ATOMIC DECOMPOSITION OF REAL JBW*-TRIPLES

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[Received 29 June 1999. Revised 16 March 2000]

Abstract

We study the natural partial ordering in the set of all tripotents of a real JB*-triple. We prove some characterizations of real-minimal tripotents and an atomic deocmpostion theorem for real JBW*-triples is given.

1. Introduction

The class of complex Banach spaces called JB*-triples has been the object of intensive investigations in the last two decades. By Kaup's Riemann mapping theorem [**14**], a holomorphically symmetric bounded domain is holomorphically equivalent to the unit ball of some complex JB*-triple and, conversely, the unit ball of any complex JB*-triple is a symmetric domain. This fact made it possible to apply complex geometric arguments very successfully in the study of complex JB*-triples as well as the standard techniques of functional analysis. In particular a deeply elaborated geometric theory of dual complex JB*-triples (the so-called complex JBW*-triples) appeared, shedding new light on von Neumann algebras and the Gelfand–Naimark theorem.

Recently, considerable attention is paid to real JB*-triples introduced in [**13**] as isometric copies of closed real subtriples of complex JB*-triples, objects which can conveniently be studied as invariant subspaces of conjugations in complex JB*-triples. In this manner several results concerning complex JB*-triples have been extended to the real case [**6, 13, 15, 17**].

In the complex case JBW*-triples split into the direct sum of an ideal free of indecomposable tripotents (called atoms) and the direct sum of a family of Cartan factors, the latter coinciding with the w*-closed linear hull of all atoms. One of the main purposes of the present paper is the real analogue of this decomposition. For this first we investigate briefly the natural ordering of tripotents with respect to orthogonal decomposability. We complete the result of [**5, 6, 8**] proving that the minimal tripotents of a real JBW*-triple are the support functionals of the extreme points of the unit ball of its predual. As a key point to the atomic decomposition we show in particular that real atoms (indecomposable tripotents in the bidual) are the same as real minimal tripotents in the terminology of [**15**] and every real atom is the real part of some complex minimal tripotent from the complexification. We conclude our work obtaining an atomic decomposition for real JBW*-triples.

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2. Tripotents in real JB*-triples

By a *complex* JB*-*triple* we mean a complex Banach space V equipped with a triple product $\{\ ,\ ,\ \}: V \times V \times V \to V$ which is bilinear symmetric in the outer variables and conjugate-linear in the middle one satisfying

(i) (*Jordan identity*) $\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$,

(ii) $L(x, x)$ is an Hermitian operator on V with non-negative spectrum, and $\| \{x, x, x\} \| = \|x\|^3$,

for all $x, y, z, a, b \in V$, where $L(a, b)$ is the linear operator defined by $L(a, b)y := \{a, b, y\}$.

A *real* JB*-*triple* is a real Banach space E with an $\mathbb{R}$-trilinear triple product whose complexification admits a (necessarily unique) complex JB*-triple structure with norm and triple product extending those of E. It is well known [**13**] that any closed real subtriple of a complex JB*-triple is a real JB*-triple. In particular complex JB*-triples are real JB*-triples at the same time.

By a (*real complex*) JBW*-*triple* we mean a real (complex) JB*-triple whose underlying Banach space is a dual Banach space in metric sense. It is known (see [**2, 17**]) that every real or complex JBW*-triple has a unique predual up to isometric linear isomorphisms and its triple product is separately w*-continuous in each variable (with respect to the unique weak*-topology induced by any predual).

Henceforth E will denote an arbitrarily fixed real JB*-triple. We write

$$\widehat{E} := E \oplus iE\ , \qquad \tau : x \oplus iy \mapsto x \oplus (-iy) \tag{2.1}$$

for the complexification of E and the *canonical conjugation* of $\widehat{E}$. We shall denote by $\|.\|$ and $\{\ ,\ ,\ \}$ the complex JB*-triple norm and the complex triple product in $\widehat{E}$ extending those of E, respectively. Notice that the conjugation τ is a conjugate-linear isometry of $\widehat{E}$ which is w*-continuous if $\widehat{E}$ is a JBW*-triple [**13**]. Given a real or complex JB*-triple U, the elements $e \in U$ with $\{e, e, e\} = e$ are called *tripotents*. We denote the set of all tripotents of U by $\mathrm{Tri}\,(U)$. Every $e \in \mathrm{Tri}\,(U)$ induces a decomposition of U into the eigenspaces of $L(e, e)$, the *Peirce decomposition*

$$U = U_0(e) \oplus U_1(e) \oplus U_2(e), \quad \text{where } U_k(e) := \left\{x \in U : \{e, e, x\} = \frac{k}{2}x\right\}$$

are the *Peirce subspaces* of U associated with the tripotent e. We have the multiplication rules

