SCUOLA NORMALE SUPERIORE

Tesi di perfezionamento

HOLOMORPHIC MAPS AND FIXED POINTS

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Introduction

The concept of holomorphic maps between infinite dimensional Banach spaces was defined in the early '40-s to fullfill some requirements of harmonic analysis (cf. [HP1]). Very soon one succeeded in proving the equivalence between Fréchet holomorphy, Gâteaux holomorphy and local representability by uniformly convergent series of homogeneous polynomials under a not too restrictive hypothesis (in the presence of local boundedness). This fact made one conjecture that the elegant and fruitful methods of the (finite dimensional) complex analysis in several variables can be mostly preserved also for infinite dimensions. However, the first in fact relevant positive results in this direction take their origin only in the years '70-s, first of all on the field of studying the geometry of infinite dimensional domains and of the theory of some topological algebras (e.g. C^* -algebras).

The major difficulties of this development seem to consist in the spectacular differences between the properties of finite and infinite dimensional Lie-algebras and in the different behaviour of the existence of fixed points of finite and infinite dimensional holomorphic maps, respectively.

This work can be considered as the summary of my researches concerning the fixed points of biholomorphic automorphisms of the closed unit ball in Banach spaces, followed during the academic years 1977/78 and '78/79 at the Scuola Normale Superiore of Pisa under the supervision of Prof. E. Vesentini.

In 1971 Hayden and Suffridge [HS1] proved that any biholomorphic automorphism of the open unit ball in a Hilbert space can be continuously extended to the closed unit ball and always admits fixed points there. This result stands in clear contrast with the fact established much earlyer by Kakutani [K1] that there can be found a diffeomorphism of the closed unit ball of any infinite dimensional Hilbert space onto itself without fixed points. In 1976 W. Kaup and H. Upmeier [KU1] have shown that, in general, if E, B(E) and Aut B(E) denote a Banach space, its open unit ball and the group of the biholomorphic automorphism of B(E), respectively, then any $F \in$ Aut B(E) can be continuously extended to $\overline{B}$(E) ($\equiv$the closure in E of B(E)). Hence the question naturally arises if, by writing Aut $\overline{B}$(E) for the group of the continuous extensions to $\overline{B}$(E) of the elements of Aut B(E), any mapping in Aut $\overline{B}$(E) has a fixed point. The essentially more complex problem of the existence of fixed points for bounded holomorphic maps has already been treated in the literature (e.g. [EH1], [HS2]). The strongest results in this setting are a contraction principle guaranteing fixed points of any holomorphic map of $\overline{B}$(E) into $\lambda \cdot \overline{B}$(E) whenever

$0 \leq \lambda < 1$ (cf. [EH1]; E denotes any Banach space), and a theorem stating that if E is a reflexive separable Banach space and F maps $\overline{B}(E)$ holomorphically into itself then for almost every $\vartheta \in \mathbb{R}$, the map $e^{i\vartheta}F$ admits a fixed point (cf. [HS2]). Although these theorems can not be directly applied to answer our original question, they provide us a good help in finding the suitable type of spaces to give counterexamples: In Chapter 1 we show that those compact spaces Ω for which any element of Aut $\overline{B}(C(\Omega))$ has a fixed point are necessarily F-spaces (def. see [GJ1]). In the next two chapters we examine the sufficiency of this condition. This problem is not only of independent interest from the view point of the theory of rings of continuous functions (cf. [GJ1]). It may be important also for the investigations of the fixed point problem of the biholomorphic automorphisms of the closed unit ball even in the most general Banach space setting: Recent finite and infinite dimensional results (cf. [Sun1], Chapter 8) concerning the description of those Banach spaces E where Aut $\overline{B}(E)$ admits at least one non-linear member (i.e. not all $F \in$ Aut $\overline{B}(E)$ have the fixed point 0) lead us to such a conjecture that if some $\tilde{F} \in$ Aut $\overline{B}(E)$ has a fixed point then it admits a fixed point also in the set $\overline{B}(E) \cap E_0$ where $E_0 = \mathbb{C} \cdot \{F(0) : F \in \text{Aut } \overline{B}(E)\}$ and the geometry of E_0 is very closely related to the geometry of some M-lattice (for definition see [Sch1]; in particular the M-lattices having an order unit can be identified with the spaces $C(\Omega)$ with compact Ω-s). Unfortunately, it turns out to be not sufficient and we can not reach a definitive answer to this question in the present work. However, we succeeded in characterizing all those M-lattices that admit a predual and whose unit ball has only biholomorphic automorphisms with fixed points (Chapter 3). Further we find (in Chapter 2) a new characterization of the compact F-spaces Ω in terms of the fixed points of the members of Aut $\overline{B}(C(\Omega))$.

It is an easy consequence of Kakutani's celebrated theorem concerning L-lattices (see [Sch1]) that any M-lattice with predual can be represented as an $L^\infty(\mu)$ space for some measure μ (cf. [R1]). Thus, in Chapter 3, we achieved the complete description of those L^∞-spaces E where any member of Aut $\overline{B}(E)$ has fixed point. On the other hand, as a corollary of his deep results concerning the Kobayashi and Carathéodory distances on subdomains of locally compact topological vector spaces, E. Vesentini [V1] resolved the dual problem by proving that all the biholomorphic automorphisms of the unit ball of an L^1-space are linear. The same result can be obtained, as it is noted also in [V1], from Suffridge's subordination principle (see [Suf1]). This fact enforced the conjecture that the behaviour from the view point of linearity of the elements of Aut $\overline{B}(E)$ for L^p-spaces E over one dimension must be the same as in the two dimensional special case described already by Thullen's classical theorem [Th1], i.e. any member of Aut $\overline{B}(L^p(\mu))$ is linear if and only if dim $L^p(\mu) > 1$ and $p \neq 2, \infty$.

However, directe applications of both Vesentini's method and the subordination principle of Suffridge seem to be rather difficult in treating the "Mittelpunkttreu" property of Aut $\overline{B}(E)$ for L^p-spaces if $p\neq 1$. In Chapter 4 we provide an alternative approach to this problem which reduces the proof of the conjecture in question to a two dimensional straightforward calculation, by using a consequence of the fact (which can be regarded as one of the major achievements also in the theory of infinite dimensional Lie algebras till now) established in [KU1] that any element of the connected component containing $id_{B(E)}$ of Aut B(E) has the form exp(X) where is X some suitable in B(E) complete polynomial vector field of second degree. The heuristical background of our method is the observation that in Vesentini's paper [V1] an analogous reduction is in effect performed but its extension for $p\neq 1$ is essentially more sophisticated than that in the case of our proof in account of difficulties related to the determination of the Bergmann metric of the non-symmetric two dimensional domain $\{(\zeta_1,\zeta_2):|\zeta_1|^p+|\zeta_2|^p<1\}$. (Recently E. Vesentini, as he friendly told me, discovered relevant new results concerning the Bergmann metric of the domains $\{(\zeta_1,\zeta_2):|\zeta_1|^{p_1}+|\zeta_2|^{p_2}<1\}$ $(0<p_1,p_2<\infty)$. Hence, following a similar way as in [V1], one can very probably obtain also a formula for the Kobayashi and Carathéodory distances of the unit ball of L^p-spaces (whence the description of the automorphisms of the unit ball quasi trivially follows)).

In the articles [St2],[V1] the lattice structure (and in particular the presence of a sufficiently large family of lattice orthogonal pairs) played one of the chief roles in treating the L^p-spaces. In Chapter 5 we clairify the more profound geometrical background of this phenomenon. We prove the following projection principle: If E is any Banach space and P denotes a contractive projection of E onto its subspace $\tilde{E}$ then we have $\{PF(0) : F \in \mathrm{Aut}\, B(E)\} \subset \{\tilde{F}(0) : \tilde{F} \in \mathrm{Aut}\, B(\tilde{E})\}$. This principle enables us to decide in many cases at once if the biholomorphic automorphism group of the unit ball consist of only linear mappings. We paid more attention to the examination of the use of contractive projections with finite rank (finite dimensional range). These investigations lead to a system of parametric partial differential equations which describes the gauge function of those finite dimensional star-shaped circular domains that admit non-linear biholomorphic automorphisms. In 1974, T. Sunada [Sun1] gave the complete description of all those groups $\mathcal{G}$ formed by biholomorphic transformations of some subdomain of $\mathbb{C}^n$ for which there exists a Reinhardt subdomain D of $\mathbb{C}^n$ such that $\{F|_D : F \in \mathcal{G}\} = \mathrm{Aut}_0 D$ ($\equiv$ the connected component containing id_D of the biholomoprhic automorphism group of D). His proofs are based upon a precise analysis of the roots of the Lie algebra of the Lie group $\mathrm{Aut}_o D$ where D is any Reinhardt domain in $\mathbb{C}^n$ (cf. [Kp1]) thus they heavily depend on the finite dimensionality of D. Our projection principle furnishes the possibility of passing from Sunada's cited results by a limiting process to a complete description of $\mathrm{Aut}_o B(E)$ for all those Banach lattices E whose finite dimensional projection bands are dense in the space. Heuristically it is worth to remark here that, easily seen, the convex finite dimensional Reinhardt domains can be identified with the unit balls of the finite dimensional Banach lattices (Chapter 7). Hence we can achieve an exact solution of the fixed point problem of Aut B(E) for the above Banach lattices E. Moreover, by a partial solution of the parametric partial differential equations (deduced in Chapter 5) on the gauge function of finite dimensional star-shaped circular domains admitting non-linear biholomorphic automorphisms, we reobtain also Sunada's theorem with more informations than in [Sun1] about the geometric shape of those finite dimensional Reinhardt domains D for which Aut D contains a non-linear member. This will be the topics of Chapter 6 (as a preliminary work for Chapter 7) but, for the sake of simplicity, we perform this programm only for convex Reinhardt domains here. (In the general case one can proceed analogously; however complications concerning the differentiability properties of the gauge function would render more sophisticated the argumentation.) It is an open problem yet to give a parametric formula (like that of Thullen for two dimensions) characterizing all the possible convex Reinhardt domains D in $\mathbb{C}^n$ for which Aut D admits a non-linear member. The achievements of Chapter 6 provide a hope that such a complete analogon of Thullen's theorem can be deduced from our conciderations.

Here I should like to express my sincere gratitude to Prof. E. Vesentini for having introduced me in this very nice branch of modern analysis, having called my attention to many important open questions and for the stimulating discussions about this work. I am very grateful to the Scuola Normale Superiore of Pisa for its hospitality and supports of my work.

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Notations and basic definitions

Troughout the whole work we shall deal only with complex Banach spaces. If E is a Banach space we shall denote its (topological) dual by E*. If $f \in E$ and $\phi \in E^*$ we set $\langle f,\phi\rangle \equiv \phi(f)$, as it is usual, and $\mathrm{Re}\phi$ stands for the real-linear functional $g(\in E) \mapsto \mathrm{Re}\langle g,\phi\rangle$ (i.e. $\langle g,\mathrm{Re}\phi\rangle \equiv \mathrm{Re}\phi(g)$ $\forall g \in E$) where Re denotes the real part operation (defined for complex numbers). Without danger of confusion, we shall write $\|\ \|$ for the norm in case of any Banach space in the text or, to be more clear, we put $\|\ \|_*$ for the dual norm when E and E* are treated together and we use also subscripts for the sake of the easier reading (e.g. we write $\|\ \|_{L^2}$ etc). If E is a Banach lattice (always complex here), E_+ denotes its positive cone and $\vee,\wedge$ the supremum and infimum operations, respectively. For any Banach space E, the unit ball is denoted by $B(E)$ (thus $B(E) \equiv \{f \in E: \|f\| < 1\}$) and its closure by $\overline{B}(E)$, respectively. If D is a subset of a Banach space E then Aut D and $\mathrm{Aut}_o D$ denote the group of the biholomorphic automorphisms of D and the connected component of Aut D containing the identity map of D (denoted by id_D), respectively. (Thus Aut D consists of all those homeomorphisms of D onto itself that admit an invertible Fréchet derivative at any inner point of D). As a standard reference concerning Banach space holomorphy we use [VF1]. For a better arranging of some formulas, we define mappings F by writing $F \equiv [X \ni x \mapsto e(x)]$ to mean that F is a mapping with domain X and for any element x of X, F assumes the value e(x). If it is obvious what is the domain, we write simply $F \equiv [x \mapsto e(x)]$ or speak of the map $x \mapsto e(x)$. $\mathbb{R}, \mathbb{C}, \Delta, \mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $(.)^+$, entier $(.)$ are the standard notations of the sets {reals}, {complex numbers}, $\{\zeta \in \mathbb{C} : |\zeta| < 1\}$, {natural numbers}, {integers}, {rational numbers} and the positive and entire part functions (defined on $\mathbb{R}$), respectively. If Ω is a topological space and $H \subset \Omega$ then $\overline{H}, \overset{\circ}{H}$ and ∂H denote the closure, the interior and the boundary of H in Ω, respectively. (If it is necessary to emphasize the fact that the topology on Ω is τ then we write $\overline{H}^{\tau}$, $\overset{\circ\tau}{H}, \partial_{\tau} H$.) $C(\Omega)$ denotes the set of all the complex valued functions on Ω and $C_b(\Omega)$ stands for the Banach space of bounded continuous $\Omega \to C$ functions endoved with the usual sup-norm. If Ω is compact, we write $C(\Omega)$ simply instead of $C_b(\Omega)$. If $\mu_1, \mu_2, \ldots$ is a sequence of (positive) measures with supporting sets $X_1, X_2, \ldots$, respectively, and if $\rho_1, \rho_2, \ldots > 0$ then $\bigoplus_{n=1}^{\infty} \rho_n \mu_n$ denotes the measure μ defined on $X \equiv \bigcup_{n=1}^{\infty} X_n \times \{n\}$ in the following way : a set $Y \subset X$ is μ-measurable if and only if there exist $Y_1 \subset X_1$, $Y_2 \subset X_2$, ... such that Y_n is μ_n-measurable for each $n \in \mathbb{N}$ and $Y = \bigcup_{n=1}^{\infty} Y_n \times \{n\}$ further the values of μ are defined by $\mu(\bigcup_{n=1}^{\infty} Y_n \times \{n\}) \equiv \sum_{n=1}^{\infty} \rho_n \mu_n(Y_n)$. In many of our considerations occure Möbius transformations i.e. the elements of $\mathrm{Aut}\overline{\Delta}$ $(\equiv \mathrm{Aut}\{\zeta \in \mathbb{C} : |\zeta| \leq 1\})$. As it is well-known, $\mathrm{Aut}\overline{\Delta} = \{[\overline{\Delta} \ni \zeta \mapsto k \frac{\zeta + u}{1 + \overline{u}\zeta}] : |k| = 1 > |u|\}$.

The topology on $\mathrm{Aut}\,\overline{\Delta}$ is defined by pointwise convergence of its elements. Thus the sets $\{N \in \mathrm{Aut}\,\overline{\Delta} : |N\zeta - M\zeta| < \varepsilon\}$ $(\varepsilon > 0)$ form a base of neighbourhoods for a generic $M \in \mathrm{Aut}\,\overline{\Delta}$. Now the map $(k,u) \to [\zeta \mapsto k \frac{\zeta + u}{1+\bar{u}\zeta}]$ constitutes a homeomorphism between $(\partial\Delta) \times \Delta$ and $\mathrm{Aut}\,\overline{\Delta}$. (Hence $\mathrm{Aut}\,\overline{\Delta}$ is a connected Lie group.) If S is any set, the symbol 1_S means its characteristic function (i.e. $1_S(x) = 1$ if $x \in S$ and $1_S(x) = 0$ if $x \notin S$; the exact domain of definition of 1_S will always clear from the context .

Chapter 1

Fixed point free biholomorphic automorphisms in $C_b(\Omega)$ spaces. Remarks on Aut $\overline{\Delta}$

Considerations in geometric functional analysis suggest (cf. [HS2]) the conjecture that even the closed unit ball of some Banach space admits a biholomorphic automorphism which has no fixed point. Heuristically, the only difficulty is to find the appropriate type of space to prove this conjecture. Fortunately it happens that the spaces of continuous functions are good condites. An extramally simple and instructive example is the following: The mapping

$$(1) \qquad F : f \to \left[\overline{\Delta} \ni \zeta \mapsto \frac{f(\zeta) + \zeta/2}{1 + \overline{\zeta} f(\zeta)/2}\right]$$

defined for the continuous functions $f : \overline{\Delta} \to \overline{\Delta}$ clearly belongs to Aut $\overline{B}$ $(C(\overline{\Delta}))$ but $Ff_o = f_o$ would imply $f_o(\zeta) = \dfrac{f_o(\zeta) + \zeta/2}{1 + \overline{\zeta} f_o(\zeta)/2}$

$\forall \zeta \in \overline{\Delta}$ whence $f_o(\zeta)^2 = \zeta/\overline{\zeta}$ $\forall \zeta \in \overline{\Delta} \setminus \{0\}$ which excludes the continuity of f_o at the point $0 (\in \overline{\Delta})$.

The construction of (1) suggests an approach promising positive results to the question: What is the necessary and sufficient topological condition on a compact space Ω to admit a member of Aut $\overline{B}(C(\Omega))$ without fixed points?

Proposition 1. Any such topological space Ω for which every $F \in \operatorname{Aut} \overline{B}(C_b(\Omega))$ has a fixed point is necessarily an F-space.[1)]

Proof. Let $t(.)$ be any continuous function on Ω, set $G \equiv \{x \in \Omega: t(x) \neq 0\}$ and consider any $\varphi \in C_b(G)$. We may assume without loss of generality that range$(t) \subset [0, \pi/2]$ (thus $G = \{x \in \Omega: t(x) > 0\}$). Define the functions $k : \Omega \to \partial\Delta$ and $u : \Omega \to \frac{1}{2}\overline{\Delta}$ by $k(.) \equiv e^{it(.)}$ and $u(x) \equiv -\dfrac{2i\varphi(x)e^{-it(x)/2}}{1 + |\varphi(x)|^2} \sin \dfrac{t(x)}{2}$

if $x \in G$, $u(x) \equiv 0$ for $x \in \Omega \setminus G$. Observe that the transformations $N(x) \equiv [\overline{\Delta} \ni \zeta \mapsto k(x)\dfrac{\zeta + u(x)}{1 + \overline{u(x)}\zeta}]$ are in Aut $\overline{\Delta}$ for all fixed $x \in \Omega$ since $|k(x)| = 1$ and $|u(x)| < \frac{1}{2} < 1$. Moreover the map $N : \Omega \to \operatorname{Aut} \overline{\Delta}$ is continuous because so are k and u. Consider now the automorphism F of $\overline{B}(C_b(\Omega))$ defined by $F(f) \equiv [x \mapsto N(x)f(x)]$. By hypothesis, for some $f_o \in \overline{B}(C_b(\Omega))$ we have $F(f_o) = f_o$. Thus $k(x)\dfrac{f_o(x) + u(x)}{1 + \overline{u(x)}f_o(x)} = f_o(x)$ $\forall x \in \Omega$ and

therefore $f_o^2 \dfrac{2i\overline{\varphi}e^{-it/2}}{1 + |\varphi|^2} \sin \dfrac{t}{2} + (1 - e^{it})f_o + \dfrac{2i\varphi e^{it/2}}{1 + |\varphi|^2} \sin\dfrac{t}{2} = 0$

on G. Dividing by $\dfrac{2ie^{it/2}}{1 + |\varphi|^2} \sin \dfrac{t}{2}$ $\left(= \dfrac{e^{it} - 1}{1 + |\varphi|^2} \neq 0\right.$ since

$0 < t(.) < \frac{\pi}{2}$ on G) we obtain $\overline{\varphi(x)}\, f_o(x)^2 - (1 + |\varphi(x)|^2)\, f_o(x) +$

1) i.e. given any cozero set G in Ω, each function $\varphi \in C_b(G)$ has a continuous extension to Ω (cf. [GJ1, 14.25 Theorem (6)]).

$+ \varphi(x) = 0$ i.e. $f_o(x) \in \{\varphi(x), \frac{1}{\varphi(x)}\}$ $\forall x \in G$. But $\|f_o\| < 1$ and hence necessarily $f_o|G = \varphi$. So f_o is a continuous extension of φ. □

In order to prove some converse of Proposition 1 and to generalize it, we go back to Aut $\bar{\Delta}$. Recall that any Möbius transformation M has a unique representation of the form $M = [\bar{\Delta} \ni \zeta \mapsto k_M \frac{\zeta + u_M}{1 + \overline{u_M}\zeta}]$ with $|k_M| = 1$ and $|u_M| < 1$, and the mapping $M \mapsto (k_M, u_M)$ establishes a homeomorphism between Aut $\bar{\Delta}$ and $(\partial\Delta) \times \Delta$. We shall reserve the notation (k_M, u_M) for this mapping.

Lemma 1. Let $id_{\bar{\Delta}} \neq M \in \text{Aut}\,\bar{\Delta}$ and $e^{it} = k_M$. Then M has

a) a unique fixed point which lies in Δ iff

$|u_M| < |\sin \frac{t}{2}|$ $(= |\frac{k_M - 1}{2}|)$

b) two distinct fixed points lying in $\partial\Delta$ iff $|u_M| > |\sin \frac{t}{2}|$

c) a unique fixed point lying in $\partial\Delta$ iff $|u_M| = |\sin \frac{t}{2}|$.

Proof. Simple computation.

Lemma 2. There are exactly two different continuous mappings from $(\text{Aut}\,\bar{\Delta}) \setminus \{id_{\bar{\Delta}}\}$ into $\bar{\Delta}$ which associate to any (non-identical) Möbius transformation one of its fixed points.

Proof. Recall that, in general, if $0 < r < 1$ and $F \in \text{Aut}\,\bar{B}(E)$ where E is any complex Banach space then the mapping rF has always a unique fixed point (cf. [EH1]). Thus we may define the function $Q : [0,1) \times \text{Aut}_{\bar{\Delta}} \to \bar{\Delta}$ by $Q(r,M) \equiv$ $\equiv$ [the fixed point of rM]. If $r_j \to r (\in [0,1))$ and $M_j \to M$ then the net $Q(r_j, M_j)$ $(= r_jM_jQ(r_j,M_j))$ tends obviously to some fixed point of rM, showing the continuity of Q. We shall prove that for every $\mathrm{id}_{\bar\Delta} \neq M \in \mathrm{Aut}\,\bar\Delta$, the sets

$$S_M \equiv \{\zeta : \exists \text{net } ((s_j,N_j) : j \in J) : (s_j,N_j) \to (1,M) \text{ and } Q(s_j,N_j) \to \zeta\}$$

contain exactly one point. In fact, on the one hand

$$S_M = \bigcap_{n=1}^{\infty} \overline{Q\{(s,N) : 1-\frac{1}{n} < s < 1 \text{ and } |k_M-k_N|, |u_M-u_N| < \frac{1}{n}\}}$$

i.e. the intersection of a decreasing sequence of non-empty connected compact subsets of $\bar\Delta$, thus $S_M \neq \emptyset$ is connected and compact. On the other hand $S_M \subset \{\zeta : M\zeta = \zeta\}$ which implies cardinality $(S_M) < 2$. But the now established fact cardinality $(S_M) = 1$ $\forall M \in (\mathrm{Aut}\,\bar\Delta) \setminus \{\mathrm{id}_{\bar\Delta}\}$ means that the function $R : (\mathrm{Aut}\,\bar\Delta) \setminus \{\mathrm{id}_{\bar\Delta}\} \to \bar\Delta$ is well-defined by $R(M) \equiv \lim_{r \uparrow 1} Q(r,M)$ and it is continuous. Since $\{R(M)\} = S_M \subset \{\zeta : M\zeta = \zeta\}$, the mapping R(.) is a continuous section of the multifunction $\phi : M \mapsto \{\zeta \in \bar\Delta : M\zeta = \zeta\}$.

If R′ denotes another continuous section of ϕ (defined on $(\mathrm{Aut}\,\bar\Delta)\setminus\{\mathrm{id}_{\bar\Delta}\}$) then, by Lemma 1 a)c), $\{M : R(M) \neq R'(M)\} \subset D \equiv$ $\equiv \{M : |u_M| > |\frac{k_M-1}{2}|\}$. Since $\phi(M) = \{\zeta \in \bar\Delta : \overline{u_M}\zeta^2 + (1-k_M)\zeta - k_M u_M = 0\}$ $\forall M \in D$, we have by Rouché's theorem on continuity of the roots of polynomials depending on their coefficients [Con1], that the mapping $\phi|D$ is continuous from D into the space of the non-empty compact subsets of $\mathbb{C}$ endowed with the Hausdorff distance. Since cardinality $\phi(M) = 2 \quad \forall M \in D$, it easily follows that $\{M \in D : R'(M) = R(M)\}$ is open-closed in D. But D is connected because it is homeomorphic to $\{(k,u) \in (\partial\Delta)\times\Delta :$

$$|u| > |\frac{k-1}{2}|\} = \{(e^{it}, re^{i\delta}) : t \in (-\pi,\pi),\ 1 > r > |\sin \frac{t}{2}|, \delta \in \mathbb{R}\}$$

which is a continuous image of the obviously connected set $\{(t,r) : t \in (-\pi,\pi),\ 1 > r > |\sin \frac{t}{2}|\}\times\mathbb{R}$. Thus if $R' \neq R$ then we necessarily have that

$$(2) \qquad \{R'(M)\} = \phi(M) \setminus \{R(M)\} \qquad \forall M \in D.$$

On the other hand, it directly follows from Rouché's mentioned theorem that if we define R′ by (2) on D and to coincide with R elswhere, them R′ is continuous. □

Lemma 3. For any $M \in \mathrm{Aut}\,\bar{\Delta}$ with $M \neq id_{\bar{\Delta}}$ there exists a Lie homomorphism $t \mapsto M^t$ of $\mathbb{R}$ into $\mathrm{Aut}\,\bar{\Delta}$ such that $M^1 = M$ and, by setting $t_o \equiv \inf\{t > 0 : M^t = id_{\bar{\Delta}}\}$ (convention: $\inf \emptyset \equiv +\infty$), we have

$$(3) \qquad \{\zeta : M^t \zeta = \zeta\} = \{\zeta : M\zeta = \zeta\} \quad \forall t \in (0, t_o).$$

Proof. Fix M arbitrarily. According to Lemma 1, there are the following possible cases: a) M has a fixed point in Δ, b) M has two fixed point on $\partial\Delta$, c) the unique fixed point of M lies in $\partial\Delta$.

a) Since Aut $\overline{\Delta}$ acts transitively on Δ, we can choose $N \in$ Aut $\overline{\Delta}$ which sends the fixed point of M ints O. Thus O is the fixed point of $K \equiv NMN^{-1}$. By the Schwarz Lemma [Con1], $\exists\, \delta \in \mathbb{R}$ $K = [\zeta \mapsto e^{i\delta}\zeta]$. Set $K^t \equiv [\zeta \mapsto e^{i\delta t}\zeta]$ (for $t \in \mathbb{R}$). Since $t \mapsto K^t$ is trivially a Lie homomorphism of $\mathbb{R}$ into Aut $\overline{\Delta}$, we may define M^t by $M^t \equiv N^{-1}K^tN$ (for $t \in \mathbb{R}$).

b) The group Aut $\overline{\Delta}$ acts doubly transitively on $\partial\Delta$. Thus we can find $N \in$ Aut $\overline{\Delta}$ such that the one fixed point of M is sent by N into 1 and the other into -1. Now the fixed points of $K \equiv NMN^{-}$ are -1 and 1. Observe that $k_K = 1$ and $u_K \in \mathbb{R}$ (for $k_K \dfrac{1+u_K}{1+\overline{u_K}} = 1$ and $k_K \dfrac{(-1)+u_K}{1-\overline{u_K}} = -1$ imply $\dfrac{1+u_K}{1-u_K} \Big/ \left(\overline{\dfrac{1+u_K}{1-u_K}}\right) = 1$ i.e. $\dfrac{1+u_K}{1-u_K} \in \mathbb{R}$). Now set $\delta \equiv \operatorname{areath}(u_M)$ and

$K^t \equiv \left[\zeta \mapsto \dfrac{\zeta + th(t\delta)}{1+\zeta\, th(t\delta)}\right]$. A directe calculation shows $K^{t+s} = K^tK^s$ $\forall\, t,s \in \mathbb{R}$. Thus also in this case we may put $M^t \equiv N^{-1}K^tN$ $(t \in \mathbb{R})$.

c) Let ζ_o denote the fixed point of M and fix $N' \in$ Aut $\overline{\Delta}$ so that $N\zeta_o = 1$. Further let N" be the Cayley transformation $\zeta \mapsto \frac{1}{i}\,\dfrac{\zeta+1}{\zeta-1}$ (acting between $\overline{\Delta}$ and $\Pi \equiv \{\infty, \zeta \in \mathbb{C} : \operatorname{Im}\zeta > 0\}$) and set $N \equiv N''N'$. Now the mapping $K \equiv NMN^{-1}$ belongs to Aut Π and satisfies $K(\infty) = \infty$. Therefore K is linear, moreover $\exists\, \alpha, \beta \in \mathbb{R}$ $K = [\zeta \mapsto \alpha\zeta + \beta]$. Since the only fixed point of M is ζ_o, K must have no other fixed point than ∞. But hence K is a translation (i.e. $\exists \beta \in \mathbb{R}$ $K = [\zeta \mapsto \zeta+\beta]$). Then by letting $K^t \equiv$ $\equiv [\zeta \mapsto \zeta+\beta t]$ and $M^t \equiv N^{-1}K^tN$ $(t \in \mathbb{R})$ we are done. □

Lemma 4. Let $R(.)$ denote any one of the continuous sections of $M \mapsto \{\zeta : M\zeta=\zeta\}$ (on $(\mathrm{Aut}\,\bar{\Delta})\setminus\{\mathrm{id}_{\bar{\Delta}}\}$). Then a) $MR(M) = R(M)$, b) $R(M^n) = R(M)$ whenever $M^n \neq \mathrm{id}_{\bar{\Delta}}$ $(n=\pm 1, \pm 2, \ldots)$, c) $R(NMN^{-1}) = NR(M)$ for all $N \in \mathrm{Aut}\,\bar{\Delta}$.

Proof. a) is trivial. b) Fix M and n and take a Lie homomorphism $t \mapsto M^t$ as in Lemma 3, set also $t_0 \equiv \inf\{t > 0 : M^t \neq \mathrm{id}_{\bar{\Delta}}\}$. From a) and (3) we deduce $R(\{M^t : t \in (0,t_0)\}) \subset \{\zeta : M\zeta = \zeta\} \ni R(M)$. Hence the function $\rho \equiv [(0,t_0) \ni t \mapsto R(M^t)]$ is constant (recall, M has at most two fixed points). Thus if $M^n = \mathrm{id}_{\bar{\Delta}}$, $R(M^n) = R(M^{\mathrm{mod}_{t_0} n}) = \rho(\mathrm{mod}_{t_0} n) = \rho(\mathrm{mod}_{t_0} 1) = R(M^{\mathrm{mod}_{t_0} 1}) = R(M^1) = R(M)$ [2].

c) Let $t \mapsto N^t$ be any Lie $\mathbb{R} \to \mathrm{Aut}\,\bar{\Delta}$ homomorphism with $N^1 = N$. Observe that $N^{-t}R(N^tMN^{-t}) \in \{\zeta : M\zeta=\zeta\}$ (since $N^tMN^{-t}\eta = \eta \Leftrightarrow M(N^{-t}\eta) = N^{-t}\eta)$ $\forall t \in \mathbb{R}$. Therefore the function $t \mapsto N^{-t}R(N^tMN^{-t})$ is constant. In particular, $N^{-1}R(NMN^{-1}) = N^0R(N^0MN^0) = R(M)$. □

Definition 1. Let δ_n denote the metric on $\mathbb{C}^n$ defined by $\delta_n((\alpha_1,\ldots,\alpha_n),(\beta_1,\ldots,\beta_n)) \equiv \max\{|\alpha_j-\beta_j| : j = 1,\ldots,n\}$. For any $N^* \equiv (N_0,\ldots,N_{n-2}) \in (\mathrm{Aut}\,\bar{\Delta})^{n-1}$ and $\zeta^* \equiv (\zeta_0,\ldots,\zeta_{n-1}) \in \bar{\Delta}^n$ set

$$(4) \qquad P_n(N^*,\zeta^*) \equiv [\text{that } \eta \in \bar{\Delta} \text{ for which } \delta_n(\zeta^*,(\eta, N_0\eta,\ldots,N_{n-2}\eta)) \text{ is minimal}].$$

---

2) $\mathrm{mod}_\alpha \beta \equiv \inf([0,\infty) \cap \{\beta + n\alpha : n \in \mathbb{Z}\})$, $\mathrm{mod}_\infty \beta \equiv \beta$ for all $\alpha > 0$, $\beta \in \mathbb{R}$.

