ON C_0-SEMIGROUPS OF HOLOMORHIC ISOMETRIES WITH FIXED POINT IN JB*-TRIPLES

L.L. STACHÓ

Communicated by

We develop an alternative self-contained approach with generalizations in the spirit of Kaup's JB*-triple theory to the results of Vesentini and Katskewitch-Reich-Shoikhet concerning the strongly continuous one-parameter semigroups (C_0-SGR) of the unit ball in infinite dimensional reflexive TROs (ternary rigns of operators). We start with a study of Hille-Yosida type arguments in the setting of bounded domains in complex Banach spaces and investigate the JB*-algebraic role of joint boundary fixed points which may result in closed explicit formulas giving a deeper insight into the structure of semigroups appearing in physical applications.

AMS 2010 Subject Classification: 47D03, 32H15, 46G20

Key words: Carathéodory distance; isometry; fixed point; holomorphic map; C_0-semigroup; infinitesimal generator; JB*-triple; Möbius transformation; Cartan factor; TRO

1. INTRODUCTION

Our aim in this paper is to extend the fixed point method developed in [21,22] in the setting of symmetric domains and investigate the structural role of joint fixed point of strongly continuous one-parameter semigroup (abbreviated with C_0-SGR (C_0-GR for groups) in the sequel) from a Jordan theoretic view point. It is well-known that the geometric actuality of the topics originates from the fact that the related results concern natural infinite dimensional generalizations for Poincaré's model of the hyperbolic (Bolyai–Lobachewski) plane giving rise to a differential geometric study of the isometries by means of complex analysis. The first natural generalization to infinite dimensions of the Poincaré plane is the unit ball of a Hilbert space with its Carathéodory distance whose invesigations were started by E. Vesentini [9,23]. Besides the problem of the algebraic description of holomorphic (Carathéodory-) isometries, a new feature appears in infinite dimensions: the possibility of non-surjective isometries along with the possibility of several different natural topologies on the semigroup of holomorphic isometries. In a celebrated paper in 1987, Vesentini [23] achieved the first deep results on C_0-SGRs of holomorphic Carathéodory isometries for the Hilbert ball using a projective linear model coupled with linear Hille-Yosida theory. However, no closed formulas were given explicitly in [23], and the results of the last section there relied heavily upon an implicit assumption: strongly continuous linear representations were used without justifying their existence among the several admissible ones. Recently, in [21,22], with joint fixed point arguments, we estabished the existence of the related strongly continuous linear representations. Hence we achieved closed formulas in terms of fixed points and fractional linear forms involving C_0-GRs of linear isometries. The involved Stone type exponential spectral resolutions gave rise even to dilations with C_0-GRs of automorphisms.

Our primary interest here will be to investigate the extendendibility of the results in [21,22] to infinite-dimensional bounded domains in Banach spaces. We pay particular attention to symmetric domains where a Harish-Chandra type representation with unit balls of JB*-triples due to W. Kaup [13] along with strong algebraic tools is available. Kaup's theory is based on an exhaustive Banach-Lie and Jordan algebraic description of *uniformly continuous groups* of ball-automorphisms. Kaup's Möbius transformations will play an essential role in this work. Notice that the category of JB*-triples includes C^*-algebras, ternary rings of operators (TRO) subspaces of bounded linear operators between two Hibert spaces and spin factors with high interest in quantum physics. As a first forerunner of this paper, later on, Vesentini [24] continued his investigations in the TRO case applying linear models with Hille-Yosida theory. He outlined methods for the solution of the related Riccati type equations, however, again with the implicit assumption of the strong continuity of the projrctive representation. He also made an attempt to spin factors [25] extending Hirzebruch's description to infinite dimensions, but with a warning negative result concerning the usual treatment by physicists of finite dimensional spin groups. In 1996, S. Reich and D. Shoikhet [19] attacked the problems from the direction of geometric functional analysis focusing to the bounday behaviour of continuously extended holomprphic isometries. Their results may be of interest concerning our problems in Remark 4.8. Toward 2000, with V. Khatskevich [14,25] they investigated the structure of C_0-SGRs on a general bounded Banach space domain. Their consideration were restricted to the locally uniformly continuous case (cf. [14, p.2]). A look at the linear case [8, II.Cor.1.6] shows that, in the setting of symmetric domains we are lead to bounded (everywhere defined) generators and hence to uniformly continuous groups as in Kaup's theory. In [25] they developed fine descriptions of C_0-SGRs of fractional linear maps with linear C_0-SGR model in Pontryagin spaces which can complement Vesentini's work and also the results of our Section 7 for non Kaup type generators.

We start with a 'compromiseless' imitation of the linear Hille-Yosida theory in Section 2 in the setting of holomorphic self-maps of bounded domains following the lines of the excellent monograph [8]. With slight modification using Cauchy estimates, we can prove the holomorphic analogs of the basic lemmas [8, II.1.1-5] except for one: the automatic density of the infinitesimal generator. Thouh it is likely that such cases are impossible, we know examples of real dynamical systems with empty generator [26]. There is another obstacle appearing in the investigation of C_0-SGRs of holomorphic isometries of the unit ball: the 0-preserving ones may be non-linear (see Remark 6.9) in contrast to the case of holomorphic automorphisms. Also their Jordan homomorphic properties may fail [5]. Section 3 is devoted to the study of this situation by means of Schwarz Lemma. Fortunately, we can establish the reqired linearity properties along with more Jordan algebraic features in reflexive JB*-triples with some geometry of tripotents (Jordan triple-idempotents) [18,1,2] discussed in Section 6 later. In Section 4 we recall the necessary material to the algebraic study of symmetric domains (unit balls without loss of generality) from Jordan theory and present some new results concerning C_0-SGRs consisting of compositions by generalized Möbius transformations and linear isometries. Section 5 contains one of our main results which can be stated in a pure geometric form as follows: if a C_0-SCR of holomorphic isometries of a bounded symmetric domain admits a common boundary fixed point the its generator is either empty or dense in the underlying domain. Section 6 is a technical preparation to cases where we can apply our previous results with restrictions to Cartan factors, namely if we have a bounded symmetric domain in a reflexive Banach space. We finish the paper in Section 7 with presenting the analogue of the first triangularization step with closed formula in [22] generalized to TRO-setting.

2. C_0-SEMIGROUPS OF HOLOMORPHIC ENDOMORPHISMS

Througout the whole work $\mathbf{E}$ denotes a complex Banach space, $\mathbf{D}$ will be a bounded domain in $\mathbf{E}$ (fixed arbitrarily), and

$$\mathrm{Hol}(\mathbf{D}) := \big\{\text{holomorphic maps } \mathbf{D} \to \mathbf{D}\big\}.$$

We shall write $d_{\mathbf{D}}$ for the Carathéodory distance on $\mathbf{D}$, that is

$$d_{\mathbf{D}}(x,y) = \sup\big\{\mathrm{artanh}\,\big|f(y)\big| : \text{for holomorphic } f : \mathbf{D} \to \Delta \text{ with } f(x) = 0\big\}$$

where $\Delta := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and $\mathbb{T} := \{\zeta \in \mathbb{C} : |\zeta| = 1\} = \partial\Delta$ are the standard notations for the unit disc and circle, respectively.

Remark 2.1. Given a holomorphic endomorphism $f \in \mathrm{Hol}(\mathbf{D})$, we know [9,11] that it is a $d_{\mathbf{D}}$-contraction, and in terms of its Taylor series

$$f(a+v) = \sum_{n=0}^{\infty} n!^{-1}[D_a^n f]v^n, \quad [D_a^n f]v^n = \big[D_{z=a}^n f(z)\big]v^n = \frac{d^n}{d\zeta^n}\Big|_{\zeta=0} f(a+\zeta v)$$

we have the Cauchy estimates

$$\big\|n!^{-1}[D_a^n f]v^n\big\| \leq \mathrm{diam}(\mathbf{D})\mathrm{dist}(a,\partial\mathbf{D})^{-(n+1)}\|v\|^n.$$

In particular f is locally Lipschitzian, and its Lipschitz constant on a convex compact subset $K \subset\subset \mathbf{D}$ can be estimated in terms of the diameter of of $\mathbf{D}$ and the distance of $\mathbf{K}$ (with respact to the norm of $\mathbf{E}$) from the boundary of $\mathbf{D}$ as follows

$$\mathrm{Lip}\big(f|K\big) \leq \mathrm{diam}(\mathbf{D})\mathrm{dist}(K,\partial\mathbf{D})^{-1}.$$

Pointwise convergent nets in Hol($\mathbf{D}$ converge uniformly on compact sets along with their derivatives [16]: $f_j \to f$ implies

$$(2.2) \qquad [D^n f_j]v^n\big|_K \rightrightarrows [D^n f]v^n\big|_K \quad (K \subset\subset \mathbf{D},\ n = 0,1,2,\dots, v \in \mathbf{E}).$$

Definition 2.3. A family $[\Phi^t : t \in \mathbb{R}_+]$ in $\mathrm{Hol}(\mathbf{D})$ is said to be a C_0-*semigroup* (C_0-SGR for short in the sequel) if $\Phi^0 = \mathrm{Id}(=$ [identity on $\mathbf{D}$]$)$, $\Phi^{t+h} = \Phi^t \circ \Phi^h$ $(t,h \in \mathbb{R}_+)$ and all the orbits $t \mapsto \Phi^t(x)$ with any starting point $x \in \mathbf{D}$ are continuous. We define the *infinitesimal generator* of $[\Phi^t : t \in \mathbb{R}_+]$ as [1]

$$\Phi' := \frac{d}{dt}\Big|_{t=0+} \Phi^t, \quad \mathrm{dom}(\Phi') = \big\{x : \exists\, v \;\; \Phi^h(x) = x + hv + o_{\mathbf{E}}(h)\big\}.$$

Henceforth $[\Phi^t : t \in \mathbb{R}_+]$ denotes an arbitrarily fixed C_0-SGR in $\mathrm{Hol}(\mathbf{D})$.

PROPOSITION 2.4. *Given any point $x \in \mathrm{dom}(\Phi')$, the orbit $t \mapsto \Phi^t(x)$ is continuously differentiable.*

Proof. By definition, $\Phi^h(x) = x + hv + o(h)$. Thus for any $t \geq 0$, $\Phi^{t+h}(x) - \Phi^t(x) = \Phi^t\big(x + hv + o(h)\big) - \Phi^t(x) = h[D_{z=x}\Phi^t(z)]v + o(h)$. In particular

[1]We use the ordr symbols o, O of Landau in normed space sense: if $(\mathbf{X}, |\cdot|)$ is a normed space, $o_{\mathbf{X}}(h)$ resp. $O_{\mathbf{X}}(h)$ mean suitable functions $\phi, \psi : \mathbb{R}_+ \to \mathbf{X}$ with $\lim_{h\to 0+} h^{-1}\phi(h) = 0$ resp. $\limsup_{h\to 0+} h^{-1}\psi(h) < \infty$. In most calculations, we omit the space indices without danger of confusion. (In most cases, clearly from the contex, $o \equiv o_{\mathbf{E}}$).

$x \in \mathrm{dom}\big(\frac{d}{ds}\big|_{s=t+0}\Phi^s\big)$ for $h \searrow 0$. That is the orbit is $t \mapsto \Phi^t(x)$ is differentiable from the right. For the left-derivatives we argue as follows. Given $t>0$ and $x \in \mathrm{dom}(\Phi')$ with $\phi^h(x) = x + hv + w_h$, $w_h = o(h)$ $(h \searrow 0)$ we have

$$\begin{aligned}&\big[\Phi^{t-h}(x) - \Phi^t(x)\big]/(-h) = \big[\Phi^{t-h}(x) - \Phi^{t-h}(x+hv+w_h)\big]/(-h) = \\ &= \big[D_x\Phi^{t-h}\big]v + \big[D_x\Phi^{t-h}\big](w_h/h) + \sum_{n>1} h^{n-1}\big[D_x^n\Phi^{t-h}\big]\big(v+w_h/h\big)^n.\end{aligned}$$

Since the singleton $\{x\}$ is compact, by (2.2), $\big[D_x\Phi^{t-h}\big]v \to \big[D_x\Phi^t\big]v$ for $h \searrow 0$. By Cauchy estimates, with $\delta := \mathrm{dist}\big(\{\Phi^s(x) : 0 \le s \le t\}, \partial D\big) > 0$, we have

$$\begin{aligned}&\big\|\big[D_x\Phi^{t-h}\big](w_h/h)\big\| \le \mathrm{diam}(D)\delta^{-1}\|w_h/h\| \to 0 \quad (h \searrow 0) \quad \text{and} \\ &\big\|\big[D_x^n\Phi^{t-h}\big](v+w_h/h)\big\| \le \mathrm{diam}(D)\delta^{n-1}\|v+w_h/h\|^n\end{aligned}$$

implying $\left\|\sum_{n>1} h^{n-1}\big[D_x^n\Phi^{t-h}\big](v+w_h/h)\right\| \to 0 \quad (h \searrow 0)$. This shows the differrnciability of $t \mapsto \Phi^t$. In course of the calculation we have seen that

$$\frac{d}{dt}\Phi^t(x) = \Phi'\big(\Phi^t(x)\big) = \big[D_x\Phi^t\big]\Phi'(x) \qquad \big(x \in \mathrm{dom}(\Phi')\big).$$

Since the singleton $\{x\}$ is compact, by (2.2), the function $t \mapsto \big[D_x\Phi^t\big]v$ is continuous for any $v \in \mathbf{E}$, in particular for $v := \Phi'(x)$ if $x \in \mathrm{dom}(\Phi')$.