$$\{U_0(e), U_2(e), U\} = \{U_2(e), U_0(e), U\} = 0\ , \quad \left\{U_i(e), U_j(e), U_k(e)\right\} \subset U_{i+j+k}(e)$$

for $i, j, k \in \{0, 1, 2\}$ with the convention that $U_\ell(e) := 0$ ($\ell \neq 0, 1, 2$). The canonical projection $P_k : U \to U_k(e)$ called the *Peirce k-projection* of e is clearly a polynomial of $L(e, e)$. If x is an element of U, we write $Q(x)$ for the mapping given by $Q(x)y := \{x, y, x\}$. According to the Jordan identity, $Q(e)^3 = Q(\{e, e, e\}) = Q(e)$ for every tripotent $e \in U$. Hence $Q(e)$ induces also a decomposition

$$U = U^1(e) \oplus U^0(e) \oplus U^{-1}(e) \qquad \text{where} \qquad U^k(e) := \{x \in U :\ Q(e)x = kx\}$$

into $\mathbb{R}$-linear subspaces with the properties

$$U_2(e) = U^1(e) \oplus U^{-1}(e)\ , \quad U^0(e) = U_1(e) \oplus U_0(e)\ ,$$
$$\left\{U^j(e), U^k(e), U^\ell(e)\right\} \subset U^{jk\ell}(e) \qquad (jk\ell \neq 0,\ j, k, \ell \in \{0, \pm 1\}).$$

We write $P^k(e)$ for the natural projection of U onto $U^k(e)$.

Two tripotents u, v in U are said to be *orthogonal* ($u \perp v$ in notation) if $L(u, v) = L(v, u) = 0$. It is well known that $u \perp v \iff \{u, u, v\} = 0 \iff \{v, v, u\} = 0$ that is if $u \in U_0(v)$ or $v \in U_0(u)$. Notice that $v + z \in \mathrm{Tri}\,(U)$ with $L(v + z, v + z) = L(v, v) + L(z, z)$ whenever $v, z \in \mathrm{Tri}\,(U)$ and $v \perp z$.

Let $u, v \in \mathrm{Tri}\,(U)$. We say that u *majorizes* v in orthogonal decomposability, $u \geqslant v$ or $v \leqslant u$ in notation, if $u - v \in \mathrm{Tri}\,(U)$ with $u - v \perp v$. Henceforth we write $\min \mathrm{Tri}\,(U) := \{u \in \mathrm{Tri}\,(U) \setminus \{0\} : \nexists\, v \in \mathrm{Tri}\,(U) \setminus \{0\}\ v \leqslant u \neq v\}$.

Notice that if $u \geqslant v$ with $u, v \in U$, then $U_2(u) \supset U_2(v)$ and $U_0(u) \subset U_0(v)$. Applying the algebraic argument of [**5**, Lemma 2.4] we get immediately that $u \geqslant v \iff Q(u)v = Q(v)u = v \iff U^1(u) \supset U^1(v)$ and $Q(v)u = v$. Therefore, the map $u \mapsto U^1(u)$ is a strictly monotonic mapping from $\mathrm{Tri}\,(U)$ into the set of all $\mathbb{R}$-linear subspaces of U. In particular, $e \in \min \mathrm{Tri}\,(U)$ if $U^1(e) = \mathbb{R}e$.

PROPOSITION 2.2 *For a non-zero tripotent e in a real* JBW*-*triple W we have $e \in \min \mathrm{Tri}\,(W)$ if and only if $W^1(e) = \mathbb{R}e$. In the latter case the Peirce space $W_2(e)$ is a real Hilbert space.*

Proof. If $e \in \mathrm{Tri}\,(W) \setminus \{0\}$ with $W^1(e) = \mathbb{R}e$, then trivially $e \in \min \mathrm{Tri}\,(W)$. Let now $e \in \min \mathrm{Tri}\,(W)$. Suppose there exists $x \in W^1(e)$ but $x \notin \mathbb{R}e$. It is well known [**12**, Lemma 3.13] that the w*-closed complex JB*-subtriple $\widehat{A}$ of $\widehat{W}$ generated by $\{e, x\}$ is a commutative complex von Neumann-*algebra (W*-algebra) whose unit element is e when equipped with the operations $u \circ v := \{u, e, v\}$, $u^* := Q(e)u$, $(u, v \in \widehat{A})$. By the separate w*-continuity of the triple product in its second variable and since $\{\widehat{W}^1(e), \widehat{W}^1(e), \widehat{W}^1(e)\} = \widehat{W}^1(e)$, $\widehat{W}^1(e)$ is a w*-closed real JB*-subtriple in $\widehat{W}$ containing $A := \{a \in \widehat{A} : a^* = a\}$. Therefore the real w*-closed JB*-subtriple of W generated by $\{e, x\}$ is necessarily A. In particular A is a commutative real W*-subalgebra of $\widehat{A}$. Hence there is a surjective linear isometry