Lemma 5. The definition of P_n makes sense (i.e. there is a unique $\eta \in \bar{\Delta}$ with $\delta_n(\zeta^*,(\eta,N_o\eta,\dots,N_{n-2}\eta)) < \delta_n(\zeta^*,(\eta',N_o\eta',\dots,N_{n-2}\eta'))$ $\forall \eta' \in \bar{\Delta}$). Furthermore, if $M_o\dots M_{n-1} = id_{\bar{\Delta}}$ and $P_n((M_o^{-1},M_1^{-1}M_o^{-1},\dots,M_{n-2}^{-1}\dots M_o^{-1}),\zeta^*) = \eta$ then $P_n((M_1^{-1},M_2^{-1}M_1^{-1},\dots,M_{n-1}^{-1}\dots M_1^{-1}),(\zeta_1,\zeta_2,\dots,\zeta_{n-1},\zeta_o)) = M_o^{-1}\eta$ (here $\zeta^* \equiv (\zeta_1,\dots,\zeta_{n-1}))$.

Proof. A standard compactness argument shows the existence of at least one minimizing η in (4). Set $\varepsilon \equiv \delta_n(\zeta^*,\{(\eta,N_o\eta,\dots,N_{n-2}\eta):\eta\in\bar{\Delta}\})$. Observe then that $\varepsilon = \min\{\varepsilon' > 0: (\zeta^*+\varepsilon'\bar{\Delta}^n) \cap \{(\eta,N_o\eta,\dots,N_{n-2}\eta):\eta\in\bar{\Delta}\} \neq \emptyset\}$. Thus for the set $Z \equiv \{(\eta,N_o\eta,N_{n-2}\eta):\eta\in\bar{\Delta}\}$ we have $Z \cap (\zeta^*+\varepsilon\Delta^n) = \emptyset$ and $\{\eta\in\bar{\Delta}:\delta_n(\zeta^*,(\eta,N_o\eta,\dots,N_{n-2}\eta)) = \varepsilon\} \subset Z \cap \partial(\zeta^*+\varepsilon\bar{\Delta}^n)$. Let Φ denote the map $\Phi : (\alpha_o,\dots,\alpha_{n-1}) \mapsto (\alpha_o,N_o^{-1}\alpha_1,\dots,N_{n-2}^{-1}\alpha_{n-1})$. Then $\Phi(Z) = \{(\zeta,\dots,\zeta)\in\mathbb{C}^n:\zeta\in\bar{\Delta}\}$ and the set $\Phi(\zeta^*+\varepsilon\Delta)$ is a set of the form $\{(\alpha_o,\dots,\alpha_{n-1}) : |\alpha_o-\beta_o|<\varepsilon_o,\dots,|\alpha_{n-1}-\beta_{n-1}|<\varepsilon_{n-1}\}$ for some $\beta^*\in\mathbb{C}^n$ and $\varepsilon^*\in[0,\infty)^n$. So it suffices to prove that if $\Delta_o,\dots,\Delta_{n-1}$ are open discs in $\mathbb{C}$ then the set $D \equiv \{(\lambda,\dots,\lambda)\in\mathbb{C}^n:\lambda\in\bar{\Delta}\}$ intersects the boundary of $C \equiv \Delta_o \times\dots\times \Delta_{n-1}$ in at most one point whenever $D\cap C = \emptyset$. Proceed by contradiction: If not, let $F_j \equiv \bar{\Delta}_o\times\dots\times\overline{\Delta_{j-1}}\times(\partial\Delta_j)\times\overline{\Delta_{j+1}}\times\dots\times\overline{\Delta_{n-1}}$ $(j=0,\dots,n-1)$. Then $\partial C = F_o\cup\dots\cup F_{n-1}$. Since C and D are convex, there exist $\lambda\in\mathbb{C}$ and $\mu\in\mathbb{C}\setminus\{0\}$ with $(\lambda,\dots,\lambda)+[-1,1]\cdot(\mu,\dots,\mu)\subset\partial C$. Therefore for some index j, the intersection $F_j \cap [(\lambda,\dots,\lambda)+$

$+ [-1,1]\cdot(\mu,\dots,\mu)]$ contains an inner point of the segment $(\lambda,\dots,\lambda) + [-1,1]\cdot(\mu,\dots,\mu)$. That is, for some j and for some $\lambda' \in \mathbb{C}$ and $\mu' \in \mathbb{C}\setminus\{0\}$ we have $(\lambda',\dots,\lambda') + (-1,1)\cdot$ $\cdot(\mu',\dots,\mu') \subset F_j$. But this would mean that $\lambda' + \tau\mu' \in \partial\Delta_j$ $\forall \tau \in (-1,1)$ which is impossible. Thus (4) makes sense.

To prove the second statement, observe that, by definitions of P_n and δ_n we have $\delta_n((\zeta_o,\dots,\zeta_{n-1}),$ $(\eta, M_o^{-1}\eta,\dots,M_{n-2}^{-1}\dots M_o^{-1}\eta)) \le \delta_n((\zeta_o,\dots,\zeta_{n-1}),(\eta', M_o^{-1}\eta',\dots$ $\dots, M_{n-2}^{-1}\dots M_o^{-1}\eta')) \ \forall \eta' \in \bar{\Delta}$. Thus for any $\eta' \in \bar{\Delta}$, $\delta_n((\zeta_1,\dots,\zeta_{n-1},\zeta_o),(M_o^{-1}\eta,\dots,M_{n-2}^{-1}\dots M_o^{-1}\eta,\eta)) \le$ $\le \delta_n((\zeta_1,\dots,\zeta_{n-1},\zeta_o),(M_o^{-1}\eta',\dots,M_{n-2}\dots M_o\eta',\eta'))$ or which is the same, $\delta_n((\zeta_1,\dots,\zeta_{n-1},\zeta_o),((M_o^{-1}\ ), M_1^{-1}(M_o^{-1}\eta),\dots$ $\dots, M_{n-2}^{-1}\dots M_1^{-1}(M_o^{-1}\eta), M_{n-1}^{-1}\dots M_1^{-1}(M_o^{-1}\eta))) \le$ [similar expression with η' in place of η]. Since $\bar{\Delta} = \{M_o^{-1}\eta' : \eta' \in \bar{\Delta}\}$, this means that the function $\lambda \mapsto \delta_n((\zeta_1,\dots,\zeta_{n-1},\zeta_o),(\lambda, M_1^{-1}\lambda,\dots,M_{n-1}^{-1}\dots M_1^{-1}\lambda))$ attains its minimum over $\bar{\Delta}$ at the point $M_o^{-1}\eta$. □

Lemma 6. The mapping $P_n : (\mathrm{Aut}\ \bar{\Delta})^{n-1} \times \bar{\Delta}^n \to \bar{\Delta}$ is continuous.

Proof. Since P_n is a map of a locally compact space into a compact space, it suffices to see that its graph is closed. To do this, examine first the function $\ \ : (\mathrm{Aut}\ \bar{\Delta})^{n-1} \times \bar{\Delta}^n \to [0,\infty)$ defined by $(N^*,\zeta^*) \equiv \delta_n(\zeta^*,\{(\eta, N_o\eta,\dots,N_{n-2}\eta) : \eta \in \bar{\Delta}\})$. Clearly, $\varphi = \inf\{\varphi_\eta : \eta \in \bar{\Delta}\}$ where $\varphi_\eta \equiv [(N^*,\zeta^*) \mapsto \delta_n(\zeta^*,(\eta, N_o\eta,\dots,N_{n-2}\eta))]$. It follows from the triangle inequality that all φ_η (with $\eta \in \bar{\Delta}$) satisfy the Lipschitz condition

$$|\varphi_\eta(N^*,\zeta^*)-\varphi_\eta(N'^*,\zeta'^*)|\le\sum_{j=o}^{n-1}|\zeta_j-\zeta_j'|+\sum_{j=o}^{n-2}\sup\{|N_j\xi-N_j'\xi|:\xi\in\overline{\Delta}\}$$

$(\forall N^*,N'^*\in(\mathrm{Aut}\,\overline{\Delta})^{n-1}\ \ \forall\zeta^*,\zeta'^*\in\overline{\Delta}^n)$. But then also their infimum satisfies the same Lipschitz condition. Thus φ is continuous. Now if $(N^{(i)*},\zeta^{(i)*})\to(N^*,\zeta^*)$ and $P_n(N^{(i)*},\zeta^{(i)*})\to\eta$ then

$$\begin{array}{c}(N^{(i)*},\zeta^{(i)*})=\delta_n(\zeta^{(i)*},(P_n(N^{(i)*},\zeta^{(i)*}),N_o^{(i)}P_n(N^{(i)*},\zeta^{(i)*}),\ldots\\ \ldots,N_{n-2}^{(i)}P_n(N^{(i)*},\zeta^{(i)*})))\\ \downarrow \qquad\qquad\qquad \downarrow\\ (N^*,\zeta^*)=\inf\{\delta_n(\zeta^*,(\eta',N_o\eta',\ldots,N_{n-2}\eta')):\eta'\in\Delta\}=\delta_n(\zeta^*,(\eta,N_o\eta,\ldots\\ \ldots,N_{n-2}\eta)).\end{array}$$

But this latter inequality is the definition of the relation $P_n\ (N^*,\zeta^*)\ =\eta.$ □

<u>Theorem 1.</u> Let Ω denote any topological space. Then the following statements are equivalent

a) All the automorphisms of $\overline{B}(C_b(\Omega))$ of the form $f\mapsto[x\mapsto M(x)\ f(x)]$ where $M(.)$ is any continuous $\Omega\longrightarrow\mathrm{Aut}\,\overline{\Delta}$ mapping have fixed point

b) All the automorphisms of $\overline{B}(C_b(\Omega))$ of the form $f\mapsto[x\mapsto M(x)\ f(Tx)]$ where $M(.)$ is a continuous $\Omega\longrightarrow\mathrm{Aut}\,\overline{\Delta}$ mapping and T is a periodic homeomorphism of Ω onto itself have a fixed point

c) Ω is an F-space.

<u>Proof.</u> b) $\Longrightarrow$ a) is evident and a) $\Longrightarrow$ c) is established by the proof of Proposition 1. To prove c) $\Longrightarrow$ b), suppose that Ω is

an F-space and let $M:\Omega\to \mathrm{Aut}\,\overline{\Delta}$ and $T:\Omega\leftrightarrow\Omega$ be continuous. Define F by $F(f)\equiv[x\mapsto M(x)f(Tx)]$ (for all $f\in\overline{B}(C_b(\Omega))$). (Clearly $F\in \mathrm{Aut}\,\overline{B}(C_b(\Omega))$). Further assume $T^n = id_\Omega$, and let $R(.)$ denote a continuous section defined on $(\mathrm{Aut}\,\overline{\Delta})\setminus\{id_{\overline{\Delta}}\}$ of the multifunction $M\mapsto\{\zeta:M\zeta=\zeta\}$ (its existence is seen in Lemma 2).

Consider the set $G\equiv\{x\in\Omega:M(x)M(Tx)\ldots M(T^{n-1}x)\neq id_{\overline{\Delta}}\}$ and define the function $g: G\to\overline{\Delta}$ by $g(x)\equiv R(M(x)\ldots M(T^{n-1}x))$. Since G is the inverse image of the open subset $(\mathrm{Aut}\,\overline{\Delta})\setminus\{id_{\overline{\Delta}}\}$ of the metrizable space $\mathrm{Aut}_{\overline{\Delta}}$ by the continuous mapping $x\mapsto M(x)\ldots M(T^{n-1}x)$, the set G is a cozero subset of Ω (namely we have is particular $G=\{x\in\Omega:|k_{M(x)\ldots M(T^{n-1}x)}|+|u_{M(x)\ldots M(T^{n-2}x)}|\neq 0\}$). On the other hand, G is also T-invariant because in case of $M(x)\ldots M(T^{n-1}x)=id_{\overline{\Delta}}$ we have $M(Tx)\ldots M(T^{n-1}x)M(x)=id_{\overline{\Delta}}$ and here the last term can be written as $M(x)=M(T^nx)=M(T^{n-1}(Tx))$. About the function $g(.)$ we can state the following:

$$g(x) = M(x)g(Tx) \qquad \forall\, x\in G. \tag{5}$$

Indeed, if $x\in G$, we have $g(Tx)=R(M(Tx)\ldots M(T^{n-1}(Tx)))=$
$=R(M(Tx\ldots M(T^{n-1}x)M(x))=R(M(x)^{-1}[M(x)\ldots M(T^{n-1}x)]M(x))=$
$=$ by Lemma 4 c) $=M(x)^{-1}R(M(x)\ldots M(T^{n-1}x))=M(x)^{-1}g(x)$.

Now let $h(.)$ be a continuous extension of $g(.)$ from G to Ω. The existence of such a function $h(.)$ is established by [GJ1, 14.25.Theorem (6)] since Ω is assumed to be an F-space. Since $|g|<1$, we any assume without any loss of generality that also $|h|<1$. Thus let $h\in\overline{B}(C_b(\Omega))$ with $h|G=g$. Define the function $f:\Omega \to \overline{\Delta}$ (which will be our candidate to be a fixed point of F) by

$$f(x) \equiv P_n((M(x)^{-1}, M(Tx)^{-1}M(x)^{-1}, \ldots, M(T^{n-2}x)^{-1} \ldots M(x)^{-1},$$
$$(h(x), h(Tx), \ldots, h(T^{n-1}x))).$$

Check that for any $x \in \Omega$, $f(x) = M(x)\ f(Tx)$:

First let $x \in G$. Then $h(x) = g(x)$. But we have $g = M\cdot(g\circ T)$ which implies $g(Tx) = M(x)^{-1}\ g(x)$, $g(T^2x) = M(Tx)^{-1}\ g(Tx) =$ $= M(Tx)^{-1}M(x)^{-1}g(x), \ldots, g(T^{n-1}x) = M(T^{n-2}x)^{-1} \ldots M(x)^{-1}\ g(x)$. Thus

(6) $f(x) = P_n((M(x)^{-1}, \ldots, M(T^{n-2}x)^{-1} \ldots M(x)^{-1}),$
$(g(x), M(x)^{-1}\ g(x), \ldots, M(T^{n-2}x)^{-1} \ldots M(x)^{-1}\ g(x)))$. A directe application of Definition 1 to the right hand side of (6) yields that $f(x) = g(x)$. Hence and by (5) we obtain $f(x) = g(x) = M(x)g(Tx) =$ applying (6) to $Tx(\in G)$ in place of $x = M(x)\ f(Tx)$.

Then let $x \in \Omega \setminus G$. Now $M(x) \ldots M(T^{n-1}x) = id_{\overline{\Delta}}$. Thus

$$f(Tx) = P_n((M(Tx)^{-1}, \ldots, M(T^{n-1}x)^{-1} \ldots M(Tx)^{-1}), (h(Tx), \ldots, h(T^n x))) =$$
$$= P_n((M(Tx)^{-1}, \ldots, M(T^{n-1}x)^{-1} \ldots M(Tx)^{-1}), (h(Tx), \ldots$$
$$\ldots, h(T^{n-1}x), h(x))).$$

Therefore, by substituting $M_j \equiv M(T^jx)$, $\zeta_j \equiv h(T^jx)$ $(j=0,\ldots n-1)$ and $\eta \equiv f(x)$ in Lemma 8, we can verify $f(Tx) = M(x)^{-1}f(x)$.

The continuity of $f(.)$ is an immediate consequence of Lemma 6. □

Chapter 2

On Aut $\overline{B}(C(\Omega))$ in case of compact F-spaces Ω

It is well-known that [VF1] for a compact topological space Ω, the automorphisms of $D\equiv\overline{B}(C(\Omega))$ are exactly those transformations $F:D\to C(\Omega)$ which can be represented in the form

$$(7) \qquad F(f) = [\Omega \ni x \mapsto M_F(x) f(T_F x)] \qquad (\forall f \in D)$$

where T_F and M_F are a uniquely by F determined homeomorphism of Ω onto itself and a continuous $\Omega \longrightarrow \text{Aut } \overline{\Delta}$ mapping, respectively. In the sequel we reserve the notations T_F, M_F to indicate the $\Omega \leftrightarrow \Omega$ homeomorphism and $\Omega \to \text{Aut } \overline{\Delta}$ mapping, respectively, defined implicitly by (7) whenever $F \in \text{Aut } \overline{B}(C(\Omega))$.

Since for any F-space Ω there exists a completely regular F-space $\tilde{\Omega}$ such that $C_b(\Omega) \approx C_b(\tilde{\Omega})$ (i.e. $C_b(\Omega)$ is isometrically isomorphic with $C_b(\tilde{\Omega})$; cf. [GJ1, 3.9. Theorem]) and since the Stone-Čech compactification of any (completely regular) F-space is an F-space (cf. [GJ1, 14.25. Theorem(10)]), it suffices to restrict our attention to compact F-spaces Ω (by Proposition 1) when looking for those Ω-s that admit an elements whith fixed points for Aut $\overline{B}(C(\Omega))$. Fortunately, in this case the description provided by (7) enables us a very precise controll of Aut $\overline{B}(C(\Omega))$. However, the complete characterization of those compact space Ω where any $F \in \text{Aut } \overline{B}(C(\Omega))$ has fixed point seems to be extremely difficult yet. Theorem 1 localizes somewhat the difficulties to one point: to the description of the topological automorphisms[3)] of the compact F-spaces.

Definition 2. If T is a mapping of some set Ω into itself and $x \in \Omega$ then we shall call the number inf $\{n \in \mathbb{N}: T^n x = x\}$ the rank of T at the point x and we shall denote it by $r_T(x)$. T will be said pointwise periodic if $r_T(x) < \infty$ (i.e. $\{n \in \mathbb{N}: T^n x = x\} \neq \emptyset$) for all $x \in \Omega$.

Lemma 7. Let Ω be a Baire space and T a pointwise periodic automorphism of Ω. For $n=1,2,\dots$ set $\Omega_n \equiv \{x \in \Omega: r_T(x) < n\}$ and let $G \equiv \bigcup_{n=1}^{\infty} (\Omega_n \setminus \Omega_{n-1})^{\circ}$ where $\Omega_o \equiv \emptyset$ ($^{\circ}$ denoting the interior). Then G is an open dense T-invariant subset of Ω. Further we have $\overline{\lim}_{y \to x} r_T(y) = \overline{\lim}_{G \ni y \to x} r_T(y) \quad \forall x \in \Omega$.

Proof. If $(x_j : j \in J)$ is such a net that $x_j \to x$ (in Ω) and $T^m x_j = x_j \quad \forall j \in J$ then obviously $T^m x = x$. Thus the function $r_T(.)$ is lower semicontinuous. Therefore $\Omega_1, \Omega_2, \dots$ are all closed. Since the pointwise periodicity of T is equivalent to $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$, this means that the set $G' \equiv \bigcup_{n=1}^{\infty} \Omega_n^{\circ}$ is dense in Ω. Consider now any open $U \subset \Omega$. By density of G' in Ω we can find an n_o with $\Omega_{n_o}^{\circ} \cap U \neq \emptyset$. Since $r_T(x) < n_o \quad \forall x \in \Omega_{n_o}$, there exists $y_o \in \Omega_{n_o}^{\circ} \cap U$ such that $r_T(y_o) = \max\{r_T(x) : x \in \Omega_{n_o}^{\circ} \cap U\}$. But

3) The (topological) automorphisms of a topological space are its homeomorphisms onto itself.

$\{x \in \Omega_{n_o} \cap U: r_T(x) = r_T(y_o)\} = U \cap \Omega_{n_o}^{o} \cap \{x \in \Omega: r_T(x) > r_T(y_o) - 1\}$ is an open neighbourhood of the point y_o. Therefore the set $G" \equiv \{y \in \Omega: \exists U$ neighbourhood of $y \ \forall x \in U \ r_T(x) = r_T(y)\}$ is dense in Ω. But $G" = \bigcup_{n=1}^{\infty} \{y \in \Omega: \exists U$ nbh.of $y \ \forall x \in U \ r_T(x) = n\} =$
$= \bigcup_{n=1}^{\infty} \{x \in \Omega: r_T(x) = n\}^{o} = \bigcup_{n=1}^{\infty} (\Omega_n \setminus \Omega_{n-1})^{o}$.

The T-invariance of G is immediate from $r_T(x) = r_T(Tx) \ \forall x \in \Omega$.

To prove the second statement, observe that by the lower semicontinuity of $r_T(.)$ we have $\underline{\lim}_{G \ni z \to y} r_T(z) \geq r_T(y) \quad \forall y \in \Omega$.

Thus $\overline{\lim}_{G \ni y \to x} r_T(y) \leq \overline{\lim}_{y \to x} r_T(y) < \overline{\lim}_{y \to x} \overline{\lim}_{G \ni z \to y} r_T(z) = \lim_{G \ni z \to x} r_T(z)$. □

Lemma 8. Let Ω be a compact space, T an automorphism of Ω, $f_1, f_2, \ldots \in C(\Omega)$ and let A denote the closed C*-subalgebra of $C(\Omega)$ (with the usual complex-conjugate involution) generated by the functions 1_Ω and $f_n \circ T^m$ $(n \in \mathbb{N}, m \in \mathbb{Z})$. Then there are a compact metric space K, a surjective continuous map ϕ: $\Omega \to K$ and a homeomorphism $\tilde{T}$ of K onto itself such that $A =$ $= C(K) \circ \phi$ [4] and $\phi \circ T = \tilde{T} \circ \phi$.

Proof. The commutative Gel'fand-Neumark theorem establishes the existence of a compact Hausdorff space K and an isometric *-isomorphism ψ between C(K) and A. Fix such a space K and a mapping ψ. Since A is separable, C(K) is also separable and therefore K is metrizable (cf. [Sch1]). Let us evaluate $\psi^* \delta_x$

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4) i.e. $\forall f \in C(\Omega) \quad f \in A \Leftrightarrow \exists \tilde{f} \in C(K) \quad f = \tilde{f} \circ \phi$

for an arbitrary $x \in \Omega$ (ψ^* and δ_x denoting the adjoint map of ψ and the Dirac-δ associated to the point x,resp.): $\psi^*\delta_x(\tilde{f}) = \langle \tilde{f}, \psi^*\delta_x \rangle = \langle \psi\tilde{f}, \delta_x \rangle \quad \forall \tilde{f} \in C(K)$. Thus $\psi^*\delta_x$ is a non-vanishing multiplicative linear functional over $C(K)$. Hence there is a unique $\tilde{x} \in K$ such that $\psi^*\delta_x = \delta_{\tilde{x}}$. Let $\phi: \Omega \to K$ be the map which sends any point $x \in \Omega$ into that $\tilde{x} \in K$ that satisfies $\psi^*\delta_x = \delta_{\tilde{x}}$. Now $(\psi\tilde{f})(x) = \langle \psi\tilde{f}, \delta_x \rangle = \langle \tilde{f}, \psi^*\delta_x \rangle =$ $= \langle \tilde{f}, \delta_{\phi(x)} \rangle = \tilde{f}(\phi(x)) \quad \forall x \in \Omega$ i.e. $\psi\tilde{f} = \tilde{f} \circ \phi \quad \forall \tilde{f} \in C(K)$. Thus $A = \psi C(K) = C(K) \circ \phi$. Further we have $\|\tilde{f}\| =$ $= \|\psi\tilde{f}\| = \|\tilde{f} \circ \phi\| \quad \forall \tilde{f} \in C(K)$, and this implies also that range $\phi = K$.

To complete the argument, consider the transformation $Q: C(K) \to C(K)$ defined by $Q\tilde{f} \equiv \psi^{-1}[(\psi\tilde{f}) \circ T]$. Observe that Q is an order preserving surjective isometry of $C(K)$. So there is a unique homeomorphism $\tilde{T}: K \leftrightarrow K$ with $Q(\tilde{f}) = \tilde{f} \circ \tilde{T} \quad \forall \tilde{f} \in C(K)$ (see [Sch1]). Defining $\tilde{T}$ in this way, we have $\tilde{f} \circ \tilde{T} \circ \phi =$
$$= (Q\tilde{f}) \circ \phi = (\psi^{-1}[(\psi\tilde{f}) \circ T]) \circ \phi = [\psi^{-1}(\tilde{f} \circ \phi \circ T)] \circ \phi = \psi[\psi^{-1}(\tilde{f} \circ \phi \circ T)] =$$
$= \tilde{f} \circ \phi \circ T \quad \forall \tilde{f} \in C(K)$. Therefore $\tilde{T} \circ \phi = \phi \circ T$. □

<u>Corollary 1.</u> If $f_1 = f_2 = \ldots = f\ (\in C(\Omega))$ then

$$\inf\{n \in \mathbb{N}: f(T^k x) = f(T^{\operatorname{mod}_n k} x) \quad \forall k \in \mathbb{Z}\} = r_{\tilde{T}}(\phi(x)).$$

<u>Proof.</u> Choose a function $\tilde{f} \in C(K)$ such that $f =$ $= \tilde{f} \circ \phi$ and set $r^*(x) \equiv \inf\{n \in \mathbb{N}: \forall k \in \mathbb{Z}\ f(T^k x) = f(T^{\operatorname{mod}_n k} x)\}$ (for $x \in \Omega$), $\tilde{r}^*(\tilde{x}) \equiv \inf\{n \in \mathbb{N}: \forall k \in \mathbb{Z}\ \tilde{f}(\tilde{T}^k \tilde{x}) = \tilde{f}(\tilde{T}^{\operatorname{mod}_n k} \tilde{x})\}$ (for $\tilde{x} \in K$).

First we shall see that $r^*(x) = \tilde{r}^*(\phi(x)) \quad \forall x \in \Omega$:

$r^*(x) < \ell$ iff for some $0<n\leq\ell \quad k \; Z \; f(T^{\mathrm{mod}_n k} x) = f(T^k x)$,

iff for some $0<n<\ell \quad \forall k \in \mathbb{Z} \; \tilde{f}(\phi(T^{\mathrm{mod}_n k} x)) = \tilde{f}(\phi(T^k x))$, iff

for some $0<n\leq\ell \quad \forall k \in \mathbb{Z} \; \tilde{f}(\tilde{T}^{\mathrm{mod}_n k} \phi(x)) = \tilde{f}(\tilde{T}\phi(x))$, iff

$\tilde{r}^*(\phi(x)) \leq \ell$. Since these equivalences hold for all $\ell \in \mathbb{N}$, indeed $r^* = \tilde{r}^* \circ \phi$.

We prove now that $\tilde{r}^* = r_{\tilde{T}}$: Since $A = C(K)\circ\phi$, for each pair $x,y \in \Omega$ with $\phi(x) \neq \phi(y)$ there exists $g \in A$ such that $g(x) \neq g(y)$. By definition of A and by the Stone-Weierstrass theorem, hence we obtain that $\exists k \in \mathbb{Z} \quad f(T^k x) \neq f(T^k y)$. Thus $\phi(x)=\phi(y)$ iff $\forall k \in \mathbb{Z} \quad f(T^k x) = f(T^k y)$. Therefore if $\tilde{x} \in K$ and $x \in \phi^{-1}(\{x\})$ then $\tilde{T}^n \tilde{x} = \tilde{x}$ iff $\tilde{T}^n \phi(x) = \phi(x)$ iff $\phi(T^n x) = \phi(x)$ iff $\forall k \in \mathbb{Z} \quad f(T^{n+k} x) = f(T^k x)$ (these equivalences hold for any $n \in \mathbb{N}$). Thus for all $n \in \mathbb{N}$, equivalent are $\tilde{T}^n \tilde{x} = \tilde{x}$ and $\forall k \in \mathbb{Z} \; f(T^k x) = f(T^{\mathrm{mod}_n k} x)$. This implies that $\inf\{n \in \mathbb{N} : \tilde{T}^n \tilde{x} = \tilde{x}\} = \inf\{n \in \mathbb{N} : \tilde{f}(\tilde{T}^{\mathrm{mod}_n k} \tilde{x}) = f(\tilde{T}^k \tilde{x}) \quad \forall k \in \mathbb{Z}\}$. □

Lemma 9. Let Ω be a compact F-space, T a pointwise periodic automorphism of Ω. Then for all $f \in C(\Omega)$, there exists $n_o \in \mathbb{N}$ such that $f = f \circ T^{n_o}$.

Proof. Set again $f_1 \equiv f_2 \equiv \ldots \equiv f$ and let A, K, ϕ and $\tilde{T}$ be as in Lemma 8. Suppose the countrary of the statement of Lemma 9, i.e. that, in wiew of Corollary 1, $\sup\{r_{\tilde{T}}(\tilde{x}) : \tilde{x} \in K\} = \infty$. Since clearly $r_{\tilde{T}}(\phi(x)) \leq r_T(x) \quad \forall x \in \Omega$, the homeomorphism $\tilde{T}: K \leftrightarrow K$ is also pointwise periodic. Hence we can apply Lemma 7 to K and $\tilde{T}$ (in place of Ω and T there). This shows, by the lower semicontinuity of the function $r_{\tilde{T}}(.)$, that there is a sequence $\tilde{x}_1, \tilde{x}_2, \ldots$ K with $r_{\tilde{T}}(\tilde{x}_n) \uparrow \infty$ $(n \to \infty)$ such that $\tilde{x}_n$ is an inner point of $\{\tilde{x} \in K : r_{\tilde{T}}\tilde{x} = r_{\tilde{T}}(\tilde{x}_n)\}$ for all n. For any $n \in \mathbb{N}$, let $V_n^o, \ldots, V_n^{r_{\tilde{T}}(\tilde{x}_n)-1}$ $\{\tilde{x} \in K : r_{\tilde{T}}(\tilde{x}) = r_T(\tilde{x}_n)\}$ be pairwise disjoint neighbourhoods of the points $\tilde{x}_n, \tilde{T}\tilde{x}_n, \ldots$ $\ldots, \tilde{T}^{r_{\tilde{T}}(\tilde{x}_n)-1}\tilde{x}_n$, respectively. (Remark: $\{\tilde{x}$ $K : r_{\tilde{T}}(\tilde{x}) = r_{\tilde{T}}(\tilde{x}_n)\} =$ $= \{\tilde{x} \in K : r_{\tilde{T}}(\tilde{x}) = r_{\tilde{T}}(\tilde{T}^k \tilde{x}_n)\}$ $\forall k \in \mathbb{Z}$.) Set $U_n^k \equiv$ $\equiv \bigcap_{\ell=0}^{r_{\tilde{T}}(\tilde{x}_n)-1} (\tilde{T}^{k-\ell} V_n^\ell)$ (for $n \in \mathbb{N}, k \in \mathbb{Z}$). Now the family $\{U_n^k : n \in \mathbb{N},$ $0 \le k < r_{\tilde{T}}(\tilde{x}_n)\}$ is disjoint and $U_n^k = \tilde{T}^k U_n^o$ (for all n and $0 \le k < r_{\tilde{T}}(\tilde{x}_n)$). Let us fix an irrational number δ and a sequence of integers $\ell_1, \ell_2, \ldots$ with $\ell_n / r_{\tilde{T}}(\tilde{x}_n) \to \delta$ $(n \to \infty)$. Define the functions $\tilde{g}(.), \tilde{h}(.)$ on $\bigcup_{n=1}^{\infty} \bigcup_{k=o}^{r_{\tilde{T}}(\tilde{x}_n)-1} U_n^k$ by $\tilde{g}(\tilde{x}) \equiv$ $\equiv \exp(2\pi i k \ell_n / r_{\tilde{T}}(\tilde{x}_n))$ and $\tilde{h}(\tilde{x}) \equiv \exp(2\pi i \ell_n / r_{\tilde{T}}(\tilde{x}_n))$ for all $\tilde{x} \in U_n^k$ $(n \in \mathbb{N},\ 0 \le k < r_{\tilde{T}}(\tilde{x}_n))$. Set $g_o \equiv \tilde{g} \circ \phi$ and $h_o \equiv \tilde{h} \circ \phi$ with domain $G \equiv \phi^{-1}(\bigcup_{n=1}^{\infty} \bigcup_{k=o}^{r_{\tilde{T}}(\tilde{x}_n)-1} U_n^k)$. Then we have $\tilde{g}(\tilde{T}\tilde{x}) =$ $= \tilde{h}(\tilde{x}) \cdot \tilde{g}(\tilde{x})$ $\forall \tilde{x}$. Therefore $g_o \circ T = h_o g_o$. Since the set G is the inverse image by a continuous mapping of an open sebset of a metric space, it is a cozero set. Thus we can find (cf. [1]) continuous extensions g,h of the functions g_o, h_o to the whole space Ω, respectively. Since $g_o(Tx) = h_o(x) g_o(x)$ $\forall x \in G$, we have $g(Tx) = h(x) g(x)$ $\forall x \in \bar{G}$ (!). In particular, if

$x_n \in \phi^{-1}(\{\tilde{x}_n\})$ $(n=1,2,\ldots)$ and $x(\in \Omega)$ is a cluster point of the sequence $(x_1,x_2,\ldots)$ then $1 = \lim_{n\to\infty} g(x_n) = g(x) = g(T^{r_T(x)}x) = $
$= h(T^{r_T(x)-1}x)g(T^{r_T(x)-1}x) = \ldots = [h(T^{r_T(x)-1}x)\ldots h(x)]\cdot g(x)$.

