ON PRIME JB*-TRIPLES

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1. Introduction

ONE of the many interesting results in a recent study of ultra-prime Jordan-Banach algebras [**4**] was that there exists a universal constant $K > 0$ such that for any prime JB*-algebra A and any $a, b \in A$, we have $\|Q_{a,b}\| \geqslant K\|a\| \cdot \|b\|$. For prime JB*-algebras representable on a complex Hilbert space, known as *JC*-algebras*, an admissible value of $K = \frac{1}{20412}$ was given for the universal constant. The demonstration of an admissible value for the prime exceptional JB*-algebra was left open.

The purpose of this paper is both to sharpen and to extend this result by showing that, for any prime JB*-triple A, we have

$$\|Q_{a,b}\| \geqslant \frac{1}{6}\|a\| \cdot \|b\|$$

for all $b \in A$. Hence for any JB*-triple A, the follow three conditions are equivalent:

(i) A is prime.
(ii) The constant $K_A = \inf\{\|Q_{a,b}\| : \|a\| = \|b\| = 1\}$ is greater than zero.
(iii) $K_A \geqslant \frac{1}{6}$.

Our methods, which are different from [**4**], involve application of representation theory to results on Cartan factors.

Historically JB*-triples arose in the study of complex holomorphy: a bounded symmetric domain is biholomorphic to the open unit ball of a JB*-*triple* is a complex Banach space A together with a continuous triple product $(a, b, c) \in A^3 \mapsto \{abc\} \in A$ such that

(i) $\{abc\}$ is symmetric and bilinear in a, c and conjugate linear in b;
(ii) $\{xy\{abc\}\} = \{\{xya\}bc\} + \{ab\{xyc\}\} - \{a\{yxb\}c\}$;
(iii) the operator $x \mapsto \{aax\}$ is hermitian with positive spectrum;
(iv) $\|\{aaa\}\| = \|a\|^3$.

The identity (ii) is referred to as the *main identity*. The maps $D_{a,b}$ and $Q_{a,b}$ are given by $D_{a,b}(x) = \{abx\}$ and $Q_{a,b}(x) = \{axb\}$. We write $Q_a = Q_{\,,}$. Crucially, surjective linear isometries between JB*-triples are the triple isomorphisms [**9**, **13**]. We refer to [**16**] for a recent survey of JB*-triples.

Every C^*-algebra is a JB*-triple via $\{abc\} = \frac{1}{2}(ab^*c + cb^*a)$. More generally, every JB*-algebra (alias *Jordan C*-algebra* [17]), with Jordan product $(a, b) \mapsto a \circ b$, is a JB*-triple with respect to $\{abc\} = (a \circ b^*) \circ c + (b^* \circ c) \circ a - (a \circ c) \circ b^*$. A JB*-triple isometric to a subtriple of a C*-algebra is called a JC*-*triple* (also known as J*-*algebra* [9]).

The second dual A^{**} of a JB*-triple A is a JB*-triple containing A as a weak*-dense JB*-subtriple [6]. A JB*-triple with predual is known as a JBW*-*triple*: the predual is unique and the triple product is separately weak*-continuous in each variable [2, 10]. Important examples of JB*-triples are the Cartan factors. Let H, K be complex Hilbert spaces and, with respect to a conjugation $h \mapsto \bar{h}$ on H, define $a^T(h) = \overline{a^*(\bar{h})}$ for each $a \in B(H)$, the space of bounded linear operators on H. Let $\mathcal{O}$ denote the complex octonions. The six types of Cartan factors are as follows:

(1) *Rectangular*, $B(H, K)$, the bounded linear operators from H to K;
(2) *Symplectic*, $\{x \in B(H) : x^T = -x\}$;
(3) *Hermitian*, $\{x \in B(H) : x^T = x\}$;
(4) *Span*, H with product $\{xyz\} = \frac{1}{2}(\langle x, y\rangle z + \langle z, y\rangle x - \langle x, \bar{z}\rangle \bar{y})$ and norm $2\|x\|^2 = \langle x, x\rangle + (\langle x, x\rangle^2 - |\langle x, \bar{x}\rangle|^2)^{\frac{1}{2}}$ and $\dim H \geqslant 3$;
(5) $B_{1,2}$ = *the* 1×2 *matrices over* $\mathcal{O}$;
(6) M_3^8 = *the hermitian* 3×3 *matrices over* $\mathcal{O}$.

The Cartan factors (1)–(4) are JC*-triples while (5) and (6) are the *exceptional* Cartan factors.

An element u in a JB*-triple A is a *tripotent* if $u = \{uuu\}$, associated with which are the *Peirce Projections*

$$P_2(u) = Q_u^2, \quad P_1(u) = 2(D_{u,u} - Q_u^2), \quad P_0(u) = I - 2D_{u,u} + Q_u^3$$

which are contractive and mutually orthogonal with sum I and have ranges

$$A_j(u) = P_j(A) = \{x : \{uux\} = \frac{j}{2}x\}$$

so that $A = A_2(u) \oplus A_1(u) \oplus A_0(u)$. By the *Peirce rules*, we have $\{A_i(u)A_j(u)A_k(u)\} \subset A_{i-j+k}(u)$ when $i - j + k \in \{0, 1, 2\}$, and $\{0\}$ otherwise, and $\{A_2(u)A_0(u)A\} = \{A_0(u)A_2(u)A\} = \{0\}$. A nonzero tripotent u is *minimal* if $\{uAu\} = \mathbb{C}u$.