COROLLARY 2.5. $\mathrm{dom}(\Phi')$ *consists of the poits* $x \in \mathbf{D}$ *with continuously differentiable orbits* $t \mapsto \Phi^t(x)$.

Remark 2.6. In classical linear Hille-Yosida theory, the continuous differentiability of differentiable orbits is trivial. Namely $\frac{d}{dt}\Phi^t(x) = \Phi^t\big(\Phi'(x)\big)$ $\big(x \in \mathrm{dom}(\Phi')\big)$ if $\Phi^t \in \mathcal{L}(\mathbf{E})$ even in real setting. However, in real Banach spaces where Cauchy type estimates are not available, there are non-linear C_0-semigroups even with empty infinitesimal generator [26].

PROPOSITION 2.7. *The graph of* Φ' *is closed.*

Proof. For $n = 1, 2, \ldots$ let $x_n \in \mathrm{dom}(\Phi')$, $v_n := \Phi'(x_n)$, and assume $x_n \to x \in \mathbf{D}$, $v_n \to v \in \mathbf{E}$. Then

$$\begin{aligned}&\frac{\Phi^h(x_n) - x_n}{h} = \int_{s=0}^h \Big[\frac{d}{ds}\Phi^s(x_n)\Big]\,ds = \int_{s=0}^h \big[D_{x_n}\Phi^s\big]v_n\,ds = \int_{s=0}^1 \big[D_{x_n}\Phi^{sh}\big]v_n\,ds, \\ &\big[D_{x_n}\Phi^{sh}\big]v_n - v = \big[D_{x_n}\Phi^{sh}\big](v_n - v) + \big(\big[D_{x_n}\Phi^{sh}\big] - \big[D_{x_n}\Phi^0\big]\big)v.\end{aligned}$$

Since the set $K := \{x\} \cup \{x_n\}_{n=1}^\infty \subset \mathbf{D}$ is compact, by (2.2) we have $[D\Phi^{sh}]v\big|K \rightrightarrows v = [D\Phi^0]v\big|K$ for $t \searrow 0$. Also $\big\|\big[D_{x_n}\Phi^t\big](v_n - v)\big\| \leq M\|v_n - v\|$ with $M := \mathrm{diam}(\mathbf{D})\mathrm{dist}(K, \partial\mathbf{D})^{-1}$. Thus the functions $f_n(t) := \big[D_{x_n}\Phi^t\big]v_n$ satisfy $\|f_n(t) - v\| \leq \max_{z\in K} \|v - D_z\Phi^t]v\| + M\|v_- v\|$. Hence $h^{-1}\big(\Phi^h(x) - x\big) = \lim_n h^{-1}\big(\Phi^h(x_n) - x_n\big) = \int_{s=0}^1 f_n(sh)\, ds \to v$ as $h \searrow 0$.

PROPOSITION 2.8. *Let* $[\Phi^t : t \in \mathbb{R}_+], [\Psi^t : t \in \mathbb{R}_+]$ *be* C_0*-SGR of holomorphic* $\mathbf{D} \to \mathbf{D}$ *maps with the same generator. Then they coincide on* $\mathrm{dom}(\Phi')\big(= \mathrm{dom}(\Psi')\big)$.

Proof. For $t, s, h \geq 0$ with $t \geq s + h$ we have

$$\frac{1}{h}\Big[\Phi^{t-(s+h)}\big(\Psi^{s+h}(x)\big) - \Phi^{t-s}\big(\Psi^s(x)\big)\Big] = \frac{1}{h}\Big[\Phi^{t-(s+h)}\big(\Psi^{s+h}(x)\big) - \\ - \Phi^{t-(s+h)}\big(\Psi^s(x)\big)\Big] - \frac{1}{h}\Big[\Phi^{t-(s+h)}\big(\Psi^s(x)\big) - \Phi^{t-s}\big(\Psi^s(x)\big)\Big];$$

$$\frac{1}{h}\Big[\Phi^{t-(s+h)}\big(\Psi^{s+h}(x)\big) - \Phi^{t-(s+h)}\big(\Psi^s(x)\big)\Big] = \frac{1}{h}\int_0^1 \Big[\frac{\partial}{\partial u}\Phi^{t-(s+h)}\big(\Psi^{s+uh}(x)\big)\Big]\, du =$$
$$= \int_{u=0}^1 \Big[\mathrm{D}_{\Psi^{s+uh}(x)}\Phi^{t-(s+h)}\Big]\Big[\frac{1}{h}\frac{\partial}{\partial u}\Psi^{s+uh}(x)\Big]\, du =$$
$$= \int_{u=0}^1 \Big[\mathrm{D}_{\Psi^{s+uh}(x)}\Phi^{t-(s+h)}\Big]\Psi'\big(\Psi^{s+uh}(x)\big)\, du \longrightarrow$$
$$\longrightarrow \Big[\mathrm{D}_{\Psi^{s+uh}(x)}\Phi^{t-(s+h)}\Big]\Psi'\big(\Psi^s(x)\big) \quad \text{as} \quad h \searrow 0;$$

$$\frac{1}{h}\Big[\Phi^{t-(s+h)}\big(\Psi^s(x)\big) - \Phi^{t-(s+h)}\big(\Psi^s(x)\big)\Big] = \frac{1}{h}\int_1^0 \Big[\frac{\partial}{\partial u}\Phi^{t-(s+h)}\Big(\Phi^h\big(\Psi^s(x)\big)\Big)\Big]\, du =$$
$$= -\int_0^1 \Big[\mathrm{D}_{\Psi^s(x)}\Phi^{t-(s+h)}\Big]\Big[\frac{1}{h}\frac{\partial}{\partial u}\Phi^{uh}\big(\Psi^s(x)\big)\Big]\, du \longrightarrow$$
$$\longrightarrow -\Big[\mathrm{D}_{\Psi^s(x)}\Phi^{t-(s+h)}\Big]\Phi'\big(\Psi^s(x)\big) \quad \text{as} \quad h \searrow 0$$

because the maps $(y, \tau, w) \mapsto \Big[\mathrm{D}_y\Phi^\tau\Big]w$ resp. $(y, \tau, w) \mapsto \Big[\mathrm{D}_y\Psi^\tau\Big]w$ are continuous on any domain $\mathbf{K} \times [0, t] \times \mathbf{W}$ with compact $\mathbf{K} \subset \mathbf{D}$ $\big($actually $\mathbf{K} := \{\Psi^s(x) : s \in [0, t]\}\big)$ and compact balanced $\mathbf{W} \subset \mathbf{E}$ with $\mathbf{K} + \mathbf{W} \subset \mathbf{D}$. It follows $\frac{d}{ds}\Phi^{t-s}\Big(\Psi^s(x)\Big) = \Psi'\big(\Psi^s(x)\big) - \Phi'\big(\Psi^s(x)\big) = 0$ implying that $[0, t] \ni s \mapsto \Phi^{t-s}\Big(\Psi^s(x)\Big)$ is constant. In particular, by considering $s = 0$ resp. $s = t$ we get $\Phi^t(x) = \Psi^t(x)$.

Remark 2.9. Once we know that $\mathrm{dom}(\Phi')$ is dense in $\mathbf{D}$ (which is well known if the maps Φ^t, Ψ^t are linear) we can conclude the coincidence $\Phi^t = \Psi^t$ $(t \in \mathbb{R}_+)$. However, it seems to be a hard open problem if this density holds in our holomorphic setting. It is also an open question if $[\Phi^t : t \in \mathbb{R}_+]$ can be chosen to admit only nowhere differentiable orbits.

3. HOLOMORPHIC ISOMETRIES OF THE UNIT BALL

Definition 3.1. Throughout this section $\mathbf{E}$ is an arbitrarily fixed complex Banach space, $\mathbf{D}$ denotes a bounded domain in $\mathbf{E}$ and $\mathbf{B} := \{x \in \mathbf{E} : \|x\| < 1\}$, $\partial\mathbf{B} := \{x \in \mathbf{E} : \|x\| = 1\}$ will be the standard notations for the unit ball and sphere in $\mathbf{E}$, respectively. We shall write

$$\mathrm{Iso}_h(\mathbf{D}) := \big\{\text{holomorphic } d_{\mathbf{D}}\text{-isometries}\big\}, \quad \delta_{\mathbf{D}}(a, v) := \frac{d}{dt}\Big|_{t=0+} d_{\mathbf{D}}(a + tv, a)$$

for the family of all *Carathéodory isometries* of $\mathbf{D}$ resp. the *infinitesimal Carathéodory metric* of $\mathbf{D}$ at a point $a \in \mathbf{D}$. In case of the unit ball we have

$$d_{\mathbf{B}}(0, x) = \mathrm{artanh}\, \|x\| \quad (x \in \mathbf{B}), \qquad \delta_{\mathbf{B}}(v) = \|v\| \quad (v \in \mathbf{E}).$$

In this section we consider a holomorphic endomorphism $\Phi \in \mathrm{Iso}(d_{\mathbf{B}})$ leaving the origin fixed: $0 = \Phi(0)$. We write its Taylor series in the form

$$\Phi(x) = Ux + \Omega(x) = Ux + \sum_{n=2}^{\infty} \Omega_n(x), \quad \Omega_n(x) := n!^{-1}\big[D_0^n\Phi\big]x^n. \tag{3.2}$$

PROPOSITION 3.3. *Φ maps the spheres $\rho\partial\mathbf{B} = \{x : \|x\| = \rho\}$ resp. the balls $\rho\mathbf{B} = \{x : \|x\| < \rho\}$ $(0 \le \rho < 1)$ into themselves.*

Proof. It is well-known [9] that the Fréchet derivatives

$$D_a\Psi = D_{z=a}\Psi(z) : v \mapsto \frac{d}{d\zeta}\Big|_{\zeta=0} \Psi(a + \zeta v)$$

of a holomorphic $\big[d_{\mathbf{D}_1} \to d_{\mathbf{D}_2}\big]$-isometry $\Psi : \mathbf{D}_1 \to \mathbf{D}_2$ between two bounded domains are (complex-linear) $\big[\delta_{\mathbf{D}_1}(a, \cdot) \to \delta_{\mathbf{D}_2}(\Psi(a), \cdot)\big]$-isometries. In particular U is necessarily an $\mathbf{E}$-isometry: $\|Ux\| = \|x\|$ $(x \in \mathbf{E}$. Furthermore, since $\Phi \in \mathrm{Iso}_{\mathbf{B}}$, for any $x \in \mathbf{B}$, we have

$$\mathrm{artanh}\, \|x\| = d_{\mathbf{B}}(0, x) = d_{\mathbf{B}}\big(\Phi(0), \Phi(x)\big) = d_{\mathbf{B}}\big(0, \Phi(x)\big) = \mathrm{artanh}\, \|\Phi(x)\|.$$

Question. 3.4. Under which hypothesis is ϕ linear (i.e. $\Phi = U$)?

LEMMA 3.5. *We have* $\Phi = U$ *if and only if* $\mathrm{range}(\Phi) \subset \mathrm{range}(U)$.

Proof. Trivially $\mathrm{range}(\Phi) \subset \mathrm{range}(U)$ if $\Phi = U$. Otherwise, by assumption, the map $\widetilde{\Phi} := U^{-1} \circ \Phi$ is a well-defined $\mathbf{B} \to \mathbf{B}$ holomophy with $\widetilde{\Phi}(0) = 0$ and $D_0\widetilde{\Phi} = U^{-1}D_0\Phi = U^{-1}U = \mathrm{id}_{\mathbf{E}}$. From the classical Cartan's Uniqueness Theorem [9,11] it follows $\widetilde{\Phi} = \mathrm{id}_{\mathbf{B}}$ whence the statement is immediate.

Definition 3.6. Given a unit vector $y \in \partial\mathbf{B}$, we write

$$\mathcal{S}(y) := \big\{L \in \mathcal{L}(\mathbf{E}, \mathbb{C}) : 1 = \langle L, y\rangle = \|L\|\big\}$$

for the family of all *supporting* $\mathbb{C}$*-linear functionals* of $\mathbf{B}$ at y.