$$\phi : L^\infty_{\mathbb{R}}(\mu) \to A \quad \text{with} \quad \phi(fg) = \{\phi(f), e, \phi(g)\}\,,\ \phi(1_\Omega) = e$$

for a positive Radon measure μ on some compact topological space Ω. Since $\phi^{-1}x \notin \mathbb{R}1_\Omega = \phi^{-1}e$, there is a μ-measurable set S such that $\mu(S), \mu(\Omega \setminus S) > 0$. Then, with $g_1 := 1_S$, $g_2 := 1_{\Omega\setminus S}$, $e_k := \phi g_k$, we have $e_1, e_2 \in \mathrm{Tri}\,(W)$, $e_1 \perp e_2$ and $e = \phi 1_\Omega = \phi(g_1 + g_2) = e_1 + e_2$ which contradicts the minimality of e.

The statement concerning the spin structure of the Peirce space $W_2(e)$ holds in a more general setting. If E is a real JB*-triple and $0 \neq e \in \mathrm{Tri}\,(E)$ with $E^1(e) = \mathbb{R}e$ then $E_2(e)$ is a real JB*-triple of *rank* 1 in the terminology of [**15**]. Then, by [**15**, Proposition 5.4], as metric space $E_2(e)$ is a real Hilbert space.

REMARK 2.3 In [**15**] the term *real minimal* tripotent is used for tripotents with $E^1(e) = \mathbb{R}e$ in real JB*-triples. This terminology is somewhat misleading from the viewpoint that $\mathrm{Tri}\,(E)$ carries a natural ordering with respect to which minimal tripotents may have $\dim(E^1(e)) > 1$ (for example, in $C_{\mathbb{R}}([0, 1])$). By Proposition 2.2 above, for real JBW*-triples the terminology of [**15**] is appropriate. We suggest that tripotents in real JB*-triples with $E^1(e) = \mathbb{R}e$ should be called *atomic* tripotents in accordance with the terminology of complex atomic decompositions [7]. Indeed, as a consequence of Proposition 2.2, an atomic tripotent is minimal (thus orthogonally indecomposable) even in the bidual embedding of the underlying real JB*-triple.

The last part of this section is devoted to showing that, analogously to the complex case described in [**5**], for a real JBW*-triple the natural ordering of its tripotents is faithfully represented by the natural ordering of the w*-semi-exposed faces of its unit ball and the norm-semi-exposed faces of the unit ball of its predual, respectively. Given any real or complex Banach space X with norm $\|.\|$, we write $B(X)$ and X^* for its closed unit ball and dual space, respectively. We denote the standard functional norm on X^* by $\|\ \|$. For polar faces of unit balls we shall use the notation

$$\begin{aligned} S' &:= \{f \in B(X^*)\ :\ f(x) = 1\ \ (x \in S)\} \qquad (S \subset B(X))\,, \\ K_\prime &:= \{x \in B(X)\ :\ f(x) = 1\ \ (f \in K)\} \qquad (K \subset B(X^*)). \end{aligned}$$

We say that $F \subseteq B(X)$ is a *norm-semi-exposed face* of $B(X)$ if $F = K_\prime$ for some $K \subset B(X^*)$. In case X has a unique predual, we write X_* for its canonical predual $X_* :=$ {w*-continuous linear functionals of X} as a subspace of X^*. Then X is identified canonically with $(X_*)^*$ as $x \equiv [X_* \ni f \mapsto f(x)]$ for $x \in X$ and we say that F is a *w*-semi-exposed face* of $B(X)$ if $F = S'$ for some $S \subset B(X_*)$. We denote by $\mathcal{S}_{\rm n}(B(X))$ (resp. $\mathcal{S}_*(B(X))$) the set of all norm-semi-exposed faces of $B(X)$ (resp. the set of all w*-semi-exposed faces of $B(X)$ if X has a unique predual).

Recently Edwards and Rüttimann [**6**] deduced the following order description of tripotents in terms of faces from the analogous results in complex setting. For the sake of completeness we include a direct independent discussion in terms of semi-exposed faces which requires a shorter proof than that in [**6**].

PROPOSITION 2.4 [**6**] *Let W be a real* JBW**-triple. Then the map*

$$\Phi(e) := e + [B(W) \cap W_0(e)] \qquad (e \in \mathrm{Tri}\,(W))$$

is an anti-order isomorphism between $\mathrm{Tri}\,(W) \setminus \{0\}$ *and* $\mathcal{S}_{\rm w*}(B(W)) \setminus \{B(W)\}$ *(with ordinary set inclusion) and* $\Phi(e) = (\{e\}_\prime)'$ $(e \in \mathrm{Tri}\,(W))$. *The map* $\Phi_*(e) := \{e\}_\prime$ $(e \in \mathrm{Tri}\,(W))$ *is a surjective order isomorphism between* $\mathrm{Tri}\,(W)$ *and* $\mathcal{S}_{\rm n}(B(W_*)) \setminus \{B(W_*)\}$.

