepsilon x_2^{\mathrm{T}} \big[\widetilde{u}(t) * W^t x_2\big] - \varepsilon \widetilde{u}(t) * [x_2^{\mathrm{T}} W^t x_1] - \varepsilon^2 \widetilde{u}(t)\Big) = \\ &= -\frac{1}{\rho_1} \sinh(\rho_1 t) * \Big(\widetilde{u}(t) * \langle W^t x_2 | x_2\rangle * \langle W^t x_1 | x_2\rangle + \langle W^t x_1 | x_2\rangle + \\ &\qquad + \varepsilon \widetilde{u}(t) * \langle W^t x_2 | x_2\rangle - \varepsilon \widetilde{u}(t) * \langle W^t x_1 | x_2\rangle - \varepsilon^2 \widetilde{u}(t)\Big). \end{aligned}$$

We can summarize the above considerations as follows.

THEOREM 6.4. *Let $e \in \mathbf{H}_0$ be an arbitrarily fixed unit vector and define $\mathbf{H}_1 := \mathbf{H}_0 \ominus (\mathbb{R}\, e)$. A generic C0-SGR $[\Psi^t : t \in \mathbb{R}_+]$ of holomorphic $d_{\mathbf{B}}$-isometries, whose continuous extension to the closed unit ball admits a common fixed point which is an extreme point on the unit sphere, can be written in the form*

$$\Psi^t = \Theta \circ \Phi_{G^t_{\mathcal{W}_1, x_1, x_2, \rho_1, \varepsilon}} \circ \Theta^{-1} \qquad (t \in \mathbb{R}_+)$$

where

(i) Θ *is any* $\mathbf{E}$-*Möbius transformation,* $\mathcal{W}_1 = [W^t : t \in \mathbb{R}_+]$ *is any C0-SGR of* $\mathbf{H}_1$*-isometries,* $x_1, x_2 \in \mathbf{H}_1$ *and* $\rho_1, \varepsilon \in \mathbb{R}$*,*

(ii) $[G^t_{\mathcal{W}_1,x_1,x_2,\rho_1,\varepsilon} : t \in \mathbb{R}_+]$ *is a C0-SGR of* $\mathcal{L}\big(\mathbb{R} \oplus [\mathbf{H}_1 \oplus (\mathbb{R}\, e)] \oplus \mathbb{R}\big)$*-type operator matrices such that*

$$G^t_{\mathcal{W}_1,x_1,x_2,\rho_1,\varepsilon} = S_1\Big(\big[T^t_{ij}\big]^3_{i,j=1}\Big)S^{-1} \quad (t \in \mathbb{R}_+)$$

with the coordinatization matrices (5.2)*, the entries* $T^t_{ij} for (i,j) \neq (2,2)$ *are described in* (6.1)*, and* T^t_{22} *is given in* $(6.3^{(i-iv)})$ *arising from a quasi triangular system with* scalar-valued *governing function.*

Case 2, triangular subcase $\varepsilon = 0$. In terms of the decomposition $\mathbf{H} \oplus \mathbb{R}^2 \simeq [\mathbb{R}u] \oplus [\mathbb{R}v] \oplus \mathbf{H}_2 \oplus \mathbb{R}^2$ where $e = (u + iv)/2,\ u, v$ unit vectors in $\mathbf{H}_0$ and the rotation group (3.9), we can write

$$S_2^{-1} G' S_2 = \begin{bmatrix} -\rho_1 + \rho_2 R' & 0 & 0 \\ [x_2, -x_1] & M_2 & 0 \\ \rho_2 - \rho_1 R' & [x_1, x_2]^{\mathrm{T}} & \rho_1 + \rho_2 R' \end{bmatrix}. \tag{6.5}$$