2. Primeness in JB*-triples

In this section, we shall briefly run through some relevant equivalent formulation of primeness.

Let J be a linear subspace of a JB*-triple A. We call J an *ideal* of A if $\{AAJ\} + \{AJA\} \subset J$. If J is a norm closed and $\{AJJ\} \subset J$, then J is an ideal of A [3].

For any ideal J of A, the annihilator $J^{\perp} = \{a \in A : \{aJA\} = \{0\}\}$ is a norm closed ideal of A as follows from the main identity.

Given norm closed ideals J and K with $x \in J \cap K$, we have $x = \{yyy\}$ for some $y \in J \cap K$ by triple functional calculus (cf. [**12**, 3]). It follows that

$$J \cap K = \{JKA\} = \{JAK\}.$$

The following result might have independent interest.

PROPOSITION 2.1. *Let X and Y be subsets of a JB*-triple A with $\{XAY\} = \{0\}$. Then there exist norm closed ideals J and K in A with $X \subset J$, $Y \subset K$ and $\{JAK\} = \{0\}$.*

Proof. Put $J = \{a \in A : \{aAY\} = \{0\}\}$ and $K = \{a \in A : \{JAa\} = \{0\}\}$. Then $X \subset J$, $Y \subset K$ and $\{JAK\} = \{0\}$. We shall show that J and K are ideals of A. By the main identity, we have

$$\{0\} = \{JA\{JAY\}\} = \{\{JAJ\}AY\} + \{JA\{JAY\}\} - \{J\{AJA\}Y\} = \{\{JAJ\}AY\}.$$

therefore J contains $\{JAJ\}$ and so is, in particular, a JB*-subtriple of A. similarly K is a JB*-subtriple of A. Let $\bar{J}$, $\bar{K}$ denote the weak*-closure of J, K in $A^{**} = M$, and let $u \in \bar{J}$, $v \in \bar{K}$ be tripotents. Then $\{uuv\} = 0$ and so $\{uvM\} = \{0\}$ by [**14**, 3.9]. But $\bar{J}$, $\bar{K}$ are JBW$*$-triples and so, as Banach spaces, are generated by their tripotents. So $(\bar{J}\bar{K}M) = \{0\}$. It follows that $\{JKA\} = \{KJA\} = \{0\}$ and the main identity gives

$$\{0\} = \{AJ\{JAY\}\} = \{\{AJJ\}AY\} + \{JA\{AJY\}\} - \{J\{JAA\}Y\} = \{\{AJJ\}AY\}.$$

Hence $\{AJJ\} \subset J$ and J is an ideal by [**3**]. Similarly K is an ideal.

A JB*-triple A is defined to be a *prime* JB*-triple if whenever J, K are norm closed ideals of A with $J \cap K = \{0\}$, then $J = \{0\}$ or $K = \{0\}$. (As is clear from the above, "norm closed ideals" in the definition may be replaced with "ideals")

PROPOSITION 2.2. *Let A be a JB*-triple. The following conditions are equivalent:*

(i) *A is prime.*
(ii) *$J^{\perp} = \{0\}$ for every nonzero ideal J of A.*
(iii) *If $x, y \in A$ with $Q_{x,y} = 0$, then $x = 0$ or $y = 0$.*

Proof. (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are immediate from the definition.

(ii) $\Rightarrow$ (iii); Assume (ii) and let $x, y \in A$ with $Q_{x,y} = 0$. Suppose that $x \neq 0$. By Proposition 2.1, there exist ideals J, K of A with $x \in J$ and $y \in K \subset J^{\perp} = \{0\}$.

Let M be a JBW*-triple and let J be a weak*-closed ideal of M. Then $M = J + J^{\perp}$ [**10**]. Recall that M is called a *factor* if it has no proper weak*-closed ideals. By Proposition 2.2, we have

COROLLARY 2.3. *A JBW*-triple is prime if and only if it is a factor.*

3. Cartan factors

In this section, we investigate the constant K_A for a Cartan factor A. We always have $K_A \leqslant \frac{1}{2}$ if $\dim A > 1$. Given a finite dimensional Cartan factor A with unit sphere $S = \{a \in A : \|a\| = 1\}$, the product $S \times S$ is compact and we have a continuous function $g : S \times S \to (0, \infty)$ given by $g(a, b) = \|Q_{a,b}\|$. Hence $K_A \geqslant \inf g(S \times S) > 0$. To estimate K_A, more elaborate methods are required.