Proof. By [**13**, Proposition 4.6], the map Φ is a bijection $\mathrm{Tri}\,(W) \setminus \{0\} \leftrightarrow \mathcal{S}_{\rm w*}(B(W)) \setminus \{B(W)\}$. Let $u, v \in \mathrm{Tri}\,(W)$ with $0 \neq u \leqslant v$. In terms of the complexification, also $u \leqslant v$ in $\widehat{W} := W \oplus iW$. Hence, by [**5**, Theorem 4.6],

$$\widehat{\Phi}(u) := u + (B_{\widehat{W}} \cap \widehat{W}_0(u)) \supset \widehat{\Phi}(v) := v + (B_{\widehat{W}} \cap \widehat{W}_0(v)). \tag{2.5}$$

Since it is easy to see that $W \cap \widehat{\Phi}(v) = \Phi(v)$, from (2.5) it follows that $\Phi(u) = W \cap \widehat{\Phi}(u) \supset W \cap \widehat{\Phi}(v) = \Phi(v)$.

Conversely, if $\Phi(v) \subset \Phi(u)$ then $v \in \Phi(v) \subset \Phi(u) = u + (B(W) \cap W_0(u))$. Hence $v = u + x$ for some $x \in W_0(u)$ and therefore $u \leqslant v$.

Given any $e \in \mathrm{Tri}\,(W)$, the set $(\{e\}_\prime)'$ is the least w*-semi-exposed face of $B(W)$ containing the tripotent e. If $e \in F \in \mathcal{S}_{\rm w*}(B(W))$ then it follows $u \leqslant e$ and $\Phi(e) \subset \Phi(u) = F$. Thus $\Phi(e) = (\{e\}_\prime)'$.

Now we can proceed to the proof of the statement concerning the map Φ_*. Notice that in general, for any Banach space X with unique predual $X_*(\subset X^*)$ we have $S_\prime = ((S_\prime)')_\prime$ and $K' = ((K')_\prime)'$ for $S \subset B(X)$, $K \subset B(X_*)$. Therefore $\Phi(e) = (\{e\}_\prime)' = [\Phi_*(e)]'$ and $\Phi_*(e) = \{e\}_\prime = [(\{e\}_\prime)']_\prime = [\Phi(e)]_\prime \in \mathcal{S}_{\rm n}(B(W_*)) \setminus \{B(W_*)\}$ for all $e \in \mathrm{Tri}\,(W)$.

If $u \leqslant v$ in $\mathrm{Tri}\,(W)$ then, as we have seen, $\Phi(u) \supset \Phi(v)$ and and hence $\Phi_*(u) = [\Phi(u)]_\prime \subset [\Phi(v)]_\prime = \Phi_*(v)$. Conversely, if $\Phi_*(u) \subset \Phi_*(v)$ then $\Phi(u) = [\Phi_*(u)]' \supset [\Phi_*(v)]' = \Phi(v)$ implying $u \leqslant v$.

Finally we show that range$(\Phi_*) \supset \mathcal{S}_n(B(W_*)) \setminus \{B(W_*)\}$. Let $K \in \mathcal{S}_n(B(W_*)) \setminus \{B(W_*)\}$ be chosen arbitrarily. By definition, $K = S_{\prime}$ for some $S \subset B(W)$. Since $(S_{\prime})'$ is the smallest w*-semi-exposed face of $B(W)$ containing S, $K' = \Phi(u)$ for suitable $u \in \mathrm{Tri}\,(W)$. It follows that $\Phi_*(K) = [\Phi(u)]_{\prime} = (K')_{\prime} = [(S_{\prime})']_{\prime} = S_{\prime} = K$, that is, $K \in$ range(Φ_*).

Concerning minimal tripotents we complete this result. We are going to prove that the minimal tripotents of a real JBW*-triple are the support functionals of the extreme points of the unit ball of the predual.

REMARK 2.6 It is well known [**7**, Corollary 1.2; **12**] that in a complex JB*-triple, the Peirce projections of a tripotent are contractive. Hence, by passing to comlexification, it follows immediately that Peirce projections of tripotents in real JB*-triples are contractive. In particular, if E is a real JB*-triple and $e \in \mathrm{Tri}\,(E)$ then the operator

$$P^1(e) := \tfrac{1}{2}(Id_E + Q(e))P_2(e)$$

is contractive, too. In [**17**, Lemma 2.9] it is shown that if E is a real JB*-triple, $e \in \mathrm{Tri}\,(E)$, $f \in E^*$ and $\|fP_2(e)\| = \|f\|$ then $f = fP_2(e)$. The following lemma generalizes slightly this result.