LEMMA 3.7. *Given a point* $x \in \partial\mathbf{B}$ *along with a vector* $v \in \mathbf{E}$ *such that* $x + \Delta v \subset \partial\mathbf{B}$*, we have*

$$\big\langle L, \Phi\big(\zeta(x + \eta v)\big)\big\rangle = 1 \quad (\zeta, \eta \in \Delta) \quad \text{for all } L \in \mathcal{S}(Ux).$$

Proof. Let $L \in \mathcal{S}(Ux)$ and consider the holomorphic map $\Phi_{x,v} : \Delta^2 \to \mathbb{C}$ defined as

$$\Phi_{x,v}(\zeta, \eta) := U(x + \eta v) + \sum_{n=2}^{\infty} \zeta^{n-1}\eta^n \Omega_n\big(\zeta(x + \eta v)\big) \quad (|\zeta|, |\eta| < 1).$$

Observe that, for any $0 \neq \zeta, \eta \in \Delta$, we have $\Phi_{x,v}(\zeta, \eta) = \zeta^{-1}\Phi\big(\zeta(x + \eta v)\big)$ implying

$$\|\Phi_{x,v}(\zeta, \eta)\| = |\zeta|^{-1}\|\Phi\big(\zeta(x + \eta v)\big)\| = |\zeta|^{-1}\|\zeta(x + \eta v)\| = \|\zeta(x + \eta v)\| = 1.$$

Thus $\Phi_{x,v.L} : (\zeta, \eta) \mapsto \langle L, \Phi_{x,v}(\zeta, \eta)\rangle$ is a holomorphic function on Δ^2 with $|\Phi_{x,v,L}(\zeta, \eta)| \leq \|L\| = 1$ and

$$\Phi_{x,v,L}(0, 0) = \lim_{0 \neq \zeta, \eta \to 0} \Phi_{x,v,L}(\zeta, \eta) = \langle L, \Phi_{x,v}(0, 0)\rangle = \langle L, Ux\rangle = 1.$$

By the Maximum Principle, $\Phi_{x,v,L} \equiv 1$ which completes the proof.

COROLLARY 3.8. $\langle L, \Omega_n(Uy)\rangle = 0$ *for all* $y \in \partial\mathbf{B}$ *and* $L \in \mathcal{S}(Uy)$.

Proof. Given $L \in \mathcal{S}(Uy)$ where $y \in \partial\mathbf{B}$, for all $\zeta \in \Delta$ (even with $\zeta = 0$) we have

$$1 \equiv \big\langle L, \zeta^{-1}\Phi(\zeta y)\big\rangle = \Phi_{\zeta,0} = \Big\langle L, Uy + \sum_{n=2}^{\infty} \zeta^{n-1}\Omega_n(Uy)\Big\rangle.$$

Notation 3.9. In terms of the Taylor expansion (3.2), let

$$F(\zeta,x) := \zeta^{-1}\Phi(\zeta x),\ F(0,x) := Ux \quad (0 \neq \zeta \in \Delta,\ x \in \mathbf{B}).$$

Notice that F is holomorphic around the origin with $\mathrm{ran}(F) \subset \partial\mathbf{B}$ and

$$F(\zeta,x)=Ux+\sum_{n=1}^{\infty}\zeta^n\Omega_{n+1}(x).$$

LEMMA 3.10. *Let* $\mathbf{K} \subset \partial\mathbf{B}$ *be a convex subset of the unit sphere. Then for its convex hull we have* $\mathrm{Conv}\big(F(\Delta,\mathbf{K})\big) \subset \partial\mathbf{B}$.

Proof. Assume $x_1,\ldots,x_k \in \mathbf{K}$, $\zeta_1,\ldots,\zeta_k \in \Delta$ and consider a convex combination

$$y := \sum_{j=1}^{k}\lambda_j F(\zeta_j,x_j) \ \text{ where } \ \sum_{j=1}^{k}\lambda_j = 1,\ \lambda_1,\ldots,\lambda_k > 0.$$

We have to see that $y \in \partial\mathbf{B}$. Consider the points

$$y_t := \sum_{j=1}^{k}\lambda_j F(e^{2\pi it}\zeta_j,x_j) \quad (t \in \mathbb{R}).$$

We have $\|y_t\| \leq 1$ $(t \in \mathbb{R})$ since F ranges in the unit sphere. On the other hand

$$\int_0^1 y_t\,dt = \sum_{j=1}^{k}\lambda_j \int_0^1 \big[Ux_j + \sum_{n=1}^{\infty} e^{2n\pi it}\Omega_{n+1}(x_j)\big]dt = \sum_{j=1}^{k}\lambda_j Ux_j = U\sum_{j=1}^{k}\lambda_j x_j.$$

By assumption $x := \sum_{j=1}^{k}\lambda_j x_j \in \mathbf{K}$ implying that $\|Ux\| = 1$ and necessarily $\|y_t\| \equiv 1$. In particular $y = y_0 \in \partial\mathbf{B}$.

Remark 3.11. The map Φ extends holomorphically to some spherical neighborhood of $\overline{\mathbf{B}}$ by a result of Braun-Kaup-Upmeier [4]. We denote the extension also by Φ without danger of confusion. An application of the the arguments of the lemma with $\zeta_j = 1$ and the extended Φ yields the following.

COROLLARY 3.12. *If* $\mathbf{F}$ *is a face of* $\overline{\mathbf{B}}$ *then* $\Phi(\mathbf{F})$ *is contained in some face of* $\overline{\mathbf{B}}$ *again.*

4. JB*-TRIPLES, MÖBIUS TRANSFORMATIONS

Assumption 4.1. Henceforth throughout the whole work we assume that $\mathbf{E}$ is a *JB*-triple*. That is the unit ball $\mathbf{B}$ of $\mathbf{E}$ is a holomorphically homogeneous (and hence symmetric) domain. It is well-known [13, 11]. that this assumption is equivalent to the existence of a (necessarily unique) continuous operation of three variables the so-called *triple product*

$$(x, y, z) \mapsto \{xy^*z\}$$

defined for all tuples from $\mathbf{E}^3$ with values in $\mathbf{E}$ and satisfying the axioms

$(J1)$ $\{xy^*z\}$ is symmetric linear in x, z and conjugate-linear in y,

$(J2)$ $\big\{ab^*\{xy^*z\}\big\} = \big\{\{ab^*x\}y^*z\big\} - \big\{x\{ba^*y\}^*z\big\} + \big\{xy^*\{ab^*z\}\big\}$,

$(J3)$ $\big\|\exp\big(\zeta\{aa^*\cdot\}\big)\big\| \leq 1$ whenever $\mathrm{Re}(\zeta) \leq 0$,

$(J4)$ $\big\|\{xx^*x\}\big\| = \|x\|^3$.

The geometric importance of JB*-triples relies upon the fact that any bounded symmetric Banach space domain is biholomorphically eqivalent to the unit ball of some JB*-triple. In this section we establish some terminology and recall some basic results concerning JB*-triples.

We reserve the notations $L(a,b), Q(a,b), B(a,b)$ for the real-linear operators

$$L(a,b)x := \{ab^*x\}, \quad Q(a,b)x := \{ax^*b\}, \quad B(a,b) := \mathrm{Id} - 2L(a,b) + Q(a,b)^2$$

with the abbreviations $L(a) := L(a,a)$, $Q(a) := Q(a,a)$, $B(a) := B(a,a)$. Usually thay are called *multiplication-, quadratic representation-* and the *Bergman* operators. Notice that $(J2)$ is equivalent to saying that each multiplication $iL(a)$ is a derivation of the triple product, while $J(2)$ means that $L(a)$ is an $\mathbf{E}$-hermitian operator with non-negative spectrum. Furthermore we can deduce the norm-identity $\|a\|^2 = \|L(a)\| = \mathrm{radSp}\big(La\big) = \max\mathrm{Sp}\big(L(a)\big)$.

Definition 4.2. A *Möbius transformation* in $\mathbf{E}$ is the holomorphic continuation of some holomorphic automorphism of the unit ball $\mathbf{B}$ to a maximal spherical neighborhood (with center 0) of $\overline{\mathbf{B}}$ (cf. Rem.3.11). W. Kaup [13] established the following canonical form

$$\Phi = M_a \circ U \quad \text{with} \quad a = \Phi(0), \quad \mathrm{dom}(\Phi) = \|a\|^{-1}\mathbf{B}$$

in terms of a surjective linear isomerty U of $\mathbf{E}$ and a *Möbius-shift*

$$M_a : x \mapsto a + B(a)^{1/2}[1 + L(x,a)]^{-1}x \qquad (a \in \mathbf{B},\ \|x\| < \|a\|^{-1}).$$

In the sequel we reserve the notation M_a for Möbius shifts. Two maps $\Phi, \Psi : \mathbf{B} \to \mathbf{B}$ are said to be *Möbius equivalent* if

$$\Psi = \Theta \circ \Phi \circ \Theta^{-1} \qquad \text{for some } \ \Theta \in \mathrm{Aut}(\mathbf{B}).$$

Remark 4.3. The use of Möbius equivalence relies upon the fact that any C_0-SGR $[\Phi^t : t \in \mathbb{R}_+]$ of $\mathrm{Iso}(d_\mathbf{B})$ with $\mathrm{dom}(\Phi') \neq \emptyset$ is Möbius equivalent to some where the orbit of the origin is differentiable, e.g. $[M_{-a} \circ \Phi^t \circ M_a : t \in \mathbb{R}_+]$ with any choice of $a \in \mathrm{dom}(\Phi')$. In Kaup's theory for uniformly continuous one-parameter groups of Möbius transformations, a crucial role was played by the linearity of the isotropy subgroup of the origin due to Cartan' Uniqueness Theorem. However, this is not automatic for non-surjective Carathéodory isometries (see Remark 4.7 later). Next we start the study of the algebraically well behaving situation

$$\Phi^t = M_{a_t} \circ U_t, \quad t \mapsto a_t \text{ differentiable}, \ U_t \text{ linear } \mathbf{E}\text{-isometry}. \tag{4.4}$$

LEMMA 4.5. *Under* (4.4)*, the following statements are equivalent:*
(i) *the orbit $t \mapsto \Phi^t(x)$ is differentiable,* (ii) *$t \mapsto U_t x$ is differentiable,*
(iii) *$U_t x = x + tu' + o(t)$ $(t \searrow 0)$ for some $u' \in \mathbf{E}$.*

Proof. From Proposition 2.4 and Corollary 2.5 we know that $x \in \mathrm{dom}(\Phi')$ iff the orbit $t \mapsto \Phi^t(x)$ is differentiable which is equivalent to the right sided differentiability $(*)$ $\Phi^t(x) = z + tv' + o(t)$ $(t \searrow 0)$ for some $v' \in \mathbf{E}$. Thus it suffices to see the equivalence of $(*)$ to $(**)$ $U_t(x) = x + tu + o(t)$ for some $u \in \mathbf{E}$.

Since $U_t x = M_{a_t}^{-1}\big(\Phi^t(x)\big) = M_{-a_t}\big(\Phi^t(x)\big)$ $(t \in \mathbb{R}_+)$, $a_0 = \Phi^0(0) = 0$ and $a_t = ta' + o(t)$ $(t \searrow 0)$ with $a' := \frac{d}{dt}\big|_{t=0+} a_t$, both implications $(*) \Rightarrow (**)$ and $(**) \Rightarrow (*)$ are immediate from the observation below.

LEMMA 4.6. *The mapping $(a,z) \mapsto M_a(z)$ is real-analytic on the domain $\big\{(a,z) \in \mathbf{E}^2 : \|a\| < 1,\ \|z\| < 1/\|a\|\big\}$. For any $c \in \mathbf{B}$, $u \in \overline{\mathbf{B}}$, $v, w \in \mathbf{E}$ we have*

$$\begin{aligned} &M_{c+hv+o(h)}\big(u + hw + o(h)\big) = \\ &= M_c(u) - h\big(L(w,c) + L(u,v)\big)u + h\big(1 + L(u,c)\big)^{-1} w + o(h) \quad (h \searrow 0). \end{aligned}$$

Proof. The real analyticity of $(a,z) \mapsto M_a(z)$ on the mentioned domain is proved in [13]. Its power series around 0 converges locally uniformly on the bi-balls $\big[\rho\mathbf{B}\big] \times \big[\rho^{-1}\mathbf{B}\big]$ $(\rho < 1/3)$ as a direct consequence of the norm relations $\|L(x,y)\|, \|Q(x,y)\| \leq \|x\| \cdot \|y\|$ in the binomial expansion $B(a_t)^{1/2} =$

$\sum_{n=0}^{\infty}\binom{1/2}{n}\big[-2L(a)+Q(a)^2\big]^n$ and the series $\big(1+L(u,c)\big)^{-1}=\sum_{n=0}^{\infty}(-1)^nL(u,c)^n$.
Hence, for $\|c\|<1/3$, $\|u\|<3$ we have

$$\begin{aligned}&M_{c+hv+o(h)}\big(u+hw+o(h)\big)=\big(c+hv+o(h)\big)+\\&+B\big(c+hv+o(h)\big)^{1/2}\big(1+L(u+hw+o(h),c+hv+o(h))\big)^{-1}\big(u+hw+o(h)\big)=\\&=M_c(u)-h\big(L(w,c)+L(u,v)\big)u+h\big(1+L(u,c)\big)^{-1}w+o(h).\end{aligned}$$

This is a poynomial relation concerning the directional derivatives of the map $(a,z)\mapsto M_a(z)$ which is valid on a neighborhood of the origin. With analytic continuation, it holds on the whole (connected) domain of analyticity.