LEMMA 3.1. *Let A be a finite dimensional Cartan factor, let u be a nonzero tripotent in A and let $b_0 \in A_0(u)$. Then there exists $x \in A_1(u)$ with $\|x\| = 1$ such that $\|\{uxb_0\}\| \geqslant \frac{1}{2}\|b_0\|$.*

Proof. The conclusion being trivial if $b_0 = 0$, we may suppose that $\|b_0\| = 1$. By the spectral decomposition of b_0 in $A_0(u)$ we have $b_0 = v + c$ where v is a nonzero tripotent in $A_0(u)$ and c is orthogonal to v. For any $x \in A$ and $z \in A_0(u)$ we have $\{xv\{uuz\}\} = 0$, so that by the main identity

$$\{uu\{xvz\}\} = \{u\{vxu\}z\}.$$

Put $D = \{uAv\}$. We note from Proposition 2.1 that $D \neq \{0\}$ as A is a factor. Consider the map $T : D \to A$ given by $T(d) = \{udb_0\}$. For $x \in A$, the above equation gives

$$T(\{vxu\}) = \{u\{vxu\}b_0\} = \{uu\{xvb_0\}\} = \{uu\{xvv\}\} = \{u\{vxu\}v\}.$$

This shows that $T(D) \subset D$ and, as $D \subset A_1(u) \cap A_1(v)$ by the Peirce rules, it shows that

$$T^2(d) = \{uu\{dvv\}\} = \frac{1}{4}d$$

for all $d \in D$. So T attains its norm on the unit sphere of D and $\|T\| \geqslant \frac{1}{2}$.

PROPOSITION 3.2. *Let A be a finite dimensional Cartan factor. Then $\|Q_{a,b}\| \geqslant \frac{1}{6}\|a\| \cdot \|b\|$ for all $a, b \in A$.*

Proof. Let $a, b \in A$ with $\|a\| = \|b\| = 1$. By spectral decomposition, $a = u + a_0$ where u is a minimal tripotent of A and $a_0 \in A_0(u)$. By Pierce decomposition with respect to u, we have $b = \lambda u + b_1 + b_0$ where $\lambda \in \mathbb{C}$, $b_1 \in A_1(u)$ and $b_0 \in A_0(u)$. We have

$$\|Q_{a,b}\| \geqslant \|\{aub\}\| = \|\{uub\}\| = \|\lambda u + \frac{1}{2}b_1\| \geqslant \max(|\lambda|, \frac{1}{2}\|b_1\|)$$

where the last inequality follows from the contractiveness of $P_j(u)$, $j = 0, 1$.

The required conclusion follows if $|\lambda| \geqslant \frac{1}{6}$ or $\|b_1\| \geqslant \frac{1}{3}$. Otherwise we have $|\lambda| < \frac{1}{6}$ and $\|b_1\| < \frac{1}{3}$. In which case, by orthogonality of u and b_0 [7, 1.3(a)]

$$\max(|\lambda|, \|b_0\|) = \|\lambda u + b_0\| \geqslant 1 - \|b_1\| > \frac{2}{3}.$$

By Lemma 3.1, choose $x \in A_1(u)$ with $\|x\| = 1$ and $\|\{uxb_0\}\| > \frac{1}{3}$. By Peirce arithmetic, we have $\{axb\} = x_2 + x_1 + x_0$ with $x_j \in A_j(u)$, $j = 0, 1, 2$ and

$$x_2 = \{uxb_1\}, \quad x_1 = \{uxb_0\} + \lambda\{a_0xu\}, \quad x_0 = \{a_0xb_1\}.$$

Hence, as $P_1(u)$ is contractive, we obtain

$$\|Q_{a,b}\| \geqslant \|x_1\| \geqslant \|\{uxb_0\}\| - |\lambda|\|\{a_0xu\}\| > \frac{1}{3} - |\lambda| > \frac{1}{6}.$$

We note that Proposition 3.2 also solves the problem posed in [4] of exhibiting a positive lower bound of K_A for $A = M_3^8$.

We shall proceed to a close examination of special Cartan factors and we shall make use of the following.

REMARK 3.3. For a (complex) Hilbert space H with conjugation $h \mapsto \bar{h}$ and $a \in B(H)$, the operator $a^T \in B(H)$ is defined as before. The rank one operator $v \mapsto \langle v, k\rangle h$ is denoted by $h \otimes k$. In particular, for $h, k, h', k' \in H$, we have

$$\langle (h \otimes k + h' \otimes k')(k), h\rangle = \|h\|^2\|k\|^2 + \langle k, k'\rangle\langle h', h\rangle.$$

Hence, for $\|h\| = \|k\| = 1$, we have

(i) $\|h \otimes k + h' \otimes k'\| \geqslant |1 + \langle k, k'\rangle\langle h', h\rangle|$;
(ii) $\|h \otimes \bar{k} + k \otimes \bar{h}\| \geqslant 1 + |\langle h, k\rangle|^2 \geqslant 1$;
(iii) $\|h \otimes \bar{k} - k \otimes \bar{h}\| = 1$ if h and k are orthogonal;
(iv) if $a^T = a$, then $a^T(\bar{h}) = \overline{a(h)}$;
(v) if $a^T = -a$, then $\langle ah, \bar{k}\rangle = -\langle ak, \bar{h}\rangle$ and so $\langle ah, \bar{h}\rangle = 0$;
(vi) if $\|ah\| = 1$, then $a^*ah = h$ since $\|a^*ah - h\|^2 = \|a^*ah\|^2 - 1 \leqslant 0$.

Recall that a Hilbert space H is a JB*-triple via $\{abc\} = \frac{1}{2}(\langle a, b\rangle c + \langle c, b\rangle a)$ and is isometric to $B(H, \mathbb{C})$. In the following, for $n < \infty$, $M_{n,n}(\mathbb{C})$ denotes the full algebra of $n \times n$ matrices; for $n \geqslant 4$, $A_n(\mathbb{C}) = \{x \in M_{n,n}(\mathbb{C}) : x^T = -x\}$; for $n \geqslant 2$, $S_n(\mathbb{C}) = \{x \in M_{n,n}(\mathbb{C}) : x^T = x\}$. The JB*-triples $S_2(\mathbb{C})$, $M_{2,2}(\mathbb{C})$ and $A_4(\mathbb{C})$ are spin factors.