LEMMA 2.7 *Let E be a real* JB*-*triple, $u \in \mathrm{Tri}\,(E)$ and $f \in E^*$ such that $f(u) = \|f\| = 1$. Then $f = f \circ P^1(u)$.*

Proof. By [**17**, Lemma 2.9] we have $f = P_2(u)^* f (:= f \circ P_2(u))$. Let $y \in E^{-1}(u)$ ($\{u, u, y\} = y$, $\{u, y, u\} = -y$). We may assume without loss of generality that $f(y) \geqslant 0$. By induction we get

$$(u + ty)^{3^n} = u + ty + O(|t|^2) \qquad (t > 0,\ n = 1, 2, \dots).$$

Therefore, for $t > 0$,

$$\begin{aligned} (1 + tf(y))^{3^n} &\leqslant \|u + ty\|^{3^n} = \|(u + ty)^{3^n}\| = \|u + ty + O(|t|^2)\| \\ &\leqslant 1 + t\|y\| + O(|t|^2), \\ 1 + 3^n tf(y) &+ O(|t|^2) \leqslant 1 + t\|y\| + O(|t|^2), \\ 3^n f(y) &+ O(|t|) \leqslant \|y\| + O(|t|). \end{aligned}$$

Thus, for $t \downarrow 0$, we obtain $f(y) \leqslant (1/3^n)\|y\|$ $(n = 1, 2, \dots)$. It follows that $f(y) = 0$ for every $y \in E^{-1}(u)$. Since $E_2(u) = E^1(u) \oplus E^{-1}(u)$ and $f = fP_2(u)$, we conclude that $f = f \circ P^1(u)$.

LEMMA 2.8 *Let X be a Banach dual space with unique predual X_*, let M be a weak*-closed subspace of X and let $f \in$ ext$(B(X_*))$. Assume that P is a (w*-w*)-continuous contractive projection of X onto M such that $f = P^* f$. Then $f|_M \in$ ext$(B(M_*))$ (with the usual identifications $X_* \equiv \{f \in X^* :\ f$ w*-continuous$\}$, $M_* \equiv \{f|_M :\ f \in X_*\}$).*

Proof. Suppose that $f|_M = \frac{1}{2}(g + h)$ where $g, h \in B(M_*)$, that is, $g, h : M \to \mathbb{R}$ are w*-continuous bounded linear functionals. Consider the functionals $\widetilde{g} := g \circ P$ and $\widetilde{h} := h \circ P$. Since P is (w*-w*)-continuous with $\|P\| \leqslant 1$ and $g, h \in B(M_*)$, we have $\widetilde{g}, \widetilde{h} \in B(X_*)$. On the other hand $f = f \circ P = (f|_M) \circ P = \frac{1}{2}(g + h) \circ P = \frac{1}{2}(\widetilde{g} + \widetilde{h})$. Since $f \in$ ext$(B(X_*))$, we get $f = \widetilde{g} = \widetilde{h}$ which entails $g = h = f|_M$, that is, $f|_M \in$ ext$(B(M_*))$.

PROPOSITION 2.9 *Let W be a real* JBW*-*triple with predual W_* and let $f \in \mathrm{ext}(B(W_*))$. Then* $\{f\} \in \mathcal{S}_{\mathrm{n}}(B(X_*))$.

Proof. Using the same argument as in the proof of [**17**, Lemma 2.10], we see the existence of some $u \in \mathrm{Tri}\,(W)$ such that $f(u) = 1$. By Lemma 2.7, $f = f \circ P^1(u)$. Since the subspaces $W^1(u)$ and $\ker P^1(u) (= W^0(u) \oplus W^{-1}(u))$ are w*-closed in W and since (see Remark 2.6) we have $\|P^1(u)\| = 1$, we can apply Lemma 2.8 to conclude that $f|_{W^1(u)} \in \mathrm{ext}(B((W^1(u))_*))$. It is well known [**13**] that $W^1(u)$ is a JB-algebra in a canonical manner. As being a w*-closed subspace of W, $W^1(u)$ has a predual and hence we can regard it as a JBW-algebra. By assumption $f|_{W^1(u)}$ is a pure state (extremal point of the unit ball of the predual) of the JBW-algebra $W^1(u)$. Thus, by [**1**, Proposition 3.3], $\{f|_{W^1(u)}\}$ is a norm-exposed face of $B((W^1(u))_*)$, that is, there exists $a \in W^1(u)$ such that $f(a) = 1$ and $g(a) < 1$ for all $g \in B((W^1(u))_*) \setminus \{f|_{W^1(u)}\}$. Consider any $h \in B(W_*)$. Then $h|_{W^1(u)} \in B((W^1(u))_*)$. Thus we have the alternatives $h(a) < 1$ or $h|_{W^1(u)} = f|_{W^1(u)}$. Assume that $h|_{W^1(u)} = f|_{W^1(u)}$. Then $h(u) = f(u) = 1$ and, by Lemma 2.7, $h = hP^1(u)$ whence

$$h = h \circ P^1(u) = [h|_{W^1(u)}] \circ P^1(u) = [f|_{W^1(u)}] \circ P^1(u) = f \circ P^1(u) = f.$$

This shows that $\{f\}$ is a norm-semi-exposed face of $B(W_*)$.