PROPOSITION 4.7. *Under* (4.4)*, the infinitesimal generator is of* Kaup's type*: for some $a'\in\mathbf{E}$ and a not necessarily bounded closed linear $\mathbf{E}$-operator U' with* $\mathrm{dom}(\Phi')=\mathrm{dom}(U')\cap\mathbf{B}$ *we have*

$$\Phi'(x)=a'+U'x-\{x[a']^*x\}\qquad\big(x\in\mathrm{dom}(\Phi')\big).$$

Proof. By assumption $a_t=ta'+o(t)$ with $a':=\frac{d}{dt}\big|_{t\to0+}a_t$. Suppose $x\in\mathrm{dom}(\Phi')$. According to Lemma 4.5, we can also write $U_tx=x+tU'x$ where $U'x:=\frac{d}{dt}\big|_{t\to0+}U_tx$. An application of Lemma 4.6 with $c:=0$, $h:=t$, $v:=a'$, $u:=x$, $w:=U'x$ yields

$$\begin{aligned}&\Phi^t(x)=M_{a_t}\big(U_tx\big)=M_{t+a'+o(t)}\big(x+tU'x+o(t)\big)=\\&=x-t\big[L(U'x,0)+L(x,a')\big]x+tU'x+o(t)=x-tU'x+t\big\{x[a']^*x\big\}+o(t).\end{aligned}$$

The set $\mathbf{U}:=\big\{z:\frac{d}{dt}\big|_{t\to0+}U_tz\text{ exists}\big\}$ is a linear submanifold of $\mathbf{E}$ and the mapping $\widetilde{U}':z\mapsto\frac{d}{dt}\big|_{t\to0+}U_tz$ is linear due to the linearity of the maps U_t. Also $\mathrm{dom}(\Phi')\subset\mathbf{U}\cap\mathbf{B}$.

Remark 4.8. Open problems: Let $\big[\Phi^t:t\in\mathbb{R}\big]$ be any C_0-SGR of holomorphic Carathéodory isometries of the unit ball in a JB*-triple. (1) Is

(4.9) $\qquad\Phi^t=M_{a_t}\circ U_t$ with linear $\{..^*.\}$-homomorphic isometries U_t

valid without further assumptions? (2) Is Φ' defined on a dense subset of $\mathbf{B}$? (3) Is U' in 4.7 necessarily the generator of a C_0-SGR of linear isometries?

5. C_0-SGR WITH COMMON FIXED POINT IN JB*-TRIPLES

Throughout this section we assume that $\big(\mathbf{E},\{..^*.\}\big)$ is a JB*-triple and $[\Phi^t : t\in\mathbb{R}_+]$ is a C_0-SGR of Carathéodory isometries of the unit ball $\mathbf{B}$ with the property (4.9). We shall use the canonical decomposition

$$\Phi^t = M_{a_t}\circ U_t \quad \text{with} \quad a_t := \Phi^t(0) \quad \text{and} \quad U_t\in\mathcal{U}(\mathbf{E}) := \big\{\text{linear } \mathbf{E}-\text{isometries}\big\}.$$

Furhermore we assume that $\mathrm{dom}(\Phi')\neq\emptyset$, moreover the origin belongs to the domain of the generator and the holomorphic extensions of the maps Φ^t admit a common fixed point in closed unit ball, that is

$$(5.1') \qquad t\mapsto a_t := \Phi^t(0) \ \text{ is differentiable}, \quad a_t = ta' + o_{\mathbf{E}}(t) \ \ (t\searrow 0).$$

$$(5.1'') \qquad M_{a_t}(U_t e) = e\in\overline{\mathbf{B}} \qquad (t\in\mathbb{R}).$$

We may assume (5.1′) without loss of generality whenever $\mathrm{dom}(\Phi')\neq\emptyset$ by passing to $\widetilde{\Phi}^t := M_c^{-1}\circ\Phi^t\circ M_c = M_{-c}\circ\Phi^t\circ M_c$ instead of Φ^t with any pont $c\in\mathrm{dom}(\Phi')$. It is folklore (for a reference see [17] e.g.) that all Möbius transformations are weak*-continuous in case $\mathbf{E}$ admits a predual. Hence the fixed point property (5.1″) is guaranteed automatically (by Schauder's fixed point theorem and the weak*-compactness of the closed unit ball) in JBW*-triples, in particular in JB*-triples of finite rank.

Definition 5.2. For the Fréchet derivatives at the fixed point, we write

$$\Lambda^t := \mathrm{D}_e\Phi^t \ \Big(: z\mapsto \frac{d}{dt}\Big|_{t=0}\Phi^t(e+tz)\Big) \quad (t\in\mathbb{R}_+).$$

LEMMA 5.3. *The family* $[\Lambda^t : t\in\mathbb{R}]$ *is a* C_0*-SGR of bounded linear operators. In particular*

$$\mathrm{dom}(\Lambda') \quad \text{is a dense linear submanifold in } \mathbf{E}.$$

Proof. Notice that the family $[\Lambda^t : t\in\mathbb{R}]$ is a one-parameter semigroup of bounded linear operators since each map Φ^t is defined on some neighborhood of e (moreover even of $\overline{\mathbf{B}}$) whence the composition property $\Phi^t\circ\Phi^s = \Phi^{t+s}$ implies $\Lambda^t\Lambda^s = \Lambda^{t+s}$ $(s,t\in\mathbb{R}_+)$. Using the estimates $\|L(a,b)\|, \|Q(a,b)\|\le\|a\|\cdot\|b\|$, a look at the power series expansion of the Möbius parts M_{a_t} ensures that Φ^t maps the ball $2\mathbf{B}$ into $4\mathbf{B}$ whenever we have $\|a_t\|<1/8$. As a consequence of Lemma 4.6, for $t\searrow 0$ we have $\Phi^t(z)\to z$ and hence

$$\Lambda^t z = (2\pi i)^{-1}\int_{|\zeta|=1}\zeta^{-1}\Phi^t(e+\zeta z)\,d\zeta \to (2\pi i)^{-1}\int_{|\zeta|=1}\zeta^{-1}z\,d\zeta = z$$

with uniformly bounded pointwise norm convergence in the integration for any $z \in \mathbf{B}$ whenever $\|a_t\| < 1/8$. Therefore the family $[\Lambda^t : t \in \mathbb{R}]$ is a C_0-SGR of bounded linear operators. The classical linear Hille-Yosida theory ensures the density of the domain of its generator.

THEOREM 5.4. *The domain of the infinitesimal generator of a C_0-SGR consisting of maps composed from Möbius transformations and linear isometries in a JB*-triple with a common fixed point in the closed unit ball is either dense in the unit ball or empty.*

Proof. As we noted, by passing to suitable a Möbius equivalent C_0-SGR, it suffices to see that $\mathrm{dom}(\Phi')$ is a dense subset of the unit ball $\mathbf{B}$ under the assumptions $(4.9' - 9'')$. We establish this density property by showing that

$$\mathrm{dom}(\Lambda') \subset \mathrm{dom}(\Phi') \tag{5.5}$$

or, which is the same by Lemma 4.5,

$$(5.5') \quad z \in \mathbf{B} \text{ with } \Lambda^t z = z + tz' + o(t), \Rightarrow U_t z = z + tu' + o(t) \ \text{ for some } u' \in \mathbf{E}.$$

Suppose $z \in \mathbf{B}$ with $\Lambda^t z = z + tz' + o(t)$ $(t \searrow 0)$. To prove $(5.5')$, let us consider any parameter $t \in \mathbb{R}_+$ being so small that $\|a_t\| < 1/4$. By writing $a := a_t,\ U := U_t,\ \Phi := \Phi^t$ for short, we have

$$\begin{aligned} &\Phi(z+e) - e = (A_z + B)^{-1} C z \\ &\text{where} \quad A_z = L(Uz, a) B(a)^{-1/2}, \quad B = [1 + L(Ue, a)] B(a)^{-1/2}, \\ &\qquad\qquad C = U + L(U\bullet, a) B(a)^{-1/2}(a - e). \end{aligned} \tag{5.6}$$

Indeed, by setting $w := \Phi(e+z) - e$,

$$\begin{aligned} w + e = \Phi(e+z) &= M_a(Uz + Ue) = \\ &= a + B(a)^{1/2} \big[1 + L(Uz{+}Ue\,, a)\big]^{-1} (Uz + Ue), \\ \big[1 + L(Uz{+}Ue\,, a)\big] &B(a)^{-1/2} \big(w + (e-a)\big) = Uz + Ue. \end{aligned}$$

On the other hand, by the fixed point property $\Phi(e) = M_a(Ue) = e$ we have $Ue = M_{-a}(e) = \big[1 + L(Ue, a)\big] B(a)^{-1/2}(e - a)$, whence we get (5.6) as follows:

$$\begin{aligned} &Uz = (U(z+e) - Ue = \\ &= \big[1{+}L(Uz{+}Ue\,, a)\big] B(a)^{-1/2} \big(w{+}(e{-}a)\big) - \big[1{+}L(Ue, a)\big] B(a)^{-1/2}(e{-}a), \\ &= \big[1{+}L(Uz{+}Ue\,, a)\big] B(a)^{-1/2} w + L(Uz, a) B(a)^{-1/2}(e{-}a), \\ &w {=} B(a)^{1/2} \big[1{+}L(Uz{+}Ue\,, a)\big]^{-1} \big[Uz {-} L(Uz, a) B(a)^{-1/2}(e{-}a)\big] {=} (A_z{+}B)^{-1} C z. \end{aligned}$$

By passing to Fréchet derivatives, from (5.6) we obtain

$$\begin{aligned}\Lambda^t z &= \Lambda z = \mathrm{D}_e \Phi = \frac{\partial}{\partial z}\Big|_{z=0} (A_z + B)^{-1} C z = \frac{d}{d\tau}\Big|_{\tau=0+} (A_{\tau z} + B)^{-1} C z = B^{-1} C z = \\ &= B(a_t)^{1/2} \big[1 + L(U_t e, a_t)\big]^{-1} \big[U_t z + L(U_t z, a_t) B(a_t)^{-1/2} (a_t - e)\big], \\ U_t z &= \big[1 + L(U_t e, a_t)\big] B(a_t)^{-1/2} \Lambda^t z - L(U_t z, a_t) B(a_t)^{-1/2} (a_t - e).\end{aligned}$$

For $t \searrow 0$ we know the convergence rates

$$a_t = ta' + o_{\mathbf{E}}(t), \quad U_t z = z + o_{\mathbf{E}}(1), \quad B(a_t)^{\pm 1/2} = \mathrm{Id} + o_{\mathcal{L}(\mathbf{E})}(t).$$

Indeed, $a_t = ta' + o(t)$ by asumption, $U_t z \to z$ by Lemma 4.5 because $t \mapsto \Phi^t(e) = e$ is differentiable trivially, while the relation $B(a)^{\pm 1/2} = 1 + o(1)$ is a consequence of the binomial expansion $B(a_t)^\kappa = \sum_{n=0}^{\infty} \binom{\kappa}{n} \big[- 2L(a_t) + Q(a_t)^2\big]^n$ where $\|L(a_t)\| = \|Q(a_t)\| = \|a_t\|^2 = O(t^2)$. It follows

$$\begin{aligned}U_t z = \Big[1 + L\big(e + te' + o(t), ta' + o(t)\big)\Big] \big[1 + o(t)\big] \big(z + tz' + o(t)\big) - \\ - L\big(z + o(1), ta' + o(t)\big) \big[1 + o(t)\big] \big(ta' + o(t) - e\big) = \\ = z + tL\big(z, a'\big)z + tL\big(z, a'\big)e + o(t) = ztu' + o(t)\end{aligned}$$

with $u' := L\big(z, a'\big)z + L\big(z, a'\big)e$ which completes the proof.