Similarly, $g(T^{r_T(x)}x_n) = [h(T^{r_T(x)-1}x_n)\ldots h(x_n)]g(x_n) =$
$= h(T^{r_T(x)-1}x_n)\ldots h(x_n) = \exp[2\pi i\, r_T(x)\ell_n / r_T(x_n)]$ $\forall n \in \mathbb{N}$.

But then $1 = g(x) = g(T^{r_T(x)}x) =$
$= \lim_{n\to\infty} g(T^{r_T(x)}x_n) = \exp[2\pi i \cdot r_T(x)\delta] \neq 1$,
contradiction. □

What we have shown in Lemma 9 means that the automorphism $f \mapsto f\circ T$ of $C(\Omega)$ is pointwise periodic whenever the underlying automorphism T of the compact F-space Ω is pointwise periodic. However, the following general Banach space principle holds:

Lemma 10. Let E be a Banach space, $T: E \to E$ a linear pointwise periodic contraction. Then T is periodic.

Proof. Assume T is not periodic. Now $\forall\, n \in \mathbb{N}$ $\exists f \in E$ $T^n f \neq f$. Therefore (and by linearity of T) we can define a sequence $f_1, f_2, \ldots \in E$ in the following manner. We choose f_1 so that $T^1 f_1 \neq f_1$. If $f_1,\ldots f_j$ are already defined then we set $\delta_j \equiv \mathrm{diam}\{T^n f_j : n \in \mathbb{N}\}$ and $\varepsilon_j \equiv \min\{\|T^n f_j - f_j\| : T^n f_j \neq f_j,\ n \in \mathbb{N}\}$ and then we choose f_{j+1} to satisfy the relations $T^{j+1} f_{j+1} \neq f_{j+1}$ and $\mathrm{diam}\,\{T^n f_{j+1} : n \in \mathbb{N}\} < \varepsilon_j/3$. Thereafter consider the vector $f \equiv \sum_{j=1}^{\infty} f_j$. Let $n(\in \mathbb{N})$ be arbitrarily

fixed and set $n_o = \min\{j: T^n f_j \neq f_j\}$. Then $T^h f - f =$
$= \sum_{j \geq n_o} (T^n f_j - f_j)$. Thus $\|T^n f - f\| \geq \|T^n f_{n_o} - f_{n_o}\| - \sum_{j>n_o} \|T^n f_j - f_j\| \geq$
$\geq \varepsilon_{n_o} - \sum_{j>n_o} \delta_j$. But we have $\delta_j < \frac{1}{3}\varepsilon_{j-1} < \frac{1}{3}\delta_{j-1}$ $\forall j \in \mathbb{N}$ whence
$\sum_{j>n_o} \delta_j \leq \delta_{n_o+1} \sum_{k=0}^{\infty} 3^{-k} = \frac{3}{2}\delta_{n_o+1} \leq \frac{1}{2}\varepsilon_{n_o}$. Thus $\|T^n f - f\| \geq$
$\geq \varepsilon_{n_o}/2 > 0$ $\forall n \in \mathbb{N}$, i.e. T is not pointwise periodic. □

Hence it readily follows:

Theorem 3. Let Ω be a compact F-space and T a pointwise periodic automorphism of Ω. Then T is necessarily periodic.

Proof. Lemma 9 and Lemma 10 directly yield that we can find n such that $f \circ T^n = f$ $\forall f \in C(\Omega)$. Hence necessarily $T^n = id_\Omega$ (since $T^n x \neq x$ would imply $f \circ T^n \neq f$ whenever $f(x) = 0 \neq f(T^n x)$, and $C(\Omega)$ separates the points of Ω by its compactness).

Theorem 1'. The following two conditions are equivalent for a compact space Ω:
a) Every $F \in \mathrm{Aut}\,\overline{B}(C(\Omega))$ with pointwise periodic T_F has fixed points; b) Ω is an F-space.

Proof. Immediate from Theorem 2 and Theorem 3. □

Chapter 3

The case of M-lattices with predual

Having established Theorem 1', it is natural to ask whether the condition on a compact Ω of being an F-space ensures the existence of fixed points for every $F \in \operatorname{Aut} \overline{B}(C(\Omega))$. The question can be stated equivalently in the following way: Consider any commutative C*-algebra with unit whose maximal ideal space is an F-space. Does any biholomorphic automorphism of the unit ball have a fixed point? In the latter setting, we can expect a negative answer. In fact, as we shall see, the space $E \equiv L^\infty(0,1)$ admits an $F \in \operatorname{Aut} \overline{B}(E)$ of the form $F: f \mapsto [x \mapsto M(x) f(Tx)]$ with an ergodic transformation of the interval $(0,1)$ and a Borel measurable function $M:(0,1) \to \operatorname{Aut} \overline{\Delta}$ without fixed point. (The maximal ideal space of $L^\infty(0,1)$ is hyperstonian (see |Sem1|, |W1|) hence obviously an F-space).

Throughout this Chapter, let M_1, M_2 denote the transformations $[\mathbb{C} \ni \zeta \mapsto -\zeta]$ and $[\mathbb{C} \ni \zeta \mapsto \dfrac{\zeta + \operatorname{th}(1)}{1 + \zeta \operatorname{th}(1)}]$, respectively. (Note: $M_1|_{\overline{\Delta}}, M_2|_{\overline{\Delta}} \in \operatorname{Aut} \overline{\Delta}$. The reason for the constant th(1) is the simple convenience that $M_2^t: \zeta \mapsto \dfrac{\zeta + \operatorname{th}(t)}{1 + \zeta \operatorname{th}(t)}$ $\forall t \in \mathbb{Z}$ [cf. Proof b) in Lemma 3].) Let λ be the normed Lebesgue measure on the unit circle $\partial\Delta$ of $\mathbb{C}$ (i.e. $\lambda \equiv \frac{1}{2\pi} \operatorname{length}|_{\partial\Delta}$). Further we fix an irrational number $\delta \in (0,1)$ and denote by T the clockwise rotation of $\partial\Delta$ by the angle $2\pi\delta$, i.e. $T: x \mapsto \exp(-2\pi i \delta) \cdot x$.

The space $L^{\infty}(\partial\Delta,\lambda)$ is considered, as usually, as $\{\tilde{\varphi}:\varphi$ is a bounded Borel $\partial\Delta\to\mathbb{C}$ function$\}$ where $\tilde{\varphi}\equiv\{\psi(:\partial\Delta\to\mathbb{C}):$ $\lambda\{x\in\partial\Delta:\psi(x)\neq\varphi(x)\}=0\}$. Finally, let $M:\partial\Delta\to\mathrm{Aut}\,\overline{\mathbb{C}}$ be the function $\exp(2\pi i\tau)\mapsto\begin{cases}M_1 & \text{if } 0\leq\tau<\delta\\ M_2 & \text{if } \delta\leq\tau<1\end{cases}$, and define $F:\overline{B}(L^{\infty}(\partial\Delta,\lambda))\to$ $\to L^{\infty}(\partial\Delta,\lambda)$ by $F(\tilde{\varphi})\equiv[x\mapsto M(x)\varphi(Tx)]^{\sim}$ for all Borel measurable $\varphi:\partial\Delta\to\overline{\Delta}$. Clearly, $F\in\mathrm{Aut}\,\overline{B}(L^{\infty}(\partial\Delta,\lambda))$.

Theorem 4. The transformation F (defined above) has no fixed point.

The proof is divided into eight steps

1) Let G be the subgroup of Aut $\mathbb{C}$ generated by M_1 and M_2. Since

$$(8)\qquad M_2M_1=M_1M_2^{-1}\quad(\text{and}\quad M_1M_2=M_2^{-1}M_1),$$

we have $G=\{M_1^sM_2^t : s=0,1;\ t\in\mathbb{Z}\}$. This representation of G is unique in the sense that if $s,s'\in\{0,1\}$ and $t,t'\in\mathbb{Z}$ with $M_1^sM_2^t=M_1^{s'}M_2^{t'}$ then $s=s'$ and $t=t'$ (since $\mathrm{id}_{\mathbb{C}}=M_1^{s-s'}M_2^{t'-t}=$ $=[\zeta\mapsto(-1)^{s-s'}\dfrac{\zeta+\mathrm{th}(t'-t)}{1+\zeta\,\mathrm{th}(t'-t)}]$).

2) In the following we shall argue by contradiction assuming that Theorem 4 does not hold. Denote by f_o a fixed point of F and let $\varphi_o:\partial\Delta\to\overline{\Delta}$ be a representant of f_o (thus $f_o=\tilde{\varphi}_o$). The symbol $\forall_\lambda$ will indicate "λ-almost everywhere". Now $\varphi_o(Tx)=M(x)^{-1}\varphi_o(x)$ $\quad\forall_\lambda\ x\in\partial\Delta\quad\forall n\in\mathbb{N}$, and therefore

(9) $\varphi_0(T^n x) = M(T^{n-1}x)^{-1}\dots M(x)^{-1}\varphi_0(x) \quad \forall_\lambda x \in \partial\Delta \quad \forall n \in \mathbb{N}.$

Thus range $(\varphi_0 \circ T^n) \subset G\cdot(\text{range } \varphi_0) \quad \forall n \in \mathbb{N}.$

3) It is well-known that the transformation T is ergodic (cf. [Ha2]). Hence it follows that if $S \subset \partial\Delta$ is such that $T(S)$ differs just in a O-set (wrtλ) from S (i.e. $\lambda([S \cup T(S)] \setminus [S \cap T(S)]) = 0$) then either $\lambda(S) = 0$ or $\lambda(S) = 1$.

Thus if for a Borel set $\Gamma \subset C$ we have $N(\Gamma) = \Gamma \quad \forall N \in G$, then $\varphi_0^{-1}(\Gamma)$ is either a O-set or the complementary set in $\partial\Delta$ of some O-set (wrt λ).

4) If $\zeta, \eta \in \overline{\Delta}\setminus\{-1,1\}$ and $\eta \notin G(\zeta)$ then there exist G-invariant neighbourhoods U,V of ζ and η, respectively, that are disjoint.

Proof: Observe that for any $t \in \mathbb{Z}$, $M_2^t: 1 \mapsto 1, (-1) \mapsto (-1)$, $[-1,1] \leftrightarrow [-1,1]$, circle $\longrightarrow$ (other) circle. So from the conformity of Aut $\mathbb{C}$ it easily follows that, for every $t \in \mathbb{Z}$, M_2^t maps the bounded domain $D \equiv \{\zeta \in \mathbb{C}: |\zeta - i| < \sqrt{2}\} \cap \{\zeta \in \mathbb{C}: |\zeta + i| < \sqrt{2}\}$ onto itself. Thus $ND = D \quad \forall N \in G$ (cf. 1)). Let d_D denote the Kobayashi distance on D (for its definition see [VF1], [Ko1]) and consider the orbit $G(\zeta)$. From (8) we deduce that $G(\zeta) = \{\pm M_2^t\zeta : t \in \mathbb{Z}\} \subset \Delta\setminus\{-1,1\} \subset D$. Since $M_2^t\zeta = \frac{\zeta + th(t)}{1+\zeta th(t)} \to \pm 1$ according to $t \to \pm\infty$, the set $G(\zeta)$ has no cluster point in D. Hence $d_D(\eta, G(\zeta)) > 0$. Thus the choices $U \equiv \{\zeta' \in D: d_D(\zeta', G(\zeta)) < \frac{1}{2} d_D(\eta, G(\zeta))\}$ and $V \equiv \{\eta' \in D: d_D(\eta', G(\zeta)) > \frac{1}{2} d_D(\eta, G(\zeta))\}$ suit our requirements.

We show now that $\lambda(\varphi_o^{-1}(G(\zeta_o))) = 1$ for some $\zeta_o \in \overline{\Delta}$.

Proof: The last remark and 3) exclude that for every pair $\zeta,\eta \in \overline{\Delta}\setminus\{-1,1\}$ and for all neighbourhoods U,V of $G(\zeta)$ and $G(\eta)$, respectively, we have $\lambda(\varphi_o^{-1}(U))>0$ and $\lambda(\varphi_o^{-1}(V))>0$ in the same time. If for any $\zeta \in \overline{\Delta}\setminus\{-1,1\}$, one can find a neighbourhood U of $G(\zeta)$ such that $\lambda(\varphi_o^{-1}(U)) = 0$ then the separability of $\mathbb{C}$ implies that $\lambda(\varphi_o^{-1}(\overline{\Delta}\setminus\{-1,1\})) = 0$, whence $\lambda(\varphi_o^{-1}(\{-1,1\})) = \lambda(\varphi_o^{-1}(\overline{\Delta})) - \lambda(\varphi_o^{-1}(\overline{\Delta}\setminus\{-1,1\})) = 1$. Now we can choose e.g. $\zeta_o \equiv 1$. If for some $\zeta_1 \in \overline{\Delta}\setminus\{-1,1\}$, any neighbourhood U of $G(\zeta_1)$ satisfies $\lambda(\varphi_o^{-1}(U))>0$ then for any G-invariant neighbourhood of this ζ_1 we necessarily have by 3) that $\lambda(\varphi_o^{-1}(U)) = 1$. Therefore $1 = \lambda(\varphi_o^{-1}(\{\zeta \in D : d_D(\zeta, G(\zeta_1)) < \frac{1}{n}\})) \longrightarrow \lambda(\varphi_o^{-1}(G(\zeta_1))$ $(n \to \infty)$. Thus, in this case, $\zeta_o \equiv \zeta_1$ suits.

Henceforth we assume that

range $\varphi_o = \{c_1, c_2, \ldots\} \subset G(c) \subset \overline{\Delta}$ (where $c, c_1, c_2, \ldots$ are given constants). Our previous observation ensures that this can be done without loss of generality.

5) Step 1) directly implies the existence of a unique pair of Borel functions $s_n : \partial\Delta \to \{0,1\}$ and $t_n : \partial\Delta \to \mathbb{Z}$ for each $n \in \mathbb{N}$, such that

$$M_1^{s_n(x)} M_2^{t_n(x)} = M(T^{n-1}x)^{-1} \ldots M(x)^{-1} \qquad \forall x \in \partial\Delta .$$

Thus by (9) we have

$$(9') \qquad \varphi_o(T^n x) = M_1^{s_n(x)} M_2^{t_n(x)} \varphi_o(x) \qquad \forall_\lambda x \in \partial\Delta \qquad \forall n \in \mathbb{N}.$$

Introducing the functions $s \equiv 1_{\{\exp(2\pi i\tau : 0 \le \tau < \delta\}}$ and $t \equiv 1_{\partial\Delta} - s$, we also have $M(x) = M_1^{s(x)} M_2^{t(x)}|_{\overline{\Delta}}$ $\forall x \in \partial\Delta$. Now (8) enables us to express s_n and t_n in terms of s and t. In particular, one sees by induction on n that $s_n(x) = \mathrm{mod}_2[s(x)+\ldots+s(T^{n-1}x)]$. Thus

$$M_1^{s_n(x)} = [\zeta \mapsto (-1)^{s(x)+\ldots+s(T^{n-1}x)}\zeta] \quad \forall x \in \partial\Delta \quad \forall n \in \mathbb{N}.$$

6) We achieve a stronger control over the functions $(-1)^{s_n(.)}$: Consider the function $\tilde{s}: \mathbb{R} \to \{0,1\}$ defined by $\tilde{s}(\tau) \equiv s(\exp(2\pi i\tau))$. Thus $\tilde{s}(\tau) = \sum_{m=-\infty}^{\infty} 1_{[0,\delta)}(\tau+m)$ $\forall \tau \in \mathbb{R}$.
Introducing the functions $\tilde{s}_n(\tau) \equiv s(\exp(2\pi i\tau)) + s(T\exp(2\pi i\tau)) + \ldots + s(T^{n-1}\exp(2\pi i\tau))$, we have $\tilde{s}_n(\tau) = \tilde{s}(\tau) + \tilde{s}(\tau-\delta) + \ldots + \tilde{s}(\tau-(n-1)\delta) = \sum_{m=-\infty}^{\infty} \sum_{k=0}^{n-1} 1_{[0,\delta)}(\tau+m-k\delta) =$

$$= \sum_{m=-\infty}^{\infty} \sum_{k=0}^{n-1} 1_{[k\delta,(k+1)\delta)}(\tau+m) = \sum_{m=-\infty}^{\infty} 1_{[0,n\delta)}(\tau+m).$$

Therefore, $\tilde{s}_n$ is a periodic continuation (with period-length 1) of the function

$$\tau \mapsto \begin{cases} \mathrm{entier}(n\delta+1) \text{ if } 0 \le \tau < n\delta - \mathrm{entier}(n\delta) \\ \mathrm{entier}(n\delta) \quad \text{if } n\delta - \mathrm{entier}(n\delta) \le \tau < 1 \end{cases}.$$

Since $\int_{\partial\Delta} (-1)^{s_n} d\lambda = \int_0^1 (-1)^{\tilde{s}_n(\tau)} d\tau$, this means that if $n_m \to \infty$ is a sequence in $\mathbb{N}$ such that $\mathrm{dist}\,(n_m\delta, \{2k-1 : k \in \mathbb{N}\}) \to 0$ $(m \to \infty)$ then $\int_{\partial\Delta} (-1)^{s_{n_m}} d\lambda \to -1$, i.e. the sequence of the functions $(-1)^{s_{n_m}}$ converges in measure to the identically -1 function on $\partial\Delta$ (wrt λ). So, by the classical Riesz-Weyl Lemma, there is a subsequence $(n_{m_j} : j \in \mathbb{N})$ with $(-1)^{s_{n_{m_j}}(x)} \longrightarrow -1 \quad (j \to \infty) \quad \forall_\lambda \; x \in \partial\Delta$, or which is the same, $s_{n_{m_j}}(x) \to 1 \quad (j \to \infty) \quad \forall_\lambda \; x \in \partial\Delta$.

Similarly, dist $(n'_m \delta, \{2k : k \in \mathbb{N}\}) \to 0 \quad (m \to \infty)$ implies the existence of a subsequence $(n'_{m'_j} : j \in \mathbb{N})$ with $s_{n'_{m'_j}}(x) \to 0 \; (j \to \infty) \quad \forall_\lambda x \in \partial\Delta$.

7) A sequence $n_m \to \infty$ for which dist $(n_m \cdot \delta, \{2k-1 : k \in \mathbb{N}\}) \to 0$ $(m \to \infty)$ certainly exists. (Proof: The set $\{\exp(\pi i n \delta) : n \in \mathbb{N}\}$ is dense in $\partial\Delta$ and the relation dist $(n_m \delta, \{2k-1 : k \in \mathbb{N}\}) \to 0$ is equivalent to $\exp[2\pi i (\delta/2) n_m] \to -1$.) Clearly, for any such a sequence $(n_m : m \in \mathbb{N})$ we have $\exp(2\pi i \delta_{n_m}) \to 1$ i.e. $T^{n_m} \to id_{\partial\Delta}$ (if $m \to \infty$).

From now on, let $(n'_m : m \in \mathbb{N})$ denote a fixed sequence in $\mathbb{N}$ such that $n'_m \to \infty$, $T^{n'_m} \to id_{\partial\Delta}$ and $\forall_\lambda \; x \in \partial\Delta \quad s_{n'_m}(x) \to 1 \quad (m \to \infty)$.

Suppose then that $(n_m : m \in \mathbb{N})$ is a sequence with $n_m \to \infty$, $T^{n_m} \to id_{\partial\Delta}$, and $\forall_\lambda x \in \partial\Delta \quad s_{n_m}(x) \to 0 \quad (m \to \infty)$. Since range $\varphi_0 \subset G(c) = \{\pm M_2^t : t \in \mathbb{Z}\}$ (cf. conclusion of 4)) and since $G(c)$ has two cluster points outside of itself whenever $c \neq \pm 1$ (namely the points -1 and 1), range φ_0 is a discrete subset of $\mathbb{C}$. By the Lebesgue Shift Theorem, the fact $T^{n'_m} \to id_{\partial\Delta}$ implies $\varphi_0(T^{n'_m} x) \to \varphi_0(x) \quad \forall_\lambda \; x \in \partial\Delta$. Similarly, $\varphi_0(T^{n_m} x) \to \varphi_0(x) \quad \forall_\lambda \; x \in \partial\Delta$. By the discreteness of range φ_0, we have then

(10) $\forall_\lambda x\in\partial\Delta \quad \exists m_o(x) \qquad \forall m>m_o(x)$

$$\varphi_o(x)=\varphi_o(T^{n'_m}x)= M_1^{s_{n'_m}(x)} M_2^{t_{n'_m}(x)}\varphi_o(x)= M_1M_2^{t_{n'_m}(x)}\varphi_o(x)$$

and

$$\varphi_o(x)=\varphi_o(T^{n_m}x)= M_1^{s_{n_m}(x)} M_2^{t_{n_m}(x)}\varphi_o(x)= M_2^{t_{n_m}(x)}\varphi_o(x).$$

Thus $\forall_\lambda x\in\partial\Delta \quad \exists t',t''\in\mathbb{Z} \quad M_1M_2^{t'}\varphi_o(x)=M_2^{t''}\varphi_o(x)=\varphi_o(x)$. Since for each $t''\neq 0$ and $\zeta\in\mathbb{C}$, from $M_2^{t''}\zeta=\zeta$ it follows $\zeta=-1$ or $\zeta=1$, (10) can hold only if $\forall_\lambda x\in\partial\Delta \quad \exists m_o(x)$ $\forall m>m_o(x) \quad t_{n_m}(x)=0$. Thus

(10′) If $n_m\to\infty$ is a sequence with $T^{n_m}\to id_{\partial\Delta}$ and $\forall_\lambda x\in\partial\Delta \quad s_{n_m}(x)\to 0$ then $t_{n_m}(x)\to 0 \quad \forall_\lambda x\in\partial\Delta$.

8) We shall arrive at a contradiction, by showing that (10′) is impossible. In fact, we shall prove that

a) There exists a sequence $n_m\to\infty$ consisting of <u>odd</u> numbers such that $T^{n_m}\to id_{\partial\Delta}$ and $\forall_\lambda x\in\partial\Delta \quad s_{n_m}(x)\to 0$,

b) $\mathrm{mod}_2\,[s_n(x)+t_n(x)]=\mathrm{mod}_2 n \qquad \forall x\in\partial\Delta \quad \forall n\in\mathbb{N}$.

By b), for any sequence $(n_m: m\in\mathbb{N})$ as in a), we have that $t_{n_m}(x)$ is odd for all $m\in\mathbb{N}$ and $x\in\partial\Delta$. But hence $t_{n_m}(x)\not\to 0$ $\forall x\in\partial\Delta$. This contradiction proves the theorem.

Proof of a): The conclusion of 6) tells as that a) is equivalent to the existence of a sequence $n_m^*\to\infty$ of odd numbers such that dist $(n_m^*\cdot\delta,\{2k:k\in\mathbb{N}\})\to 0$ $(m\to\infty)$. But this latter property is equivalent to $\exp[2\pi i n_m^*(\delta/2)]\to 1$ which can be easily satisfied by some odd sequence $(n^*_m : m \in \mathbb{N})$, since the set $\{\exp[2\pi i(2\ell+1)(\delta/2)] : \ell \in \mathbb{N}\}$ is dense in $\partial\Delta$ (for δ is irrational).

Proof of b): Proceed by induction on n. For $n=1$, $M_1^{s_1(x)} M_2^{t_1(x)} = M(x)^{-1}$ ($= M_1^{-1}$ or M_2^{-1}). Thus either $s_1(x) = 1$ and $t_1(x) = 0$ or $s_1(x) = 0$ and $t_1(x) = 1$. Anyway, $s_1(x)+t_1(x)$ is odd, similarly to $1(=n)$ for all $x \in \partial\Delta$.

To perform the inductive step, observe that $M_1^{s_{n+1}(x)} M_2^{t_{n+1}(x)} = M(T^n x)^{-1} M(T^{n-1}x)^{-1} \ldots M(x)^{-1} =$ $= M(T^n x)^{-1} M_1^{s_n(x)} M_2^{t_n(x)}$. Now there are three cases:

i) If $M(T^n x) = M_1$ then $M_1^{s_{n+1}(x)} M_2^{t_{n+1}(x)} = M_1^{s_n(x)-1} M_2^{t_n(x)} =$ $= M_1^{\mathrm{mod}_2[s_n(x)-1]} M_2^{t_n(x)}$, i.e. $\mathrm{mod}_2 [s_{n+1}(x)+t_{n+1}(x)] =$ $= \mathrm{mod}_2 [s_n(x)-1+t_n(x)] =$ by the induction hypotheses $=$ $= \mathrm{mod}_2(n-1) = \mathrm{mod}_2(n+1)$.

ii) If $M(T^n x) = M$ and $s_n(x) = 0$ then $M_1^{s_{n+1}(x)} M_2^{t_{n+1}(x)} =$ $= M_2^{-1} M_2^{t_n(x)}$, i.e. $0 = s_{n+1}(x)$ and $t_{n+1}(x) = t_n(x)-1$. Thus $\mathrm{mod}_2[s_{n+1}(x)+t_{n+1}(x)] = \mathrm{mod}_2[s_n(x)+t_n(x)-1] = \mathrm{mod}_2(n-1) =$ $= \mathrm{mod}_2(n+1)$.

iii) If $M(T^n x) = M_2$ and $s_n(x) = 1$ then $M_1^{s_{n+1}(x)} M_2^{t_{n+1}(x)} =$ $= M_2^{-1} M_1 M_2^{t_n(x)} =$ by (8) $= M_1 M_2^{t_n(x)+1}$, i.e. $\mathrm{mod}_2[s_{n+1}(x)+t_{n+1}(x)] =$ $= \mathrm{mod}_2[s_n(x)+t_n(x)+1] = \mathrm{mod}_2(n+1)$.

The proof of Theorem 4 is complete. □

The construction of the counterexample occuring in Theorem 4 may seem to be too much particular. However, a theorem of D. Maharam (cf. [Sem1], [Mah1]) asserts that for any σ-finite measure μ, there exists a sequence $\rho_1, \rho_2, \ldots > 0$ and a sequence of cardinalities $\mathfrak{a}_1, \mathfrak{a}_2, \ldots$ such that $L^1(\mu) \approx L^1(\bigoplus_{n=1}^{\infty} \rho_n \lambda^{\mathfrak{a}_n})$ (for $\mathfrak{a} > 0$, $\lambda^{\mathfrak{a}}$ denotes the $\mathfrak{a}$-th power of the measure λ; $\lambda^0 \equiv$ an atom with weight 1). This fact enables us an application of Theorem 4 to decide the fixed point problem of Aut $\overline{B}(E)$ even for the most general L^∞-spaces E (and hence, by a theorem of M. Rieffel [R1], for all M-lattices admitting a predual).

Lemma 11. Let X be a discrete topological space. Then for all $F \in \text{Aut}\,\overline{B}(C_b(X))$ there exists a (unique) permutation T of X and a function $M: X \to \text{Aut}\,\overline{\Delta}$ such that $F =$
$= [f \mapsto [x \mapsto M(x) f(Tx)]]$.

Proof. Let ϕf denote the (unique) continuous extension to βX (the Stone-Čech compactification of X) of any $f \in C_b(X)$. Now the map $\hat{F} \equiv \phi F \phi^{-1}$ is a biholomorphic automorphism of $\overline{B}(C(\beta X))$. Since the isolated points of βX are exactly the points of X and since any automorphism of a topological space sends the set of its isolated points onto itself, we have $T_{\hat{F}}(X) = X$. Hence $(Ff)(x) = (\phi^{-1}\hat{F}\phi f)(x) = (\hat{F}\phi f)|_X(x) =$
$= (\hat{F}\phi f)(x) = [\hat{F}(\phi f)](x) = M_{\hat{F}}(x)[(\phi f)(T_{\hat{F}}x)] =$ since $T_{\hat{F}}x \in X$ $=$
$= M_{\hat{F}}(x) f(T_{\hat{F}}x) \quad \forall x \in X.$ □

Corollary 2. For a discrete space X, all the members of Aut $\overline{B}(C_b(X))$ have fixed point.

Proof. Let τ denote the topology of pointwise convergence on $C_b(X)$ (i.e. by definition, $f_j \xrightarrow{\tau} f$ iff $\forall x \in X$ $f_j(x) \to f(x)$, for every net $(f_j : j \in J)$ and function f in $C_b(X)$). Observe that $\overline{B}(C_b(X))$ endowed with the topology τ coincides (set theoretically) with the topological product space $\overline{\Delta}^X$ which is compact by Tychonoff's Product Space Theorem. On the other hand, from Lemma 11 it readily follows that any $F \in \text{Aut}\, \overline{B}(C_b(X))$ is also $\tau \longrightarrow \tau$ continuous (the definition of F requires only its $\| \|$-topology $\to$ $\| \|$-topology continuity). Hence the Schauder-Tychonoff Fixed Point Theorem establishes (cf. [DS1]) that each $F \in \text{Aut}\, \overline{B}(C_b(X))$ has fixed point. □

Theorem 5. Let E be an M-lattice (for definition see [Sch1], [R1]) having a predual *E. Then the following properties are equivalent:

a) Any $F \in \text{Aut}\, \overline{B}(E)$ has a fixed point

b) $E \simeq C_b(X)$ for some discrete topological space X.