LEMMA 3.4. *Let H be a Hilbert space and let $a, b \in H$. Then*

$$\|Q_{a,b}\| \geqslant \frac{1}{2}(\|a\|^2 + \|b\|^2 + 3|\langle a, b\rangle|^2)^{\frac{1}{2}} \geqslant \frac{1}{2}\|a\| \cdot \|b\|.$$

Moreover the constant $\frac{1}{2}$ is best possible for dim $H > 1$.

Proof. Let $\|a\| = \|b\| = 1$. Then

$$4\|Q_{a,b}\|^2 \geqslant 4\|\{aab\}\|^2 \geqslant \|b + \langle b, a\rangle a\|^2 = 1 + 3|\langle a, b\rangle|^2 \geqslant 1,$$

and the inequality follows.

If $\langle a, b\rangle = 0$, then $\|Q_{a,b}(x)\| = \frac{1}{2}(|\langle a, x\rangle|^2 + |\langle b, x\rangle|^2)^{\frac{1}{2}} \leqslant \frac{1}{2}\|x\|$ for all $x \in H$, so that $\|Q_{a,b}\| = \frac{1}{2}$.

LEMMA 3.5. *Let $A = M_{n,n}(\mathbb{C})$ and let $a, b \in A$. Then*

$$\|Q_{a,b}\| \geqslant (\sqrt{2} - 1)\|a\| \cdot \|b\|.$$

Proof. Put $K_1 = \sup\{|\langle ah, bh\rangle| : \|h\| = 1\}$ and $K_2 = \sup\{|\langle a^*h, b^*h\rangle| : \|h\| = 1\}$. Let $\|a\| = \|b\| = 1$ and choose $h, k \in \mathbb{C}^n$ with $\|ah\| = \|b^*k\| = \|h\| = 1$. Then

$$\begin{aligned}
2\|Q_{a,b}\| &\geqslant \|a(h \otimes k)b + b(h \otimes k)a\| \\
&= \|ah \otimes b^*k + bh \otimes a^*k\| \\
&\geqslant |1 + \langle b^*k, a^*k\rangle\langle bh, ah\rangle| \quad \text{by Remark 3.3(i)} \\
&\geqslant 1 - |\langle ah, bh\rangle| \cdot |\langle a^*k, b^*k\rangle| \\
&\geqslant 1 - K_1K_2.
\end{aligned}$$

Now let $\eta \in \mathbb{C}^n$ with $\|\eta\| = 1$ and consider the minimal projection $p = \eta \otimes \eta$. Then $P(x) = px$ defines a continuous projection $P : A \to A$ with $P(A)$ isometric to the Hilbert space $\mathbb{C}^n$. Further $PQ_{a,b}P = Q_{P(a),P(b)}P$ and $P(a)\eta = a^*\eta$. Hence, by the middle estimate in Lemma 3.4,

$$\begin{aligned}
2\|Q_{a,b}\| &\geqslant 2\|Q_{P(a),P(b)}P\| \\
&\geqslant (\|a^*\eta\|^2\|b^*\eta\|^2 + 3|\langle a^*\eta, b^*\eta\rangle|^2)^{\frac{1}{2}} \\
&\geqslant 2|\langle a^*\eta, b^*\eta\rangle|.
\end{aligned}$$

Hence $\|Q_{a,b}\| \geqslant K_2$. In turn $\|Q_b\| = \|Q_{a^*,b^*}\| \geqslant K_1$. So, if K_1 or $K_2 \geqslant \sqrt{2} - 1$, the result follows. Otherwise $K_1, K_2 < \sqrt{2} - 1$ so that

$$\|Q_{a,b}\| \geqslant \frac{1}{2}(1 - K_1K_2) > \frac{1}{2}(1 - (\sqrt{2} - 1)^2) = \sqrt{2} - 1.$$

LEMMA 3.6. *Let $a, b \in S_n(\mathbb{C})$ where $n < \infty$. Then $\|Q_{a,b}\| \geqslant \frac{1}{4}\|a\| \cdot \|b\|$.*

Proof. Let $\|a\| = \|b\| = 1$ and let $h \in \mathbb{C}^n$ with $\|h\| = 1$. As $\bar{h} \otimes h$ has norm 1 and is in $S_n(\mathbb{C})$ we have, using Remark 3.3 (ii),

$$2\|Q_{a,b}\| \geqslant \|a(h \otimes \bar{h})b + b(h \otimes \bar{h})a\| = \|ah \otimes \overline{bh} + bh \otimes \overline{ah}\| \geqslant \|ah\|.\|bh\|.$$

Suppose now that $\|ah\| = 1$ and choose $k \in \mathbb{C}^n$ with $\|bk\| = \|k\| = 1$. Multiplying k by a suitable constant of modulus 1, we may suppose that $Re\langle h, k\rangle = 0$. In which case, by Remark 3.3 (vi),

$$Re\langle ah, ak\rangle = Re\langle a^*ah, k\rangle = Re\langle h, k\rangle = 0.$$

similarly $Re\langle bh, bk\rangle = 0$. Hence, by the above inequality,

$$\begin{aligned}4\|Q_{a,b}\| &= 2\|Q_{a,b}\| \cdot \|h+k\|^2\\ &\geqslant \|a(h+k)\| \cdot \|b(h+k)\|\\ &\geqslant (1+\|ak\|^2)^{\frac{1}{2}}(1+\|bh\|^2)^{\frac{1}{2}} \geqslant 1.\end{aligned}$$