6. JB*-TRIPLES WITH FINITE RANK

In JB*-triple theory, an analogous role to projectors in C^*-algebras is played by the family of *tripotents* (idempotents of 3rd degree)

$$\mathrm{Trip}(\mathbf{E}) := \big\{e \in \mathbf{E} : \{ee^*e\} = e\big\}.$$

Notice that non-zero tripotents are unit vectors due to $(J4)$. It is an important geometrical feature of tripotents [6,7,18,1] that if $\mathbf{E}$ JBW*-triple (that is $\mathbf{E}$ admits a norm predual analogously to $W*$-algebras) and $\mathbf{B} \neq \mathbf{F}$ is a norm-exposed face of $\overline{\mathbf{B}}$ then for some $e \in \mathrm{Trip}(\mathbf{E}$ we have

$$\mathbf{F} = \big\{x \in \partial\mathbf{B} : \ x - e \perp^{\mathrm{Jordan}} e\big\} = \big\{M_c(e) : \ c \perp^{\mathrm{Jordan}} e, \ \|c\| \leq 1\big\}$$

with the concept of *Jordan-orthogonality*: $a \perp^{\mathrm{Jordan}} b$ if $L(a, b) = L(b, a) = 0$. It is well-known that $e \perp^{\mathrm{Jordan}} x \iff L(e)x = 0$ whenever e is a tripotent.

Assumption 6.1. Throughout this section we assume that

$$(\mathbf{E},\{\ldots\}) \text{ is a JB*triple with } \quad \mathrm{rank}(\mathbf{E}) = r < \infty.$$

We are goint to establish (4.9) in this case. This is contained implicitly in [2] by Apazoglou-Peralta (even for real setting). Here we present a simple geometric argument based on the following well-known facts.

Remark 6.2. It is well-known [12,17] that $\mathbf{E}$ is reflexive, as being an ℓ^∞-direct sum of finitely many Cartan factors of which only the types $\mathcal{L}(\mathbf{H}_1,\mathbf{H}_2)$ and Spin factors can be infinite dimensional. According to [6,18], the norm exposed faces of the unit ball $\mathbf{B}$ are in a natural one-to-one correspondance with the *tripotents* of $\mathbf{E}$ as being of the form

$$\begin{aligned}\mathrm{Face}(\mathbf{B},e) = \big\{y\in\partial\mathbf{B} : \langle L,y\rangle = 1 \ \text{ for all } L\in\mathcal{S}(e)\big\} = \\ = \big\{e+v : v\perp^{\mathrm{Jordan}} e,\ \|v\|\le 1\big\} \qquad (e\in\mathrm{Trip}(\mathbf{E})).\end{aligned}$$

LEMMA 6.3. *Let $a,b\in\partial\mathbf{B}$ be unit vectors with $\|\alpha a+\beta b\| = \max\{|\alpha|,|\beta|\}$ $(\alpha,\beta\in\mathbb{C})$. Then*

$$a = e+a_0,\quad a_0,b\perp^{\mathrm{Jordan}} e,\quad b = f+b_0,\quad b_0,a\perp^{\mathrm{Jordan}} f,\quad e\perp^{\mathrm{Jordan}} f$$

with suitable tripotents $e,f\in\mathrm{Trip}(\mathbf{E})$ and vectors $a_0,b_0\in\overline{\mathbf{B}}$.

Proof. Since $a,b\in\partial\mathbf{B}$, we have

$$\begin{aligned}a\in\mathrm{Face}(\mathbf{B},e),\ a = a_0+e, a_0\perp^{\mathrm{Jordan}} e,\\ b\in\mathrm{Face}(\mathbf{B},f),\ b = b_0+e, b_0\perp^{\mathrm{Jordan}} f\end{aligned}$$

with suitable $e,f\in\mathrm{Trip}(\mathbf{E})$ and vectors $a_0,b_0\in\overline{\mathbf{B}}$. By assumption $\|a{+}\beta b\| = 1$ whenever $|\beta|\le 1$. That is the disc $a+\Delta b = a+a_0+\Delta b$ is also contained in the face $\mathrm{Face}(\mathbf{B},e)$ of the point a. Similarly (with the chages $a\leftrightarrow b, e\leftrightarrow f, a_0\leftrightarrow b_0$), $b+\Delta a\subset\mathrm{Face}(\mathbf{B},f)$. It follows

$$e\perp^{\mathrm{Jordan}} b = f+b_0,\quad f\perp^{\mathrm{Jordan}} a = e+a_0$$

implying $\big($with the standard notation $L(x,y): z\mapsto\{xy^*z\}\big)$

$$\begin{aligned}&L(e,f{+}b_0) = L(f{+}b_0,e) = 0 \ \text{ i.e. } \ L(e,f) = -L(e,b_0),\ L(f,e) = -L(b_0,e);\\ &L(f,e{+}a_0) = L(e{+}a_0,f) = 0 \ \text{ i.e. } \ L(f,e) = -L(f,a_0),\ L(e,f) = -L(a_0,f);\\ &L(e,f) = -L(e,b_0) = -L(a_0,f),\quad L(f,e) = -L(f,a_0) = -L(b_0,e).\end{aligned}$$

Since $a_0 \perp^{\text{Jordan}} e$, hence we get

$$-L(f,e)e = -L(f,a_0)e = \{fa_0e\} = \{ea_0f\} = L(e,a_0)f = 0$$

which means the Jordan-orthogonality $\{fee\} = 0$ of the tripotents e, f.

COROLLARY 6.4. *If $a_1, \ldots, a_r \in \mathbf{E}$ have the property*

$$\left\| \sum_{k=1}^{r} \alpha_k a_k \right\| = \max_{k=1}^{r} |\alpha_k| \quad (\alpha_1, \ldots, \alpha_m \in \mathbb{C}),$$

then, necessarily, $a_1, \ldots, a_r$ are pairwise Jordan-orthogonal tripotents.

Proof. Recall that $r = \operatorname{rank}(\mathbf{E})$ is the maximal number of pairwise Jordan-ortogonal non-zero vectors in $\mathbf{E}$. By the previous lemma, we can write

$$a_k = e_k + a_{k0}, \quad a_k \perp^{\text{Jordan}} e_j \ (j \neq k)$$

with a maximal Jordan-orthogonal family of tripotents $\{e_1, \ldots, e_r\}$ and suitable vectors $a_{10}, \ldots, a_{r0} \in \overline{\mathbf{B}}$ such that $a_{k0} \perp^{\text{Jordan}} e_k$ $(k = 1, \ldots, r)$. The property $a_k \perp^{\text{Jordan}} e_j$ $(j \neq k)$ along with the maximality of $\{e_1, \ldots, e_r\}$ implies that, for any index k, necessarily $a_k \in \mathbb{C}e_k$ and hence even $a_k = \varepsilon_k e_k \in \operatorname{Trip}(\mathbf{E})$ with $|\varepsilon_k| = 1$ (because $\|a_k\| = 1$).

PROPOSITION 6.5. *The 0-preserving holomorphic Carathéodory isometries of the unit ball of a JB*-triple with finite rank are linear triple product homomorphisms. We have the decompostion $(4,9)$ for C_0-SGRs in* $\operatorname{Iso}(d_{\mathbf{B}})$.

Proof. Let $(\mathbf{E}, \{\ldots\})$ be a JB*-triple with rank $r < \infty$ and let $\Phi = U + \Omega \in \operatorname{Iso}(d_{\mathbf{B}})$ with $U := D_0\Phi$ and $\Omega(0) = 0$. According to the results of the previous section, the linear term U is a $\mathbf{E}$-isometry. Consider a maximal family $x_1, \ldots, x_r \in \operatorname{Trip}(\mathbf{E})$ of pairwise orthogonal tripotents. It is well-known that $\|\sum_{k=1}^r \alpha_k x_k\| = \max_{k=1}^r |\alpha_k|$ $(\alpha_1, \ldots, \alpha_r \in \mathbb{C})$ in this case. Thus the vectors $a_k := Ux_k$ satisfy the hypothesis of Lemma 6.3 and its corollary, giving rise to the conclusion that $Ux_1, \ldots, Ux_r$ form also a maximal family of (minimal) tripotents in $\mathbf{E}$. Therefore (by Kaup's description of the extreme points of $\mathbf{B}$), all the vectors $u_{\zeta_1,\ldots,\zeta_r} := \sum_{k=1}^r \zeta_k U x_k$ with $|\zeta_k| = 1$ are extreme points of $\mathbf{B}$ with

$$\operatorname{Face}(\mathbf{B}, u_{\zeta_1,\ldots,\zeta_r}) - u_{\zeta_1,\ldots,\zeta_r} = \left\{ v \in \mathbf{E} : v \perp^{\text{Jordan}} u_{\zeta_1,\ldots,\zeta_r} \right\} = \bigcap_{L \in \mathcal{S}(u_{\zeta_1,\ldots,\zeta_r})} \ker(L) \ = \{0\}.$$

According to Corollary 3.12, we have $\Omega(u_{\zeta_1,\dots,\zeta_r}) = \sum_{n=0}^{\infty} \Omega_n(u_{\zeta_1,\dots,\zeta_r}) \in$ $\in \bigcap_{L\in\mathcal{S}(u_{\zeta_1,\dots,\zeta_r})} \ker(L) = \{0\}$ implying even

$$\Omega\left(\sum_{k=1}^{r} \zeta_k U x_k\right) = 0 \qquad \big(|\zeta_1|, \dots, |\zeta_r| \leq 1\big).$$

Since every point of the ball $\mathbf{B}$ is a *finite* linear combination of extreme points (because $\mathbf{E}$ is of finite rank), necessarily $\Phi = U$ is a linear isometry with range $U\mathbf{E} = \mathrm{Span}\big\{Ux : x \in \mathrm{ext}(\mathbf{B})\big\}$ which is a subtriple of $\mathbf{E}$. It is well-known [12, 3] that linear isometries between JB*-triples are triple product homomorphisms.

LEMMA 6.6. *An endomorphism $U \in \mathcal{L}(\mathbf{E})$ of the triple product maps Cartan factors of $\mathbf{E}$ into Cartan factors.*

Proof. First observe that any minimal tripotent (atom) e of $\mathbf{E}$ is mapped into a minimal tripotent by U and Ue belongs to some Cartan factor of $\mathbf{E}$. Indeed, we can find a maximal Jordan-orthogonal system $e_1, \dots, e_r$ (where $r = \mathrm{rank}(\mathbf{E})$) of minial tripotents with $e = e_1$. The vectors Ue_k form again a maximal Jordan-orthogonal system of (necessarily minimal) tripotents by the definition of $\mathrm{rank}(\mathbf{E})$. The stetement follows hence because the factor components of any tripotent form a Jordan-orthogonal system of tripotents.

Let $\mathbf{F}$ be a Cartan factor of $\mathbf{E}$ and consider two minimal tripotents in $e_1, e_2 \in \mathbf{F}$. It suffices to see that Ue_1 and Ue_2 belong to the same Cartan factor of $\mathbf{E}$. Suppose the contrary. Then we would have $Ue_1 \in \mathbf{F}_1 \perp \mathrm{Jordan}\mathbf{F}_2 \ni Ue_2$ with some Cartan factors $\mathbf{F}_1 \neq \mathbf{F}_2$. However, even if $e_1 \perp^{\mathrm{Jordan}} e_2$, there exists a minimal tripotent $f \in \mathbf{F}$ with $f \not\perp^{\mathrm{Jordan}} e_1, e_2$. (this can be seen elementarily, knowing the structures of Cartan factors) and the relations lead to the contradiction $Ue_k \not\perp^{\mathrm{Jordan}} Uf$ implying $Ue_k, f \in \mathbf{F}_k$ $(k = 1, 2)$.

COROLLARY 6.7. *Given a strongly continuous one-parameter family* (*not necessarily C_0-SGR*) $[U_t : t \in \mathbb{R}_+]$ *of linear maps in* $\mathrm{Iso}(d_{\mathbf{B}})$ (*thus necessarily* $\{\dots\}$*-homomorphisms*)*, there exists $\varepsilon > 0$ such that $U_t\mathbf{F}$ $t \in [0, \varepsilon]$ for every Cartan factor of* $\mathbf{E}$*.*

Proof. $\mathbf{E}$ is a finite Jordan-orthogonal direct (and hence ℓ^∞-direct) sum of its Cartan factors. Let $\mathbf{F}$ be any of them and consider any minimal tripotent $(0 \neq) e \in \mathbf{F}$. Since each U_t is a $\{\dots\}$-homomorphism, the vectors $U_t e$ are minimal tripotents. By assumption $U_t e \to e = U_0 e$ $(t \searrow 0)$.

Therefore there exists $\varepsilon_{\mathbf{F},e} > 0$ with $U_t e \not\perp^{\text{Jordan}} e$ $(t \in [0, \varepsilon_{\mathbf{F},e}])$. Proof: $\{[U_t e][U_t e]e\} \to \{eee\} = e \neq 0$ as $t \searrow 0$. As we have noticed, non-orthogonal minimal tripotents belong to the same Cartan factor. In particular $U_t e \in \mathbf{F}$ $(t \in [0, \varepsilon_{\mathbf{F},e}])$. Since each U_t maps Cartan factors into Cartan factors, hence also $U_t\mathbf{F} \subset \mathbf{F}$ $(t \in [0, \varepsilon_{\mathbf{F},e}])$.