Proof. By a theorem of M. Rieffel [R1], the M-lattices with predual are exactly the L^∞-spaces. Thus we may assume without loss of generality that $*E = L^1(X,\mu)$ and $E = L^\infty(X,\mu)$ for some fixed measure space (X,μ). If the measure μ is atomic then obviously b) holds and hence Corollary 2 implies a). Suppose μ is non-atomic. Then b) is false, thus it suffices to find an $F \in \operatorname{Aut} \overline{B}(L^{\infty}(X,\mu))$ free of fixed points. Fix a μ-measurable subset $X' \subset X$ such that the measure $\mu|_{X'}$ be non-atomic and we have $0 < \mu(X') < \infty$. By Maharam's Isomorphism Theorem (cf. [Sem1],[Mah1]; cited also before Lemma 11), there exists a μ-measurable subset $Y \subset X'$ and a cardinality $\mathfrak{a} > 0$ such that $\mu(Y) > 0$ and $L^1(Y,\mu|_Y) \approx L^1(\mu(Y) \cdot \lambda\ ) (\sim L^1(\lambda^{\mathfrak{a}})$ by the mapping $f \mapsto \mu(Y)f$. Therefore $L^{\infty}(X,\mu)$ is isometrically isomorphic with the direct sum of $L^{\infty}(\lambda^{\mathfrak{a}})$ and some other L^{∞}-space $\hat{E}$ where the norm of a generic element (f,g) (f in $L^{\infty}(\lambda^{\mathfrak{a}})$, g in $\hat{E}$) is defined by $\|(f,g)\| \equiv \max\{\|f\|,\|g\|\}$. Hence, to prove Theorem 5, it suffices to show that some $F \in \operatorname{Aut} \overline{B}(L^{\infty}(\lambda^{\mathfrak{a}}))$ has no fixed point. But it follows from Theorem 4 that the mapping $F_0 : \overline{B}(L^{\infty}(\lambda^{\mathfrak{a}})) \to L^{\infty}(\lambda^{\mathfrak{a}})$ defined by

$$F_0 : f \mapsto \left[(\partial\Delta)^{\mathfrak{a}} \ni (\xi_\alpha : \alpha < \mathfrak{a}) \mapsto M(\xi_0)\varphi_f((T\xi_0), \xi_\alpha : 0 < \alpha < \mathfrak{a}) \right]^{\sim}$$

where $M : \partial\Delta \to \operatorname{Aut} \mathbb{C}$ and $T : \partial\Delta \to \partial\Delta$ are the same as in Theorem 4 and φ_f denotes a (fixed) Borel measurable representant with range in $\overline{\Delta}$ of f, for any $f \in \overline{B}(L^{\infty}(\lambda^{\mathfrak{a}}))$, has no fixed point and belongs to $\operatorname{Aut} \overline{B}(L^{\infty}(\lambda^{\mathfrak{a}}))$. □

Chapter 4

The linearity of Aut $\overline{B}$ in L^p-spaces if $p\neq 2,\infty$

It was the first result concerning the fixed point of infinite dimensional holomorphic maps that [HS1] the biholomorphic automorphisms of the closed unit ball in a Hilbert space and hence in L^2-spaces have fixed point. In the previous chapter we characterized all those L^∞-spaces where any member in Aut $\overline{B}$ admits a fixed point. In both cases it was easy to prove an exhaustive generic formula for the elements of Aut $\overline{B}$, and the difficulties of finding those spaces where all the mappings in Aut $\overline{B}$ have fixed points arose from the complicated topological behaviour of these formulas. What happens in the other L^p-spaces? A look at the two dimensional special case[4)] suggests the conjecture that the auswer must be contained in the fact that now Aut $\overline{B}$ consists only of linear mappings unless the space has dimension 1. However, this linearity of Aut $\overline{B}(L^p)$ is much harder to prove than to discover and justify the algebraically more

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4) By Thullen's classical theorem [Th1], the only bounded Reinhardt domains in $\mathbb{C}^2$ whose biholomorphic automorphism group is not completely linear are $\{(\zeta_1,\zeta_2):|\zeta_1|,|\zeta_2|<\rho\}$ and $\{(\zeta_1,\zeta_2):|\zeta_1|^2+|\zeta_2|^p<\rho\}$, $\{(\zeta_1,\zeta_2):|\zeta_1|^p+|\zeta_2|^2<\rho\}$ where ρ and p range (independently) over $(0,\infty)$.

sophisticated formulas describing the elements of Aut $\overline{B}(L^2)$ and Aut $\overline{B}(L^\infty)$, respectively. In finite dimensions it readily follows from a theorem of T. Sunada [Sun1] which is somewhat the n-dimensional analogue of Thullen's mentioned theorem. Further has been established for all L^1-spaces merely recently in a paper of E. Vesentini [V1] as a by-product of a fare reaching study of Aut D-invariant distances on subdomains D of locally convex vector spaces. There is indicated in [V1] also an alternative approach of proving the linearity of Aut $\overline{B}(L^1)$ which goes back to a result of T. Suffridge [Suf1] concerning holomorphic mappings with convex range. It can be expected that both these ways are suitable in obtaining the complete description of Aut $\overline{B}$ for all L^p-spaces in a common framework. However, in order to perform the necessary generalizations, we face enormous problems of algebraic character whose solution seems to require a further development of the general theory rather than a directe attack.

Here we present a new approach (cf. also [St2]) which applies to any L^p-space and which stands, as we shall point out it in the next chapter (Remark 3), in a close relation with an extended version of Vesentin's following lemma [V1, Lemma 4.3]:

For each Banach space E and vector $v \in \partial B(E)$, the mapping $[\Delta \ni \zeta \mapsto \zeta v]$ determines a complex geodetic curve with respect to both the Carathéodory and Kobayashi distances[5] associated to $B(E)$.

Our starting point for the remaining part of this work is a description due to W. Kaup - H. Upmeier [KU1] of Aut D (as infinite dimensional Lie group) based on the identification of its Lie algebra with the usual Lie algebra of the in D complete holomorphic vector fields for bounded balanced Banach space domains D. Before stating it, we establish some terminology:

Definition 3. By a vector field v on a subset S of a Banach space E we simply mean an $S \to E$ map. We define the exponential image (denoted by exp(v)) of a vector field $S \to E$ as that (necessarily unique) map F whose domain is the set $S_v \equiv \{f \in S: \exists! \varphi: [0,1] \to S$ differentiable map $\varphi(0) = f$ and $\varphi'(t) = v(\varphi(t)) \quad \forall t \in (0,1)\}$ and satisfies $F(f) =$ $=$ [the value at 1 of the unique $\varphi: [0,1] \to S$ diffeomorphism with $\varphi(0) = f$ and $\varphi'(.) = v(\varphi(.))$] for each $f \in S_v$. If $H \subset S$ and $\mathrm{dom}\, \exp(tv) \supset H \quad \forall t \in \mathbb{R}$ then we shall say that the vector field v is complete in H. If $D \subset E$ (E being a Banach space) then we shall denote the connected component (wrt. the topology of uniform convergence) containing id_D of Aut D by $\mathrm{Aut}_o D$. Further we set $\mathrm{Aut}^o D \equiv \{F \in \mathrm{Aut}\, D: \exists L: E \to E$ linear operator $F = L|_D\}$.

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5) For definitions see e.g. [V1] or [VF1].

We summarize the main results of [KU1] (with J.-P. Vigué's note; cf. [KU1, Remark] and [Vig1, Corollaire]) in the following theorem:

Theorem 6 (Kaup-Upmeier). Let E be a Banach space and D a bounded balanced domain in E. Then

a) $\text{Aut } D = (\text{Aut}^{\circ} D)(\text{Aut}_{\circ} D)$.

b) $\text{Aut}_{\circ} D = \exp \mathcal{P}$ $(\equiv\{\exp(v) : v \in \mathcal{P}\})$ where $\mathcal{P}$ is the family of those vector fields on D that are complete in D.

c) $(\text{Aut } D)\{0\}$ $(\equiv\{F(0) : F \in \text{Aut } D\})$ is the intersection of some (closed) subspace of E with D. The group $\text{Aut}_{\circ} D$ acts transitively on $(\text{Aut } D)\{0\}$.

d) There exists a (unique) conjugate-linear continuous mapping $c \mapsto q_c(.,.)$ from the subspace $\mathbb{C}\cdot(\text{Aut } D)\{0\}$ into the space of the symmetric $E\times E \to E$ bilinear forms (for definition see [VF1]) such that $\mathcal{P}=\{[D \ni f \mapsto c+\ell(f)+q_c(f,f)] : c \in \mathbb{C}\cdot(\text{Aut } D)\{0\}, \ell$ is a continuous in D complete linear vector field on E}. □

Definition 4. We shall write log*Aut D for the set of those holomorphic vector fields on E whose restriction to D is complete in D, whenever E is a Banach space and D is a bounded balanced subdomain of E. It is clear from Theorem 6 b)d) that log*Aut D is an $\mathbb{R}$-linear submanifold of $\{E \to E$ polynomials of second degree$\}$ and furthermore $\text{Aut}_{\circ} D = \{\exp(v|_D) : v \in \text{log*Aut } D\}$.

Lemma 12. Suppose $v \equiv [f \mapsto c+\ell(f)+q(f,f)] \in \log^*\mathrm{Aut}\, D$ where D is a bounded balanced domain in a Banach space E and c,ℓ,q are a constant in E, $E \to E$ linear and $E \times E \to E$ symmetric bilinear form, respectively. Let $f_o \in E$ be arbitrarily fixed and $\varphi(.)$ denote the maximal solution of the initial value problem $\left.\begin{matrix} \frac{d}{dt}x = v(x) \\ x(0) = f_o \end{matrix}\right\}$ and set $\rho \equiv \sup\{1, \|g\| : g \in D\}$.

Then $\mathrm{dom}\,\varphi$ contains the interval $\{t : |t| < \frac{\log[1+\mathrm{dist}(f_o,D)^{-1}]}{\|\ell\| + 2\rho\|q\|}\}$ and for each $g \in D$ we have

$$(11) \quad \|\varphi(t) - \exp(tv)(g)\| < [(1+\|f_o - g\|^{-1})e^{-(\|\ell\|+2\rho\|q\|)|t|} - 1]^{-1}$$
$$\text{whenever } |t| < \frac{\log[1+\|f_o - g\|^{-1}]}{\|\ell\| + 2\rho\|q\|} .$$

Proof. Fix an arbitrary $g \in D$. Write $\psi(t) \equiv \exp(tv)(g)$ (for $t \in \mathbb{R}$; cf. Definition 4) and $\delta(t) \equiv |\varphi(t) - \psi(t)|$ (for $t \in \mathrm{dom}\,\varphi$), respectively. From the definition of the exponential map, it readily follows that $\psi(0) = g$ and $\psi'(t) = v(\psi(t))$ $\forall t \in \mathbb{R}$. It is also well-known from the elementary theory of ordinary differential equations (cf. [D1]) that the function δ is absolutely continuous and admits left-and right-hand-side semi-derivatives, respectively, since it is the composition of $\|.\|$ with a continuously differentiable $\mathbb{R} \to E$ function. Now we have $\delta(t_1) - \delta(t_2) = \|\varphi(t_1) - \psi(t_1)\| - \|\varphi(t_2) - \psi(t_2)\| < \|(\varphi(t_1) - \psi(t_1)) - (\varphi(t_2) - \psi(t_2))\|$ $\forall t_1, t_2 \in \mathrm{dom}\,\varphi$. Hence, for any $t \in \mathrm{dom}\,\varphi$,

$$\frac{d^+}{dt}\delta(t) \equiv \lim_{\tau\downarrow 0} \frac{\delta(t+\tau)-\delta(t)}{\tau} \leq \lim_{\tau\downarrow 0} \left\| \frac{\varphi(t+\tau)-\psi(t+\tau)-\varphi(t)+\psi(t)}{\tau} \right\| =$$

$$= \|v(\varphi(t))-v(\psi(t))\| = \|\ell(\varphi(t)-\psi(t))+q(\varphi(t)+\psi(t),\varphi(t)-\psi(t))\| \leq$$

$$\leq \|\ell\|\,\delta(t)+\|q\|\cdot\|\varphi(t)+\psi(t)\|\,\delta(t)=\delta(t)\,[\,\|\ell\|+\|q\|\cdot\|2\psi(t)+(\varphi(t)-\psi(t))\|\,] \leq$$

$$\leq \delta(t)\,[\,\|\ell\|+\|q\|\,(2\rho+\delta(t))\,] < (\|\ell\|+2\rho\|q\|)\,(\delta(t)+\delta(t)^2).$$

Thus we have $(\delta(t)+\delta(t)^2)^{-1}\frac{d^+}{dt}\delta(t) \leq C \equiv \|\ell\|+2\rho\|q\|$ i.e. $\frac{d^+}{dt}[\log\delta(t)-$ $-(\log(1+\delta(t))] \leq C$ $\forall t \in$ dom . Therefore $\log\frac{\delta(t)}{1+\delta(t)} \leq$ $\leq \log\frac{\delta(0)}{1+\delta(0)} + Ct$ if $t>0$ whence

(11′)
$$\delta(t) \leq [(1+\delta(0)^{-1})\cdot e^{-Ct}-1]^{-1} \quad \text{whenever}$$
$$t>0;\ (1+\delta(0)^{-1})\cdot e^{-Ct}-1>0 \quad \text{and} \quad t \in \operatorname{dom}\varphi .$$

On the other hand, if $\operatorname{dom}\varphi \not\supset \mathbb{R}_+$ then from the maximality of φ we obtain (cf. [D1]) that $\overline{\lim}_{t\uparrow t^*} \|\varphi(t)\| = \infty$ where $t^* \equiv$ $\equiv \sup \operatorname{dom}\varphi (<\infty)$.[6] Thus if $t^*<\infty$ then $\lim_{t\uparrow t^*} \delta(t)=\infty$ since $\delta(t)=$ $= \|\varphi(t)-\psi(t)\| \geq \|\varphi(t)\|-\|\psi(t)\| \geq \|\varphi(t)\|-\rho$ $\forall t \in \operatorname{dom}\varphi$. But now (11′) establishes $\operatorname{dom}\varphi \supset [0,C^{-1}\log(1+\delta(0)^{-1}))$. Similarly, by considering the vector field $-v(\in \log^*\mathrm{Aut}\, D)$, we obtain $\frac{d}{dt}\varphi(-t)=$ $=[-v(\varphi(-t))]$ and $\frac{d}{dt}\psi(-t)=[-v(\psi(-t))]$ $\forall t \in \operatorname{dom}\varphi$ and hence dom $[t\mapsto\varphi(-t)] \supset [0,C^{-1}\log(1+\delta(0)^{-1}))$. Therefore, also dom $(-C^{-1}\log(1+\delta(0)^{-1}),0]$ holds. Then (11) is immediate from (11′) and its application to the field $-v$. The relation $\operatorname{dom}\varphi \supset$ $\supset \{t: |t| < \frac{\log(1+\operatorname{dist}(f_o,D)^{-1})}{\|\ell\|+2\rho\|q\|}\}$ follows from arbitrariness of g in D. □

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[6] For since, given $\rho>\|f_o\|$, the vector field v is Lipschitzian on $2\rho B(E)$ $(\supset \rho\overline{B}(E)\ni f_o)$ and hence $t^*<\infty$ implies $\exists t\in(0,t^*)$ $\varphi(t)\in\partial(\rho B(E))$.

Corollary 3. a) $\operatorname{dom}\exp(tv) \supset \{ f \in E : \operatorname{dist}(f,D) < (\exp|\,|t|\,(\|\ell\|+2\rho\|q\|)\,|-1)^{-1}\} \quad \forall t \in \mathbb{R}$.

b) v is complete in ∂D. Moreover $\exp(tv)(\partial D)=\partial D \quad \forall t \in \mathbb{R}$.

Proof. a) Since the field v is locally Lipschitzian, the maximal solution of $\left.\begin{array}{l}\frac{d}{dt}x = v(x)\\ x(0) = f_o\end{array}\right\}$ is unique. Hence, by definition, $\varphi(t)=\exp(tv)(f_o) \quad \forall t \in \operatorname{dom}\varphi$. Thus $f_o \in \operatorname{dom}\exp(tv)$ iff $|t| < \dfrac{\log(1+\operatorname{dist}(f_o,D)^{-1})}{|\ell|+2\rho|q|}$ $(\forall f_o \in E, t \in \mathbb{R})$, for we have $\operatorname{dom}\varphi \supset$

$$\supset \{t : |t| < \frac{\log(1+\operatorname{dist}(f_o,D)^{-1})}{\|\ell\|+2\rho\|q\|}\}.$$

b) From a) we obtain $\operatorname{dom}\exp(tv) \supset \overline{D} (=\{f \in E : \operatorname{dist}(f,D)=0\})$ $\forall t \in \mathbb{R}$. Fix $f_o \in \partial D$, $t_o \in \mathbb{R}$ arbitrarily and for each $\varepsilon > 0$ choose a vector g_ε in D such that $\|f_o - g_\varepsilon\| < [(\varepsilon^{-1}+1)e^{(\|\ell\|+2\rho\|q\|)|t_o|}-1|^{-1}$. Then (11) implies $\lim_{\varepsilon\downarrow 0}\|\exp(t_o v)(f_o) - \exp(t_o v)(g_\varepsilon)\| = 0$ i.e., by completeness of v in D, $f_o \in \overline{D}$. Hence (since the exponential image of a locally Lipschitzian vector field is clearly one-to-one) $\exp(t_o v)(\partial D) = [\exp(t_o v)(\overline{D})] \setminus [\exp(t_o v)(D)] \subset \overline{D}\setminus D = \partial D \quad \forall t_o \in \mathbb{R}$. On the other hand, if $f \in \partial D$ then $\exp(t_o v)[\exp(-t_o v)(f)] = f$ whence $\partial D \supset \exp(t_o v)(\partial D) \quad \forall t_o \in \mathbb{R}$. □

Remark 1. By the Campbell-Hausdorff formula (see [Hoc1]) the exponential image of a holomorphic vector field restricted to an open set is always holomorphic. Hence Corollary 3 a) yields the following sharpening of [KU1,Corollary]:

Every member of Aut B(E) is the restriction to B(E) of an injective holomorphic map of some <u>spherical</u> neighbourhood of $\overline{B}(E)$ whenever E is a Banach space.

<u>Remark 2.</u> From Corollary 3 b) we see that Aut D = $= \{F|_D : F \in \text{Aut } \overline{D}\}$. Hence Theorem 6 a)b)d) hold also for $\overline{D}$ in place of D. However, $(\text{Aut } \overline{D})\{0\} = (\text{Aut } D)\{0\}$ (thus Theorem 6c) may not be modified).

<u>Lemma 13.</u> Let E,D,v denote a Banach space, a bounded balanced domain in E and a holomorphic vector field on E. Then $v \in \log^*\text{Aut } D$ if and only if v is complete in ∂D.

<u>Proof.</u> The necessity part of the proof is contained in Corollary 3b). Sufficiency: Assume v is complete in ∂D. By Theorem 6b), it suffices to show the completeness of v in D, i.e. that given $f_0 \in D$, the maximal solution φ of $\left.\begin{array}{l}\frac{d}{dt}x = v(x)\\ x(0) = f_0\end{array}\right\}$ is defined on $\mathbb{R}$ and $\varphi(t) \in D \quad \forall t \in \mathbb{R}$. If not, by boundedness of D and maximality of φ, there exists $t_0 \in \mathbb{R}$ such that $\varphi(t_0) \in \partial D$. Observe that, by writing ψ for the maximal solution of $\left.\begin{array}{l}\frac{d}{dt}x = v(x)\\ x(0) = \varphi(t_0)\end{array}\right\}$, we have $\text{dom}\varphi = (\text{dom}\psi) - t_0$ and $\forall t \in \text{dom}\varphi \;\; \varphi(t) = \psi(t-t_0)$. But, by hypothesis, $\text{dom}\,\psi = \mathbb{R}$ and range $\psi \subset \partial D$. This fact contradicts to $\psi(-t_0) = \varphi(0) = f_0 \in D$. □

Proposition 2. Let E be a Banach space, $D \equiv \{f \in E: p(f) < 1\}$ a bounded balanced domain in E where p is a given $E \to \mathbb{R}$ function. Further let $f_o \in \partial D$ and $v \equiv [f \mapsto c + \ell(f) + q(f,f)] \in \log^* \mathrm{Aut}\, D$ where $c \in E$ and $\ell: E \to E, q: E \times E \to E$ denote a (continuous) linear and symmetric bilinear mapping, respectively. Then for each $\phi \in E^*$,

$$(12) \quad \mathrm{Re}\langle \ell(f_o), \phi \rangle = 0 \quad \text{and} \quad \overline{\langle c, \phi \rangle} + \langle q(f_o, f_o), \phi \rangle = 0$$

$$\text{whenever} \quad \mathrm{Re}\,\phi \in \mathrm{subgrad}|_{f_o} p.$$ [7)]

In particular, if D is star-shaped from the point 0 and if $p = $ $= \mathrm{gauge}\, D (\equiv [f \mapsto \inf\{\rho > 0: f \in \rho D\}])$,

$$(12') \quad \mathrm{Re}\langle \ell(f), \phi \rangle = 0 \quad \text{and} \quad [\mathrm{gauge} D(f)]^2 \, \overline{\langle c, \phi \rangle} + \langle q(f,f), \phi \rangle = 0$$

$$\text{whenever} \quad \mathrm{Re}\,\phi \in \mathrm{subgrad}|_{f_o} \mathrm{gauge}\, D.$$

Proof. Let $\vartheta \in \mathbb{R}$ and $\phi \in \{\psi \in E^*: \mathrm{Re}\,\psi \in \mathrm{subgrad}|_{f_o} p\}$ be arbitrarily fixed. Set $\varphi(t) \equiv e^{-i\vartheta} \exp(tv)(e^{i\vartheta} f_o)$ (for $t \in \mathbb{R}$). Now we have $\varphi(0) = f_o$ and $p(\varphi(t) = 1 \quad \forall t \in \mathbb{R}$. Hence $0 = \frac{1}{t}[p(\varphi(t)) - -p(\varphi(0))] \geq \mathrm{Re}\left\langle \frac{\varphi(t) - \varphi(0)}{t}, \phi \right\rangle - o(1)$ and $0 = \frac{1}{t}[p(\varphi(-t)) - p(\varphi(0))] \geq$ $\geq \mathrm{Re}\left\langle \frac{\varphi(-t) - \varphi(0)}{t}, \phi \right\rangle - o(1) \quad \forall t > 0$. By letting $t \downarrow 0$, we obtain

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7) The subgradient of p at the point f_o is defined as the (possibly empty) set of all such real-linear continuous $E \to R$ functionals Λ that satisfy $[p(f_o + v_n) - (p(f_o) + \Lambda(v_n))]^- / \|v_n\| \to 0$ (where $^-$ denotes the negative part operation) for each sequence $v_1, v_2, \ldots, v_n \in E \setminus \{0\}$ tending to 0. It is well-known that (see e.g. [Hol1]) every real-linear $E \to \mathbb{R}$ functional can be represented as $\mathrm{Re}\phi$ for some (unique) $\phi \in E^*$.

$0 \geq \mathrm{Re}\langle e^{-i\vartheta} v(e^{i\vartheta} f_o), \phi\rangle$ and $0 \geq \mathrm{Re}\langle e^{-i\vartheta} v(e^{i\vartheta} f_o), \phi\rangle$. Thus $\mathrm{Re}\langle e^{-i\vartheta} v(e^{i\vartheta} f_o), \phi\rangle = \mathrm{Re}\langle c + e^{i\vartheta}\ell(f_o) + e^{2i\vartheta} q(f_o, f_o), e^{-i\vartheta}\phi\rangle = 0 \quad \forall \vartheta \in \mathbb{R}$. That is, $0 = \mathrm{Re}[e^{-i\vartheta}\langle c, \phi\rangle + \ell(f_o) + e^{i}\langle q(f_o, f_o), \phi\rangle] = \mathrm{Re}[\langle \ell(f_o), \phi\rangle + e^{i\vartheta}(\overline{\langle c, \phi\rangle} + \langle q(f_o, f_o), \phi\rangle)] \quad \forall \vartheta \in \mathbb{R}$. But this is possible only if (12) holds.

If D is star-shaped from O and p = gauge D then easily seen $\mathrm{subgrad}|_f\, p = \mathrm{subgrad}|_{\rho f}\, p \quad \forall \rho > 0 \quad \forall f \in E$ (cf. e.g., for convex D, with [Hol1]). Thus if $f \neq 0$ and $\phi \in \mathrm{subgrad}|_f\, p$ are arbitrarily chosen, we have $p(\frac{f}{p(f)}) = 1$ and $\phi \in \mathrm{subgrad}|_{f/p(f)}\, p$ whence (12) implies $\mathrm{Re}\langle \frac{1}{p(f)}\ell(f), \phi\rangle = \overline{\langle c, \phi\rangle} + \langle \frac{1}{p(f)^2} q(f, f), \phi\rangle = 0$. □

In the special case $D \equiv B(E)$ and $p(.) \equiv \|.\|$ (= gauge B (E) (.)), the set $\{\Phi \in E^* : \mathrm{Re}\phi \in \mathrm{subgrad}|_f\, p\}$ has the familiar expression $\{\phi \in \partial B(E^*) : \langle f, \phi\rangle = \|f\|\}$ (for each $f \neq 0$)[8]. Therefore (12') yields

<u>Corollary 4.</u> If $[f \mapsto c + \ell(f) + q(f, f)] \in \log^* \mathrm{Aut}\, B(E)$ (E, c, ℓ, q as in Proposition 2) then

(12") $\mathrm{Re}\langle \ell(f), \phi\rangle = 0$ and $\|f\|^2\, \overline{\langle c, \phi\rangle} + \langle q(f, f), \phi\rangle = 0$

whenever $\langle f, \phi\rangle = \|f\| \cdot \|\phi\|_*$ $\quad f \in E, \phi \in E^*$. □

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[8] It is shown in [Hol1] that $\mathrm{subgrad}|_f \|.\| = \{\Lambda \in E^{\mathbb{R}*} : \langle f, \Lambda\rangle = \|f\|, \|\Lambda\|_{\mathbb{R}*} = 1\} \quad \forall f \in E \setminus \{0\}$ where $E^{\mathbb{R}*}$ denotes the space of the real-linear $E \to \mathbb{R}$ functionals equipped with the usual norm $\|.\|_{\mathbb{R}*}$: $\Lambda \mapsto \sup\{|\langle f, \Lambda\rangle| : f \in \overline{B}(E)\}$. On the other hand, if $\Lambda \in E^{\mathbb{R}*}$ and $\phi \in E^*$ we have $\Lambda = \mathrm{Re}\phi$ if and only if $\langle f, \phi\rangle = \langle f, \Lambda\rangle - i\langle if, \Lambda\rangle \quad \forall f \in E$ (see also [Hol1]). Hence $\mathrm{subgrad}|_f \|.\| = \{\mathrm{Re}\phi : \phi \in E^*, \|\mathrm{Re}\phi\|_{\mathbb{R}*} = 1, \langle f, \phi\rangle = 1\} \quad \forall f \in E \setminus \{0\}$. But $\|\phi\|_* = \|\mathrm{Re}\phi\|_{\mathbb{R}*} \quad \forall \phi \in E^*$ (see e.g. [Ber1]). Thus $\mathrm{subgrad}|_f \|.\| = \{\mathrm{Re}\phi : \phi \in \partial B(E^*), \langle f, \phi\rangle = \|f\|\}$ for each $f \in E \setminus \{0\}$.

At this point we can return to the unit ball of L^p-spaces:

Theorem 7. Every biholomorphic automorphism of $B(L^p(X,\mu))$ is the restriction to $B(E)$ of an E-unitary linear mapping whenever $p \neq 2,\infty$ and $\dim L^p(X,\mu)>1$.

Proof. Let p,X,μ be such that $p\in[1,\infty)\setminus\{2\}$ and $\dim L^p(X,\mu)>$ >1 and suppose that $[f\mapsto c+\ell(f)+q(f,f)]\in \log^*\mathrm{Aut}\, B(E)$ where $c\in E, \ell:E\to E, q:E\times E\to E$ are a linear and symmetric bilinear map, respectively. By Theorem 6a)d), it suffices to show that we necessarily have $c = 0$.

As usually, let as identify $L^p(X,\mu)^*$ with $L^{p^*}(X,\mu)$ where $p^*\equiv\frac{p}{p-1}$ by defining the pairing operation $\langle\, ,\rangle$ by $\langle f,\phi\rangle\equiv$ $\equiv\int_X f\cdot\phi\, d\mu$ $(f\in L^p(X,\mu), \phi\in L^{p^*}(X,\mu))$. For all $f\in L^p(X,\mu)$, set $f^*\equiv$ $\equiv\bar{f}|f|^{p-2}$. Now we have $\int_X |f^*|^{p^*}d\mu= \int_X |f|^p d\mu<\infty$ whence $f^*\in L^{p^*}(X,\mu)$ and $\langle f,f^*\rangle=\int_X |f|^p d\mu=\|f\|^p=\|f\|\cdot\|f^*\|_*$ $\forall f\in L^p(X,\mu)$.

Let X_1,X_2 be any two disjoint μ-measurable subsets of X with finite positive μ-measure, $\rho\geq 0$ and $\vartheta\in\mathbb{R}$. Consider the function $f\equiv(1_{X_1}+\rho e^{i\vartheta}1_{X_2})^\sim$ $(\equiv\{\varphi\ \mu$-measurable $X\to\mathbb{C}$ function: $\mu(\{x:\varphi(x)\neq 1_{X_1}(x)+\rho e^{i\vartheta}1_{X_2}(x)\})=0\})$. Clearly we have $f\in L^p(X,\mu)$, $\|f\|=[\mu(X_1)+\rho^p\mu(X_2)]^{1/p}$ and $f^*=(1_{X_1}+\rho^{p-1}e^{-i\vartheta}1_{X_2})$. Thus, by writing $\alpha^m_{jk}\equiv\langle q(\tilde{1}_{X_j},\tilde{1}_{X_k}),\tilde{1}_{X_m}\rangle, \mu_j\equiv\mu(X_j)$ and $\gamma_j\equiv\langle c,\tilde{1}_{X_j}\rangle$ (for $j,k,m=$ $= 1,2$), from (12") we obtain

$$[\mu_1+\rho^p\mu_2]^{2/p}(\gamma_1+\rho^{p-1}\cdot\gamma_2)+\alpha^1_{11}+2\alpha^1_{12}\rho e^{i\vartheta}+\alpha^1_{22}\rho^2 e^{2i\vartheta}+$$
$$+\alpha^2_{11}\rho^{p-1}e^{-i\vartheta}+2\alpha^2_{12}\rho^p+\alpha^2_{22}\rho^{p+1}e^{i\vartheta}=0$$

for each $\rho\in\mathbb{R}_+$ and $\vartheta\in\mathbb{R}$. That is

$e^{2i\vartheta}\overset{1}{\alpha}_{22}\rho^2+e^{i\vartheta}\{\overset{2}{\alpha}_{22}\rho^{p+1}+2\overset{1}{\alpha}_{12}\rho+\rho^{p-1}\gamma_2[\mu_1+\rho^p\mu_2]^{2/p}\}+\{\overset{1}{\alpha}_{11}+2\overset{2}{\alpha}_{12}\rho^p+$ $+\gamma_1[\mu_1+\rho^p\mu_2]^{2/p}\}+e^{-i\vartheta}\overset{2}{\alpha}_{11}\rho^{p-1} = 0$ for any $\vartheta\in\mathbb{R}$ whenever $\rho\in\mathbb{R}_+$ is arbitrarily fixed. Therefore

$$0 = \overset{1}{\alpha}_{22}=\overset{2}{\alpha}_{22}\rho^{p+1}+2\overset{1}{\alpha}_{12}\rho+\rho^{p-1}\gamma_2[\mu_1+\rho^p\mu_2]^{2/p} = \overset{1}{\alpha}_{11}+2\overset{2}{\alpha}_{12}\rho^p + +\gamma_1[\mu_1+\rho^p\mu_2]^{2/p} = \overset{2}{\alpha}_{11}= 0 \qquad \forall\rho\in\mathbb{R}_+.$$

In particular, $0 = = \lim_{\rho\uparrow\infty} \frac{1}{\rho^2}\{\overset{1}{\alpha}_{11}+2\overset{2}{\alpha}_{12}\rho^p+\gamma_1[\mu_1+\rho^p\mu_2]^{2/p}\}$. But since $\overset{1}{\alpha}_{11}/\rho^2\to 0$, $\gamma_1[\mu_1+\rho^p\mu_2]^{2/p}/\rho^2\to\gamma_1\mu_2^{2/p}$ $(\rho\uparrow\infty)$ and $p\neq 2$, it follows $\overset{2}{\alpha}_{12}= =\gamma_1 = 0$.

Thus (by definition of γ_1 and by the arbitrariness of the disjoint pair X_1,X_2) we have

(13) $\quad 0 = \langle c,1_{X_1}\rangle = \int_{X_1} c \, d\mu$ whenever $\mu(X_1)<\infty$ and
there exists X_2 such that $X_1\cap X_2=\emptyset$ and $0<\mu(X_2)<\infty$.

Hence we readily obtain that $c=0$. (Indeed: Since dim $L^p(X,\mu)>$ >1, we can fix X_1^0,X_2^0 μ-measurable $\subset X$ so that $X_1^0\cap X_2^0 = \emptyset$ and $0<\mu(X_1^0),\mu(X_2^0)<\infty$. Then (13) implies that $\int_Y c\,d\mu=0$ whenever $0\leq$ $\leq\mu(Y)<0$ and $Y\cap X_1^0= \emptyset$ or $Y\cap X_2^0=\emptyset$. This is sufficient to conclude $\int_Y c\,d\mu= 0$ for each $Y\subset X$ with finite μ-measure.) By Theorem 6d), this means that log*Aut B $(L^p(X,\mu))$ consists only of linear mappings. Therefore any element of $\mathrm{Aut}_0 B(L^p(X,\mu))$ is linear. Now the linearity of Aut B $(L^p(X,\mu))$ is immediate from Theorem 6a). □

Chapter 5

A projection principle

In Proposition 2 we obtained a condition that is suited for explicite calculations when we want to describe log*Aut D (and hence $\mathrm{Aut}_0 D$) in terms of the geometric parameters (like the gauge function) of a bounded balanced Banach space domain D in many cases if ∂D is sufficiently smooth. However, the direct application of Proposition 2 to determine $\mathrm{Aut}_0 D$ seems hopelessly complicated even if the space E (that supports D) is supposed to be finite dimensional and D to have a C^∞-smooth boundary. On the other hand, we have seen the successful application of (12") in the special case of L^p-spaces due to the fact that for disjoint (measurable) subsets X_1, X_2 of the underlying measure space, the gradient of the form at a linear combination of the functions $\tilde{1}_{X_1}, \tilde{1}_{X_2}$ is a well-controllable linear combination of $\mathrm{grad}|_{\tilde{1}_{X_1}} \|\cdot\|$ and $\mathrm{grad}|_{\tilde{1}_{X_2}} \|\cdot\|$. In this chapter we look for the deeper geometrical background of the proof of Theorem 7 in a more general abstract setting.