The integration of (6.5) is immediate with (1.6):

$$G^t = \begin{bmatrix} e^{-\rho_1 t} R^{\rho_2 t} & 0 & 0 \\ G^t_{21} & W^t_2 & 0 \\ G^t_{31} & G^t_{32} & e^{\rho_1 t} R^{\rho_2 t} \end{bmatrix} \tag{$6.5^{(\mathrm{i})}$}$$

where

$$\begin{aligned} G^t_{21} &= \big(W^t_2 [x_2, -x_1]\big) * \big(e^{-\rho_1 t} R^{\rho_2 t}\big), \\ G^t_{32} &= \big(e^{-\rho_1 t} R^{\rho_2 t} [x_1, x_2]^{\mathrm{T}}\big) * W^t_2, \\ G^t_{31} &= \big(e^{\rho_1 t} R^{\rho_2 t} (\rho_2 - \rho_1 R')\big) * \big(e^{-\rho_1 t} R^{\rho_2 t}\big) + \\ &\quad + \big(e^{\rho_1 t} R^{\rho_2 t} [x_1, x_2]^{\mathrm{T}}\big) * \big(W^t_2 [x_2, -x_1]\big) * \big(e^{-\rho_1 t} R^{\rho_2 t}\big). \end{aligned} \tag{$6.5^{(\mathrm{ii})}$}$$

Case 2 quasitriangular case $\varepsilon \neq 0$. We can improve the coordinatization (5.7) as

$$\begin{aligned} \widetilde{G}' &= Z^{-\tau}\big[S_2^{-1} G' S_2\big] Z^{\tau} = \\ &= \begin{bmatrix} 0 & \varepsilon - \rho_2 & 0 & 2\varepsilon & 0 \\ \rho_2 - \varepsilon & 0 & 0 & 0 & 2\varepsilon \\ x_2 & -x_1 & M_2 & 0 & 0 \\ \lambda & 0 & x_1^{\mathrm{T}} & 0 & -\varepsilon - \rho_2 \\ 0 & \lambda & x_2^{\mathrm{T}} & \varepsilon + \rho_2 & 0 \end{bmatrix}, \quad \tau = \frac{\rho_1}{2\varepsilon}, \quad \lambda = \frac{\rho_1^2}{2\varepsilon} - \rho_2 \end{aligned} \tag{6.6}$$

where

$$(6.6') \qquad Z^\tau = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & I_2 & 0 & 0 \\ \tau & 0 & 0 & 1 & 0 \\ 0 & \tau & 0 & 0 & 1 \end{bmatrix}, \qquad \big[Z^\tau\big]^{-1} = Z^{-\tau}.$$

To find the convolution structure of the C0-SGR $[\widetilde{G}^t : t \in \mathbb{R}_+]$ with infinitesimal generator $\widetilde{G}'$, we apply Proposition 3.6 with the grouping (3.2) with

$$\widetilde{G}' = \begin{bmatrix} V_1' & C \\ B & V_2' \end{bmatrix}, \quad V_1' = \begin{bmatrix} (\rho_2 - \varepsilon)R' & 0 \\ [x_1, -x_2] & M_2 \end{bmatrix}, \quad C = \begin{bmatrix} 2\varepsilon R^0 \\ [0,0] \end{bmatrix},$$
$$B = \left[\lambda R^0, \begin{bmatrix} x_1^{\mathrm{T}} \\ x_2^{\mathrm{T}} \end{bmatrix}\right], \qquad V_2' = (\rho_2 + \varepsilon)R'$$

in terms of the rotation group (3.9). According to (1.6), by writing $\omega = \rho_2 - \varepsilon$ for short, the C0-SGR $[V_1^t : t \in \mathbb{R}_+]$ with infinitesimal generator V_1' has the form

$$V_1^t = \begin{bmatrix} R^{\omega t} & 0 \\ W_2^t * [x_2, -x_1]R^{\omega t} & W_2^t \end{bmatrix} = \begin{bmatrix} R^{\omega t} & 0 \\ \big[X(t), Y(t)\big] & W_2^t \end{bmatrix}$$

where $[W_2^t : t \in \mathbb{R}_+]$ is the C0-SGR with infinitesimal generator M_2 and