As a consequence of Proposition 2.9, if W is a real JBW*-triple and $f \in \mathrm{ext}(B(W_*))$ then $\{f\}$ is norm-exposed, thus $(\{f\}')_\prime = \{f\}$ is a minimal norm-semi-exposed face of $B(E_*)$.

COROLLARY 2.1 *The minimal tripotents of the real* JBW*-*triple W are exactly the support tripotents of the extreme points of $B(W_*)$. That is, $f \in \mathrm{ext}B(W_*)$ if and only if $f(u) = 1$ for some $u \in \min\mathrm{Tri}\,(W)$. Moreover, if $u \in \min\mathrm{Tri}\,(W)$ then $f(u) = 1$ for some $f \in \mathrm{ext}B(W_*)$.*

Proof. Let $f \in \mathrm{ext}B(W_*)$. We have seen that the singleton $\{f\}$ is a minimal norm-semi-exposed face of $B(W_*)$. By Proposition 2.4, $\{f\} = \Phi_*(u)$ and hence $f(u) = 1$ for some minimal tripotent of W.

Suppose $u \in \min\mathrm{Tri}\,(W)$ and consider any $f \in \Phi_*(u)$. We show that $\{f\}' = \Phi(u)$ and $f \in \mathrm{ext}B(W_*)$.

By the definition of $\Phi_*(u)$, $f(u) = 1$. Hence $\{f\}' \in \mathcal{S}_{\mathrm{w}^*}(B(W))$ and $\{f\}' \supset [\Phi_*(u)]' = \Phi(u)$ (since, as a folklore result in Banach space geometry, $S \subset T \subset B(X)$ implies $S' \supset T'$ and $S_\prime = ((S_\prime)')_\prime$). By Proposition 2.4, $\Phi(u)$ is a maximal w*-semi-exposed face of $B(W)$ and $\Phi_*(u)$ is a minimal norm-semi-exposed face of $B(W)$ with $\Phi(u) = [\Phi_*(u)]'$. It follows that $\{f\}' = \Phi(u)$. Assume $f = (f_1 + f_2)/2$, where $f_1, f_2 \in B(W_*)$. Since $u \in \mathrm{Tri}\,(W)$, $\|u\| = 1$, and $\|f_k\| \leqslant 1$ we have $|f_k(u)| \leqslant 1$ $(k = 1, 2)$, necessarily

$$1 = f(u) = f_1(u) = f_2(u)\,, \quad \|f|W_2(u)\| = \|f_1|W_2(u)\| = \|f_2|W_2(u)\| = 1.$$

By Lemma 2.7, $f = f\,P^1(u)$ and $f_i = f_i\,P^1(u)$ $(i = 1, 2)$. Since $u \in \min\mathrm{Tri}\,(W)$ we have $P^1(u)(W) = \mathbb{R}u$ entailing $f_1 = f_2 = f$.

3. Atomic decomposition

As previously W denotes a real JBW*-triple. First we establish a relationship between the minimal tripotents of W and those of its complexification $\widehat{W}$. The term *(complex) minimal tripotent* is well established in the literature of complex JB*-triples: if V is a complex JB*-triple then $u \in \mathrm{Tri}(V) \setminus \{0\}$ is called a (complex) minimal tripotent if

$$V_2(u) = \mathbb{C}u \tag{3.1}$$

without any reference to the ordering $\leqslant$ of orthogonal decomposability. Notice that, if V is a complex JB*-triple and $u \in \mathrm{Tri}(V)$, then $V_2(u) = \mathbb{C}u$ is equivalent to $V^1(u) = \mathbb{R}u$. Similar arguments to that in the proof of Proposition 2.2 show that indeed $\min \mathrm{Tri}(V) = \{u \in \mathrm{Tri}(V) : V_2(u) = \mathbb{C}u\}$ in any complex JBW*-triple V.