First of all we need some converse of Proposition 2. It is an interesting problem whether the converse of Proposition 2 holds without any additional condition or even under less restrictive hypothesis than the everywhere non-emptyness of the subgradient of p. We prove here only a weaker version, a slightly generalized form of the converse of Corollary 4:

Lemma 14. Let E be a Banach space, D a from O star shaped bounded balanced domain in E such that the function $p(.)\equiv$gauge D(.)is locally Lipschitzian and admits a non-empty subgradient (cf. footnote [6]) at every point of E, and let $v\equiv$ $\equiv[f\mapsto c+\ell(f)+q(f,f)]$ denote a polynomial vector field of second degree on E (c,ℓ,q as in Proposition 2). Then we have $v\in\log^*\mathrm{Aut}\,D$ if and only if (12') holds.

Proof. $v\in\log^*\mathrm{Aut}\,D \Rightarrow$(12') is contained in Proposition 2. We turn to prove (12')$\Rightarrow \log^*\mathrm{Aut}\,D\ni v$:

Suppose (12') and let us fix $f_0\in\partial D$ arbitrarily. By Lemma 13, it suffices to show that the maximal solution $\varphi(.)$ of the initial value problem $\left.\begin{array}{l}\frac{d}{dt}x = v(x)\\ x(0) = f_0\end{array}\right\}$ is defined on the whole $\mathbb{R}$ and satisfies $\varphi(t)\in\partial D \quad \forall t\in\mathbb{R}$. It is well-known that $\mathrm{dom}\,\varphi\neq\mathbb{R}$ implies the existence of a sequence $t_1,t_2,\ldots\in\mathrm{dom}\,\varphi$ such that $\|\varphi(t_n)\|\to\infty$ (cf. the proof of Lemma 12). Hence, since the domain D is bounded and since $\varphi(0)= f_0\in\partial D$, the statement "$\mathrm{dom}\,\varphi\neq\mathbb{R}$ or $\varphi(t_0)\in\partial D$ for some $t_0\in\mathrm{dom}\,\varphi$" is equivalent to "$\exists t_0'\in\mathrm{dom}\,\varphi \;\; \varphi(t_0')\in\partial D$ but $\forall\varepsilon>0\;\exists t\in\mathrm{dom}\,\varphi\;|t-t_0'|<\varepsilon$ and $\varphi(t)\notin\partial D$". We show that such point t_0' can not exist, by constructing a local solution ψ^* of the initial value problem $\left.\begin{array}{l}\frac{d}{dt}x = v(x)\\ x(t_0')=\varphi(t_0')\end{array}\right\}$ that ranges in ∂D.

Thus consider any $t_0'\in\mathrm{dom}\,\varphi$. We may assume without any loss of generality that $t_0' = 0$. Define the mapping $\tilde{P} : E\setminus\{0\} \to \partial D$ by $\tilde{P}(f) \equiv p(f)^{-1} f$ (for all $f \neq 0$). Observe that $\tilde{P}$ is locally Lipschitzian (for the function p is positive and locally Lipschitzian on $E\setminus\{0\}$) and $\tilde{P}|_{\partial D} = id_{\partial D}$. Let us fix such an $\varepsilon > 0$ that for the closed neighbourhood $U \equiv \{f \in E : \|f - f_0\| < \varepsilon$ we have $\text{Lip}\ \tilde{P}|_U (\equiv \sup\{\frac{\|\tilde{P}(f) - \tilde{P}(g)\|}{\|f-g\|} : f,g \in U\}) < \infty$. Similarly as in the Piccard-Lindelöf Theorem, we introduce the complete metric space (M,d) where $M \equiv \{$continuous $(-\delta,\delta) \to U$ functions$\}$ where $\delta \equiv \min\{(2 \cdot \text{Lip}\ \tilde{P}|_U \cdot \text{Lip}\ v|_U)^{-1}, \varepsilon \cdot (\text{Lip}\ \tilde{P}|_U \cdot \sup\|v(U)\|)^{-1}\}$ and the metric d is defined by $d(\psi_1,\psi_2) \equiv \sup\{\|\psi_1(t) - \psi_2(t)\| : t \in (-\delta,\delta)\}$ (for each $\psi_1,\psi_2 \in M$). Let T denote the transformation of M defined by $T(\psi) \equiv [t \mapsto \tilde{P}(\int_0^t v(\psi(\tau))d\tau + f_0)]$.

We prove that T is a $\frac{1}{2}$ contraction of (M,d): To show range $T \subset M$, let $\psi \in M$ and $t \in (-\delta,\delta)$ arbitrarily fixed. Then $\|\psi(t) - f_0\| = \|\tilde{P}(f_0 + \int_0^t v(\psi(\tau))d\tau) - \tilde{P}(f_0)\| \leq \text{Lip}\ \tilde{P}|_U \cdot \|\int_0^t v(\psi(\tau)d\tau\| \leq$ $\leq \text{Lip}\ \tilde{P}|_U \left|\int_0^t \|v(\psi(\tau))\| d\tau\right| \leq \text{Lip}\ \tilde{P}|_U \cdot |t| \sup\|v(U)\| \leq \text{Lip}\ \tilde{P}|_U \cdot \sup\|v(U)\| \cdot \delta < \varepsilon$, establishing $T(\psi) \in M$. To show the $\frac{1}{2}$-contractive property of T, let $\psi_1,\psi_2 \in M$ and $t \in (-\delta,\delta)$. Now $\|T(\psi_1)(t) - T(\psi_2)(t)\| = \|\tilde{P}(f_0 + \int_0^t v(\psi_1(\tau))d\tau) - \tilde{P}(f_0 + \int_0^t v(\psi_2(\tau))d\tau)\| \leq \text{Lip}\ \tilde{P}|_U \cdot \|\int_0^t [v(\psi_1(\tau)) - v(\psi_2(\tau))]d\tau\| \leq \text{Lip}\ \tilde{P}|_U \cdot \left|\int_0^t \text{Lip}\ v|_U \cdot \|\psi_1(\tau) - \psi_2(\tau)\| d\tau\right| \leq \text{Lip}\ \tilde{P}|_U \cdot \text{Lip}\ v|_U \cdot \delta \cdot d(\psi_1,\psi_2) \leq \frac{1}{2} d(\psi_1,\psi_2)$ whence $d(T(\psi_1),T(\psi_2)) \leq$ $\leq \frac{1}{2} d(\psi_1,\psi_2)$.

Therefore the transformation T admits a unique fixed point $\psi^* (\in M)$. We have $\psi^*(t) = \tilde{P}(f_o + \int_o^t v(\psi^*(\tau))d\tau) \quad \forall t \in (-\delta,\delta)$. Hence range $\psi^* \subset$ range $\tilde{P} = \partial D$. Thus to complete the proof of Lemma 14, it suffices to show that $f_o + \int_o^t v(\psi^*(\tau))d\tau \in \partial D$ or which is the same, $p(f_o + \int_o^t v(\psi^*(\tau))d\tau) = 1 \quad \forall t \in (-\delta,\delta)$.

Since the mapping p is locally Lipschitzian, the function $s: t \mapsto p(f_o + \int_o^t v(\psi^*(\tau))d\tau$ is absolutely continuous. Hence, to prove $s(t) = 1 \ \forall t \in (-\delta,\delta)$, it suffices to see that $s'(t) = 0$ whenever $s'(t)$ exists. Fix $t \in (-\delta,\delta)$ and assume that $s'(t)$ exists. Let us choose any $\phi \in E^*$ with $\mathrm{Re}\,\phi \in \mathrm{subgrad}|_{f_o + \int_o^t v(\psi^*(\tau))d\tau}\, p$. Then for each $\lambda \in (0, \delta - |t|)$,

$$\frac{1}{\lambda}[s(t+\lambda) - s(t)] = \frac{1}{\lambda}[p(f_o + \int_o^{t+\lambda} v(\psi^*(\tau))d\tau) - p(f_o + \int_o^t v(\psi^*(\tau))d\tau)] \geq$$

$$\geq \mathrm{Re}\left\langle \frac{1}{\lambda}[f_o + \int_o^{t+\lambda} v(\psi^*(\tau))d\tau - f_o - \int_o^t v(\psi^*(\tau))d\tau], \phi \right\rangle - o(1) \quad \text{and similarly}$$

$$\frac{1}{(-\lambda)}[s(t-\lambda) - s(t)] \leq \mathrm{Re}\left\langle \frac{1}{(-\lambda)} \int_t^{t-\lambda} v(\psi^*(\tau))d\tau, \phi \right\rangle + o(1).$$

By passing to $\lambda \downarrow 0$, we obtain $s'(t) = \mathrm{Re}\langle v(\psi^*(t)), \phi \rangle$. However, by the homogeneity of p, $\mathrm{subgrad}|_{f_o + \int_o^t v(\psi^*(\tau))d\tau}\, p =$ $= \mathrm{subgrad}|_{\tilde{P}(f_o + \int_o^t v(\psi^*(\tau))d\tau)}\, p = \mathrm{subgrad}|_{\psi^*(t)}\, p$ holds. Therefore $\phi \in \mathrm{subgrad}|_{\psi^*(t)}\, p$, whence $s'(t) = 0$ is immediate from (12') and the fact that $\psi^*(t) \in \partial D$. □

Corollary 5. $v \in \log^*\text{Aut } B(E)$ if and only if (12") holds.

Proof. By the triangle inequality, the norm function ($\equiv$gauge $B(E)$) is Lipschitzian. □

At this point we are prepared to establish the following basic relation between the biholomorphic automorphism groups of Banach space domains and those of their sections with linear subspaces:

Theorem 8. (Projection principle). Let E denote a Banach space, D a bounded balanced domain in E whose gauge function is locally Lipschitzian and has a non-empty subgradient at every point of E. Assume that P is such a continuous projection of E onto a subspace E_1 of E that maps D onto $E_1 \cap D$. Then for all $v \in \log^*\text{Aut } D$ we have $[E_1 \ni f \mapsto Pv(f)] \in \log^*\text{Aut}(E_1 \cap D)$[9]. Moreover, $(\text{Aut}(E_1 \cap D))\{0\} \supset P((\text{Aut } D)\{0\})$.

Proof. Set $p(.) \equiv$gauge $D(.)$. Observe that gauge $E_1 \cap D = p|_{E_1}$ whence the function gauge $D|_{E_1}$ is locally Lipschitzian and has non-empty subradient everywhere on E_1. Therefore, by Lemma 14, $Pv|_{E_1} \in \log^*\text{Aut}(E_1 \cap D)$ if and only if $\text{Re}\langle P\ell(f_1), \phi_1 \rangle = 0$

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9) $E_1 \cap D$ being considered as a domain in E_1.

and $\langle\overline{Pc,\phi_1}\rangle\, p(f_1)^2+\langle Pq(f_1,f_1),\phi_1\rangle = 0$ (where c,ℓ,q stand for the constant, linear and quadratic part of v, respectively; cf. Theorem 6d)) for each $f_1\in E_1$ and $\phi_1\in E_1^*$ with $\mathrm{Re}\phi_1\in \mathrm{subgrad}|_{f_1}(p|_{E_1})$. Thus it suffices to see that $\mathrm{Re}(\phi_1\circ P)\in \mathrm{subgrad}|_{f_1}p$ whenever $\phi_1\in E_1^*$ with $\mathrm{Re}\phi_1\in\mathrm{subgrad}|_{f_1}(p|_{E_1})$ (because this implication directly establishes $\{Pv|_{E_1}: v\in\log^*\mathrm{Aut}\,D\}\subset\log^*\mathrm{Aut}(E_1\cap D)$ and hence Theorem 6c)d) yield $(\mathrm{Aut}(E_1\cap D))\{0\}\supset P((\mathrm{Aut}\,D)\{0\})$.)

Thus let $f_1\in E_1$ and $\phi_1\in E_1^*$ be such that $\mathrm{Re}\phi_1\in\mathrm{subgrad}|_{f_1}(p|_{E_1})$. For any $\varepsilon>0$, denote by U_ε such a neighbourhood of f_1 in E_1 that $p(f_1')-p(f_1)\geq \mathrm{Re}\langle f_1'-f_1,\phi_1\rangle-\varepsilon\|f_1'-f_1\|$ $\forall f_1'\in U_\varepsilon$. Fix an arbitrary $\varepsilon>0$ and consider any $f\in P^{-1}U_\varepsilon$. Since the projection P maps D into itself, $f\in\rho D\Rightarrow Pf\in\rho D$ $\forall\rho>0$. Therefore $p(Pf) =$ $= \inf\{\rho>0: Pf\in\rho D\}\leq\inf\{\rho>0: f\in\rho D\}=p(f)$. Hence $p(f)-p(f_1)=p(f)-$ $-p(Pf_1)\geq p(Pf)-p(Pf_1)\geq\mathrm{Re}\langle P(f-f_1),\phi_1\rangle-\varepsilon\|f-f_1\|\cdot\|P\|=\mathrm{Re}\langle f-f_1,\phi\circ P\rangle-$ $-\varepsilon\|P\|\|f-f_1\|$ $\forall f\in P^{-1}U_\varepsilon$. Since P was supposed to be continuous, the set $P^{-1}U_\varepsilon$ is open in E for all $\varepsilon>0$, establishing $\phi_1\circ P\in$ $\in\mathrm{subgrad}|_{f_1}p$. □

Henceforth we restrict our attention mainly only to the unit ball (or which is essentially the same, to convex bounded balanced domains). This is the most illustrative case with the additional advantages that it enables us a simpler formulation of our statements and it eliminates most of the difficulties of topological and geometric measure theoretic character which one has to face in a more general setting, while, from the geometrical and algebrical view point, it seems to be no loss of generality.

For $D \equiv B(E)$, Theorem 8 reads as follows:

Theorem 8'. If E is a Banach space and $P:E \to E$ is a contractive projection then

(14) $\{Pv|_{PE} : v \in \log^* \mathrm{Aut}\, B(E)\} \subset \log^* \mathrm{Aut}\, B(PE)$ and

$P(\mathrm{Aut}\, B(E)\{0\} \subset \mathrm{Aut}\, B(PE)\{0\}$. □

Corollary 6. If E is a Banach lattice then for any band projection $P:E \to E$, (14) holds. □

Corollary 7. If $f_1, \dots, f_n \in E$ and $\phi_1, \dots, \phi_n \in E^*$ are such that

(15) $\forall (\zeta_1, \dots \zeta_n) \in \mathbb{C}^n \setminus \{0\} \quad \exists (\zeta_1^*, \dots, \zeta_n^*) \in \mathbb{C}^n$

$$\left\langle \sum_{j=1}^{n} \zeta_j f_j, \sum_{j=1}^{n} \zeta_j^* \phi_j \right\rangle = \left\| \sum_{j=1}^{n} \zeta_j f_j \right\| \cdot \left\| \sum_{j=1}^{n} \zeta_j^* \phi_j \right\|_* \neq 0$$

(16) $\langle \mathrm{Aut}\, B(E)\{0\}, \phi_j \rangle \neq \{0\}$ for some $j (\in \{1, \dots, n\})$

then $B(\sum_{j=1}^{n} \mathbb{C} f_j)$ admits a non-linear biholomorphic automorphism.

Proof. From the condition (15) it follows immediately that $f_1, \dots, f_n$ are linearly independent and that $\|\sum_{j=1}^{n} \zeta_j f_j\| \cdot B(E) \cap [\sum_{j=1}^{n} \zeta_j f_j + \bigcap_{j=1}^{n} \{g : \langle g, \phi_j \rangle = 0\}] \subset \|\sum_{j=1}^{n} \zeta_j f_j\| \cdot B(E) \cap [\sum_{j=1}^{n} \zeta_j f_j + \{g : \langle g, \sum_{j=1}^{n} \zeta_j^* \phi_j \rangle = 0\}] = \emptyset$ for some $\zeta_1^*, \dots \zeta_n^*$ in case of any given $(\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n \setminus \{0\}$. Hence

$$(15')\quad B(E)\cap\Big[f+\bigcap_{j=1}^{n}\{g:\langle g,\phi_j\rangle=0\}\Big]=\emptyset \Leftarrow \|f\|=1 \quad \forall f\in\sum_{j=1}^{n}\mathbb{C}f_j .$$

We also obtain from (15) that $\phi_1,\dots,\phi_n$ are linearly independent and that $(\sum_{j=1}^{n}\mathbb{C}f_j)\cap\bigcap_{j=1}^{n}\{g:\langle g,\phi_j\rangle=0\}=\{0\}$ and $E=$ $=(\sum_{j=1}^{n}\mathbb{C}f_j)+\bigcap_{j=1}^{n}\{g:\langle g,\phi_j\rangle=0\}$. Therefore there exists the projection P of E onto $\sum_{j=1}^{n}\mathbb{C}f_j$ along the subspace $\bigcap_{j=1}^{n}\{g:$ $\langle g,\phi_j\rangle=0\}$. From (15′) we see that $P(B(E))\subset B(E)$ i.e. $\|P\|=1$. Thus if (16) holds then, by Theorem 8′, $\mathrm{Aut}\,B(\sum_{j=1}^{n}\mathbb{C}f_j)\{0\}\neq$ $\neq\{0\}=\{F(0): F \text{ is linear}\}$. □

Since in many cases we know the complete description of the biholomorphic automorphism group of finite dimensional convex balanced domains (cf. [Sun1]), Corollary 7 provides us an efficient aid to determine Aut B(E) from the biholomorphic automorphism groups of some finite dimensional sections of B(E).

Example. Thullen's Theorem implies that all the biholomorphic automorphisms of the unit ball of an at least two dimensional L^p-space are linear unless $p=2,\infty$.

Proof. Let $p\neq 2,\infty$ be fixed and let (X,μ) denote a measure space. Set $E\equiv L^p(X,\mu)$. Assume $\dim E>1$ and $\mathrm{Aut}\,B(E)\{0\}\neq\{0\}$. As in the proof of Theorem 7, we identify E^* with $L^{p^*}(X,\mu)$ where $p^*\equiv\frac{p}{p-1}$ and the pairing operation $\langle .,.\rangle$ with $\langle f,\phi\rangle\equiv$ $\equiv\int_X f(x)\phi(x)\,d\mu(x)$ $(\forall f\in E,\phi\in E^*)$, respectively, and introduce the mapping $*:E\to E^*$ defined by $f^*\equiv\bar{f}|f|^{p-2}$.

Then let us fix any element $c \neq 0$ from $\mathrm{Aut}\,B(E)\{0\}$. Since $\dim L^p(X,\mu) = \dim E>1$, we can choose two disjoint subsets X_1, X_2 of X such that $0<\mu(X_1),\mu(X_2)<\infty$ and $\int_X c\, d\mu \neq 0$. Now we have $\forall \zeta_1,\zeta_2 \in \mathbb{C}$ $\sum_{j=1}^{2} \zeta_j \tilde{1}_{X_j} \in E$, $(\sum_{j=1}^{2} \zeta_j \tilde{1}_{X_j})^* = \sum_{j=1}^{2} \bar{\zeta}_j |\zeta_j|^{p-2} \tilde{1}_{X_j} = \sum_{j=1}^{2} \bar{\zeta}_j |\zeta_j|^{p-2} (\tilde{1}_{X_j})^*$ and $\langle \sum_{j=1}^{2} \zeta_j \tilde{1}_{X_j}, (\sum_{j=1}^{2} \zeta_j \tilde{1}_{X_j})^* \rangle = \| \sum_{j=1}^{2} \zeta_j \tilde{1}_{X_j} \| \cdot \cdot \| (\sum_{j=1}^{2} \zeta_j \tilde{1}_{X_j})^* \|_*$. Then Corollary 7 ensures that $\mathrm{Aut}\,B(\mathbb{C}\tilde{1}_{X_1} + \mathbb{C}\tilde{1}_{X_2})\{0\} \neq \{0\}$. Since $\|\zeta_1 \tilde{1}_{X_1} + \zeta_2 \tilde{1}_{X_2}\| = (\sum_{j=1}^{2} \mu(X_j) \cdot |\zeta_j|^p)^{1/p}$ $\forall \zeta_1,\zeta_2 \in \mathbb{C}$, this means that the bounded Reinhardt domain $\{(\zeta_1,\zeta_2) : \mu(X_1)|\zeta_1|^p + \mu(X_2)|\zeta_2|^p < 1\}$ admits a non-linear biholomorphic automorphism. But, according to Thullen's classical theorem [Th1], it is impossible. □

<u>Remark 3.</u> Remembering Vesentini's proof for L^1 [V1], in this context it is natural to ask what is the particular behaviour of the Kobayashi and Carathéodory distances in the situation of the projection principle, or more specifically: what is the relation between c_D, c_{PD}, d_D, d_{PD} (c_U and d_U standing for the Carathéodory and Kobayashi distances associated with a manifold U) if D denotes a domain in some Banach space E and P is such a contractive projection of E that maps D into itself? The answer is simply $c_D|_{PD} = c_{PD}$ and $d_D|_{PD} = d_{PD}$. Proof: Since PD can be considered as a submanifold of D, we directly have $c_D|_{PD} \le c_{PD}$ and $d_D|_{PD} \le d_{PD}$ (cf. [V1,p.42]). On the other

hand, the map $P|_D$ is holomorphic (being linear) whence (see also [V1,p.42]) $c_{PD}(Pf_1,Pf_2) \le c_D(f_1,f_2)$ and $d_{PD}(Pf_1,Pf_2) \le$ $\le d_D(f_1,f_1)$ $\forall f_1,f_2 \in D$. Since $P|_{PD} = id_{PD}$, this latter fact implies $c_{PD} \le c_D|_{PD}$ and $d_{PD} \le d_D|_{PD}$. $\square$ The previous reasoning is a slight generalization of a part of [V1,Lemma 4.3] stating that the mapping $[\Delta \ni \zeta \mapsto \zeta v]$ determines a complex geodesic wrt. both $c_{B(E)}$ and $d_{B(E)}$ whenever E is a Banach space and $v \in \partial B(E)$, moreover $c_{B(E)}|_{\Delta \cdot v} = c_{\Delta \cdot v} = d_{\Delta \cdot v} = d_{B(E)}|_{\Delta \cdot v}$. In fact, since $\Delta \cdot v$ is holomorphically equivalent to Δ and since $d_\Delta = c_\Delta$, we have $c_{\Delta \cdot v} = d_{\Delta \cdot v}$. On the other hand, the Hahn-Banach Theorem establishes the existence of some $\phi \in \partial B(E^*)$ with $\langle v,\phi \rangle = 1$. Now the map $P_v : f \mapsto \langle f,\phi \rangle v$ is a contractive projection, hence $P_v B(E) = \Delta \cdot v$. Thus we can conclude also $c_{B(E)}|_{\Delta \cdot v} = c_{\Delta \cdot v}$ and $d_{B(E)}|_{\Delta \cdot v} = d_{\Delta \cdot v}$. $\square$

A comparison of the two proofs of Theorem 7 reveals the importance of calculating explicitly the values of $\zeta_1^*, \dots \zeta_n^*$ in Corollary 7 in terms of $\zeta_1, \dots, \zeta_n$ and the norm function. This computation can be carried out even in a more general geometric situation:

Lemma 15. Let E be a Banach space, D a from 0 star shaped domain in E, $P : f \mapsto \sum_{j=1}^{n} \langle f,\phi_j \rangle f_j$ (where $f_1, \dots, f_n \in E$ and $\phi_1, \dots, \phi_n \in E^*$ are given) a projection of E that maps D into itself. Set $q \equiv$ gauge D and define $\tilde{q} : \mathbb{R}^{2n} \to \mathbb{R}_+$ by $\tilde{q}(\xi_1, \eta_1, \dots, \xi_n, \eta_n) \equiv$

$\equiv q(\sum_{j=1}^{n}(\xi_j+i\eta_j)f_j)$. Then $\{\phi\in\sum_{j=1}^{n}\mathbb{C}\phi_j : \mathrm{Re}\,\phi\in\mathrm{subgrad}|_g q\}=\{[f\mapsto$

$\mapsto\sum_{j=1}^{n}(\pi_j-i\sigma_j)\langle Pf,\phi_j\rangle] : (\pi_1,\sigma_1,\ldots,\pi_n,\sigma_n)\in\mathrm{subgrad}|_{(\mathrm{Re}\langle g,\phi_1\rangle,}$

$_{\mathrm{Im}\langle g,\phi_1\rangle,\ldots,\mathrm{Re}\langle g,\phi_n\rangle,\mathrm{Im}\langle g,\phi_n\rangle)}\tilde{q}\}\quad \forall g\in\sum_{j=1}^{n}\mathbb{C}f_j$.

<u>Proof.</u> Set $E_1\equiv\sum_{j=1}^{n}\mathbb{C}f_j$. Then $E_1^*=\sum_{j=1}^{n}\mathbb{C}(\phi_j|_{E_1})$. Similarly as at the ending of the proof of Theorem 8, we can see that if $\psi_1\in E_1^*$ and $\mathrm{Re}\,\psi_1\in\mathrm{subgrad}|_{g_1}(q|_{E_1})$ then $\mathrm{Re}(\psi_1\circ P)\in\mathrm{subgrad}|_{g_1} q$ $(\forall g_1\in E_1)$. Thus, since easily seen $\phi\in\sum_{j=1}^{n}\mathbb{C}\phi_j$ iff $\phi=\phi\circ P$, we have $\{\phi\in\sum_{j=1}^{n}\mathbb{C}\phi_j : \mathrm{Re}\,\phi\in\mathrm{subgrad}|_g q\} = \{\psi_1\circ P : \mathrm{Re}\,\psi_1\in\mathrm{subgrad}|_g(q|_{E_1})\}=$
$=\{[f\mapsto\langle Pf,\psi_1\rangle_{E_1}] : \mathrm{Re}\,\psi_1\in\mathrm{subgrad}|_g (q|_{E_1})\}=\{[f\mapsto\Lambda(Pf)-i\Lambda(iPf)] :$
$\Lambda\in\mathrm{subgrad}|_g (q|_{E_1})\}$ $(\forall g\in E_1)$. Hence we can conclude, by remarking that $\Lambda\in\mathrm{subgrad}|_g (q|_{E_1})$ iff there exists $(\pi_1,\sigma_1,\ldots$ $\ldots,\pi_n,\sigma_n)\in\mathrm{subgrad}|_{(\mathrm{Re}\langle g,\phi_1\rangle,\mathrm{Im}\langle g,\phi_1\rangle,\ldots,\mathrm{Re}\langle g,\phi_n\rangle,\mathrm{Im}\langle g,\phi_n\rangle)}\tilde{q}$ such that $\Lambda(\sum_{j=1}^{n}(\xi_j+i\eta_j)f_j)=\sum_{j=1}^{n}(\pi_j\xi_j+\sigma_j\eta_j)\quad\forall\xi_1,\eta_1,\ldots,\xi_n,\eta_n\in$ $\in\mathbb{R}$. □

<u>Corollary 8.</u> If $q(\sum_{j=1}^{n}\zeta_j f_j)=q(\sum_{j=1}^{n}e^{i\vartheta_j}\zeta_j f_j)$ $(\forall\vartheta_1,\ldots,\vartheta_n\in\mathbb{R}$ $\forall\zeta_1,\ldots,\zeta_n\in\mathbb{C})$ then we have $\mathrm{Re}\sum_{j=1}^{n}\zeta_j^*\phi_j\in\mathrm{subgrad}|_{\sum_{j=1}^{n}\zeta_j f_j} q$ if and only if $\mathrm{Re}\sum_{j=1}^{n}e^{-i\vartheta_j}\zeta_j^*\phi_j\in\mathrm{subgrad}|_{\sum_{j=1}^{n}e^{i\vartheta_j}\zeta_j f_j} q$.