$$X_1(t) = W_2^t * \big[(\cos(\omega t)x_2 - (\sin \omega t)x_1\big] = (\cos \omega t) * W_2^t x_2 - (\sin \omega t) * W_2^t x_1,$$
$$X_2(t) = W_2^t * \big[-(\sin \omega t)x_2 - (\cos \omega t)x_1\big] = -(\sin \omega t) * W_2^t x_2 - (\cos \omega t) * W_2^t x_1$$

since convolutions commute in case of a scalar valued term. By introducing the lower triangular C0-SGR

$$T^t = \begin{bmatrix} V_1^t & 0 \\ W_2^t * \big([x_2, -x_1]R^{\omega t} & W_2^t \end{bmatrix} \quad \text{with generator} \quad T' = \begin{bmatrix} V_1' & 0 \\ B & V_2' \end{bmatrix},$$

the C0-SGR $[\widetilde{G}^t : t \in \mathbb{R}_+]$ is the solution of the Volterra equation

$$\widetilde{G}^t = \begin{bmatrix} 0 & V_1^t C \\ 0 & V_2^t * (BV_1^t C) \end{bmatrix} * \widetilde{G}^t + T^t$$

whence

$$\widetilde{G}_{11}^t = (V_1^t C) * \widetilde{G}_{21}^t + V_1^t, \qquad \widetilde{G}_{12}^t = (V_1^t C) * \widetilde{G}_{22}^t,$$
$$\widetilde{G}_{21}^t = w * \widetilde{G}_{21}^t + V_2^t * (BV_1^t), \quad \widetilde{G}_{22}^t = w * \widetilde{G}_{22}^t + V_2^t,$$

where

$$w(t) = V_2^t * (BV_1^t C).$$

The governing function u associated with w assumes values as real 2×2-matrices and it is the solution of the convolution equation $u = w + u * w$ where w is the 2×2-matrix valued function defined in 3.5). Actually, since $\rho_2 + \varepsilon = \omega + 2\varepsilon$,

$$(6.7) \qquad w(t) = V_2^t * (BV_1^tC) = (V_2^tB) * (V_2C) =$$

$$= \begin{bmatrix} \cos(\omega{+}2\varepsilon)t & -\sin(\omega{+}2\varepsilon)t \\ \sin(\omega{+}2\varepsilon)t & \cos(\omega{+}2\varepsilon)t \end{bmatrix} \begin{bmatrix} \lambda & 0 & x_1^{\mathrm{T}} \\ 0 & \lambda & x_2^{\mathrm{T}} \end{bmatrix} * \begin{bmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ X(t) & Y(t) & W_2^t \end{bmatrix} \begin{bmatrix} 2\varepsilon & 0 \\ 0 & 2\varepsilon \\ 0 & 0 \end{bmatrix}.$$

In view of Proposition 3.6 we get

$$(6.8) \qquad \widetilde{G}^t = \begin{bmatrix} 1 & V_1^tC \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} V_1^t & 0 \\ V_2^t * (BV_1^t) + u(t) * V_2^t * (BV_1^t) & V_2^t + u * V_2^t \end{bmatrix}.$$

Here we have

$$(6.8^{(\mathrm{i})}) \qquad \begin{aligned} V_1^tC &= \begin{bmatrix} R^{\omega t} & 0 \\ W_2^t * \big([x_2, -x_1]R^{\omega t}\big) & W_2^t \end{bmatrix} \begin{bmatrix} 2\varepsilon R^0 \\ [0,0] \end{bmatrix} = \\ &= \begin{bmatrix} 2\varepsilon R^{\omega t} \\ 2\varepsilon W_2^t * \big([x_2, -x_1]R^{\omega t}\big) \end{bmatrix}, \end{aligned}$$

$$(6.8^{(\mathrm{ii})}) \qquad \begin{aligned} BV_1^t &= \left[\lambda R^0, \begin{bmatrix} x_1^{\mathrm{T}} \\ x_2^{\mathrm{T}} \end{bmatrix}\right] \begin{bmatrix} R^{\omega t} & 0 \\ W_2^t * \big([x_2, -x_2]R^{\omega t} & W_2^t \end{bmatrix} = \\ &= \left[\lambda R^{\omega t} + \begin{bmatrix} x_1^{\mathrm{T}} \\ x_2^{\mathrm{T}} \end{bmatrix} \Big[X_1(t), X_2(t)\Big], \begin{bmatrix} x_1^{\mathrm{T}} W_2^t \\ x_2^{\mathrm{T}} W_2^t \end{bmatrix}\right]. \end{aligned}$$

We can summarize the results concerning Case 2 in a context free manner as follows.