LEMMA 3.2 *Let $e \in \mathrm{Tri}(W)$ be a tripotent with $W^1(e) = \mathbb{R}e$. Then e is the sum of at most two orthogonal (minimal) tripotents in $\widehat{W}$.*

Proof. If e is minimal in $\widehat{W}$ there is nothing to prove. If e is not minimal, $W_2(e)$ is a real spin factor (see the last paragraph of the proof of Proposition 2.2). Hence $\widehat{W}_2(e) = W_2(e) \oplus i W_2(e)$ is a complex spin factor. It is well known **[16]** that in a complex spin factor each element of the real part is the sum of two orthogonal (complex) minimal tripotents being conjugates of each other. Thus $e = v + \tau v$ where $v \in \mathrm{Tri}(\widehat{W}_2(e)) (\subset \mathrm{Tri}(\widehat{W}))$, $[\widehat{W}_2(e)]_2(v) = \mathbb{C}v$ and $v \perp \tau v$. In particular $v \leqslant e$ in $\mathrm{Tri}(\widehat{W})$. Therefore, $\widehat{W}_2(v) \subset \widehat{W}_2(e)$ whence even $\widehat{W}_2(v) = [\widehat{W}_2(e)]_2(v) = \mathbb{C}v$.

By **[7**, Theorem 2], the complexification $\widehat{W}$ of any real JBW*-triple W decomposes into the orthogonal direct sum (or equivalently ℓ^∞-direct sum) of the JBW*-ideals $\widehat{A} := \overline{\mathrm{Span}}^{w*} \min \mathrm{Tri}(\widehat{W})$ and $\widehat{N} := \{x \in \widehat{W} : L(a,a)x = 0 \ (a \in \widehat{A})\}$. Here $\min \mathrm{Tri}(\widehat{N}) = \emptyset$ and

$$\widehat{A} = \overline{\mathrm{Span}_{\mathbb{C}}}^{w*} \bigcup_{F \in \mathcal{F}} F \quad \text{for} \quad \mathcal{F} := \{\text{ minimal w*-closed complex ideals of } \widehat{W}\}.$$

Since the conjugation τ preserves the triple product and is w*-continuous, $\tau(\min \mathrm{Tri}(\widehat{W})) = \min \mathrm{Tri}(\widehat{W})$, $\tau(\widehat{A}) = \widehat{A}$ and $\tau(\widehat{N}) = \widehat{N}$. Hence W decomposes into the orthogonal direct sum

$$W = \widehat{A}^\tau \oplus^\infty \widehat{N}^\tau, \quad \widehat{A}^\tau := \{x \in \widehat{A} : \tau(x) = x\}, \quad \widehat{N}^\tau := \{x \in \widehat{N} : \tau(x) = x\}.$$

We set $\mathcal{F}_0 := \{F \in \mathcal{F} : \tau(F) = F\}$ and fix any maximal subset $\mathcal{F}_1$ of $\mathcal{F} \setminus \mathcal{F}_0$ with the property $\tau(F) \perp F$ $(F \in \mathcal{F}_1)$. We define $\mathcal{F}_{-1} := \{\tau(F) : F \in \mathcal{F}_1\}$. By **[15**, Lemma 6.2] we have $\widehat{A} = \oplus^\infty(\mathcal{F}_{-1} \cup \mathcal{F}_0 \cup \mathcal{F}_1)$.

REMARKS 3.3

1. If $e \in \min \mathrm{Tri}(W)$, by Lemma 3.2, e is either minimal in $\widehat{W}$ or $e = v + \tau(v)$ for some $v \in \min \mathrm{Tri}(\widehat{W})$ such that $v \perp \tau(v)$. It is well known **[8]** that each $v \in \min \mathrm{Tri}(\widehat{W})$ belongs to a unique minimal w*-closed complex ideal (complex Cartan factor) of $\widehat{W}$. Therefore, given any $e \in \min \mathrm{Tri}(W)$, there exists a complex Cartan factor $F \in \mathcal{F}$ of $\widehat{W}$ and a complex-minimal tripotent $v \in F$ such that either $e = v \in F$, or $F = \tau(F)$, $e = v + \tau(v)$, or $F \perp \tau(F)$, $e = v + \tau(v)$.

2. If W is reflexive, there is no infinite sequence $e_1, e_2, \ldots$ of pairwise orthogonal non-zero tripotents in W (because their span would be a subspace isomorphic to c_0). Therefore, by spectral decomposition, *any element is a finite linear combination of a family of minimal tripotents in a reflexive real* JB*-*triple.*

3. According to Kaup's classification in [**15**, Theorem 4.1], the real form of a complex Cartan factor of type < 4 can be written as

$$\{x \in \mathcal{L}(H,K) : Px = x\}, \quad P \in \mathcal{L}_{\mathbb{R}}(\mathcal{L}(H,K), \mathcal{L}(H,K)), \quad P^2 = P$$

such that the real-linear projection P is w*-continuous and maps finite rank operators to finite rank operators.