We can summarize the above results in the following structure description.

THEOREM 6.8. *Let $\mathbf{\Phi} := [\Phi^t : t \in \mathbb{R}_+]$ be a C_0-SGR of holomorphic Carathéodory isometries of the unit ball $\mathbf{B}$ in a reflexive JB*-triple $\mathbf{E}$ being the (necessarily finite) direct sum $\mathbf{E} = \oplus_{k=1}^N \mathbf{F}_k$ of its Cartan factors. Then $\mathbf{\Phi}$ is the direct sum of its factor-restrictions which are Möbius transformations composed with linear isometries preserving the triple product whose continuous extensions to the closed unit ball admit common fixed point.*

Remark 6.9. It is natural to ask if we can extend the arguments to ℓ^∞-sums of finite rank Cartan factors? Unfortunately, the answer is negative already in the setting of Proposition 6.5.

Counter-example: $\Phi(\zeta_0, \zeta_1, \ldots) := (\zeta_0^2, \zeta_0, \zeta_1, \ldots)$ in

$$\mathbf{E} := c_0\Big(= \big\{(\zeta_0, \zeta_1, \ldots) : \mathbb{C} \ni \zeta_n \to 0\big\}\Big), \quad \big\|(\zeta_0, \zeta_1, \ldots)\big\| := \max_n |\zeta_n|$$

with $d_{\mathbf{B}}\big((\zeta_0, \zeta_1, \ldots), (\eta_0, \eta_1, \ldots)\big) = \max_n d_\Delta(\zeta_n, \eta_n)$. Clearly Φ maps the ball $\mathbf{B}$ into itself holomorphically with $\Phi(0) = 0$. Since $\zeta \mapsto \zeta^2$ is d_Δ-contractive,

$$d_{\mathbf{B}}\big(\Phi(\zeta_0, \zeta_1, \ldots), \Phi(\eta_0, \eta_1, \ldots)\big) = \max\big\{d_\Delta(\zeta_0^2, \eta_0^2), \max_n d_\Delta(\zeta_n, \eta_n)\big\} =$$
$$= \max_n d_\Delta(\zeta_n, \eta_n) = d_{\mathbf{B}}(\zeta_0, \zeta_1, \ldots), (\eta_0, \eta_1, \ldots)\big).$$

7. THE CASE OF REFLEXIVE TRO FACTORS

According to Theorem 6.8, the study of C_0-SGR of holomorphic isometries of the unit ball is reduced to the classical balls in TRO- and Spin-factors along with those in finite-dimensional factors as symmetric resp. antisymmetric matrices and the exceptional 16- resp. 27-dimensional factors with octonion matrices. As a first illustration of our results, we outline an approach to the case of a reflexive TRO factor using an extension of the fixed point technics applied to Hilbert balls in our previous works [21, 22].

Notation 7.1. Throughout this section let $\mathbf{H}_1, \mathbf{H}_2$ denote two Hilbert spaces with the inner products $\langle x|y\rangle_k$ being linear in x and conjugate-linear in y and the norms $\|x\|_k := \langle x|x\rangle_k^{1/2}$ $(k = 1, 2)$, respectively. We omit the indices $1, 2$ in most cases without danger of confusion. As for a typical reflexive TRO-factor, we let

$$\mathbf{E} := \mathcal{L}(\mathbf{H}_1, \mathbf{H}_2) := \big\{\text{bded. lin. } \mathbf{H}_1 \leftarrow \mathbf{H}_2 \text{ operators}\big\} \quad \text{with} \quad r := \dim(\mathbf{H}_2) < \infty$$

equipped with the usual operator norm and the corresponding JB*-triple product $\{XY^*Z\} := \big(XY^*Z + ZY^*X\big)/2$. We are going to develop algebraic formulas for an arbitrarily fixed C_0-SGR

$$\mathbf{\Phi} := [\Phi^t : t \in \mathbb{R}] \text{ with common fixed point } \Phi^t(E) = E \in \overline{\mathbf{B}}$$

of holomorphic Carathéodory isometries of the open unit ball $\mathbf{B}$ of $\mathbf{E}$ with continuous extension to $\overline{\mathbf{B}}$. According to Theorem 6.8, we have

$$\Psi^t = \mathrm{M}_{a(t)} \circ \mathrm{U}_t \quad \text{with} \quad a(t) = \Psi^t(0), \ \mathrm{U}_t : \mathbf{E} \to \mathbf{E} \text{ lin. isometry.}$$

It is well-known that the Möbius transformations above are fractional linear maps with Potapov's formula [11, p.157], while the (necessarily $\{..^*.\}$-homomorphic) linear isometries of $\mathbf{E}$ are tensorial products of linear $\mathbf{H}_1$-isometries with $\mathbf{H}_2$-unitary operators by Vesentini [24, Thm. 4.3]). Following Vesentini's treatment in [24] (which goes back to Hirzebruch's ideas [10] in finite dimensions) we study $\mathfrak{P}$ by means of the projective linear representation

$$\mathfrak{P}\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : X \mapsto (AX + B)(CX + D)^{-1}$$

for $A \in \mathcal{L}(\mathbf{H}_1)$, $B \in \mathcal{L}(\mathbf{H}_1, \mathbf{H}_2) = \mathbf{E}$, $C \in \mathcal{L}(\mathbf{H}_2, \mathbf{H}_1) = \mathbf{E}^*$, $D \in \mathcal{L}(\mathbf{H}_2)$ with the representation identity $\mathfrak{P}(\mathcal{AB}) = \mathfrak{P}(\mathcal{A})\,\mathfrak{P}(\mathcal{B})$. Thus we have

$$\text{(7.2)} \qquad \Phi^t = \mathfrak{P}(\mathcal{A}_t), \quad \mathcal{A}_t = \begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} = \mathcal{M}_{a(t)}\mathcal{U}_t, \quad a(0) = 0, \ \ \mathcal{U}_0 = \mathrm{Id}$$

with the standard notation

$$\text{(7.3)} \qquad \mathcal{M}_a := \begin{bmatrix} (1 - aa^*)^{-1/2} & 0 \\ 0 & (1 - a^*a)^{-1/2} \end{bmatrix} \begin{bmatrix} 1 & a \\ a^* & 1 \end{bmatrix}, \quad \mathcal{U}_t := \begin{bmatrix} U_t & 0 \\ 0 & V_t \end{bmatrix}$$

where $U_t^* U_t = U_0 = 1 (= \mathrm{Id}_{\mathbf{H}_1})$ and $V_t^* V_t = V_t V_t^* V_0 = 1 (= \mathrm{Id}_{\mathbf{H}_2})$.

Remark 7.4. The representation (7.2) is far from being unique. Namely we have $U_t \otimes V_t^* = \mathfrak{P}\big(\mathrm{diag}(U_t, V_t)\big) = \mathfrak{P}\big(\kappa(t)\,\mathrm{diag}(U_t, V_t)\big)$ with arbitrary multipliers $\kappa(t) \in \mathbb{T}$. In [24] Vesentini investigates $[\Phi^t : t \in \mathbb{R}_+]$ immediately in the form (7.2) with the assumpion that the representation $[\mathcal{A}_t : t \in \mathbb{R}_+]$ is a C_0-SGR in $\mathcal{L}(\mathbf{H}_1 \oplus \mathbf{H}_2)$. The norm continuity of $t \mapsto \mathcal{M}_{a(t)}$ as a map $\mathbb{R}_+ \to \mathcal{L}(\mathbf{H}_1 \oplus \mathbf{H}_2)$ is immediate. However, apriori the map $t \mapsto \mathcal{U}_t \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ may be discontinuous even for all x, y. Our first goal is to fill in this gap:

PROPOSITION 7.5. *We can find a continuous function* $t \mapsto \mu(t) \in \mathbb{C}\backslash\{0\}$ *with* $\mu(0) = 1$ *such that* $[\mu(t)\mathcal{A}_t : t \in \mathbb{R}_+]$ *is a* C_0*-SGR in* $\mathcal{L}(\mathbf{H}_1 \oplus \mathbf{H}_2)$*. As a consequence, the domain of the infinitesimal generator of* $\mathbf{\Phi}$ *is dense in* $\mathbf{B}$*.*

COROLLARY 7.6. *Assume* $0 \in \mathrm{dom}(\Phi')$ *and the let the representation* $[\mathcal{A}_t : t \in \mathbb{R}_+]$ *associated with the decomposition* $(7.2-3)$ *be a* C_0*-SGR in* $\mathcal{L}(\mathbf{H}_1 \oplus \mathbf{H}_2)$*. Then its generator* $\mathcal{A}'$ *is a possibly unbounded closed linear operator of* $\mathbf{H}_1 \oplus \mathbf{H}_2$*-split matrix form with dense domain and we have*

$$\mathcal{A}' = \begin{bmatrix} U' & b \\ b^* & V' \end{bmatrix}, \quad \mathrm{dom}(\mathcal{A}') = \mathrm{dom}(U') \oplus \mathbf{H}_2$$

where $U' : X(\in \mathbf{D}_1) \mapsto \frac{d}{dt}\big|_{t=0+} U_t X$ *resp.* $V' : Y(\in \mathbf{H}_1) \mapsto \frac{d}{dt}\big|_{t=0+} V_t Y$ *are generators of* C_0*-SGRs of* $\mathbf{H}_1$*- resp.* $\mathbf{H}_2$*-isometries and* $b := \frac{d}{dt}\big|_{t=0+} a(t) \in \mathbf{E}$*.*

Once the existence of a projective C_0-SGR representation $[\mathcal{A}_t : t \in \mathbb{R}_+]$ of $\mathbf{\Phi}$ is established, the method outlined in [24] for the integration of the Riccati type equation corresponding to a Kaup type generator works. Also the application of the techniques elaborated by Khatskevich-Reich-Shoikhet [25] is justified. Nevertheless, with our projective shift argument in [22, 3.5-8] we can achieve the following algebraically more informing results in terms of the fixed point E:

THEOREM 7.7. *By assuming up to Möbius equivalence that* $0 \in \mathrm{dom}(\Phi')$*, for all* $X \in \mathbf{B}$ *we have*

$$\Phi^t(X) = E + W_t(X-E)\Big[\int_0^t S_{t-h} b^* W_h(X-E)\, dh + S_t\Big]^{-1} =$$
$$= \mathfrak{P} \begin{bmatrix} W_t + EJ_t & ES_t - (W_t + EJ_t)E \\ J_t & S_t - J_t E \end{bmatrix} (X)$$

where $[W^t : t \in \mathbb{R}_+] \subset \mathcal{L}(\mathbf{H}_1)$*,* $[S^t : t \in \mathbb{R}_+] \subset \mathcal{L}(\mathbf{H}_2)$ *are* C_0*-SGRs with generators* $U' - Eb^*$ *and* $V' + b^*E$*, respectively and* $J_t := \int\limits_0^t S_{t-h} b^* W_h\, dh$*.*

Next we proceed to the proofs of 7.5-7. We borrow a crucial step from one of our earlier works [20]:

LEMMA 7.8. *There exists a function* $\kappa : \mathbb{R}_+ \to \mathbb{T}$ *such that the operator valued function* $t \mapsto U_t$, $t \mapsto V_t$ *in* (7.2−3) *are strogly continuous.*

Proof. The adjusted strong continuity argumets in [20, Cor.2.6] can be applied even with linear isomeries instead of unitary operators.

Assumption 7.9. Henceforth we assume without loss of generality that

$$\begin{aligned}
&(1) \qquad t \mapsto U_t x,\ t \mapsto V_t y \ \text{ are continuous for any } x \in \mathbf{H}_1,\ y \in \mathbf{H}_2;\\
&(2) \qquad \mathcal{A}_t \mathcal{A}_h = \lambda(t,h)\mathcal{A}_{t+h} \ \ (t,h \in \mathbb{R}_+), \quad \lambda : \mathbb{R}_+^2 \to \mathbb{C}\backslash\{0\} =: \mathbb{C}_0;\\
&(3) \qquad \Phi^t(E) = E \ \ (t \in \mathbb{R}_+), \quad \|E\| = 1.
\end{aligned}$$

Notice that Assumption (2) is equivalent to the semigroup property $\Phi^t \circ \Phi^h = \Phi^{t+h} \iff \mathfrak{P}(\mathcal{A}_t) \circ \mathfrak{P}(\mathcal{A}_t) = \mathfrak{P}(\mathcal{A}_{t+h}) \iff \mathcal{A}_t \mathcal{A}_h = \lambda \mathcal{A}_{t+h}$ for some $\lambda \in \mathbb{C}_0$. In (3), we assume the common fixed point to be located in the boundary of the unit ball since the case of inner fixed points is of no interest: the maps $\Theta^t = \mathrm{M}_{-E} \circ \Phi^t \circ \mathrm{M}_E$ are 0-preserving and hence linear isometries by Prop. 6.5.