Proof. We need only to observe that given $\vartheta_1,\ldots,\vartheta_n \in \mathbb{R}$ and $\Lambda \in \mathrm{subgrad}|_g\, q$ where $g \in E_1$, by defining the linear transformation $Q : \sum_{j=1}^{n} \alpha_j f_j \mapsto \sum_{j=1}^{n} e^{i\vartheta_j} \alpha_j f_j$ of E_1 onto itself we have $q \circ Q = q$ whence the statement $\Lambda \in \mathrm{subgrad}|_g (q|_{E_1})$ is equivalent to $\Lambda \circ Q^{-1} \in \mathrm{subgrad}|_{Qg}\, q\circ Q = \mathrm{subgrad}|_{Qg}\, q$, i.e. $\mathrm{Re}\psi_1 \in \mathrm{subgrad}|_{g_1} (q|_{E_1})$ iff $\mathrm{Re}(\psi_1 \circ Q^1) = (\mathrm{Re}\psi_1)\circ Q^{-1} \in \mathrm{subgrad}|_{Qg}$ $(q|_{E_1})$. □

Taking Lemma 14 into consideration, Lemma 15 and Corollary 8 may have particular interest when $E = \sum_{j=1}^{n} \mathbb{C}\cdot f_j$:

Proposition 3. Suppose $E = \sum_{j=1}^{n} \mathbb{C}\cdot f_j$ where $f_1,\ldots,f_n$ form a base for E and let $\phi_1,\ldots,\phi_n$ denote the dual base of $\{f_1,\ldots,f_n\}$ in E^* (i.e. we have $\langle f_j,\phi_k\rangle = \delta_{jk}$ where $\delta_{jk} \equiv \begin{cases} 0 & \text{if } j\neq k \\ 1 & \text{if } j=k \end{cases}$). Let D denote such a from the 0 star shaped balanced domain in E whose gauge function $\mathbb{r}(.)$ is Lipschitzian[9] and let $v \equiv [f \mapsto c + \ell(f) + q(f,f)]$ be a polynomial vector field on E of second degree. Then $v \in \log^*\mathrm{Aut}\, D$ if and only if

$$(17') \qquad \sum_{j,m=1}^{n} \mathrm{Re}\left[\zeta_j (\pi_m - i\sigma_m) \langle \ell(f_j), \phi_m\rangle\right] = 0$$

$$(17'') \qquad \mathbb{r}\Big(\sum_{j=1}^{n} \zeta_j f_j\Big)^2 \sum_{m=1}^{n} (\pi_m + i\sigma_m) \overline{\langle c, \phi_m\rangle} + {}$$

$$+ \sum_{j,k,m=1}^{n} \zeta_j \zeta_k \cdot (\pi_m - i\sigma_m) \langle q(f_j, f_k), \phi_m\rangle = 0$$

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[9] By locally compactness of E and homogeneity of $\mathbb{r}$, this is equivalent to the locally Lipschitzianity of $\mathbb{r}$.

for any fixed $(\zeta_1,\dots,\zeta_n)\in\mathbb{C}^n$ and for each $(\pi_1,\sigma_1,\dots,\pi_n,\sigma_n)\in$ $\in \text{subgrad}|_{(\text{Re}\zeta_1,\text{Im}\zeta_1,\dots,\text{Re}\zeta_n,\text{Im}\zeta_n)}\tilde{r}$ where $\tilde{r}\equiv$

$$\equiv\left[\mathbb{R}^{2n}\ni(\xi_1,\eta_1,\dots,\xi_n,\eta_n)\mapsto \bar{r}\left(\sum_{j=1}^{n}(\xi_j+i\eta_j)f_j\right)\right].$$

<u>Proof.</u> It is an immediate combination of Lemma 14 and Lemma 15. □

<u>Proposition 4.</u> (hypothesis and notations as in Proposition 3.) If, in addition, $\bar{r}\left(\sum_{j=1}^{n}\zeta_j f_j\right)=\bar{r}\left(\sum_{j=1}^{n}|\zeta_j| f_j\right)\ \forall\zeta_1,\dots,\zeta_n\in\mathbb{C}$ then $v\in\log^*\text{Aut } D$ if and only if the function $p:\mathbb{R}^n\ni(\rho_1,\dots,\rho_n)\mapsto$ $\mapsto\bar{r}\left(\sum_{j=1}^{n}\rho_j f_j\right)$ satisfies

$$(18')\quad \rho_j\pi_m\overline{\langle \ell(f_j),\phi_m\rangle}+\rho_m\pi_j\langle \ell(f_m),\phi_j\rangle=0\quad\text{whenever } j\neq m$$

$$(18'')\quad \text{Re}\sum_{j=1}^{n}\rho_j\pi_j\langle \ell(f_j),\phi_j\rangle=0$$

$$(19')\quad \langle q(f_j,f_k),\phi_m\rangle=0\qquad\text{whenever } m\notin\{j,k\}$$

$$(19'')\quad \pi_m\cdot\left[p(\rho_1,\dots,\rho_n)^2\overline{\langle c,\phi_m\rangle}+\rho_m^2\langle q(f_m,f_m),\phi_m\rangle\right]+$$

$$+2\sum_{\substack{j=1\\ j\neq m}}^{n}\rho_j\rho_m\pi_j\langle q(f_m,f_j),\phi_j\rangle=0\qquad(m=1,\dots,n)$$

for any fixed $(\rho_1,\dots,\rho_n)\in\mathbb{R}^n$ and for each $(\pi_1,\dots,\pi_n)\in$ $\in\text{subgrad}|_{(\rho_1,\dots,\rho_n)}p.$

Proof. Consider any $\rho_1,\vartheta_1,\ldots,\rho_n,\vartheta_n\in\mathbb{R}$ and $(\pi_1,\sigma_1,\ldots,\pi_n,\sigma_n)\in \operatorname{subgrad}|_{(\rho_1\cos\vartheta_1,\rho_1\sin\vartheta_1,\ldots,\rho_n\cos\vartheta_n,\rho_n\sin\vartheta_n)}\tilde{\bar{r}}$. By Corollary 10, there exists $(\pi_1^0,\sigma_1^0,\ldots,\pi_n^0,\sigma_n^0)\in$
$\in \operatorname{subgrad}|_{(\rho_1,0,\rho_2,0,\ldots,\rho_n,0)}\tilde{\bar{r}}$ such that $\sum_{j=1}^{n}(\pi_j-i\sigma_j)\phi_j =$
$= \sum_{j=1}^{n}(\pi_j^0-\sigma_j^0)e^{-i\vartheta_j}\phi_j$ i.e. (since $\phi_1,\ldots,\phi_n$ are linearly independent) $\pi_j-i\sigma_j=(\pi_j^0-i\sigma_j^0)e^{-i\vartheta_j}$ $(j=1,\ldots,n)$. Since $\tilde{q}(\xi_1,\eta_1,\ldots,\xi_n,\eta_n)=p(\sqrt{\xi_1^2+\eta_1^2},\ldots,\sqrt{\xi_n^2+\eta_n^2})$ and since the Fréchet derivative of the mapping $Q:(\xi_1,\eta_1,\ldots,\xi_n,\eta_n)\mapsto$
$\mapsto(\xi_1\frac{\xi_1}{\sqrt{\xi_1^2+\eta_1^2}},\xi_1\frac{\eta_1}{\sqrt{\xi_1^2+\eta_1^2}},\ldots,\xi_n\frac{\xi_n}{\sqrt{\xi_n^2+\eta_n^2}},\xi_n\frac{\eta_n}{\sqrt{\xi_n^2+\eta_n^2}})$ is easily seen $\mathrm{id}_{\mathbb{R}^{2n}}$ at each point of the form $(\rho_1,0,\rho_2,0,\ldots,\rho_n,0)$ (if $\rho_1,\ldots,\rho_n\neq 0$) and since $Q(\rho_1,0,\ldots,\rho_n,0)=(\rho_1,0,\ldots,\rho_n,0)$ $\rho_1,\ldots\rho_n\in\mathbb{R}\setminus\{0\}$,
we have $\operatorname{subgrad}|_{(\rho_1,0,\ldots,\rho_n,0)}\tilde{\bar{r}} = \operatorname{subgrad}|_{(\rho_1,0,\ldots,\rho_n,0)}\tilde{\bar{r}}\circ Q =$
$= \operatorname{subgrad}|_{(\rho_1,0,\ldots,\rho_n,0)}[(\xi_1,\eta_1,\ldots,\xi_n,\eta_n)\mapsto p(|\xi_1|,\ldots,|\xi_n|)]=$
$= \operatorname{subgrad}|_{(\rho_1,0,\ldots,\rho_n,0)}[(\xi_1,\eta_1,\ldots,\xi_n,\eta_n)\mapsto p(\xi_1,\ldots,\xi_n)] =$
$=\{(\pi_1^0,0,\ldots,\pi_n^0,0):(\pi_1^0,\ldots,\pi_n^0)\in\operatorname{subgrad}|_{(\rho_1,\ldots,\rho_n)}p\}$. Therefore
$\{(\pi_1-i\sigma_1,\ldots,\pi_n-i\sigma_n):(\pi_1,\sigma_1,\ldots,\pi_n,\sigma_n)\in\operatorname{subgrad}|_{(\rho_1\cos{}_1,\rho_1\sin{}_1,\ldots)}\tilde{\bar{r}}\}=$
$= \{(\pi_1^0 e^{-i\vartheta_1},\ldots,\pi^0 e^{-i\vartheta_n}):(\pi_1^0,\ldots,\pi_n^0)\in\operatorname{subgrad}|_{(\rho_1,\ldots,\rho_n)}p\}$.
Substituting this expression into (17′) and (17"), we obtain that $v\in\log{}^*\mathrm{Aut}\,D$ if and only if for every fixed $(\rho_1,\ldots,\rho_n)\in\mathbb{R}^n$ and $(\pi_1,\ldots,\pi_n)\in\operatorname{subgrad}|_{(\rho_1,\ldots,\rho_n)}p$,

(17*) $$\sum_{j,m=1}^{n} \rho_j \pi_m \operatorname{Re}[\langle \ell(f_j),\phi_m\rangle e^{i\vartheta_j} e^{-i\vartheta_m}]=0 \quad \forall \vartheta_1,\dots,\vartheta_n \in \mathbb{R}$$

(17**) $$p(\rho_1,\dots,\rho_n)^2 \sum_{m=1}^{n} \pi_m \langle \overline{c,\phi_m}\rangle e^{i\vartheta_m} +$$

$$+ \sum_{j,k,m=1}^{n} \rho_j\rho_k\pi_m \langle q(f_j,f_k),\phi_m\rangle e^{i\vartheta_j} e^{i\vartheta_k} e^{-i\vartheta_m} = 0$$

$$\forall \vartheta_1,\dots,\vartheta_n \in \mathbb{R}.$$

Here the case of (17**) is easy to settle: by multiplying by $e^{i\vartheta_1}\dots e^{i\vartheta_m}$, we see that the polynomial $\mathfrak{p}:(z_1,\dots z_n)\mapsto$ $\mapsto \sum_{m=1}^{n} p(\rho_1,\dots,\rho_n)^2\pi_m\langle c,\phi_m\rangle(z_m z_1\dots z_n) + \sum_{j,k,m=1}^{n}\rho_j\rho_k\pi_m\langle q(f_j,f_k),\phi_m\rangle\cdot$ $\cdot(z_j z_k z_m^{-1} z_1\dots z_n)$ vanishes on $\{(z_1,\dots,z_n):|z_n|=\dots=|z_n|=1\}$ i.e. on the distinguished boundary of the Reinhardt domain Δ^n. Hence (cf. [GF1]) the coefficients of various multipowers of $(z_1,\dots,z_n)$ must also vanish in $\mathfrak{p}$. But $\mathfrak{p}(z_1,\dots,z_n) =$

$$= \sum_{m=1}^{n}[p(\rho_1,\dots,\rho_n)^2\pi_m\langle\overline{c,\phi_m}\rangle+\rho_m^2\pi_m\langle q(f_m,f_m),\phi_m\rangle+2\sum_{\substack{j=1\\ j\neq m}}^{n}\rho_j\rho_m\pi_j\cdot$$

$$\cdot\langle q(f_m,f_j),\phi_j\rangle]z_m z_1\dots z_n+\sum_{(j,k,m):m\notin\{j,k\}}\rho_j\rho_k\pi_m\cdot$$

$\cdot\langle q(f_j,f_k),\phi_m\rangle\, z_j z_k z_m^{-1} z_1\dots z_n$. Now (19") is immediate. To obtain (19') we need only to remark that $\rho_1,\dots,\rho_n$ may be arbitrarily fixed.

To treat (17*), observe that for any fixed $m\in\{1,\dots,n\}$ and $\vartheta_1,\dots,\vartheta_{m-1},\vartheta_{m+1},\dots,\vartheta_n$ we have $0 = \operatorname{Re}[e^{-\vartheta_m}\sum_{\substack{j=1\\ j\neq m}}^{n} e^{i\vartheta_j}$.

$$\cdot\langle \ell(f_j),\phi_m\rangle \rho_j\pi_m + e^{i\vartheta_m}\sum_{\substack{k=1\\k\neq m}}^{n} e^{-i\vartheta_k}\langle \ell(f_m),\phi_k\rangle \rho_m\pi_k]+$$

$$+\ \mathrm{const}(\vartheta_1,\dots,\vartheta_{m-1},\vartheta_{m+1},\dots,\vartheta_n) = \mathrm{const}(\vartheta_1,\dots,\vartheta_{m-1},\vartheta_{m+1},\dots,\vartheta_n)+$$

$$+\mathrm{Re}[e^{-i\vartheta_m}\sum_{\substack{j=1\\j\neq m}}^{n} e^{i\vartheta_j}(\rho_j\pi_m\langle \ell(f_j),\phi_m\rangle+\rho_m\pi_j\overline{\langle \ell(f_m),\phi_j\rangle})] \quad \forall\vartheta_m\in\mathbb{R}.$$

This is possible only if $\sum_{\substack{j=1\\j\neq m}}^{n} e^{i\vartheta_j}(\rho_j\pi_m\langle \ell(f_j),\phi_m\rangle+\rho_m\pi_j\overline{\langle \ell(f_m),\phi_j\rangle})=0$

$\forall m\in\{1,\dots,n\}$ $\forall\,\vartheta_1,\dots,\vartheta_{m-1},\vartheta_{m+1},\dots,\vartheta_n\in\mathbb{R}$. Hence (18"). But

then $0 = \mathrm{Re}\sum_{j,m=1}^{n} e^{-i\vartheta_m} e^{i\vartheta_j}\rho_j\pi_m\langle \ell(f_j),\phi_m\rangle = (\mathrm{Re}\sum_{j=1}^{n}\rho_j\pi_j\langle \ell(f_j),\phi_j\rangle)+$

$$+\sum_{j=1}^{n-1}\sum_{m>j}\mathrm{Re}[e^{i\vartheta_j}e^{-i\vartheta_m}\rho_j\pi_m\langle \ell(f_j),\phi_m\rangle+e^{-i\vartheta_j}e^{i\vartheta_m}\rho_m\pi_j\langle \ell(f_m),\phi_j\rangle]=$$

$= \mathrm{Re}\sum_{j=1}^{n}\rho_j\pi_j\langle \ell(f_j),\phi_j\rangle$ i.e. (18'). □

Chapter 6

Description of Aut B for finite dimensional atomic Banach lattices

Beyond the L^p-spaces, there is an other wide class of Banach lattices where we can exhibit a sufficiently large family of contractive projections with finite rank. These spaces are the atomic Banach lattices. Recall that any atomic Banach lattice E can be represented as a sublattice $\tilde{E}$ of $\{X \to \mathbb{C}$ functions$\}$ for some abstract set X having the property $1_x \in \tilde{E} \quad \forall x \in X$ [11] and endowed with such a norm that assumes the value 1 on each function 1_x $(x \in X)$ (cf. [Sch1,p.143,Ex.7(b)]).

From now on, throughout Chapters 6,7, X will denote an arbitrarily fixed non-empty set, E such a Banach lattice formed by $X \to \mathbb{C}$ functions that satifies $1_x \in E$ and $\|1_x\| = 1 \quad \forall x \in X$. Further we set $E_0 \equiv \mathbb{C} \cdot (\text{Aut}\,B\,(E)\{0\})$ and for every $c \in E_0$, we shall write q_c for that unique (see Theorem 6) symmetric bilinear $E \times E \to E$ mapping which fulfills $[f \mapsto c + q_c(f,f)] \in \log^*\text{Aut}\,B\,(E)$. In the sequel we often shall treat generalized partial differential equations concerning convex functions $p:\mathbb{R}^n \to \mathbb{R}$ of the form

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[11] Without danger of set theoretic paradoxons, we use the notation 1_x to mean the function $1_{\{x\}}$ if $x \in X$.

$$(*)\qquad \varphi(\rho_1,\dots,\rho_n,p(\rho_1,\dots,\rho_n),\ \sum_{j=1}^{n} a_j(\rho_1,\dots,\rho_n)\cdot\pi_j) = 0$$

$$\forall(\pi_1,\dots,\pi_n)\in \text{subgrad}|_{(\rho_1,\dots,\rho_n)}\ p \quad \forall(\rho_1,\dots,\rho_n)\in \mathcal{D}\ .$$

For convenience, we shall abbreviate the statement (*) by the following more suggestive (but less rigorous) form

$$(*')\qquad \varphi(\rho_1,\dots,\rho_n,p,\ \sum_{j=1}^{n}\frac{\partial^* p}{\partial\rho_j}\cdot a_j) = 0 \qquad (\forall(\rho_1,\dots,\rho_n)\in D).$$

Finally, we shall denote the linear functional (in E*) $[f\mapsto f(x)]$ by 1^*_x (for any $x\in X$).

Since for any $Y\subset X$, the mapping $[E\ni f\mapsto 1_Y\cdot f]$ is a band projection and since every $X\to\mathbb{C}$ function of finite support belongs to E, the Projection Principle and Proposition 4 immediately yield.

<u>Proposition 5.</u> Let $v\equiv[f\mapsto c+\ell(f)+q(f,f)]$ be a polynomial vector field of second degree on E. Let $\mathfrak{S}$ denote the set of the finite sequences formed by distinct members of X, and for any $Y\equiv(y_1,\dots,y_N)\in\mathfrak{S}$, set $p_Y\equiv[\mathbb{R}^N\ni(\rho_1,\dots,\rho_N)\mapsto\|\sum_{j=1}^{N}\rho_j 1_{y_j}\|]$. Then $v\in\log^*\mathrm{Aut}\,B(E)$ implies

$$(21')\qquad \rho_j\,\frac{\partial^* p_Y}{\partial\rho_m}\cdot\overline{\langle\ell(1_{y_j}),1^*_{y_m}\rangle}+\rho_m\frac{\partial^* p_Y}{\partial\rho_j}\langle\ell(1_{y_m}),1^*_{y_j}\rangle = 0$$

whenever $j\neq m$,

(21") $\quad \mathrm{Re} \sum_{j=1}^{N} \rho_j \frac{\partial^* p_Y}{\partial \rho_j} \cdot \langle \ell(1_{y_j}), 1^*_{y_j} \rangle = 0$

(21''') $\quad \langle q(1_{y_j}, 1_{y_k}), 1^*_{y_m} \rangle = 0$ whenever $m \notin \{j,k\}$

(21^{IV}) $\quad \frac{\partial^* p_Y}{\partial \rho_m} [p_Y^2 \cdot \overline{\langle c, 1_{y_m} \rangle} + \rho_m^2 \langle q(1_{y_m}, 1_{y_m}), 1^*_{y_m} \rangle] +$

$$+2 \sum_{\substack{j=1 \\ j \neq m}}^{N} \rho_j \rho_m \frac{\partial^* p_Y}{\partial \rho_j} \langle q(1_{y_m}, 1_{y_j}), 1^*_{y_j} \rangle = 0 \quad (m=1,\dots,N)$$

$(\forall (\rho_1,\dots,\rho_n) \in \mathbb{R}^n,\ Y \equiv (y_1,\dots,y_N) \in \mathfrak{Y})$. Moreover, if E is finite dimensional then (21'), (21"), (21'''), (21^{IV}) for each $\rho_1,\dots,\rho_n \in \mathbb{R}$ and $Y \in \mathfrak{Y}$ also implies $v \in \log^* \mathrm{Aut}\, B(E)$. □

The system (21'),...,(21^{IV}) enables us to reconstruct the linear part of log*Aut B (E) and the mapping $c \mapsto q_c$ from the biholomoprhic automorphism groups of the finite dimensional projectional band sections of B(E) if the functions 1_x ($x \in X$) span the whole space E. Namely, if we know Aut B for any <u>three</u> dimensional atomic Banach lattice then, by (21'),...,(21^{IV}), we can dispose (for any given $c \in E_0$) the value of the function $q_c(1_x, 1_y)$ $(: E \to \mathbb{C})$ at any point $z \in X$ by applying the 3 dimensional solution to the equations (21'''), (21^{IV}) in the special case $Y \equiv (y_1, y_2, y_3) \equiv (x,y,z)$. In 1974, T.Sunada [Sun1] described all the possible Lie algebras $\mathcal{L}$ of polynomial vector fields on $\mathbb{C}^n$ that can be considered as log*Aut D for some bounded complete

Reinhardt domain[12] $D\subset\mathbb{C}^n$. He calculated also $\exp(\mathcal{L})$ for these Lie algebras $\mathcal{L}$, however, without having furnished relevant informations concerning the geometric shape of those bounded complete Reinhardt domains that admit a non-linear biholomorphic automorphism. The equations $(21'),\dots,(21^{IV})$ are even linear partial differential equations on the gauge function of the unit ball. Moreover, from Proposition 4 we directly see that Proposition 5 holds without any modifications for all such finite dimensional bounded complete Reinhardt domains $D\subset\{X\to\mathbb{C}$ functions$\}$ (X being finite) whose gauge function is Lipschitzian and has a non-empty subgradient everywhere on E (when replacing B(E) by D and the norm function by gauge D, respectively). So it may have some interest to review the complete finite dimensional solution of the system $(21'),\dots,(21^{IV})$. This will be the subject of the present chapter and we shall consider the general case in Chapter 8. This approach, maybe longer than a directe infinite dimensional treatment, offer the advantage of separating the

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[12] An open subset D of $\mathbb{C}^n$ is called a complete Reinhardt domain if $\forall(\zeta_1,\dots,\zeta_n)\in D\ \{(\eta_1,\dots,\eta_n)\in\mathbb{C}^n:(|\eta_1|,\dots,|\eta_n|)<(|\zeta_1|,\dots\dots,|\zeta_n|)\}\subset D$ (cf. [GF1,p.6,Def.1.8]). If D is a bounded convex complete Reinhardt domain then the normed vector lattice $(\mathbb{C}^n,\|\cdot\|_D)$ where $\|\cdot\|_D\equiv$gauge $D(.)$ (with the usual vector lattice structure of $\mathbb{C}^n$) is an atomic Banach lattice whose open unit ball coincides with D. Conversely, any n-dimensional atomic Banach lattice is isometrically vector lattice isomorphic to one of the spaces $(\mathbb{C}^n,\|\cdot\|_D)$ for some bounded convex complete Reinhardt domain D in $\mathbb{C}^n$.

algebraic and vector space topological considerations. We shall state our results only for the unit ball, however, we remark without proof that a geometric measure theoretical theorem of Stepanoff (see [Fed1,p.218,3.1.9.]) concerning differentiability of $\mathbb{R}^n \to \mathbb{R}$ functions enables us to give such a generalization of Proposition 4 that applies to all bounded finite dimensional complete Reinhardt domains and hence to extend our results to any finite dimensional bounded complete Reinhardt domain.

<u>One dimensional bands</u>

From Proposition 5 we obtain in particular that

$$\mathrm{Re}\left(\frac{\partial^* p_{(y)}}{\partial \rho} \cdot \langle \ell(1_y), 1_y^* \rangle \right) = 0 \qquad \text{and}$$

$$\frac{\partial^* p_{(y)}}{\partial \rho} \cdot \left[p_{(y)}^2 \overline{\langle c, 1_y^* \rangle} + \rho^2 \cdot \langle q(1_y, 1_y) 1_y^* \rangle \right] = 0 \qquad \forall y \in X$$

whenever the polynomial vector field $[f \mapsto c + \ell(f) + q(f,f)]$ belongs to $\log^* \mathrm{Aut}\, B(E)$. But $p_{(y)}(\rho) = \|\rho 1_y\| = \rho \quad \forall \rho \in \mathbb{R}$ whence (being $\frac{\partial^* p_{(y)}}{\partial \rho} = 1$) we have $\forall y \in X \quad \mathrm{Re}\langle \ell(1_y), 1_y^* \rangle = \overline{\langle c, 1_y^* \rangle} + \langle q(1_y, 1_y), 1_y^* \rangle = 0$ if $[f \mapsto c + \ell(f) + q(f,f)] \in \log^* \mathrm{Aut}\, B(E)$. Therefore we can simplify Proposition 5 as follows:

<u>Proposition 5'.</u> (notations as in Proposition 5). Assume E is finite dimensional. Now $\ell \in \log^* \mathrm{Aut}\, B(E)$ if and only if $\mathrm{Re}\langle \ell(1_y), 1_y^* \rangle = 0 \quad \forall y \in X$ and for each $Y \equiv (y_1, \ldots, y_n) \in \mathfrak{I}$ we have

$$(21^*) \qquad \rho_1 \frac{\partial^* p_Y}{\partial \rho_1} \cdot \overline{\langle \ell(1_{y_1}), 1^*_{y_2} \rangle} + \rho_2 \frac{\partial^* p_Y}{\partial \rho_1} \cdot \langle \ell(1_{y_2}), 1^*_{y_1} \rangle = 0.$$

We have $q = q_c$ if and only if $\langle q(1_y, 1_y), 1^*_y \rangle = \overline{-c(y)}$ $\forall y \in X$, $0 = \langle q(1_{y_1}, 1_{y_2}), 1^*_{y_3} \rangle$ whenever $y_3 \notin \{y_1, y_2\}$ and

$$(22^*) \quad \frac{\partial^* p_Y}{\partial \rho_m} \cdot (p_Y^2 - \rho_m^2) \overline{c(y_m)} + 2 \sum_{\substack{j=1 \\ j \neq m}}^{n} \rho_m \rho_j \frac{\partial^* p_Y}{\partial \rho_j} \langle q(1_{y_m}, 1_{y_j}), 1^*_{y_j} \rangle = 0$$

$$(m=1, \dots, N)$$

for each $Y \equiv (y_1, \dots, y_N) \in \mathfrak{S}$. Furthermore $[f \mapsto c + \ell(f) + q(f,f)] \in \log^* \mathrm{Aut}\, B(E)$ if and only if $\ell \in \log^* \mathrm{Aut}\, B(E)$ and $q = q_c$. □

Two dimensional bands

For $Y \equiv (y_1, y_2) \in \mathfrak{S}$ in Proposition 5′, we can provide the complete solution of (21*) and (21**). This fact is of great importance even for the most general case: By resolving (21*) and (21**) for all pairs $(y_1, y_2) \in \mathfrak{S}$, we achieve almost all informations (in view of the relations $q_c(1_x, 1_y)(z) = 0 \Leftarrow$ $\Leftarrow z \notin \{x, y\}$) concerning the functions $\ell(1_x)$, $q_c(1_x, 1_y)$ for any given $x, y \in X$ and $c \in E_o$.

Lemma 16. Let $p: \mathbb{R}^n \to \mathbb{R}_+$ be a lattice norm on $\mathbb{R}^n$. Set $K \equiv \{(\rho_1, \dots, \rho_{n-1}) \in \mathbb{R}^{n-1} : p(\rho_1, \dots, \rho_{n-1}, 0) < 1\}$. Then there exists a unique function $t: K \to \mathbb{R}_+$ such that $p(\rho_1, \dots, \rho_{n-1}, t(\rho_1, \dots, \rho_{n-1})) = 1$ $\forall (\rho_1, \dots, \rho_{n-1}) \in K$. This function t is necessarily concave and satisfies $\text{subgrad}\Big|_{(\rho_1,\dots,\rho_{n-1})}(-t) = \{(\frac{\pi_1}{\pi_n},\dots,\frac{\pi_{n-1}}{\pi_n}):$

$(\pi_1,\dots,\pi_n) \in \text{subgrad}\Big|_{(\lambda\rho_1,\dots,\lambda\rho_{n-1},\lambda t(\rho_1,\dots,\rho_{n-1}))} p\}$

$\forall(\rho_1,\dots,\rho_{n-1}) \in K, \lambda\neq 0.$

Proof. Existence, uniqueness and concaveness of $t(.)$ is trivial. The homogeneity property $p(\lambda\hat{\xi}) = |\lambda|p(\hat{\xi})$ $\forall\hat{\xi}\in\mathbb{R}^n, \lambda\in\mathbb{R}$ readily implies $\text{subgrad}|_{\lambda\hat{\xi}}p = (\text{sgn}\lambda)\cdot\text{subgrad}|_{\hat{\xi}}p$ $\forall\hat{\xi}\in\mathbb{R}^n, \lambda\in\mathbb{R}\setminus\{0\}$. Then let us fix an arbitrary $\hat{\rho}\equiv(\rho_1,\dots,\rho_{n-1})\in K$. Consider any $(\pi_1,\dots,\pi_n)\in\text{subgrad}|_{(\rho_1,\dots,\rho_{n-1}t(\rho_1,\dots,\rho_{n-1}))}p$. Since the function p is convex, we have $0 > p(\rho_1,\dots,\rho_{n-1},0)-1 =$
$= p(\rho_1,\dots,\rho_{n-1},0)-p(\rho_1,\dots,\rho_{n-1},t(\rho_1,\dots,\rho_{n-1})) \geq \pi_n\cdot(-t(\rho_1,\ ,\rho_{n-1}))$.
Hence $\pi_n > 0$, thus the formula for $\text{subgrad}|_{(\rho_1,\dots,\rho_{n-1})}(-t)$ makes sense. Furthermore, by convexity of p and t, $\forall\hat{\eta} \equiv$
$\equiv(\eta_1,\dots,\eta_{n-1})\in\mathbb{R}^{n-1}$ $\quad 0 = \frac{d^+}{d\tau}\Big|_0 p(\hat{\rho}+\tau\hat{\eta},t(\hat{\rho}+\tau\hat{\eta}))$ $(\equiv$

$$\equiv\lim_{\tau\downarrow 0}\frac{p(\hat{\rho}+\tau\hat{\eta},t(\hat{\rho}+\tau\hat{\eta}))-p(\hat{\rho},\hat{\eta})}{\tau}) = \frac{d^+}{d\tau}\Big|_0 p(\hat{\rho}+\tau\hat{\eta},t(\hat{\rho})+\frac{d^+}{d\tau'}\Big|_0 t(\hat{\rho}+\tau'\hat{\eta})) \geq$$

$$\geq\sum_{j=1}^{n-1}\pi_j\eta_j+\pi_n\cdot\frac{d^+}{d\tau'}\Big|_0 t(\hat{\rho}+\tau'\hat{\eta}) \text{ i.e. } \frac{d^+}{d\tau}\Big|_0(-t(\hat{\rho}+\tau\hat{\eta}))\geq\sum_{j=1}^{n-1}\frac{\pi_j}{\pi_n}\eta_j.$$

But therefore $(\frac{\pi_1}{\pi_n},\dots,\frac{\pi_{n-1}}{\pi_n}) \in \text{subgrad}|_{\hat{\rho}}(-t)$.

Now let $\hat{\sigma}\equiv(\sigma_1,\dots,\sigma_{n-1}) \in \text{subgrad}|_{\hat{\rho}}(-t)$. To complete the proof of the lemma, we have to show that for some $(\pi_1,\dots,\pi_n)\in$ $\in \operatorname{subgrad}\big|_{(\hat\rho,t(\hat\rho))} p$ we have $\hat\sigma = (\frac{\pi_1}{\pi_n},\dots,\frac{\pi_{n-1}}{\pi_n})$. Consider any $\hat\eta \equiv (\eta_1,\dots,\eta_{n-1})\in \mathbb{R}^{n-1}$. For some $\varepsilon > 0$, we have $\hat\rho+\tau\hat\eta\in K$ if $|\tau|<\varepsilon$ whence $-t(\hat\rho)+\tau\sum_{j=1}^{n}\sigma_j\eta_j \le -t(\hat\rho+\tau\hat\eta) \le 0$ for $|\tau| < \varepsilon$. That is $1 = p(\hat\rho+$ $+\tau\hat\eta, t(\hat\rho+\tau\hat\eta)) \le p(\hat\rho+\tau\hat\eta, t(\hat\rho)-\tau\sum_{j=1}^{n-1}\sigma_j\eta_j)$ for sufficiently small τ.

But the function $\tau\mapsto p(\hat\rho+\tau\hat\eta, t(\hat\rho)-\tau\sum_{j=1}^{n-1}\sigma_j\eta_j)$ is convex and assumes the value 1 for $\tau = 0$. Hence $1\le p(\hat\rho+\tau\hat\eta, t(\hat\rho)-\tau\sum_{j=1}^{n-1}\eta_j\sigma_j)$ $\forall\tau\in\mathbb{R}$ i.e. (since $\hat\eta\in\mathbb{R}^{n-1}$ is arbitrary) $1\le p(\hat\rho+\hat\eta, t(\hat\rho)-\sum_{j=1}^{n-1}\sigma_j\eta_j)$ $\forall\hat\eta\equiv(\eta_1,\dots,\eta_{n-1})\in\mathbb{R}^{n-1}$. This means that the hyperplane $L \equiv$ $\equiv\{(\hat\rho+\hat\eta, t(\hat\rho)-\sum_{j=1}^{n-1}\sigma_j\eta_j : \hat\eta\equiv(\eta_1,\dots,\eta_{n-1})\in\mathbb{R}^{n-1}\}$ supports the unit ball of p $(\equiv\{\hat\xi\in\mathbb{R}^n : p(\hat\xi)<1\})$ at the point $(\hat\rho, t(\hat\rho))$. Then, by the Hahn-Banach Theorem, there exists $(\pi_1,\dots,\pi_n)\in\mathbb{R}^n$ such that $\sum_{j=1}^{n}\xi_j\pi_j \le p(\hat\xi)$ $\forall\hat\xi\equiv(\xi_1,\dots,\xi_n)\in\mathbb{R}^n$ and $\sum_{j=1}^{n}\xi_j\pi_j = 1$ $\forall\hat\xi\in L$. Since $(\hat\rho, t(\hat\rho))\in L$, hence it follows $(\pi_1,\dots,\pi_n)\in \operatorname{subgrad}\big|_{(\hat\rho,t(\hat\rho))} p$ (cf. footnote [7]). On the other hand $\sum_{j=1}^{n-1}\pi_j\cdot(\rho_j+\eta_j)+\pi_n\cdot(t(\hat\rho)-$ $-\sum_{j=1}^{n-1}\sigma_j\eta_j) = 1$ $\forall\eta_1,\dots,\eta_{n-1}\in\mathbb{R}$ (by definition of L and since $\sum_{j=1}^{n}\xi_j\pi_j = 1$ $\forall\hat\xi\in L$) whence $\pi_j=\pi_n\sigma_j$ $(j=1,\dots,n-1)$ is immediate. □

Corollary 10. Let $\alpha_1,\ldots,\alpha_n \in \mathbb{C}$. The function p satisfies $\alpha_1\rho_1\frac{\partial^* p}{\partial\rho_n} + \alpha_n\rho_n\frac{\partial^* p}{\partial\rho_1} = 0 \quad \forall(\rho_1,\ldots,\rho_n)\in D$ where $D \equiv$

$\equiv\{(\rho_1,\ldots,\rho_n)\in\mathbb{R}^n : p(\rho_1,\ldots,\rho_n)^{-1}\cdot(\rho_1,\ldots,\rho_{n-1})\in K\}$ if and only

if $\alpha_1\rho_1+\alpha_n t\cdot\frac{\partial^*(-t)}{\partial\rho_1} = 0 \quad \forall(\rho_1,\ldots,\rho_{n-1})\in K$. Similarly, we

have $\frac{\partial^* p}{\partial\rho_1}(p^2-\rho_1^2)\cdot\alpha_1+2\rho_1\rho_n\frac{\partial^* p}{\partial\rho_n}\cdot\alpha_2 = 0 \quad \forall(\rho_1,\ldots,\rho_n)\in D$ iff

$\frac{\partial^*(-t)}{\partial\rho_1}(1-\rho_1^2)\cdot\alpha_1+2\rho_1 t\cdot\alpha_n = 0 \quad \forall(\rho_1,\ldots,\rho_{n-1})\in K$ and

$\sum_{j=1}^{n-1}\frac{\partial^* p}{\partial\rho_j}\rho_j\rho_n\cdot\alpha_j+\frac{\partial^* p}{\partial\rho_n}(p^2-\rho_n^2)\cdot\alpha_n = 0 \quad \forall(\rho_1,\ldots,\rho_n)\in D$ iff

$\sum_{j=1}^{n-1}\frac{\partial^*(-t)}{\partial\rho_j}\rho_j t\alpha_j+(1-\rho_n^2)\cdot\alpha_n = 0 \quad \forall(\rho_1,\ldots,\rho_{n-1})\in K$, respectively.