LEMMA 3.7. *Let $a, b \in A_n(\mathbb{C})$ where $4 \leqslant n < \infty$. Then $\|Q_{a,b}\| \geqslant \frac{1}{4}\|a\|\cdot\|b\|$.*

Proof. Let $\|a\| = \|b\| = 1$ and let $h \in \mathbb{C}^n$ with $\|h\| = 1$. Choose $k \in \mathbb{C}^n$ with $\|k\| = 1$ and $\langle h, k\rangle = 0$. Then $x = \bar{k} \otimes h - \bar{h} \otimes k \in A_n(\mathbb{C})$, $\|x\| = 1$ and

$$\begin{aligned}y &= ax^*b + bx^*a\\ &= ah \otimes b^*\bar{k} + bh \otimes a^*k - ak \otimes b^*\bar{h} - bk \otimes a^*\bar{h}\\ &= -(ah \otimes \overline{bk} + bh \otimes \overline{ak}) + (ak \otimes \overline{bh} + bk \otimes \overline{ah}).\end{aligned}$$

So, using Remark 3.3 (v), we have

$$\begin{aligned}2\|Q_{a,b}\| &\geqslant |\langle yh, \bar{k}\rangle|\\ &= |\langle h, \overline{bk}\rangle\langle ah, \bar{k}\rangle + \langle h, \overline{ak}\rangle\langle bh, \bar{k}\rangle|\\ &= 2|\langle ah, \bar{k}\rangle\langle bh, \bar{k}\rangle|.\end{aligned}$$

We claim that $2\|Q_{a,b}\| \geqslant \|ah\|.\|bh\|$. To see this, we suppose that ah and bh are nonzero and put $h' = \frac{\overline{ah}}{\|ah\|}$, $k' = \frac{\overline{bh}}{\|bh\|}$. Then $\langle h', h\rangle = \langle k', h\rangle = 0$ by Remark 3.3 (v). Further, as $\|Q_{a,b}\|$ is unaffected, multiplying a by a suitable constant of modulus 1, we may suppose that $\langle h', k'\rangle \geqslant 0$. The above inequality implies

$$\begin{aligned}2(1+\langle h', k'\rangle)\|Q_{a,b}\| &= \|h'+k'\|^2\|Q_{a,b}\|\\ &\geqslant |\langle ah, \overline{h'+k'}\rangle\langle bh, \overline{h'+k'}\rangle|\\ &= \|ah\|(1+\langle h', k'\rangle)\|bh\|(1+\langle h', k'\rangle)\\ &\geqslant \|ah\| \cdot \|bh\|.\end{aligned}$$

Finally, pick $\alpha, \beta \in \mathbb{C}^n$ such that $\|a\alpha\| = \|b\beta\| = \|\alpha\| = \|\beta\| = 1$. Multiplying α by a suitable constant of modulus 1, we may suppose that $\langle \alpha, \beta\rangle \geqslant 0$. Then, with $\eta = (\alpha+\beta)/\|\alpha+\beta\|$, we have

$$\begin{aligned}\|\alpha\eta\|^2 &= \frac{\|ah+ak\|^2}{\|h+k\|^2}\\ &= \frac{1+\|ak\|^2+2Re\langle ah, ak\rangle}{\|h+k\|^2}\\ &= \frac{1+\|ak\|^2+2\langle h, k\rangle}{2+2\langle h, k\rangle}\\ &\geqslant \frac{1}{2}.\end{aligned}$$

Similarly $\|b\eta\|^2 \geqslant \frac{1}{2}$. Hence, by the above, we get

$$\|Q_{a,b}\| \geqslant \frac{1}{2}\|a\eta\| \cdot \|b\eta\| \geqslant \frac{1}{4}.$$

We are now in a position to establish positive lower bounds of K_A for all Cartan factors.

THEOREM 3.8. *Let A be a Cartan factor. Then we have*

(i) $K_A \geqslant \sqrt{2}-1$ *if A is a spin factor with dim* $A \geqslant 4$ *or A is rectangular;*
(ii) $K_A \geqslant \frac{1}{4}$ *if A is hermitian or symplectic;*
(iii) $K_A \geqslant \frac{1}{6}$ *if* $A = B_{1,2}$ *or* M_3^8.

Proof. (i) Let A be a spin factor of dimension at least 4. We refer to Section 1 for the definition of a spin factor and the following notation. Let $a, b \in A$ with $\|a\| = \|b\| = 1$. The linear subspace generated by $a, \bar{a}, b, \bar{b}$ is conjugate invariant of dimension at most 4. If necessary, by adding to this list of generators sufficient conjugate invariant elements, we obtain a conjugate invariant subspace V of dimension 4 containing a and b. As $V = \bar{V}$, V is a JB*-subtriple of A as follows from the spin factor triple product rule

$$\{xyz\} = \frac{1}{2}(\langle x, y\rangle z + \langle z, y\rangle x - \langle x, \bar{z}\rangle \bar{y}).$$

Hence V is a 4-dimensional spin factor which must be isometric to $M_{2,2}(\mathbb{C})$. So $\|Q_{a,b}\| \geqslant \|Q_{a,b}|v\| \geqslant \sqrt{2}-1$ by Lemma 3.5.