LEMMA 3.4 *Let H, K be complex Hilbert spaces and let $H_1 \subset H$, $K_1 \subset K$ be subspaces. Suppose Z is a real subtriple of $\mathcal{L}(H,K)$ (with the triple product $\{x,y,z\} := (xy^*z + zy^*x)/2$). Then* $\min \mathrm{Tri}(Z_1) \subset \min \mathrm{Tri}(Z)$ *for the real subtriple $Z_1 := \{z \in Z : z(H) \subset K_1,\ z^*(K) \subset H_1\}$.*

Proof. Let $e \in \min \mathrm{Tri}(Z_1)$. Since the Peirce projection $P_2(e)$ has the form $P_2(e)z = Q(e)^2 z = ee^*ze^*e$ $(z \in Z)$, it is easy to prove that $Z_2(e) \subset Z_1$.

Assume $e = e_1 + e_2$ where e_1, e_2 are orthogonal tripotents in Z. Then $e_1, e_2 \in Z_2(e) \subset Z_1$ (since $e \geqslant e_1, e_2$ in Z). However, by the minimality of e in Z_1, either $e_1 = 0$ or $e_2 = 0$.

COROLLARY 3.5 *If* $\dim(H_1), \dim(K_1) < \infty$ *then any element of Z_1 is a finite real-linear combination from* $\min \mathrm{Tri}(Z) \cap Z_1$.

Proof. Observe that $\dim(Z_1) \leqslant \dim(H_1)\dim(K_1) < \infty$. Thus, according to Remark 3.3 (b), any element of Z_1 is a finite real-linear combination of minimal tripotents of Z_1 and the latter are automatically minimal tripotents of Z.

Now we are in a position to prove our main result.

THEOREM 3.6 *Let W be a real* JBW*-*triple. Then*

$$W = A \oplus^{\infty} N, \tag{3.7}$$

where $A := \overline{\mathrm{Span}_{\mathbb{R}}}^{\mathrm{w}*} \min \mathrm{Tri}(W)$, and $N := \{x \in W : L(a,a)x = 0 \ (a \in A)\}$ are w*-*closed ideals in W and* $\min \mathrm{Tri}(N) = \emptyset$. *In terms of the complexifications $\widehat{W} := W \oplus iW$, $\widehat{A} := A \oplus iA$, $\widehat{N} := N \oplus iN$ we have $\widehat{A} = \overline{\mathrm{Span}_{\mathbb{C}}}^{\mathrm{w}*} \min \mathrm{Tri}(\widehat{W})$ and $\widehat{N} = \{x \in \widehat{W} : L(a,a)x = 0 \ (a, \in \widehat{A})\}$.*

Proof. According to Remark 3.3 (a), if $e \in \min \mathrm{Tri}(W)$ then either $e \in F$ for some Cartan factor $F \in \mathcal{F}_0$ or $e \in F \oplus \tau(F)$ for some $F \in \mathcal{F}_1$. Observe that $e + \tau(e) \in \min \mathrm{Tri}\{x + \tau(x) : x \in F\}$ whenever $F \in \mathcal{F}_1$ and $e \in \min \mathrm{Tri}(F)$. Since $F = \overline{\mathrm{Span}_{\mathbb{R}}}^{\mathrm{w}*} \min \mathrm{Tri}(F)$ for any Cartan factor $F \in \mathcal{F}$ and since the conjugation τ is w*-continuous, (3.7) is immediate in this case.

Let $F \in \mathcal{F}_0$. If F is reflexive, that is if F is of Type 4 or finite dimensional (in particular of Type 5,6) then Remark 3.3 (b) ensures (3.6). Assume F is of Type $\leqslant 3$. Then, by Remark 3.3 (c), we may assume without loss of generality that F is a w*-closed real subtriple of the form $F = \{x \in \mathcal{L}(H,K) : \theta(x) = x\}$ for some complex Hilbert spaces H, K and a w*-continuous (complex-linear) projection θ mapping finite rank operators to finite rank operators. Also the conjugation τ maps finite rank operators to finite rank operators. Thus $Z := \{z \in F : \tau(z) = z\}$ is a w*-closed real subtriple of $\mathcal{L}(H,K)$. Given any element z of Z, there is a net $(z_i : i \in I)$ consisting of finite rank operators converging to z in w*-sense in $\mathcal{L}(H,K)$. Then $f_i := (\frac{1}{2} + \frac{1}{2}\tau)(\frac{1}{2} + \frac{1}{2}\theta)z_i \overset{w*}{\to} z$ and $f_i \in \{z \in Z : \mathrm{rank}(z) < \infty\}$ $(i \in I)$. Applying the corollary of the previous lemma with the finite-dimensional spaces $H_i := f_i(H)$ respectively $K_i := f_i^*(K)$ in place of H_1 resp. K_1, we see that any term f_i is a finite real-linear combination from $\min \mathrm{Tri}(F)$. The proof of (3.6) and hence the proof of (3.7) is complete.

Acknowledgement

The authors were supported by the Bilateral Spanish-Hungarian Research Cooperation grant TET E-3/97, the Programa Nacional F.P.I. of the Ministry of Education of Spain and the grant OTKA T26532 of the Hungarian Scientific Research Foundation.

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