Furthermore $\{(\rho_1,\rho_2,\ldots,\rho_n) : p(\rho_1,\ldots,\rho_n)<1\}=\{(\rho_1,\ldots,\rho_n) : 0 < \rho_n < t(\rho_1,\ldots,\rho_{n-1})$ and $(\rho_1,\ldots,\rho_{n-1})\in K\}$ and $p = \text{gauge}\{(\rho_1,\ldots,\rho_n) : p(\rho_1,\ldots,\rho_n) < 1\}$. □

Lemma 17. Let $p:\mathbb{R}^2\to\mathbb{R}_+$ be a lattice norm on $\mathbb{R}^2$ such that $p(1,0) = p(0,1) = 1$ and let $\alpha_1,\alpha_2\in\mathbb{C}, \alpha_1\neq 0$. Then we have

a) $\alpha_1\rho_1\frac{\partial^* p}{\partial\rho_2}+\alpha_2\rho_2\frac{\partial^* p}{\partial\rho_1} = 0$ if and only if $\alpha_1+\alpha_2 = 0$

and $p(\rho_1,\rho_2) = \sqrt{\rho_1^2+\rho_2^2} \quad \forall\rho_1,\rho_2\in\mathbb{R}$.

b) $\frac{\partial^* p}{\partial\rho_1}(p^2-\rho_1^2)\cdot\alpha_1+2\rho_1\rho_2\frac{\partial^* p}{\partial\rho_2}\alpha_2 = 0$ iff $\frac{\alpha_2}{\alpha_1}\in[-1,0]$

and $p = \text{gauge}\{(\rho_1,\rho_2) : \rho_1^2+|\rho_2|^{-\frac{\alpha_1}{\alpha_2}} < 1\}$[14].

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[14] If $\alpha_2=0$, we define $\{(\rho_1,\rho_2) : \rho_1^2+|\rho_2|^{\infty}<1\}\equiv\{(\rho_1,\rho_2) : |\rho_1|,|\rho_2|<1\}$.

Proof. a) Consider the function $t:(-1,1)\to\mathbb{R}_+$ defined implicitly by $p(\rho,t(\rho))\equiv 1$ $\forall\rho\in(-1,1)$. Lemma 17 ensures that this definition makes sense and it is not ambigous. By Corollary 10, since convex functions are absolutely continuous, we have now $\alpha_1\rho+\alpha_2 t(\rho)[-t'(\rho)]=0$ for almost every $\rho\in(-1,1)$. By integrating, we obtain $\alpha_1\rho^2+\alpha_2[t(0)^2-t(\rho)^2]=0$ i.e. $\alpha_1\rho^2 = =\alpha_2[t(\rho)^2-1]$ $\forall\rho\in(-1,1)$. Hence we see $\alpha_2\neq 0$ (since $\alpha_1\neq 0$). Thus $t(\rho)^2=1+\frac{\alpha_1}{\alpha_2}\rho^2$ $\forall\rho\in(-1,1)$. Since range $t\subset(0,1]$, this is possible only if $\frac{\alpha_1}{\alpha_2}\in[-1,0)$. Moreover, since the equation in a) is symmetric in $(\alpha_1,\rho_1),(\alpha_2,\rho_2)$, we have also $\frac{\alpha_2}{\alpha_1}\in[-1,0)$. Therefore $\frac{\alpha_1}{\alpha_2}=-1$, i.e. $t(\rho)=\sqrt{1-\rho^2}$. Thus $\{(\rho_1,\rho_2):p(\rho_1,\rho_2)<1\}=$ = by Corollary 10 = $\{(\rho_1,\rho_2):\ 0\leq|\rho_1|<1,\ 0\leq|\rho_2|<\sqrt{1-\rho^2}\}$ = $=\{(\rho_1,\rho_2):\rho_1^2+\rho_2^2<1\}$ whence $p(\rho_1,\rho_2)=\sqrt{\rho_1^2+\rho_2^2}$ $\forall\rho_1,\rho_2$. But this function is trivially a lattice norm on $\mathbb{R}^2$ satisfying $\alpha_1\rho_1\frac{\partial^* p}{\partial\rho_2}+\alpha_2\rho_2\frac{\partial^* p}{\partial\rho_1}=0$ whenever $\alpha_1+\alpha_2=0$.

b) Introduce the same function as previously. Again by Corollary 10, $-t'(\rho)\cdot(1-\rho^2)\alpha_1+2\rho t(\rho)\cdot\alpha_2=0$ i.e. $[\log t(\rho)]' = -\frac{\alpha_2}{\alpha_1}[\log(1-\rho^2)]'$ for almost every $\rho\in(-1,1)$. Hence, by integrating, we deduce $\log t(\rho)-\log t(0)=-\frac{\alpha_2}{\alpha_1}\log(1-\rho^2)$ i.e. $\frac{\alpha_2}{\alpha_1}\in\mathbb{R}$ and $t(\rho)=(1-\rho^2)^{-\frac{\alpha_2}{\alpha_1}}$ $\forall\rho\in(-1,1)$. Thus, by Corollary 10, the set $B\equiv\{(\rho_1,\rho_2):p(\rho_1,\rho_2)<1\}$ must have the form $B=\{(\rho_1,\rho_2): |\rho_1|<1,|\rho_2|<(1-\rho_1^2)^{-\frac{\alpha_1}{\alpha_1}}\}=\{(\rho_1,\rho_2):\rho_1^2+|\rho_2|^{-\frac{\alpha_1}{\alpha_2}}<1\}$. Now B is convex (and it is the unit ball of some lattice norm in the same time) iff

$\frac{\alpha_2}{\alpha_1} \in [-1,0]$. For $\alpha_2 = 0$, $p = \text{gauge}\{(\rho_1,\rho_2) : |\rho_1|,|\rho_2|<1\} =$

$= [(\rho_1,\rho_2) \mapsto |\rho_1| \vee |\rho_2|]$ holds which obviously fullfills

$\frac{\partial^* p}{\partial \rho_1}(p^2-\rho_1^2) = 0$. If $\frac{\alpha_2}{\alpha_1} \in [-1,0)$ then it is hard to give a closed formula for $p = \text{gauge}\{(\rho_1,\rho_2) : \rho_1^2 + \rho_2^{-\frac{\alpha_1}{\alpha_2}} < 1\}$. However, since the function $t(:\rho \mapsto (1-\rho^2)^{-\frac{\alpha_2}{\alpha_1}}$ satisfies $-t'(\rho)(1-\rho^2)\alpha_1 +$ $+ 2\rho t(\rho)\alpha_2 = 0 \quad \forall \rho \in (-1,1)$, from Corollary 10 we infer that $\frac{\partial^* p}{\partial \rho_1}(p^2-\rho_1^2)\alpha_1 + 2\rho_1\rho_2 \frac{\partial^* p}{\partial \rho_2}\alpha_2 = 0 \quad \forall (\rho_1,\rho_2) \in D$ where $D \equiv$ $\equiv \{(\rho_1,\rho_2) : p(\rho_1,\rho_2)^{-1} \cdot \rho_1 \in (-1,1)\} = \{\lambda \cdot (\rho_1,\rho_2) : |\rho_1|<1,\ p(\rho_1,\rho_2) = 1,$ $\lambda \neq 0\} = \{\lambda \cdot (\rho_1,\rho_2) : |\rho_1|<1, |\rho_2| = (1-\rho_1^2)^{-\frac{\alpha_2}{\alpha_1}}, \lambda \neq 0\} = \mathbb{R}^2 \setminus (\mathbb{R}\times\{0\})$. Thus, to conclude, we need to verify $\frac{\partial^* p}{\partial \rho_1}(p^2-\rho_1^2) = 0 \quad \forall (\rho_1,\rho_2) \in \mathbb{R}\times\{0\}$. But this is true since $p(\rho,0) = |\rho| p(1,0) = |\rho| \quad \forall \rho \in \mathbb{R}$. □

<u>Proposition 6.</u> Let E be finite dimensional and ℓ a linear vector field on E. $\ell \in \log^*\text{Aut}\, B(E)$ if and only if for each $y_1,y_2 \in X$, $\overline{\langle \ell(1_{y_1}), 1^*_{y_2}\rangle} + \langle \ell(1_{y_2}), 1^*_{y_1}\rangle = 0$ and $\forall f_1 f_2 \in E \ (f_1|_{X\setminus\{y_1,y_2\}} =$ $= f_2|_{X\setminus\{y_1,y_2\}}$ and $\sum_{j=1}^{2} |f_1(y_j)|^2 = \sum_{j=1}^{2} |f_2(y_j)|^2) \Rightarrow \|f_1\| = \|f_2\|$ whenever $y_1 \neq y_2$ and $\langle \ell(1_{y_1}), 1^*_{y_2}\rangle \neq 0$.

<u>Proof.</u> Necessity: Suppose $\ell \in \log^*\text{Aut}\, B(E)$, $y_1,y_2 \in X$. By Proposition 5′, $y_1 = y_2$ implies $0 = \text{Re}\langle \ell(1_{y_1}), 1^*_{y_2}\rangle =$ $= \frac{1}{2}(\overline{\langle \ell(1_{y_1}), 1^*_{y_2}\rangle} + \langle \ell(1_{y_2}), 1^*_{y_1}\rangle)$. Then let $y_1 \neq y_2$ and $\langle \ell(1_{y_1}), 1^*_{y_2}\rangle \neq 0$.

By setting $Y\equiv(y_1,y_2)$, $p\equiv p_Y, \alpha_1\equiv\langle \ell(1_{y_1}),1^*_{y_2}\rangle, \alpha_2\equiv\langle \ell(1_{y_2}),1^*_{y_1}\rangle$, an application to (21*) of Lemma 17a) yields $\overline{\langle \ell(1_{y_1}),1^*_{y_2}\rangle} + {}$ $+\langle \ell(1_{y_2}),1^*_{y_1}\rangle = 0$. Now consider any functions $f_1,f_2\in E$ such that $f_1|_{X\setminus\{y_1,y_2\}} = f_2|_{X\setminus\{y_1,y_2\}}$ and $\rho\equiv\sqrt{\sum_{j=1,2}|f_1(y_j)|^2} = \sqrt{\sum_{j=1,2}|f_2(y_j)|^2}$. Since $\|f\|=\||f|\|$ $\forall f\in E$, we may assume $f_1,f_2\in E_+$. Then, by setting $f_o\equiv 1_{X\setminus\{y_1,y_2\}}f_1$, $\{y_3,\dots,y_n\}\equiv$ $\equiv X\setminus\{y_1,y_2\}$ and $Y\equiv(y_1,\dots,y_n)$, first we see that $f_j =$ $=f_o+\rho\cos\vartheta_j 1_{y_1}+\rho\sin\vartheta_j 1_{y_2}$ $(j=1,2)$ for some $\vartheta_1,\vartheta_2\in\mathbb{R}$. Hence, to conclude this part of the proof, it suffices to show that the function $\varphi:\mathbb{R}\ni\vartheta\longmapsto\|f_o+\rho\cos\vartheta\, 1_{y_1}+\rho\sin\vartheta\, 1_{y_2}\|=$ with notations of Proposition 5 $= p_Y(\rho\cos\vartheta, \rho\sin\vartheta, f_o(y_1),\dots,f_o(y_n))$ is constant. The function p_Y is Lipschitzian, hence φ is also Lipschitzian. Thus it suffices to see that $\varphi'(\vartheta)=0$ whenever $\varphi'(\vartheta)$ exists. Assume $\varphi'(\vartheta_o)$ exists. Then, since p_Y is Lipschitzian, $\varphi'(\vartheta_o) = \frac{d^+}{d\tau}\Big|_o p_Y(\rho\cos\vartheta_o-\tau\rho\sin\vartheta_o, \rho\sin\vartheta_o +$ $+\tau\rho\cos\vartheta_o, f_o(y_3),\dots,f_o(y_n)) = -\frac{d^+}{d\tau}\Big|_o p_Y(\rho\cos\vartheta_o+\tau\rho\sin\vartheta_o,$ $\rho\sin\vartheta_o-\tau\rho\cos\vartheta_o, f_o(y_3),\dots,f_o(y_n))$. Therefore, for any $(\pi_1,\dots,\pi_n)\in \text{subgrad}\big|_{(\rho\cos\vartheta_o,\rho\sin\vartheta_o,f_o(y_3),\dots,f_o(y_n))} p_Y$, we have $\varphi'(\vartheta_o)=\pi_1\cdot(-\rho\sin\vartheta_o)+\pi_2\cdot(\rho\cos\vartheta_o)$. However, from (21*) we obtain $(\rho\cos\vartheta_o)\cdot\pi_2\overline{\langle \ell(1_{y_1}),1^*_{y_2}\rangle}+(\rho\sin\vartheta_o)\pi_1\langle \ell(1_{y_2}),1^*_{y_1}\rangle = 0$ whence $-\pi_1\rho\sin\vartheta_o+\pi_2\rho\cos\vartheta_o = 0$ (for $0\neq\overline{\langle \ell(1_{y_1}),1^*_{y_2}\rangle} =$

$= -\langle \ell(1_{y_2}), 1^*_{y_1}\rangle)$.

Sufficiency: Clearly we have $\mathrm{Re}\,\langle \ell(1_y), 1^*_y\rangle = 0 \quad \forall y \in X$. Therefore, by Proposition 5′, it suffices to prove that given distinct $y_1, \ldots, y_m \in X$ such that $0 \neq \overline{\langle \ell(1_{y_1}), 1^*_{y_2}\rangle}$ (= $= -\langle \ell(1_{y_2}), 1_{y_1}\rangle$ by assumption) we have (21*) or which is the same now, $\rho_1 \frac{\partial^* p_Y}{\partial \rho_2} - \rho_2 \frac{\partial^* p_Y}{\partial \rho_1} = 0$. Fixing arbitrary $\rho, \rho_3, \rho_4, \ldots \rho_m \in \mathbb{R}$ and introducing the function $\varphi: \mathbb{R} \ni \vartheta \mapsto p_Y(\rho\cos\vartheta, \rho\sin\vartheta, \rho_3, \ldots, \rho_m)$, we see that

$$\text{(23)} \qquad 0 = \text{by assumption} = \varphi'(\vartheta_o) = \pi_1(-\rho\sin\vartheta_o) + \pi_2 \cdot (\rho\cos\vartheta_o)$$

$$\forall \vartheta_o \in \mathbb{R}, \quad (\pi_1, \ldots, \pi_m) \in \text{subgrad}\Big|_{(\rho\cos\vartheta_o, \rho\sin\vartheta_o, \rho_3, \, ,\rho_m)} p_Y.$$

Since each $(\rho_1, \rho_2) \in \mathbb{R}$ can be written in the form $(\rho\cos\vartheta_o, \rho\sin\vartheta_o)$ for some $\rho, \vartheta_o \in \mathbb{R}_+$, the statement of (23) is equivalent to

$$\rho_1 \frac{\partial^* p_Y}{\partial \rho_2} - \rho_2 \frac{\partial^* p_Y}{\partial \rho_2} = 0 \qquad (\forall (\rho_1, \ldots, \rho_m) \in \mathbb{R}^m). \square$$

Theorem 10. If $\dim E < \infty$ then there exists a (necessarily unique) partition $\{S_1, \ldots, S_r\}$ of the set X such that the members of $\{\exp(\ell) : \ell \in \log^*\mathrm{Aut}\, B(E), \ell \text{ is linear}\}$ are exactly those linear $E \to E$ mappings that are reduced by the subspaces $\sum_{x \in S_j} \mathbb{C} \cdot 1_x$ $(j=1, \ldots, r)$ [14] and leave invariant the forms $E \ni f \mapsto$

[14] If $E_1, \ldots, E_r$ denote subspaces of E, we say that a linear mapping $L: E \to E$ is reduced by them if $E_j \cap E_k = \{0\}$ whenever $j \neq k$ $(\in \{1, \ldots, r\})$, $E = E_1 + \ldots + E_r$ and $L(E_j) \subset E_j$ $(j=1, \ldots, r)$.

$\mapsto \sum_{x\in S_j} |f(x)|^2 \quad (j=1,\dots,r).$

Proof. Let the binary relation ~ on X defined by

$x \sim y \overset{def}{\Longleftrightarrow} \exists \ell \in \log^*\mathrm{Aut}\,B(E) \cap \{\text{linear } E\to E \text{ maps}\}\ \langle \ell(1_x), 1_y^*\rangle \neq 0.$

From Proposition 6 we directly see the symmetry of ~ . Moreover, Proposition 6 implies also reflexivity of ~ since the field $[E \ni f \mapsto if]$ always belongs to $\log^*\mathrm{Aut}\,B(E)$ (for $\exp[f\mapsto if] = [f \mapsto e^i f]$). Hence the transitive hull $\approx$ of ~ (i.e. the binary relation on X defined by $x \approx y \overset{def}{\Longleftrightarrow} x \sim z_1 \sim \dots \sim z_m \sim y$ for some finite sequence $z_1,\dots,z_m$ in X) is an equivalence on X. Let $S_1,\dots,S_r$ be the equivalence classes in X with respect to $\approx$.

Let $j\in\{1,\dots,r\}$, $x\in S_j$ and $\ell\in\log^*\mathrm{Aut}\,B(E)$ be arbitrarily fixed. Consider any $y\in X\setminus S_j$. Since $x \not\sim y$, we have $\langle \ell(1_x), 1_y^*\rangle = 0$. Therefore $\ell(1_x)\in \sum_{z\in S_j} \mathbb{C}\cdot 1_z$ whence we deduce (by arbitrariness of x in S_j) $\ell(\sum_{z\in S_j}\mathbb{C}\cdot 1_z) \subset \sum_{z\in S_j}\mathbb{C}\cdot 1_z$. Thus (by arbitrariness of j in $\{1,\dots,r\}$) ℓ is reduced by the subspaces $\sum_{z\in S_j}\mathbb{C}\cdot 1_z$ $(k=1,\dots,r)$. It is well-known (cf [Hoc1]) that $\exp(\ell) = [f \mapsto \sum_{n=1}^{\infty} \frac{1}{n} \ell^n(f)\,|$. Hence $\sum_{z\in S_k}\mathbb{C}\cdot 1_z$ $(k=1,\dots,r)$ reduce also $\exp(\ell)$. On the other hand, since we have $\overline{\langle \ell(1_{y_1}), 1_{y_2}^*\rangle} + \langle \ell(1_{y_2}), 1_{y_1}^*\rangle = 0 \quad \forall y_1, y_2 \in X$ (cf. Proposition 6), $i\ell$ can be considered as a self adjoint linear operator in the space $L^2(X)$ [15]. Therefore $\exp(\ell)$ is an $L^2(X)$-unitary operator i.e. it leaves fixed the form $f \mapsto \sum_{y\in X} |f(y)|^2$. However, hence we obtain $\sum_{z\in S_k} |f(z)|^2 = \sum_{z\in X} |(1_{S_k}(f)(z)|^2 =$

$= \sum_{z\in X} |[\exp(\ell)(1_{S_k} f)](z)|^2 =$ since the subspaces $\sum_{z\in S_m} \mathbb{C}1_z$ reduce $\exp(\ell)$ $= \sum_{z\in X} |[1_{S_k} \cdot (\exp(\ell) f)](z)|^2 = \sum_{z\in S_k} |[\exp(\ell) f](z)|^2$ $\forall f\in E$ $(k=1,\ldots,r)$.

It is an easy conseque of the Spectral Decomposition Theorem that any $L^2(X)$-unitary operator U can be written as $U=\exp(iA)$ for some $L^2(X)$-self-adjoint operator A (cf. [DS 1]). That is, to complete the proof, it suffices to show that there exists a function $\varphi:\mathbb{R}^r \to \mathbb{R}_+$ such that $\|f\| = \varphi(\sum_{z\in S_1} |f(z)|^2, \ldots, \sum_{z\in S_r} |f(z)|^2)$ $\forall f\in E$, i.e. we have $\|f_1\| = \|f_2\|$ whenever $\sum_{z\in S_k} |f_1(z)|^2 = \sum_{z\in S_k} |f_2(z)|^2$ $(k=1,\ldots,r;\ f_1, f_2 \in E)$.

Let $k\in\{1,\ldots,r\}$, $f_1, f_2 \in E$ be arbitrarily fixed so that we have $f_1|_{X\setminus S_k} = f_2|_{X\setminus S_2}$ and $\sum_{z\in S_k} |f_1(z)|^2 = \sum_{z\in S_k} |f_2(z)|^2$.

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[15] If Y is any abstract set and $1 \le p < \infty$ then $L^p(Y)$ is defined as the vector space of those $Y \to \mathbb{C}$ functions f that satisfy $\sum_{y\in Y} |f(y)|^p < \infty$ equipped with the norm $f \mapsto \|f\|_{L^p(Y)} \equiv (\sum_{y\in Y} |f(y)|^p)^{1/p}$. $L^\infty(Y) \equiv [\{\text{bounded } Y \to \mathbb{C} \text{ functions}\}, \|f\|_{L^\infty(Y)} \equiv \sup_{y\in Y} |f(y)|]$.

We prove by induction on cardinality$(\{z\in S_k: f_1(z)\neq f_2(z)\})$ that $\|f_1\|=\|f_2\|$. Indeed: If f_1 and f_2 do not coincide only at one point then clearly $|f_1|=|f_2|$ whence $\|f_1\|=\|f_2\|$. Then suppose we have proved $\|h_1\|=\|h_2\| \Leftarrow [h_1|_{X\setminus S_k} = h_2|_{X\setminus S_k}$ & cardinality$(\{z\in S_k: h_1(z)\neq h_2(z)\})\leq n$ & $\sum_{z\in S_k}|h_1(z)|^2 = \sum_{z\in S_k}|h_2(z)|^2]$ $\forall h_1,h_2\in E$ and assume cardinality$(\{z\in S_k: f_1(z)\neq f_2(z)\})=n+1$ where $n\geq 1$ is given. Pick any two points $x_1,x_2\in S_k$ such that $f_1(x_j)\neq f_2(x_j)$ $(j=1,2)$. Since $x_1\approx x_2$, we can find a chain $y_1,y_2,\ldots,y_m$ of distinct elements of S_k such that $x_1=y_1\sim y_2\sim\ldots\sim y_m=x_2$. Now consider the functions $g_s^{(j)}\equiv\left[X\ni z\mapsto\begin{cases} f_j(z) & \text{if } z\neq y_1,\ldots,y_{s+1}\\ 0 & \text{if } z=y_1,\ldots,y_s\\ \sqrt{|f(y_1)|^2+\ldots+|f(y_s)|^2} & \text{if } z=y_{s+1}\end{cases}\right]$

$(j=1,2;\ S=1,\ldots,m-1)$. Observe that, by setting $g_0^{(j)}\equiv f_j$, we have $g_s^{(j)}|_{X\setminus\{y_{s+1},y_{s+2}\}} = g_{s+1}^{(j)}|_{X\setminus\{y_{s+1},y_{s+2}\}}$ and $|g_s^{(j)}(y_{s+1})|^2 + |g_s^{(j)}(y_{s+2})| = |g_{s+1}^{(j)}(y_{s+1})|^2+|g_{s+1}^{(j)}(y_{s+2})|^2$ and $\langle\ell(1_{y_{s+1}}),1^*_{y_{s+2}}\rangle\neq 0$ for some $\ell\in\log^*\mathrm{Aut}\,B(E)$ $(j=1,2;\ s=0,\ldots,m-2)$. Hence, applying Proposition 6, we deduce $\|f_j\|=\|g_0^{(j)}\|=\|g^{(j)}\|=\ldots=\|g_{m-1}^{(j)}\|$ $(j=1,2)$. However, $\{z\in X: g_{m-1}^{(1)}(z)\neq g_{m-1}^{(2)}(z)\}\subset\{x_2\}\cup[\{z\in X: f_1(z)\neq f_2(z)\}\setminus\{y_1,\ldots,y_{m-1}\}]$ i.e. cardinality $\{z\in S_k: g_{m-1}^{(1)}(z)\neq g_{m-1}^{(2)}(z)\}\leq$cardinality $\{z\in S_k\setminus\{x_1\}: f_1(z)\neq f_2(z)\}= n < n+1$ and $g_{m-1}^{(1)}|_{X\setminus S_k} = g_{m-1}^{(2)}|_{X\setminus S_k}$. Thus our inductional hypothesis establishes

$(\|f_1\|=)\,\|g^{(1)}_{m-1}\|=\|g^{(2)}_{m-1}\|\ (=\|f_2\|)$.

Then consider any $h_1, h_2 \in E$ such that $\sum_{z\in S_k} |h_1(z)|^2 =$
$= \sum_{z\in S_k} |h_2(z)|^2$ $(k=1,\dots,r)$. Now, by setting $g_s \equiv \sum_{k=1}^{s} 1_{S_t} h_1 +$
$+ \sum_{k=s+1}^{r} 1_{S_k} h_2$ $(S=1,\dots,r-1)$, we immediately see $\|h_1\| =$
$=\|g_1\|, \|g_1\|=\|g_2\|, \dots, \|g_{r-2}\|=\|g_{r-1}\|, \|g_r\|=\|h_2\|$. □

<u>Corollary 11.</u> There exists a function $\varphi: \mathbb{R}^m_+ \to \mathbb{R}_+$ such that $\|f\|=\varphi\left(\sum_{z\in S_1} |f(z)|^2, \dots, \sum_{z\in S_r} |f(z)|^2\right)$ $\forall f\in E$. □

<u>Corollary 12.</u> There exists a subset $\mathcal{M}_o \subset \{1,\dots,r\}$ such that $E_o = \sum_{z\in \bigcup_{j\in\mathcal{M}_o} S_j} \mathbb{C}\cdot 1_z$.

<u>Proof.</u> First observe that, by Theorem 6c), $E_o=$ $=$ Span (Aut B (E){O}). Then consider any $F\in$Aut B (E), such a $k\in$ $\in\{1,\dots r\}$ that $F(0)\big|_{S_k} \neq 0$ and let $x\in S_k$. Choose any $U\in$ $\in\{L^2(S_k)$-unitary operators$\}$ with the property $U(F(0)\big|_{S_k}) =$ $=\|F(0)\big|_{S_k}\|_{L^2(S_k)} 1_x$ (this can be always done, as it is well-known; cf [Hal2]) and set L_1: $f \mapsto \left[z \mapsto \begin{cases} f(z) & \text{if } z\notin S_k \\ U(f\big|_{S_k})(z) \end{cases}\right]$
and L_2: $f \mapsto \left[z \mapsto \begin{cases} -f(z) & \text{if } z\notin S_k \\ U(f\big|_{S_k})(z) & \text{else} \end{cases}\right]$. Theorem 10 establishes

$L_1, L_2 \in \operatorname{Aut} B(E)$. Thus $1_x = \frac{1}{2} L_1 F(0) + \frac{1}{2} L_2 F(0) \in E_o$. But (by arbitrariness of k and x im S_k), this means that $\{1_x : \exists k \in \{1,\ldots,r\} \exists F \in \operatorname{Aut} B(E) \quad x \in S_k$ and $F(0)|_{S_k} \neq 0\} \subset E_o$. Hence $\sum_{z \in \bigcup_{k \in \mathcal{M}_o} S_k} \mathbb{C} \cdot 1_z = E_o$ is immediate for $\mathcal{M}_o \equiv \{k : \exists F \in \operatorname{Aut} B(E) \ F(0)|_{S_k} \neq 0\}$. □

Next we apply Lemma 17b) to Proposition 5'(22*).

<u>Lemma 18.</u> Suppose $\dim E < \infty$, $c \in E$ and let $q: E \times E \to E$ be a symmetric bilinear map. We have $c \in E_o$ and $q = q_c$ (i.e. $[f \mapsto c + q(f,f)] \in \log^* \operatorname{Aut} B(E)$) if and only if there exists a symmetric matrix $(\gamma^{(c)}_{x,y})_{x,y \in X}$ consisting of numbers belonging to $[0,1]$ such that

$$(24) \qquad q(1_x, 1_y) = -\gamma^{(c)}_{xy} (\overline{c(x)} 1_y + \overline{c(y)} 1_x) \quad \forall x,y \in X$$

$$(25) \qquad B(E) \cap (\mathbb{C} \cdot 1_x + \mathbb{C} \cdot 1_y) = \{\zeta_1 1_x + \zeta_2 1_y : |\zeta_1|^2 + |\zeta_2|^{1/\gamma^{(c)}_{xy}} < 1\} \quad \text{whenever } c(x) \neq 0 \text{ and } x \neq y$$

and such that the functions $p_Y : \mathbb{R}^m \to \mathbb{R}_+$ where $Y \equiv (y_1, \ldots, y_N) \in X^N$ defined by $p_Y(\rho_1, \ldots, \rho_m) \equiv \| \sum_{j=1}^{m} \rho_j 1_{y_j} \|$ satisfy

$$(26) \qquad \frac{\partial^* p_Y}{\partial \rho_N} (p_Y^2 - \rho_N^2) - 2 \sum_{j=1}^{N-1} \gamma^{(c)}_{y_j y_N} \rho_j \rho_N \frac{\partial^* p_Y}{\partial \rho_j} = 0$$

$$\text{whenever } c(y_N) \neq 0 \text{ and } y_j \neq y_k \text{ for } j \neq k.$$

<u>Proof.</u> It is immediate from Proposition 5' that (24) and (26) imply $c \in E_o$ and $q = q_c$.