Let $A = B(H, K)$ be rectangular and let $a, b \in A$ with $\|a\| = \|b\| = 1$. Let $0 < \varepsilon < 1$ and choose $\alpha, \beta \in H$ with $\|\alpha\| = \|\beta\| = 1$ and $\|\alpha_1\| > 1-\varepsilon$, $\|\beta_1\| > 1-\varepsilon$ where $\alpha_1 = a\alpha$, $\beta_1 = b\beta$. By Lemma 3.4, we may suppose that both H and K have dimension at least 2. Now let U, V be respectively, a 2-dimensional subspace of H, K with $\alpha, \beta \in U$ and $\alpha_1, \beta_1 \in V$, and let p, q be the corresponding orthogonal projections from H, K onto U, V. Define $P: A \to A$ by $P(x) = qxp$. Then $P(A) = qAp$ is isometric to $M_{2,2}(\mathbb{C})$, P is a contractive projection with $PQ_{a,b}P = Q_{P(a),P(b)}P$ and $\|P(a)\| \geqslant \|qaP(\alpha)\| = \|\alpha_1\| > 1-\varepsilon$. Similarly $\|P(b)\| > 1-\varepsilon$. Now, using Lemma 3.5, we have

$$\begin{aligned}\|Q_{a,b}\| &\geqslant \|Q_{P(a),P(b)}P\|\\ &\geqslant (\sqrt{2}-1)\|P(a)\|.\|P(b)\|\\ &> (\sqrt{2}-1)(1-\varepsilon)^2.\end{aligned}$$

Hence $\|Q_{a,b}\| \geqslant \sqrt{2}-1$.

(ii) By Lemma 3.6 and Lemma 3.7, we may suppose that $A = \{x \in B(H) : x^T = x\}$ or $A = \{x \in B(H) : x^T = -x\}$, where H is an infinite dimensional complex Hilbert space with conjugation $h \mapsto \bar{h}$.

In either case, let $a, b \in A$ with $\|a\| = \|b\| = 1$ and let $0 < \varepsilon < 1$. Choose $\alpha, \beta \in H$ with $\|\alpha\| = \|\beta\| = 1$ and $\|\alpha_1\| > 1-\varepsilon$, $\|\beta_1\| > 1-\varepsilon$, where

$\alpha_1 = a\alpha$, $\beta_1 = b\beta$. Now choose an 8-dimensional subspace V of H containing $\alpha, \beta, \alpha_1, \beta_1$, and satisfying $\bar{V} = V$. Let p be the orthogonal projection of H onto V. Then $p^T = p$. Consequently $P(x) = pxp$ defines a contractive projection $P : A \to A$ in both cases. The induced conjugation on pH, $ph \mapsto p\bar{h}$ for $h \in H$, is precisely the conjugation on V and we see that, via the isometry $pxp \mapsto x|_V$, where $x \in B(H)$, $P(A) = pAp$ is isometric to $S_8(\mathbb{C})$ in the first case, and isometric to $A_8(\mathbb{C})$ in the latter case. Finally, just as in (i), $PQ_{a,b}P = Q_{P(a),P(b)}P$ and $\|P(a)\| > 1 - \varepsilon$, $\|P(b)\| > 1 - \varepsilon$. Therefore, by the same calculation, this time using Lemma 3.6 and Lemma 3.7, we have $\|Q_{a,b}\| \geqslant \frac{1}{4}$.

(iii) This follows from Lemma 3.2.

4. The main theorem

We prove our main result in this section, showing the existence of a universal constant $K > 0$ such that for any prime JB*-triple A,

$$\|Q_{a,b}\| \geqslant K\|a\| \cdot \|b\|$$

for all $a, b \in A$. In fact, more sharply, we shall show that $K \geqslant \frac{1}{6}$. We shall use structure space techniques and representation theory of JB*-triples given below.

Let A be a JB*-triple and let $\partial_e(A_1^*)$ be the set of extreme points of the dual ball A_1^*. For each $\rho \in \partial_e(A_1^*)$, there is a unique minimal tripotent u in A^{**} for which $\rho(u) = 1$, called the *support* u_ρ of ρ. The map $\rho \mapsto u_\rho$ is a bijection from $\partial_e(A_1^*)$ onto the set of all minimal tripotents of A^{**}. For $\rho \in \partial_e(A_1^*)$, let A_ρ^{**} denote the weak*-closed ideal of A generated by u_ρ. Then A_ρ^{**} is a Cartan factor [**5**,**11**] and is an M-summand of A^{**} [**10**]. The natural weak* continuous contractive projection $P_\rho : A^{**} \to A_\rho^{**}$ restricts to a triple homomorphism $\pi_\rho : A \to A_\rho^{**}$ with weak* dense range. We call π_ρ the Cartan factor *representation* of A associated with ρ. By [**2**, Proposition 3.6] and its proof, we obtain

(i) $\|a\| = \sup\{\|\pi_\rho(a)\| : \rho \in \partial_e(A_1^*)\}$ for $a \in A$;
(ii) $\ker \pi_\rho$ is the largest M-ideal in $\ker \rho$ for $\rho \in \partial_e(A_1^*)$.