Definition 7.10. Henceforth we write

$$\mathcal{A}_t = \begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix}, \quad S_t := A_t E + B_t \qquad (t \in \mathbb{R}_+).$$

LEMMA 7.11. *We have* $\mathcal{A}_t \left[\begin{smallmatrix} E \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} E \\ 1 \end{smallmatrix}\right] S_t,\ S_t S_h = \lambda(t,h) S_{t+h} \quad (t,h \in \mathbb{R}_+).$

Proof. By assumption, $\Psi^t(E) = E$, that is $E = \mathcal{F}(\mathcal{A}_t)(E) = (A_t E + B_t)(C_t E + D_t)^{-1} = (A_t E + B_t) S_t^{-1}$. It follows

$$\mathcal{A}_t \begin{bmatrix} E \\ 1 \end{bmatrix} = \begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} \begin{bmatrix} E \\ 1 \end{bmatrix} = \begin{bmatrix} A_t E + B_t \\ C_t E + D_t \end{bmatrix} = \begin{bmatrix} E S_t \\ S_t \end{bmatrix} = \begin{bmatrix} E \\ 1 \end{bmatrix} S_t.$$

Hence $\lambda(t,h) \left[\begin{smallmatrix} E \\ 1 \end{smallmatrix}\right] S_{t+h} = \lambda(t,h) \mathcal{A}_{t+h} \left[\begin{smallmatrix} E \\ 1 \end{smallmatrix}\right] = \mathcal{A}_t \mathcal{A}_h \left[\begin{smallmatrix} E \\ 1 \end{smallmatrix}\right] = \mathcal{A}_t \left[\begin{smallmatrix} E \\ 1 \end{smallmatrix}\right] S_h = \left[\begin{smallmatrix} E \\ 1 \end{smallmatrix}\right] S_t S_h$.

COROLLARY 7.12. *The operators* S_t *are invertible and form an Abelian family. The function* $(t,h) \mapsto \lambda(t,h)$ *is continuous,* $\lambda(t,h) = \lambda(h,t)$ $(t,h \in \mathbb{R}_+)$.

Proof. The invertibility of S_t is a implicit in the existence of the matrix representation $\mathfrak{P}$. We have $\lambda(t,h) \left[\begin{smallmatrix} E \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} E \\ 1 \end{smallmatrix}\right] S_t S_h S_{t+h}^{-1}$ whence the continuity of λ is immediate by assumption 7.9(1) implying the strong continuity of $t \mapsto A_t, B_t, C_t, D_t$ and hence also $t \mapsto S_t = C_t E + D_t$ (the latter even with norm continuity since $\dim(\mathbf{H}_2) < \infty$). We prove the relations $S_t S_h = S_t S_h$

along with $\lambda(t,h)=\lambda(h,t)$ as follows. We have $\mathrm{trace}\,AB=\mathrm{trace}\,BA$ in finite dimensional operator algebras. Thus

$$\begin{aligned}&\mathrm{trace}(S_tS_h)=\lambda(t,h)\mathrm{trace}(S_{t+h}),\quad \mathrm{trace}(S_hS_t)=\lambda(h,t)\mathrm{trace}(S_{t+h}),\\ &\big[\lambda(t,h)-\lambda(h,t)\big]\mathrm{trace}(S_{t+h})=0\end{aligned}$$

Therefore $\mathrm{trace}(S_tS_h)\to\mathrm{trace}(S_0)=\mathrm{trace}\,\mathrm{Id}_{\mathbf{H}_2}=\dim(\mathbf{H}_2)=r\quad(t,h\to 0)$. In particular there exists $\varepsilon>0$ such that $\lambda(t,h)=\lambda(h,t)$ $(0\le t,h<\varepsilon)$. It follows $S_t\smile S_h$ for any $0\le t,h<\varepsilon$. Consider any $u,v\in\mathbb{R}$ with $u/m,v/m\in[0,\varepsilon)$. Then $S_u=\widetilde{\lambda}S^m_{u/m}$, $S_v=\widetilde{\mu}S^m_{v/m}$ for some $\widetilde{\lambda},\widetilde{\mu}\in\mathbb{C}_0$ whence the commutation $S_u\smile S_v$ is immediate.

Remark 7.13. In infinite dimensions, the relation $AB=\lambda BA$ does not imply $\neq 0\not\Rightarrow A\smile B$ even if $\lambda\in\mathbb{T}$. Counter-example: the bilateral shift $A:e_n\mapsto e_{n+1}$ $(n=0,\pm1,\ldots)$X with $B:e_n\mapsto\lambda^ne_n$.

Even in $r<\infty$ dimensions and with $\lambda^r=1$, we can find A,B such that $AB=\lambda BA\neq 0$ but $A\not\smile B$. Counter-example: Take an orthonormed basis $e_0,\ldots,e_{r-1}$, and let $A:e_0\mapsto e_1\mapsto e_2\mapsto\cdots e_{r-1}\mapsto e_0$, $B:e_k\mapsto\lambda^ke_k$.

Proof of Proposition 7.5. We can find a continuous function $t\mapsto\mu(t)\in\mathbb{C}_0$ with $\mu(0)=1$ such that both $[\mu(t)S_t:t\in\mathbb{R}_+]$, $[\mu(t)\mathcal{A}_t:t\in\mathbb{R}_+]$ are C_0-SGRs.

Proof. In view of Corollary 7.12, $\mathcal{S}:=\mathrm{Span}\{S_t:t\in\mathbb{R}_{(+)}\}$ is a finite dimensional Abelian subalgebra of $\mathcal{L}(\mathbf{H}_2)$ with unit $1=S_0$. Let us take a non-trivial multiplicative functional $M:\mathcal{S}\to\mathbb{C}$. (Actually, there exists $0\neq x\in\mathbf{H}_2$ with $Sx=M(S)x$ $(x\in\mathcal{S})$). For any parameter $\in\mathbb{R}_+$, we have $M(S_t)M(S_h)=M(S_tS_h)=\lambda(t,h)M(S_{t+h})$ where $M(S_t)\neq 0$ since the operator S_t is invertible. Define

$$\mu(t):=1/M(S_t)\qquad(t\in\mathbb{R}_+).$$

Notice that the function $t\mapsto\mu(t)$ is continuous with $\mu(0)=1$. We complete the proof with the observation

$$\begin{aligned}\mu(t)S_t\mu(h)S_h&=\frac{1}{M(S_t)M(S_h)}S_tS_h=\frac{\lambda(t,h)}{M(S_t)M(S_h)}S_{t+h}=\\ &=\frac{M(S_t)M(S_h)/M(S_{t+h})}{M(S_t)M(S_h)}S_{t+h}=\frac{1}{M(S_{t+h})}S_{t+h}=\mu(t+h)S_{t+h}.\end{aligned}$$

Proof of Corollary 7.6.

By passing to $\mu(t)S_t$ resp. $\mu(t)\mathcal{A}_t = \mathcal{M}_{a(t)}\mathrm{diag}\big[\big[\begin{smallmatrix}\kappa(t)u_t\\ \kappa(t)v_t\end{smallmatrix}\big]$ for S_t resp. $\mathcal{A}_t$, to the description of $[\Psi^t : t \in \mathbb{R}_+]$ in the form $\Psi^t = \mathfrak{P}(\mathcal{A}_t)$, we may assume without loss of generality 7.9(1-3) and

(4) $[\mathcal{A}_t : t \in \mathbb{R}_+], [S_t : t \in \mathbb{R}_+]$ are C_0-SGRs where $S_t := C_tE + D_t$.

According to (7.2-3), $\mathcal{A}_t = \big[\begin{smallmatrix}A_t & B_t\\ C_t & D_t\end{smallmatrix}\big]$ where

$$\begin{aligned} A_t &= [1-a(t)a(t)^*]^{-1/2}U_t, \qquad & B_t &= [1-a(t)^*a(t)^*]a(t)V_t,\\ C_t &= [1-a(t)^*a(t)]^{-1/2}a(t)^*U_t, \qquad & D_t &= [1-a(t)^*a(t)]V_t. \end{aligned} \tag{7.16}$$

Since $\mathrm{dom}(\mathcal{A}') = \Big\{\big[\begin{smallmatrix}X\\ Y\end{smallmatrix}\big] : t \mapsto \big[\begin{smallmatrix}A_tX+B_tY\\ C_tX+D_tY\end{smallmatrix}\big] \text{ is differentiable}\Big\}$ and $\mathrm{dom}(\Phi') = \big\{Z : t \mapsto (A_tZ + B_t)(C_tZ + D_t)^{-1} \text{ is differentiable}\big\}$, we have

$$Z \in \mathrm{dom}(\Phi') \ \text{ whenever } \ Z = XY^{-1} \text{ for some } \big[\begin{smallmatrix}X\\ Y\end{smallmatrix}\big] \in \mathrm{dom}(\mathcal{A}').$$

In particular $\mathrm{dom}(\Phi')$ *is dense in the ball* $\mathbf{B}$. Indeed, given any $Z_0 \in \mathbf{B}$, we can write $Z_0 = X_0Y_0^{-1}$ with suitable $X_0 \in \mathbf{B}$ and $Y_0 \in \mathcal{L}(\mathbf{H}_2)$ such that $\|1 - Y_0\| < 1$. Then, given any $\varepsilon > 0$, the density of $\mathrm{dom}(\mathcal{A}')$ in $\mathcal{L}(\mathbf{H}_1 \oplus \mathbf{H}_2)$ (guaranteed by linear Hille-Yosida theory) ensures the existence of $X_\varepsilon \in \mathbf{B}$ and $Y_\varepsilon \in \mathcal{L}(\mathbf{H}_2)$ such that $\big[\begin{smallmatrix}X_\varepsilon\\ Y_\varepsilon\end{smallmatrix}\big] \in \mathrm{dom}(\mathcal{A}')$, Y_ε is invertible and $\|X_\varepsilon Y_\varepsilon^{-1} - Z_0\| < \varepsilon$.

Henceforth assume (without loss of generality up to Möbius equivalence)

(5) $0 \in \mathrm{dom}(\Phi')$, i.e. $t \mapsto a(t) = \Phi^t(0) = \mathfrak{P}\mathcal{A}_t(0) = B_tD_t^{-1}$ is differentiable.

From the real-analyticity of the maps $a \mapsto \mathcal{M}_a, \mathcal{M}_a^{-1} = \mathcal{M}_{-a}$, we see that

$$\mathrm{dom}(\mathcal{A}') = \Big\{\big[\begin{smallmatrix}X\\ Y\end{smallmatrix}\big] : t \mapsto U_tX, t \mapsto V_tY \text{ are differentiable}\Big\}. \tag{7.17}$$

Since $\dim(\mathbf{H}_2) < \infty$, it follows that $t \mapsto V_t, D_t, B_t$ are differentiable. Indeed the density of $\mathrm{dom}(\mathcal{A}')$ along with the fact that $Y \in \mathrm{GL}(\mathbf{H}_2)\big(:= \{\text{invertible elements in } \mathcal{L}(\mathbf{H}_2)\}\big)$ implies the existence of $Y_0 \in \mathrm{GL}(\mathbf{H}_2)$ with differentiable $t \mapsto V_tY_0$ and $t \mapsto V_t = [V_tY_0]Y_0^{-1}$. Hence the differentiability of $t \mapsto D_t, B_t$ is immediate by (7.16). As a first consequence, we obtain the differentiability of $t \mapsto \mathcal{A}_t\big[\begin{smallmatrix}0\\ 1\end{smallmatrix}\big] = \big[\begin{smallmatrix}B_t\\ D_t\end{smallmatrix}\big]$. That is we have $\big[\begin{smallmatrix}0\\ 1\end{smallmatrix}\big] \in \mathrm{dom}(\mathcal{A}')$ and hence also $\big[\begin{smallmatrix}0\\ Y\end{smallmatrix}\big] = \big[\begin{smallmatrix}0\\ 1\end{smallmatrix}\big]Y \in \mathrm{dom}(\mathcal{A}')$ $\big(Y \in \mathrm{GL}(\mathbf{H}_2)\big)$. Since $\dim(\mathbf{H}_2) < \infty$ and since $\mathrm{dom}(\mathcal{A}')$ is a linear submanifold of $\mathcal{L}(\mathbf{H}_1, \mathbf{H}_2)$, it follows $0 \oplus \mathbf{H}_2 \subset \mathrm{dom}(\mathcal{A}')$. Thus, given any couple $\big[\begin{smallmatrix}X\\ Y\end{smallmatrix}\big] \in \mathrm{dom}(\mathcal{A}')$, also $\big[\begin{smallmatrix}X\\ 0\end{smallmatrix}\big] = \big[\begin{smallmatrix}X\\ Y\end{smallmatrix}\big] - \big[\begin{smallmatrix}0\\ Y\end{smallmatrix}\big] \in \mathrm{dom}(\mathcal{A}')$, i.e. $\mathcal{A}'$ is a $(\mathbf{H}_1, \mathbf{H}_2)$-split operator matrix and $\mathrm{dom}(\mathcal{A}') = \{X : t \mapsto U_t \text{ is differentiable}\} \oplus \mathbf{H}_2$.