THEOREM 6.9. *Let $u, v \in \mathbf{H}_0$ be an arbitrarily fixed orthonormed couple and define $\mathbf{H}_2 := \mathbf{H}_0 \ominus [(\mathbb{R}\, u) \oplus (\mathbb{R}\, v)]$. A generic C0-SGR $[\Psi^t : t \in \mathbb{R}_+]$ of holomorphic $d_{\mathbf{B}}$-isometries, whose continuous extension to the closed unit ball admits a common fixed point on the unit sphere which is not an extreme point of the unit ball, can be written in the form*

$$\Psi^t = \Theta \circ \Phi_{G^t_{\mathcal{W}_2, x_1, x_2, \rho_1, \rho_2, \varepsilon}} \circ \Theta^{-1} \qquad (t \in \mathbb{R}_+)$$

where

(i) Θ *is any* $\mathbf{E}$*-Möbius transformation,* $\mathcal{W}_2 = [W_2^t : t \in \mathbb{R}_+]$ *is any C0-SGR of (necessarily linear)* $\mathbf{H}_2$*-isometries,* $x_1, x_2 \in \mathbf{H}_2$ *are arbitrary vectors and* $\rho_1, \rho_2, \varepsilon \in \mathbb{R}$ *are arbitrary real constants,*

(ii) *the factors* $G^t_{\mathcal{W}_2,x_1,x_2,\rho_1,\rho_2,\varepsilon}$ *are* $\mathcal{L}\big(\mathbb{R}^2 \oplus [\mathbf{H}_2 \oplus (\mathbb{R}\,u) \oplus (\mathbb{R}\,v)]\big)$*-type operator matrices of the form*

$$G^t_{\mathcal{W}_2,x_1,x_2,\rho_1,\rho_2,\varepsilon} = Z^{\tau(\rho_1,\varepsilon)} S_2\Big(\widetilde{G}^t\Big) S_2^{-1} Z^{-\tau(\rho_1,\varepsilon)}$$

in terms of the coordinatization matrices (5.6) *resp.* (6.6′) *with* $\tau(\rho,\varepsilon) := \big[0 \text{ if } \varepsilon = 0,\ \rho_1/(2\varepsilon) \text{ else}\big]$ *where* $\widetilde{G}^t\big(= \widetilde{G}^t_{\mathcal{W}_2,x_1,x_2,\rho_1,\rho_2,\varepsilon}\big)$ *is given as*

(iii_1) *in the case* $\varepsilon = 0$ *we can write* $\widetilde{G}^t_{\mathcal{W}_2,x_1,x_2,\rho_1,\rho_2,0} = \big[G^t \text{ given in } (6.5^{(\mathrm{i-ii})}\big]$.

(iii_2) *in the case of* $\varepsilon \neq 0$*, the entries of* $\widetilde{G}^t_{\mathcal{W}_2,x_1,x_2,\rho_1,\rho_2,\varepsilon}$ *are described in* (6.8) *with the subentries* $\big(6.8^{(i)-(ii)}\big)$ *arising from a quasi triangular system with the* 2×2 matrix-valued *governing function u which is the solution of the Volterra convolution equation* $u = w + w * u$ *where w is given in* (6.7) *with* $\omega := \rho_2 - \varepsilon$*.*

QUESTION 6.10. Is it possible to find a coordinatization in Case 2 which gives rise to a treatment with a unique or more but independently defined scalar-valued governing functions?

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