In particular, $\{\ker \pi_\rho : \rho \in \partial_e(A_1^*)\}$, denoted by Prim A, is the set of all *primitive* M-ideals of A (cf. [**1**]) which we shall assume to be equipped with the usual hull-kernel topology. Also, there is a bijection

$$J \mapsto h(J) = \{P \in PrimA : J \subset P\}$$

from the norm-closed ideals onto the closed subsets of $Prim\ A$.

Given an M-ideal M in A, by [**1**, p.116], the polar M^0 in the dual A^* is a so-called *L-ideal*, that is, it is the range of an L-projection $E : A^* \to A^*$. The L-projections on A^* generate the *Cunnigham algebra* of A^* which is a commutative unital Banach algebra isomorphic to $C(\Omega)$ where the spectrum Ω of the algebra is hyperstonean. The L-projections form a complete lattice in $C(\Omega)$ [**1**, p.130].

LEMMA 4.1. *Let $\{M_\alpha\}$ be a family of M-ideals of a JB*-triple A. Then*

$$(\cap_\alpha M_\alpha)^0 \cap A_1^* = \overline{co} \cup_\alpha (M_\alpha^0 \cap A_1^*).$$

where $\overline{co}$ denotes the weak closed convex hull.*

Proof. By taking finite intersections, we may assume that the family $\{M_\alpha\}$ is a decreasing net. Let $M_\alpha^0 = E_\alpha A_1^*$ for some L-projection E_α on A_1^*. Then $\{E_\alpha\}$ is an increasing net of L-projections and has a least upper bound E (cf. [**1**, p.135]). By [**1**, Lemma 1.9], E_α converges strongly to E, that is, $E_\alpha f$ is norm-convergent to Ef for each $f \in A^*$. It follows that $EA^* = \overline{\overline{\Sigma M_\alpha^0}}$ where on the right we consider sums of finitely many elements and "$=$" denotes the norm-closure. We also have

$$\overline{\overline{\sum M_\alpha^0}} \cap A_1^* = \overline{\overline{co}} \cup_\alpha (M_\alpha^0 \cap A_1^*)$$

by strong convergence of E_α to E.

For a set $S \subset A^*$, we denote by S_0 the polar of S in A. Let $J = \cap_\alpha M_\alpha$. Then $J^0 = \overline{\Sigma M_\alpha^0}$ where "—" denotes the weak*-closure. We have

$$\begin{aligned}(J^0 \cap A_1^*)_0 &= \left(\overline{\sum M_\alpha^0} \cap A_1^*\right)_0 \\ &= \left(\left(\left(\sum M_\alpha^0\right)_0 \cup (A_1^*)_0\right)^0\right)_0 \\ &= \left(\left(\left(\overline{\overline{\sum M_\alpha^0}}\right)_0 \cup (A_1^*)_0\right)^0\right)_0 \\ &= \left(\overline{\overline{\sum M_\alpha^0}} \cap A_1^*\right)_0 \\ &= (\overline{\overline{co}} \cup_\alpha (M_\alpha^0 \cap A_1^*))_0 \\ &= (\overline{co} \cup_\alpha (M_\alpha^0 \cap A_1^*))_0.\end{aligned}$$

Hence we have

$$J^0 \cap A_1^* = \overline{co} \cup_\alpha (M_\alpha^0 \cap A_1^*).$$

Since $(A/\ker \pi_\rho)^*$ is isometrically linearly isomorphic to $(\ker \pi_\rho)^0$, we have, for each $a \in A$,

$$\|\pi_\rho(a)\| = \|a + \ker \pi_\rho\| = \sup\{|\psi(a)| : \psi \in (\ker \pi_\rho)^0, \|\psi\| \leqslant 1\}.$$

LEMMA 4.2. *Let A be a JB*-triple and let $a \in A$. Then the map $Prim\ A \to [0, \infty)$ given by $P \mapsto \|a + P\|$ is lower semicontinuous.*

Proof. Let $r \in \mathbb{R}$. We show that the set $\mathcal{S} = \{P \in Prim\ A : \|a + P\| \leqslant r\}$ is closed. Let $hk(\mathcal{S})$ be the hull-kernel of $\mathcal{S}$ and let $Q \in hk(\mathcal{S})$. Then $Q\ supset\ k(\mathcal{S}) = \cap\{P : P \in \mathcal{S}\}$. So, by the above lemma, we have

$$Q^0 \cap A_1^* \subset \overline{co} \cup \{P^0 \cap A_1^* : P \in \mathcal{S}\}.$$

It follows that $|\psi(a)| \leqslant r$ for every $\psi \in Q^0$ with $\|\psi\| \leqslant 1$. Hence $\|a + Q\| \leqslant r$ and $Q \in \mathcal{S}$. This shows that $\mathcal{S} = hk(\mathcal{S})$ is closed.