Consider any couple $\left[\begin{smallmatrix}X\\Y\end{smallmatrix}\right] \in \mathrm{dom}(\mathcal{A}')$. According to (5), we can write $a_t = a(0)+t\frac{d}{dt}\big|_{\tau=0+}a(\tau)+o(t) = b+o(t)$. Hence we get $\frac{d}{dt}\big|_{t=0+}A_tX = \frac{d}{dt}\big|_{t=0+}U_tX$, $\frac{d}{dt}\big|_{t=0+}B_tY = \frac{d}{dt}\big|_{t=0+}bY$, $\frac{d}{dt}\big|_{t=0+}C_tX = \frac{d}{dt}\big|_{t=0+}b^*X$, $\frac{d}{dt}\big|_{t=0+}D_tX = \frac{d}{dt}\big|_{t=0+}V_tX$. To complete the proof, we remark that the map $\left[\begin{smallmatrix}X\\Y\end{smallmatrix}\right] \mapsto \frac{d}{dt}\big|_{t=0+}\mathcal{U}_t\left[\begin{smallmatrix}X\\Y\end{smallmatrix}\right]$ is the generator of a C_0-SGR in $\mathcal{L}(\mathbf{H}_1\oplus\mathbf{H}_2)$ due to the Bounded Perturbation Theorem [8], as being the difference of $\mathcal{A}' = \mathrm{gen}[\mathcal{A}_t : t\in\mathbb{R}_+]$ and $\left[\begin{smallmatrix}0 & b\\ b^* & 0\end{smallmatrix}\right]$. Notice that the integrated C_0-SGRs $[U_0^t : t\in\mathbb{R}_+]$, $[V_0^t : t\in\mathbb{R}_+]$ with

$$U' = \mathrm{gen}[U_0^t : t\in\mathbb{R}_+], \quad V^t = \exp(tV')$$

consist of isometries, since $U\prime, V'$ are skew symmetric operators. Indeed, by definition $U' = \frac{d}{dt}\big|_{t=0+}U_t$ and for any $X\in\mathrm{dom}(U')$ we have $1 = \langle U_tX|U^tX\rangle$ i.e. $0 = \frac{d}{dt}\big|_{t=0+}\langle U_tX|U^tX\rangle = \langle U'X|X\rangle+\langle X|U'X\rangle$. Similarly also $\langle U'Y|Y\rangle + \langle Y|U'V\rangle = 0 \ \big(Y\in\mathcal{L}(\mathbf{H}_2)\big)$.

Proof of Theorem 7.7.

We continue the previous arguments with the established notations. Recall in particular that $\mathcal{A}_t\left[\begin{smallmatrix}E\\1\end{smallmatrix}\right] = \left[\begin{smallmatrix}A_t & B_t\\ C_t & D_t\end{smallmatrix}\right]\left[\begin{smallmatrix}E\\1\end{smallmatrix}\right] = \left[\begin{smallmatrix}ES_t\\ S_t\end{smallmatrix}\right] = \left[\begin{smallmatrix}E\\1\end{smallmatrix}\right]S_t$ where $[S_t : t\in\mathbb{R}]$ is a (finite dimensional) C_0-SGR in $\mathcal{L}(\mathbf{H}_2)$ with $S' := \frac{d}{dt}\big|_{t=0}S_t = \mathrm{gen}[S_t : t\in\mathbb{R}]$. Furthermore, by writing $\mathbf{D}_1 := \mathrm{dom}(U')$, for any vector $y\in\mathbf{H}_2$. the function $t\mapsto\mathcal{A}_t\left[\begin{smallmatrix}Ey\\y\end{smallmatrix}\right] = \left[\begin{smallmatrix}E\\1\end{smallmatrix}\right]S_ty$ is differentiable and $\left[\begin{smallmatrix}Ey\\y\end{smallmatrix}\right]\in\mathrm{dom}(\mathcal{A}')$ with $Ey\in\mathbf{D}_1$.

Extending slightly an idea from [22], we introduce the following *projective translation* along with its chart transform $[\mathcal{B}_t : t\in\mathbb{R}_+]$ to $[\mathcal{A}_t : t\in\mathbb{R}_+]$ as

$$\mathcal{T} := \left[\begin{smallmatrix}1 & E\\ 0 & 1\end{smallmatrix}\right], \quad T := \mathfrak{P}\mathcal{T} : X\mapsto X+E, \quad \mathcal{B}_t := \mathcal{T}^{-1}\mathcal{A}_t\mathcal{T}, \quad \mathcal{B}' := \mathcal{T}^{-1}\mathcal{A}'\mathcal{T}$$

where $\mathcal{A}'{=}\mathrm{gen}\big[\mathcal{A}_t\colon t{\in}\mathbb{R}\big]$ and $\mathcal{B}'{=}\mathrm{gen}\big[\mathcal{B}_t\colon t{\in}\mathbb{R}\big]$. Observe that

$$\mathrm{dom}\big(\mathcal{B}'\big) = \mathcal{T}^{-1}\big(\mathbf{D}_1\oplus\mathbf{H}_2\big) = \Big\{\left[\begin{smallmatrix}d-Ey\\ y\end{smallmatrix}\right]\colon d\in\mathbf{D}_1,\ y\in\mathbf{H}_2\Big\} = \mathbf{D}_1\oplus\mathbf{H}_2 = \mathrm{dom}(\mathcal{A}');$$

$$\mathcal{B}_t = \mathcal{T}^{-1}\left[\begin{smallmatrix}A_t & B_t\\ C_t & D_t\end{smallmatrix}\right]\mathcal{T} = \mathcal{T}^{-1}\left[\begin{smallmatrix}A_t & A_tE+B_t\\ C_t & C_tE+D_t\end{smallmatrix}\right] = \left[\begin{smallmatrix}1 & -E\\ 0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}A_t & ES_t\\ C_t & S_t\end{smallmatrix}\right] = \left[\begin{smallmatrix}A_t-EC_t & \mathbf{0}\\ C_t & S_t\end{smallmatrix}\right];$$

$$\mathcal{B}' = \mathcal{T}^{-1}\mathcal{A}'\mathcal{T} = \left[\begin{smallmatrix}A'-EC' & 0\\ C' & S'\end{smallmatrix}\right] = \left[\begin{smallmatrix}U'-Eb^* & 0\\ b^* & b^*E+V'\end{smallmatrix}\right].$$

Due to the triangularity of the matrices $\mathcal{B}_t$, the diagonal entries

$$W_t := \big[\mathcal{B}_t\big]_{11} = A_t - EC_t, \quad S_t := \big[\mathcal{B}_t\big]_{22} = C_tE + D_t$$

form C_0-SGRs with the infinitesimal generators

$$W' = \mathrm{gen}[W_t : t\in\mathbb{R}] = A' - EC' = U' - Eb^*,$$
$$S' = \mathrm{gen}[S_t : t\in\mathbb{R}] = C'E + D' = b^*E + V'.$$

Therefore, from the triangularization lemma [22, Lemma 3.8] it follows that

$$\mathcal{B}_t = \begin{bmatrix} W_t & 0 \\ \int_0^t S_{t-h} C' W_h \, dh & S_t \end{bmatrix} \qquad (t \in \mathbb{R}_+).$$

Thus the chart semigroup $[\Psi^t : t \in \mathbb{R}_+]$ with $\Psi^t := \mathfrak{P}\mathcal{B}_t$ consists of maps of the form $\Psi^t : X \mapsto W_t X \big[\int_0^t S_{t-h} C' W_h X \, dh + S_t \big]^{-1}$ and hence

$$\Phi^t = \mathfrak{P}\mathcal{A}_t = \mathfrak{P}\big(\mathcal{T}\mathcal{B}_t\mathcal{T}^{-1}\big) = T \circ \Psi_t \circ T^{-1}$$

yieldig the closed algebraic forms for Φ^t stated in the theorem.

REFERENCES

[1] M .Apazoglou, *Facial and ideal structure on real C*-algebras*, Math. Proc. R. Ir. Acad., **112A1,** (2012), 1-7.

[2] M. Apazoglou and A. Peralta, *Linear isometries between real JB*-triples and C*-algebras*, Q. J. Math., **65/2** (2014), 485-503.

[3] T.J. Barton, T. Dang and G. Horn, *Normal representations of Banach Jordan triple systems*, Proc. Amer. Math. Soc. **102/3,** (1988), 551-555.

[4] H. Braun, W. Kaup and H. Upmeier, *On the automorphisms of circular and Reinhardt domains in complex Banach spaces*, Manuscripta Math., **25** (1978), 97-133.

[5] C.-H. Chu and M. Mackey, *Isometries between JB*-triples*, Math. Z. **251/3** (2005), 615-633.

[6] C.M. Edwards and G.T. Rüttimann, *Exposed faces of the unit ball in a JBW*-triple*, Math. Scand. **82/2,** (1998), 287-304.

[7] C.M. Edwards and G.T. Rüttimann, *The facial and inner ideal structure of a real JBW*-triple*, Math. Nachr. **222,** (2001), 159-184.

[8] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics Vol. 194, Springer-Verlag, New York, 2000.

[9] T. Franzoni and E. Vesentini, *Holomorphic Maps and Invariant Distances*, North Holland Math. Studies 40, North Holland Publ. Co., Amsterdam-New York-Oxford, 1980.

[10] U. Hirzebruch, *Halbräume und ihre holomorphen Automorphismen*, Math. Annalen **153/5,** (1964), 395-417.

[11] J.-M. Isidro and L.L. Stachó, *Holomorphic Automorphism Groups in Banach Spaces*, North Holland Math. Studies 105, North Holland Publ. Co., Oxford–New York–Amsterdam, 1985.

[12] W. Kaup, *Über die Klassifikation der symmetrischen hermiteschen Mannigfaltigkeiten unendlicher Dimension I.*, Math. Ann., **357** (1981), 463–481.

[13] W. Kaup, *A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces*, Math. Z., **83** (1983), 503–529.

[14] V. Khatskevich, S. Reich and D. Shoikhet, *Complex dynamical systems on bounded symmetric domains*, Electronic J. of Diff. Equations **19,** (1997), 1-9.

[25] V. Katshkevich, S. Reich and D. Shoikhet, *One-parameter semigroups of fractional-linear transformations*, Operator Theory Advances and Applications, **123** (2001) 401–411.

[16] L. Nachbin, *Holomorphic functions, domains of holomorphy and local properties*, North-Holland Pub. Co., Oxford–New York–Amsterdam, 1970.

[17] E. Neher, *Jordan triple systems by the grid approach*, Lecture Notes in Math. 1280, Springer-Verlag, Berlin, 1987.

[18] A. Peralta and L.L. Stachó, *Atomic decomposition of real JBW * -triples*, Q. J. Math., **52/1,** (2001), 79-87.

[19], S. Reich and D. Shoikhet, *Generation theory for semigroups of holomorphic mapping in Banach spaces*, Abstr. Appl. Anal. **1/1,** (1996). 1-44.

[20] L.L. Stachó, *On strongly continuous one-parameter groups of automorphisms of multilinear functionals*, J. Math. Anal. Appl. **363,** (2010), 419430.

[21] L.L. Stachó, *A fixed point approach to C0-groups of ball automorphisms in Hilbert space*, Rev. Roum. **61/4,** (2016), 241-159.

[22] L.L. Stachó, *On the structure of C0-semigroups of holomorphic Carathéodory isometries in Hilbert space*, J. Math. Anal. Appl. **445.** (2017), 139-150.

[23] E. Vesentini, *Semigroups of holomorphic isometries*, Advances in Mathemetics, **65** (1987), 272–306.

[24] E. Vesentini, *Semigroups of holomorphic isometries*, in: P.M. Gauthier (ed.), Complex Potential Theory, pp. 475–548. Kluwer Academic Pulbl., Netherlands, 1994.

[25] E. Vesentini, *Semigroups of holomorphic isometries of spin-factors*, Rend. Sem. Mat. Univ. Pol. Torino **50/4,** (1992), 427-455.

[26] G.F. Webb, *Theory of nonlinear age-dependent population dynamics*, Monographs and Textbooks in Pure and App. Math., 89. Marcel Dekker, New York, 1985.

Received 27 November 2017

University of Szeged
"Bolyai" Institute
Aradi Vértanúk tere 1,
H-6720 Szeged,
Hungary
stacho@math.u-szged.hu