THEOREM 4.3. *Let A be a prime JB*-triple and let $a, b \in A$. Then*

$$\|Q_{a,b}\| \geqslant \frac{1}{6}\|a\| \cdot \|b\|.$$

Further, if A is

(i) *a JC^*-triple, then $\|Q_{a,b}\| \geqslant \frac{1}{4}\|a\| \cdot \|b\|$;*
(ii) *a C^*-algebra, then $\|Q_{a,b}\| \geqslant (\sqrt{2} - 1)\|a\| \cdot \|b\|$.*

Proof. Let $\rho \in \partial_e(A_1^*)$ and let P_ρ, π_ρ and A_ρ be as above. Then there exists $K > 0$ such that

$$\|Q_{x,y}\| \geqslant K\|x\| \cdot \|y\|$$

for all $x, y \in A_\rho^{**}$ and all $\rho \in \partial_e(A_1^*)$. Let $a, b \in A$ with $\|a\| = \|b\| = 1$. By separate weak* continuity of the triple product, $Q_{a,b}^{**} : A^{**} \to A^{**}$ is given by $Q_{a,b}^{**}(x) = \{axb\}$ for all $x \in A^{**}$. We have, for $\rho \in \partial_e(A_1^*)$,

$$\begin{aligned} \|Q_{a,b}\| &= \|Q_{a,b}^{**}\| \geqslant \|P_\rho Q_{a,b}^{**}\| \\ &= \sup\{\|\{\pi_\rho(a)P_\rho(x)\pi_\rho(b)\}\| : x \in A^{**}, \|x\| \leqslant 1\} \\ &= \sup\{\|\{\pi_\rho(a)x\pi_\rho(b)\}\| : x \in A_\rho^{**}, \|x\| \leqslant 1\} \\ &\geqslant K\|\pi_\rho(a)\| \cdot \|\pi_\rho(b)\| \quad \text{by assumption.} \end{aligned}$$

Let $0 < \varepsilon < 1$. By Lemma 4.1, the sets $U = \{\ker \pi_\rho \in \mathit{Prim}\, A : \|\pi_\rho(a)\| > 1 - \varepsilon\}$ and $V = \{\ker \pi_\rho \in \mathit{Prim}\, A : \|\pi_\rho(b)\| > 1 - \varepsilon\}$ are nonempty and open subsets of $\mathit{Prim}\, A$. As A is prime, $U \cap V$ must be nonempty too. Hence there exists $\tau \in \partial_e(A_1^*)$ such that $\|\pi_\tau(a)\| > 1 - \varepsilon$ and $\|\pi_\tau(b)\| > 1 - \varepsilon$. So $\|Q_{a,b}\| \geqslant K(1 - \varepsilon)^2$ by the above inequality. Therefore $\|Q_{a,b}\| \geqslant K$. All parts of the statement are now consequences of Theorem 3.8.

REMARK 4.4. It has been shown in **[15]** that for a prime C^*-algebra A, $\|Q_{a,b}\| \geqslant \frac{1}{3}\|a\| \cdot \|b\|$ for every $a, b \in A$. The above result gives a negative answer to the question of sharpness of the constant $\frac{1}{3}$ raised in **[15]**.

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REFERENCES

1. E. M. Alfsen and E. G. Effros, 'Structure in real Banach spaces', *Ann. of Math.* **96** (1972), 98–173.
2. T. J. Barton and R. M. Timoney, 'Weak* continuity of Jordan triple product and applications', *Math. Scand.* **59** (1986), 177–191.
3. L. J. Bunce and C.-H. Chu, 'Compact operations, multipliers and the Radon-Kikodym property in JB*-triples', *Pacific J. Math.* **53** (1992), 249–265.
4. M. Cabrera and A. Rodriguez, 'Nondegenerately ultraprime Jordan Banach algebras: a Zelmanovian treatment', *Proc. London Math. Soc.* **69** (1994), 576–604.
5. T. Dang and Y. Friedman, 'Classification of JBW*-factors and applications', *Math. Scand.* **61** (1987), 292–330.
6. S. Dineen, *The second dual of a JB*-triple system*, Complex analysis, functional analysis and approximation theory, North-Holland Math. Studies 125 (1986), 67–69.
7. Y. Friedman and B. Russo, 'Structure of a predual of a JBW*-triple', *J. Reine Angew.* **356** (1985), 67–84.
8. Y. Friedman and B. Russo, 'Solution of the contractive projection problem', *J. Funct. Analysis* **60** (1985), 56–79.
9. L. A. Harris, 'A generalisation of C^*-algebras', *Proc. London Math. Soc.* **42** (1981), 331–361.
10. G. Horn, 'Characterisation of the predual and ideal structure of a JBW*-triple', *Math. Scand.* **61** (1987), 117–133.
11. G. Horn, 'Classification of JBW*-triples of type I', *Math. Z.* **196** (1987), 271–291.
12. W. Kaup, 'Algebraic characterization of symmetric complex Banach manifolds', *Math. Ann.* **228** (1977), 39–64.
13. W. Kaup, 'A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces', *Math. Z.* **138** (1983), 503–509.
14. O. Loos, *Bounded symmetric domains and Jordan pairs*, Lecture Notes, Irvine, 1977.
15. M. Mathieu, 'More properties of the product of two derivations of a C^*-algebra', *Bull. Australia Math. Soc.* **42** (1990), 115-120.
16. B. Russo, *Structure of JB*-triples*, in Jordan Algebras (ed. W. Kaup, K. McCrimmon, H.P. Peterson), de Gruyter, Berlin, 1994.
17. J. D. M. Wright, 'Jordan C*-algebras', *Michigan Math. J.* **24** (1977), 291–302.

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