Necessity of (24), (25), (26): Assume $c \in E_o$. Since we have $\langle q_c(1_x,1_y), 1_z^* \rangle = 0$ whenever $z \notin \{x,y\}$, the function $q_c(1_x,1_y)$ is a linear combination of 1_x and 1_y for any fixed $x,y \in X$. If $x=y$ then $q_c(1_x,1_x) = \langle q_c(1_x,1_x), 1_x^* \rangle 1_x = -\overline{c(x)}\, 1_x = -\frac{1}{2}(\overline{c(x)}\, 1_y + \overline{c(y)}\, 1_x)$. Then let $x \neq y$. Now we have $q_c(1_x,1_y) = \langle q_c(1_y,1_x), 1_x^* \rangle 1_x + \langle q_c(1_x,1_y), 1_y^* \rangle 1_y$. To evaluate $\langle q_c(1_y,1_x), 1_x^* \rangle$ and $\langle q_c(1_x,1_y), 1_y^* \rangle$, we make use of the fact that, by (22*), the function $p: (\rho_1,\rho_2) \to \|\rho_1 1_x + \rho_2 1_y\|$ fullfills

$$(26') \qquad \frac{\partial^* p}{\partial \rho_1}(p^2-\rho_1^2) \cdot \overline{c(x)} + 2\rho_1\rho_2 \frac{\partial^* p}{\partial \rho_2} \langle q_c(1_x,1_y), 1_y^* \rangle = 0 =$$

$$= \frac{\partial^* p}{\partial \rho_2}(p^2-\rho_2^2) \cdot \overline{c(y)} + 2\rho_1\rho_2 \frac{\partial^* p}{\partial \rho_1} \langle q_c(1_y,1_x), 1_x^* \rangle .$$

We distinguish three cases: a) $c(x)=c(y)=0$, b) only one of $c(x)$ and $c(y)$ equals to 0, c) $c(x), c(y) \neq 0$.

a) Now we have $\frac{\partial^* p}{\partial \rho_2} \langle q_c(1_x,1_y), 1_y^* \rangle = 0 = \frac{\partial^* p}{\partial \rho_1} \langle q_c(1_y,1_x), 1_x^* \rangle$. Hence $\langle q_c(1_x,1_y), 1_y^* \rangle = 0 = \langle q_c(1_y,1_x), 1_x^* \rangle$ (i.e. $q_c(1_x,1_y) = 0$), for neither $\frac{\partial^* p}{\partial \rho_1} = 0$ nor $0 = \frac{\partial^* p}{\partial \rho_2}$ hold true.

b) We may assume $c(x) \neq 0 = c(y)$. From the second equation in (26') we see $\langle q_c(1_y,1_x), 1_x^* \rangle = 0$. An application of Lemma 17b) to the first equation of (26') establishes $\langle q_c(1_x,1_y), 1_y^* \rangle / \overline{c(x)} \in [-1,0]$ and $\{(\zeta_1,\zeta_2) : \|\zeta_1 1_x + \zeta_2 1_y\| < 1\} = \{(\zeta_1,\zeta_2) : |\zeta_1|^2 + |\zeta_2|^{-\overline{c(x)}/\langle q_c(1_y,1_x), 1_x^* \rangle} < 1\}$.

c) Lemma 17b) implies that $\langle q_c(1_x,1_y), 1_y^* \rangle / \overline{c(x)}$,

$\langle q_c(1_y,1_x),1_x^*\rangle/\overline{c(y)} \in [-1,0]$ and $\{(\zeta_1,\zeta_2):\|\zeta_1 1_x+\zeta_2 1_y\|<1\} =$

$=\{(\zeta_1,\zeta_2):|\zeta_1|^2+|\zeta_2|^{-\overline{c(x)}/\langle q_c(1_x,1_y),1_y^*\rangle}<1\} = \{(\zeta_1,\zeta_2):$

$|\zeta_1|^{-\overline{c(y)}/\langle q_c(1_y,1_x),1_x^*\rangle}+|\zeta_2|^2<1\}$. This is possible only if

$\langle q_c(1_x,1_y),1_y^*\rangle/\overline{c(x)} = \langle q_c(1_y,1_x),1_x^*\rangle/\overline{c(y)} = [\frac{1}{2}$ or $0]$. □

As a directe consequence, hence we readily obtain

<u>Proposition 7.</u> Assume dim $E<\infty$, let $\{S_m : m\in\mathcal{M}\}$ denote that partition of X which satisfies {linear elements of $\mathrm{Aut}_o B(E)\}=$ $=\{U|_{B(E)} : U\in\{L^2(X)\text{-isometries}\}$ and $U(\sum_{x\in S_m}\mathbb{C}1_x)=\sum_{x\in S_m}\mathbb{C}1_x \quad \forall m\in\mathcal{M}\}$ (cf. Theorem 10) and set $\mathcal{M}_o\equiv\{m\in\mathcal{M}: \sum_{x\in S_m}\mathbb{C}1_x\subset E_o\}$ (cf. Corollary 12). Then there exists a unique matrix $\Gamma\equiv(\gamma_{mn})_{m,n\in\mathcal{M}}$ such that

(24*) $\qquad q_c(1_x,1_y) = -\gamma_{mn}(\overline{c(x)}1_y+\overline{c(y)}1_x)$

whenever $c\in E_o, x\in S_m, y\in S_n$

(24**) $\qquad \gamma_{mn}=0 \quad$ whenever $m,n\notin\mathcal{M}_o$.

<u>Proof.</u> In view of Lemma 18 (and its proof), it suffices to show only that

i) $\langle q_{c_1}(1_{x_1},1_{y_1}),1_{y_1}^*\rangle/\overline{c_1(x_1)}$ $(=\gamma_{x_1y_1}^{(c_1)}$ in Lemma 18$)$ $=$ $=\langle q_{c_2}(1_{x_2},1_{y_2}),1_{y_2}^*\rangle/\overline{c_2(x_2)}$ whenever $c_1,c_2\in E_o$ are such that $c_1(x_1),c_2(x_2)\neq 0$ and $\exists m,n\in\mathcal{M}\ m\neq n;\ x_1,x_2\in S_m;\ y_1,y_2\in S_n$,

ii) $\langle q_c(1_x,1_y),1_y^*\rangle/\overline{c(x)} = -\frac{1}{2}$ whenever $c\in E_o, c(x)\neq 0$ and $\exists m\in\mathcal{M}_o\ y\neq x, y\in S_m$.

Ad i) By (25) and (24) we have $\{(\zeta_1,\zeta_2):\|\zeta_1 1_{x_j}+\zeta_2 1_{y_j}\|<1\}=$ $=\{(\zeta_1,\zeta_2):|\zeta_1|^2+|\zeta_2|^{-c_j(x_j)/\langle q_{c_j}(1_{x_j},1_{y_j}),1^*_{y_j}\rangle}<1\}$ $(j=1,2)$. On the other hand, by Theorem 10, there exists an $L^2(X)$-unitary operator U such that $U|_{B(E)}\in \mathrm{Aut}_o B(E)$ (i.e. U is also E-unitary) and $U(1_{x_1})=1_{x_2}, U(1_{y_1})=1_{y_2}$. Then $\{(\zeta_1,\zeta_2):\|\zeta_1 1_{x_2}+\zeta_2 1_{y_2}\|<1\}=$ $=\{(\zeta_1,\zeta_2):\|\zeta_1 U(1_{x_1})+\zeta_2 U(1_{y_2})\|<1\}=\{(\zeta_1,\zeta_2):\|U(\zeta_1 1_{x_1}+\zeta_2 1_{y_1})\|<1\}=$ $=\{(\zeta_1,\zeta_2):\|\zeta_1 1_{x_1}+\zeta_2 1_{y_1}\|<1\}$. Hence $\langle q_{c_1}(1_{x_1},1_{y_1}),1^*_{y_1}\rangle/\overline{c_1(x_1)}=$ $=\langle q_{c_2}(1_{x_2},1_{y_2}),1^*_{y_2}\rangle/\overline{c_2(x_2)}$.

Ad ii) $\{(\zeta_1,\zeta_2):|\zeta_1|^2+|\zeta_2|^{-\overline{c(x)}/\langle q_c(1_x,1_y),1^*_y\rangle}<1\}=\{(\zeta_1,\zeta_2):$ $\|\zeta_1 1_x+\zeta_2 1_y\|<1\}=$ by Corollary 13 $=\{(\zeta_1,\zeta_2):|\zeta_1|^2+|\zeta_2|^2<1\}$. □

<u>Corollary 13.</u> The matrix Γ is necessarily symmetric and has the properties $\gamma_{mm}=\frac{1}{2}, \gamma_{mn}\in[0,1]$ for all $m\in\mathcal{M}_o, n\in\mathcal{M}$ and $\gamma_{m_1m_2}\in$ $\in\{0,\frac{1}{2}\}$ if $m_1,m_2\in\mathcal{M}_o$. Further $B(E)\cap(\mathbb{C}1_x+\mathbb{C}1_y)=\{\zeta_1 1_x+\zeta_2 1_y:$ $|\zeta_1|^2+|\zeta_2|^{1/\gamma_{mn}}<1\}$ whenever $x\in S_m, y\in S_n, x\neq y, m\in\mathcal{M}_o$ and $n\in\mathcal{M}$. If $y_1\in S_{m_1},\dots,y_N\in S_{m_n}$ are distinct element of X, $y_N\in X_o$ and p denotes the function $R^N\ni(\rho_1,\dots,\rho_N)\mapsto\|\sum_{j=1}^{N}\rho_j 1_{y_j}\|$ then (by Lemma 18)

$$(26^*)\qquad \frac{\partial^* p}{\partial\rho_N}(p^2-\rho_N^2)-2\sum_{j=1}^{N-1}\gamma_{m_jm_N}\rho_j\rho_N\frac{\partial^* p}{\partial\rho_j}=0. \square$$

Three dimensional bands

Henceforth, throughout the remaining part of this chapter, we assume $\dim E<\infty$. We set $E_S \equiv \sum_{x\in S} \mathbb{C} 1_x$ (for $S\subset X$) and reserve the notations $\{S_m : m\in\mathcal{M}\}$, $\mathcal{M}_o$, $\Gamma\equiv(\gamma_{mn})_{m,n\in\mathcal{M}}$ for the partition of X satisfying $\{\exp(\ell):\ell \text{ linear} \in \log^*\mathrm{Aut}\, B(E)\} = \{U\in\{L^2(X)\text{-isometries}\}: U(E_{S_m})=E_{S_m} \ \forall m\in\mathcal{M}\}$, for the index set $\mathcal{M}_o\equiv\{m\in\mathcal{M}: E_{S_m}\subset E_o\}$ and for the matrix satisfying (24*) and (24**), respectively.

It can be conjectured that $\gamma_{mn}=0$ if m and n are distinct members of $\mathcal{M}_o$. To prove this, it turns out that it suffices to examine the solutions of (26) for $Y\equiv(y_1,y_2,y_3)$ with $y_1\in S_m, y_2\in S_n$ and y_3 arbitrary $\in X\setminus\{y_1,y_2\}$. The investigation of the three dimensional projection bands of E may have a further interest: Here begin to appear such problems concerning the geometric shape of a bounded complete Reinhardt domain having non-linear biholomorphic automorphisms that can not be treated directly by using the methods applied in [Sun1].

Lemma 19. Let $p:\mathbb{R}^n\to\mathbb{R}_+$ be a lattice norm on $\mathbb{R}^n$ such that

$$(27)\qquad 2\sum_{j=1}^{n-1} \alpha_j\rho_j\rho_n \frac{\partial^* p}{\partial\rho_j} = \frac{\partial^* p}{\partial\rho_n}\cdot(p^2-\rho_n^2)$$

where $\alpha_1,\dots,\alpha_s>0=\alpha_{s+1}=\dots=\alpha_{n-1}$ and $s>1$. Set $K\equiv\{(\rho_1,\dots\rho_{n-1})\in \mathbb{R}^{n-1}: p(\rho_1,\dots,\rho_{n-1},0)<1\}$ and let $v:K\to\mathbb{R}_+$ denote that (trivially unique) function which fullfills $p(\rho_1,\dots,\rho_{n-1},v(\rho_1,\dots,\rho_{n-1}))=1$

$\forall(\rho_1,\ldots,\rho_{n-1})\in K$. Then there exists a function $\psi:\mathbb{R}^{n-2}\to\mathbb{R}_+$ such that

$$(28)\qquad 1-v(\rho_1,\ldots,\rho_{n-1})^2=\psi\left(\frac{\log\rho_1}{\alpha_1}-\frac{\log\rho_s}{\alpha_s},\ldots,\frac{\log\rho_{s-1}}{\alpha_{s-1}}-\frac{\log\rho_s}{\alpha_s},\rho_{s+1},\ldots,\rho_{n-1}\right)\cdot\rho_s^{1/\alpha_s}$$

for all $0<(\rho_1,\ldots,\rho_{n-1})\in K$.

<u>Proof.</u> Clearly, the function $v(.)$ is concave and decreasing on $K\cap\mathbb{R}_+^{n-1}$ (i.e. $v(\rho_1,\ldots,\rho_{n-1})>v(\rho_1',\ldots,\rho_{n-1}')$ whenever $0<(\rho_1,\ldots,\rho_{n-1})<(\rho_1',\ldots,\rho_{n-1}')\in K$). Furthermore, Corollary 10 implies

$$(27')\qquad 2\sum_{j=1}^{n-1}\frac{\partial*(-v)}{\partial\rho_j}\cdot\alpha_j\rho_j v = 1-v^2\qquad \forall(\rho_1,\ldots,\rho_{n-1})\in K.$$

Since the sets $K_{\xi_1,\ldots,\xi_{n-1}}\equiv\{(\rho_1,\ldots,\rho_{n-1})\in K\cap(\mathbb{R}_+^{n-1})^o :$ $\frac{\log\rho_j}{\alpha_j}-\frac{\log\rho_s}{\alpha_s}=\xi_j$ for $j<s$ and $\rho_j=\xi_j$ for $j>s\}$ are obviously pairwise disjoint, to prove (28) it suffices to show that for every $(\xi_1,\ldots,\xi_{n-1})\in\mathbb{R}^{n-1}$ the function $(1-v^2)\rho_s^{-1/\alpha_s}$ is constant over the set $K_{\xi_1,\ldots,\xi_{n-1}}$. Observe that any point of a fixed $K_{\xi_1,\ldots,\xi_{n-1}}$ has the form $(e^{\alpha_1\xi_1}\tau^{\alpha_1/\alpha_s},\ldots,e^{\alpha_{s-1}\xi_{s-1}}\tau^{\alpha_{s-1}/\alpha_s},\tau,$ $\xi_{s+1},\ldots,\xi_{n-1})$ for some $\tau\in\mathbb{R}_+$ (for if $(\rho_1,\ldots,\rho_{n-1})\in K_{\xi_1,\ldots,\xi_{n-1}}$

then by setting $\tau\equiv\rho_s$ we have $\rho_j=\exp[\alpha_j(\frac{\log\tau}{\alpha_s}+\xi_j)]$ for $j=1,\ldots,s-1$).

Since $\alpha_1,\ldots,\alpha_s>0$, the functions $\tau\mapsto e^{\alpha_j\xi_j}\tau^{\alpha_j/\alpha_s}$ $(j=1,\ldots,s-1)$ are increasing. Hence, from the fact $K_{\xi_1,\ldots,\xi_{n-1}}=\{(e^{\alpha_1\xi_1}\tau^{\alpha_1/\alpha_s},\ldots$
$\ldots,e^{\alpha_{s-1}\xi_{s-1}}\tau^{\alpha_{s-1}/\alpha_s},\tau,\xi_{s+1},\ldots,\xi_{n-1}):p(e^{\alpha_1\xi_1}\tau^{\alpha_1/\alpha_s},\ldots$
$\ldots,e^{\alpha_{s-1}\xi_{s-1}}\tau^{\alpha_{s-1}/\alpha_s},\tau,\xi_{s+1},\ldots,\xi_{n-1})<1\}$ and the increasing property of p, we deduce $\forall\xi_1,\ldots,\xi_{n-1}\in\mathbb{R}\ \exists\tau^*_{\xi_1,\ldots,\xi_{n-1}}>0$
$K_{\xi_1,\ldots,\xi_{n-1}}=\{(e^{\alpha_1\xi_1}\tau^{\alpha_1/\alpha_s},\ldots,e^{\alpha_{s-1}\xi_{s-1}}\cdot\tau^{\alpha_{s-1}/\alpha_s}\cdot\tau,\xi_{s+1},\ldots,\xi_{n-1}):$

$\tau\in(0,\tau^*_{\xi_1,\ldots,\xi_{n-1}})\}$. Therefore we have to see that the functions $\tau\to(1-v_{\xi_1,\ldots,\xi_{n-1}}(\tau)^2)\tau^{-1/\alpha_s}$ where $v_{\xi_1,\ldots,\xi_{n-1}}\equiv[(0,\tau^*_{\xi_1,\ldots,\xi_{n-1}})\ni$
$\ni\tau\mapsto v(e^{\alpha_1\xi_1}\tau^{\alpha_1/\alpha_s},\ldots,e^{\alpha_{s-1}\xi_{s-1}}\tau^{\alpha_{s-1}/\alpha_s},\tau,\xi_{s+1},\ldots,\xi_{n-1})]$ are constant.

Let $\xi_1,\ldots,\xi_{n-1}\in\mathbb{R}$ and $\tau\in(0,\tau^*_{\xi_1,\ldots,\xi_{n-1}})$ be arbitrarily fixed. Since v(.) is concave, $v_{\xi_1,\ldots,\xi_{n-1}}$ is locally Lipschitzian and admits left-and right hand side derivatives, respectively, everywhere on $(0,\tau^*_{\xi_1,\ldots,\xi_{n-1}})$. Therefore we have $\frac{d^+}{d\tau'}\Big|_\tau v_{\xi_1,\ldots,\xi_{n-1}}(\tau')=$
$=\sum_{j=1}^{s-1}\pi_j\frac{d}{d\tau'}e^{\alpha_j\xi_j}\tau'^{\alpha_j/\alpha_s}+\pi_s$ for some suitable $(\pi_1,\ldots,\pi_{n-1})$

$$(-1)\,\mathrm{subgrad}\Big|_{(e^{\alpha_1\xi_1}\tau^{\alpha_1/\alpha_s},\ldots,e^{\alpha_{s-1}}\tau^{\alpha_{s-1}/\alpha_s},\tau,\xi_{s+1},\ldots,\xi_{n-1})}(-v).$$

Thus $\tau' \mapsto (1-v_{\xi_1,\dots,\xi_{n-1}}(\tau')^2)\tau'^{-1/\alpha_{\bar{s}}}$ is locally Lipschitzian and

$$\frac{d^+}{d\tau'}\Big|_\tau [(1-v_{\xi_1,\dots,\xi_{n-1}}(\tau')^2)\tau'^{-1/\alpha_{\bar{s}}}] = -\frac{1}{\alpha_{\bar{s}}}\tau^{(-1/\alpha_{\bar{s}})-1} +$$

$$+\frac{1}{\alpha_{\bar{s}}}\tau^{(-1/\alpha_{\bar{s}})-1} v_{\xi_1,\dots,\xi_{n-1}}(\tau)^2 - 2\tau^{-1/\alpha_{\bar{s}}} v_{\xi_1,\dots,\xi_{n-1}}(\tau)\cdot$$

$$\cdot\frac{d^+}{d\tau'}\Big|_\tau v_{\xi_1,\dots,\xi_{n-1}}(\tau') = -\frac{\tau^{-1/\alpha_{\bar{s}}}}{\alpha_{\bar{s}}\tau}|1-v_{\xi_1,\dots,\xi_{n-1}}(\tau)^2 +$$

$$+2v_{\xi_1,\dots,\xi_{n-1}}(\tau)\sum_{j=1}^{\bar{s}-1}\pi_j\alpha_j e^{\alpha_j\xi_j}\tau^{\alpha_j/\alpha_{\bar{s}}} + 2v_{\xi_1,\dots,\xi_{n-1}}(\tau)\pi_j\alpha_{\bar{s}}\tau| =$$

$$= -\frac{\rho_{\bar{s}}^{-1/\alpha_{\bar{s}}}}{\alpha_{\bar{s}}\rho_{\bar{s}}}|(1-v^2)+2v\sum_{j=1}^{n-1}\pi_j\alpha_j\rho_j| = \text{by } (27') = 0 \quad \text{where } (\rho_1,\dots,\rho_{n-1}) \equiv$$

$$\equiv (e^{\alpha_1\xi_1}\tau^{\alpha_1/\alpha_{\bar{s}}},\dots,e^{\alpha_{\bar{s}-1}\xi_{\bar{s}-1}}\tau^{\alpha_{\bar{s}-1}/\alpha_{\bar{s}}},\tau,\xi_{\bar{s}+1},\dots,\xi_{n-1}).$$

That is, the function $\tau \mapsto (1-v_{\xi_1,\dots,\xi_n}(\tau)^2)\tau^{-1/\alpha_{\bar{s}}}$ is constant. □

Remark 5. It is easy to see that if $p:\mathbb{R}_+^n \to \mathbb{R}_+$ is such a positive convex continuous function that the map $v:K \equiv \{(\rho_1,\dots,\rho_{n-1}) \in \mathbb{R}_+^{n-1} : p(\rho_1,\dots,\rho_{n-1},0)<1\} \to \mathbb{R}_+$ defined implicitly by $p(\rho_1,\dots,\rho_{n-1},v(\rho_1,\dots,\rho_{n-1}))=1$ has the form (28) for some $\psi:\mathbb{R}^{n-2}\to\mathbb{R}_+$ and if the function $\bar{v}:\bar{K}\to\mathbb{R}_+$ defined by $\bar{v}|K\equiv v$ and $\bar{v}|\partial K = 0$ is continuous then (27) holds for p.

Lemma 20. Suppose $x_j \in S_{m_j}$ $(j=1,2,3)$, $x_1,x_2 \in X_0$ and $\gamma_{m_1m_2}=\frac{1}{2}$. Then we have $\gamma_{m_1m_3}=\gamma_{m_1m_3}$.

Proof. Let us write briefly γ_{jk} instead of $\gamma_{m_j m_k}$ $(j,k = 1,2,3)$ and set $p:(\rho_1,\rho_2,\rho_3)\mapsto\|\sum_{j=1}^{3}\rho_j 1_{x_j}\|$. According to (26*), we have

(i) $\quad \frac{\partial^* p}{\partial\rho_1}(p^2-\rho_1^2)=\rho_1\rho_2\frac{\partial^* p}{\partial\rho_2}+2\gamma_{13}\rho_1\rho_2\frac{\partial^* p}{\partial\rho_3},$

(ii) $\quad \frac{\partial^* p}{\partial\rho_2}(p^2-\rho_2^2)=\rho_2\rho_1\frac{\partial^* p}{\partial\rho_1}+2\rho_2\rho_3\frac{\partial^* p}{\partial\rho_3}.$

If $\gamma_{23}=\gamma_{13}=0$, we are done. Thus we may assume without loss of generality $\gamma_{13}\neq 0$. Then from (i) and Lemma 19 it follows the existence of a unique function $\psi:R\to R_+$ such that

(i') $\quad p(\rho_1,\rho_2,\rho_3)=1 \Longleftrightarrow 1-\rho_1^2=\rho_2^2\cdot\psi(\frac{\log\rho_3}{\gamma_{13}}-2\log\rho_2)$

whenever $p(0,\rho_2,\rho_3)<1$ and $\rho_2,\rho_3>0$.

Therefore the function $\varphi\equiv\psi\circ\log$ satisfies

(i") $\quad p(\rho_1,\rho_2,\rho_3)=1 \Longleftrightarrow \rho_1^2+\rho_2^2\,\varphi(\frac{\rho_3^{1/\gamma_{13}}}{\rho_2^2})$

whenever $p(0,\rho_2,\rho_3)<1$ and $\rho_2,\rho_3>0$.

Hence, for any triplet $(\rho_1^*,\rho_2^*,\rho_3^*)>0$ with $p(0,\rho_2^*,\rho_3^*)<1=p(\rho_1^*,\rho_2^*,\rho_3^*)$, we have $\mathrm{grad}|_{(\rho_1^*,\rho_2^*,\rho_3^*)}p \,\|\, \mathrm{grad}|_{(\rho_1^*,\rho_2^*,\rho_3^*)}[\rho_1^2+\rho_2^2\,\varphi(\frac{\rho_3^{1/\gamma_{13}}}{\rho_2^2})]$ if $\mathrm{grad}|_{(\rho_1^*,\rho_2^*,\rho_3^*)}p$ exists. Since the function p (being convex) is almost everywhere totally derivable and since the multifunction $(\rho_1,\rho_2,\rho_3)\mapsto\mathrm{subgrad}|_{(\rho_1,\rho_2,\rho_3)}p$ is closed (cf. [Hol1]), for each $(\rho_1,\rho_2,\rho_3)>0$ with $p(0,\rho_2,\rho_3)<1=p(\rho_1,\rho_2,\rho_3)$ there exist $(\pi_1,\pi_2,\pi_3) \in \text{subgrad}\big|_{(\rho_1,\rho_2,\rho_3)} p$ and $\tilde{\lambda} \in \mathbb{R}^{\circ}$ such that

$$\tilde{\lambda}\cdot(\pi_1,\pi_2,\pi_3)=(2\rho_1,2\rho_2\Big[\varphi(\frac{\rho_3^{1/\gamma_{13}}}{\rho_2^2})-\frac{\rho_3^{1/\gamma_{13}}}{\rho_2^2}\varphi'(\frac{\rho_3^{1/\gamma_{13}}}{\rho_2^2})\Big],$$

$$\frac{\rho_3^{(1/\gamma_{13})-1}}{\gamma_{13}}\varphi'(\frac{\rho_3^{1/\gamma_{13}}}{\rho_2^2}))=(2\sqrt{1-\rho_2^2(\frac{\rho_3^{1/\gamma_{13}}}{\rho_2^2})},\ 2\rho_2\Big[(\frac{\rho_3^{1/\gamma_{13}}}{\rho_2^2})-$$

$$-\frac{\rho_3^{1/\gamma_{13}}}{\rho_2^2}\varphi'(\frac{\rho_3^{1/\gamma_{13}}}{\rho_2^2})\Big],\frac{\rho_3^{(1/\gamma_{13})-1}}{\rho_2^2}\varphi'(\frac{\rho_3^{1/\gamma_{13}}}{\rho_2^2}))$$ whenever

$\varphi'(\frac{\rho_3^{1/\gamma_{13}}}{\rho_2^2})$ exists. Thus from (ii) we obtain (since $\rho_1 = \sqrt{1-\rho_2^2\varphi(\frac{\rho_3^{1/\gamma_{13}}}{\rho_2^2})}$ if $p(\rho_1,\rho_2,\rho_3)=1$ and $\rho_1,\rho_2,\rho_3>0$)

(ii') $$2\rho_2[\varphi(\lambda)-\lambda\varphi'(\lambda)](1-\rho_2^2)=2\rho_2[1-\rho_2^2\varphi(\lambda)] + +2\rho_2\frac{\gamma_{23}}{\gamma_{13}}\rho_3^{1/\gamma_{13}}\varphi'(\lambda) \quad \text{where } \lambda\equiv\rho_3^{1/\gamma_{13}}/\rho_2^2$$

whenever $\varphi'(\lambda)$ exists and $0<\rho_2,\rho_3,\ p(0,\rho_2,\rho_3)<1$. Hence

(ii") $$\varphi(\lambda)-1=\lambda\varphi'(\lambda)\Big[1+(\frac{\gamma_{13}}{\gamma_{13}}-1)\rho_2^2\Big] \text{ if } \varphi'(\lambda) \text{ exists and } \rho_2,\lambda>0 \text{ with } p(0,\rho_2,(\lambda\rho_2^2)^{\gamma_{13}})<1.$$

Since p is an increasing function on $\mathbb{R}^3_+$, (i") implies that the function $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ must be also increasing and therefore almost everywhere derivable. But then from (ii") it readily follows $\frac{\gamma_{23}}{\gamma_{13}}-1=0$. □

<u>Proposition 8.</u> If $m,n\in\mathcal{M}_0$ and $m\neq n$ then $\gamma_{mn}=0$.

<u>Proof.</u> Let X consist of the points $x_1,\ldots,x_n$ where $x_j\in S_{m_j}$ $(j=1,\ldots,n)$ and assume $m_1\neq m_2, m_1\in\mathcal{M}_0$ Set $p\equiv[\mathbb{R}^n\ni(\rho_1,\ldots,\rho_n)\mapsto$

$\mapsto \| \sum_{j=1}^{n} \rho_j 1_{x_j} \|]$. From Corollary 13 (26*) we deduce

(29′) $$\frac{\partial^* p}{\partial \rho_1}(p^2-\rho_1^2) + \frac{\partial^* p}{\partial \rho_2} 2\rho_1\rho_2\gamma_{m_1m_2} + \sum_{j=3}^{n} \frac{\partial^* p}{\partial \rho_j} 2\rho_1\rho_j\gamma_{m_1m_j} = 0$$

(29″) $$\frac{\partial^* p}{\partial \rho_2}(p^2-\rho_2^2) + \frac{\partial^* p}{\partial \rho_1} 2\rho_1\rho_2\gamma_{m_1m_2} + \sum_{j=3}^{n} \frac{\partial^* p}{\partial \rho_j} 2\rho_2\rho_j\gamma_{m_2m_j} = 0.$$

Since $m_1, m_2 \in \mathcal{M}_o$, we have $\gamma_{m_1m_2} = 0$ or $\gamma_{m_1m_2} = \frac{1}{2}$ (cf. Corollary 13). Thus we have to show that $\gamma_{m_1m_2} = \frac{1}{2}$ is impossible.

Suppose $\gamma_{m_1m_2} = \frac{1}{2}$. Then by Lemma 25 we obtain $\gamma_{m_1m_j} = \gamma_{m_2m_j}$ $(j=3,\ldots,n)$ whence a subtraction of (29″) multiplied by ρ_1 from (29′) multiplied by ρ_2 yields $p^2 \cdot [\frac{\partial^* p}{\partial \rho_1}\rho_2 - \frac{\partial^* p}{\partial \rho_2}\rho_1] = 0$ i.e.

(29*) $$\frac{\partial^* p}{\partial \rho_1}\rho_2 - \frac{\partial^* p}{\partial \rho_2}\rho_1 = 0.$$

Consider now the linear vector field $\ell : \sum_{j=1}^{n} \zeta_j 1_{x_j} \mapsto \zeta_2 1_{x_1} - \zeta_1 1_{x_2}$ on E. (29*) and Proposition 5′ establish $\ell \in \log^* \mathrm{Aut}\, B(E)$. (Indeed: If Y has the form $(x_1, x_2, y_3, \ldots, y_N)$ or $(x_2, x_1, y_3, \ldots, y_N)$ then (21*) follows immediately from (29*). On the other hand, for $\{y_1, y_2\} \neq \{x_1, x_2\}$ we always have $\langle \ell(1_{y_1}), 1^*_{y_2} \rangle = 0$). However, the operator $\exp(\ell)$ is equal to $[\sum_{j=1}^{n} \zeta_j 1_{x_j} \mapsto (\zeta_1 \cos 1 + \zeta_2 \sin 1) 1_{x_1} + (-\zeta_1 \sin 1 + \zeta_2 \cos 1) 1_{x_2}]$. Therefore $\exp(\ell)(E_{S_{m_1}}) \not\subset E_{S_{m_1}}$. But this fact contradicts the definition of the partition $\{S_m : m \in \mathcal{M}\}$. □

We close the examination of the three dimensional projection bands of E by remarking that, unlike in two dimensions, the matrix Γ does not uniquely determine the geometric shape of $B(E)$ even if $\mathrm{Aut}\,B(E)$ admits a non-linear member.

Lemma 21. Suppose $X=\{x_1,\ldots,x_n\}$, $x_j\in S_{m_j}$ $(j=1,\ldots,n)$, $m_n\in \mathcal{M}_o$ and, by setting $\gamma_j\equiv\gamma_{m_j m_n}$ $(j=1,\ldots,n)$, $\gamma_1,\ldots,\gamma_s\neq 0 = =\gamma_{s+1},\ldots,\gamma_{n-1}$ where $n>s>1$. Then there is a unique function $\varphi:\mathbb{R}_+^{n-2}\to\mathbb{R}_+$ such that for all $\rho_n\in\mathbb{R}_+$ and $(\rho_1,\ldots,\rho_{n-1})\in\mathbb{R}_+^{n-1}$ with $\|\sum_{j=1}^{n-1}\rho_j 1_{x_j}\|<1$ and $\rho_s>0$ we have

$$\text{(30)}\qquad \|\sum_{j=1}^{n}\rho_j 1_{x_j}\|=1 \iff 1-\rho_n^2 =$$

$$=\rho_s^{1/\gamma_s}\,\varphi\Big(\frac{\rho_1^{1/\gamma_1}}{\rho_s^{1/\gamma_s}},\ldots,\frac{\rho_{s-1}^{1/\gamma_{s-1}}}{\rho_s^{1/\gamma_s}},\rho_{s+1},\ldots,\rho_{n-1}\Big).$$

This function φ is necessarily continuous, monotone increasing and all its directional derivatives exist at any point of $(0,\infty)^{n-2}$. In particular, $\varphi(0,\ldots,0)=1$ and if $n=3$, $s=2$ then $\lim_{\lambda\uparrow\infty}\frac{\varphi(\lambda)}{\lambda}=1$.

Proof. Define the lattice norm p on $\mathbb{R}^n$ by $p(\rho_1,\ldots,\rho_n)\equiv \equiv\|\sum_{j=1}^{n}\rho_j 1_{x_j}\|$. From Corollary 13 (26*) we deduce that $\frac{\partial^* p}{\partial\rho_n}\cdot(p^2-\rho_n^2)= =2\sum_{j=1}^{n-1}\gamma_j\rho_j\rho_n\frac{\partial^* p}{\partial\rho_j}$. Thus by Lemma 19 there exists a unique function $\psi:\mathbb{R}^{n-2}\to\mathbb{R}_+$ with

$$1-v(\rho_1,\dots,\rho_{n-1})^2=\rho_s^{1/\gamma_s}\,\psi\Big(\frac{\log\rho_1}{\gamma_1}-\frac{\log\rho_s}{\gamma_s},\dots,\frac{\log\rho_{s-1}}{\gamma_{s-1}}-$$

$-\dfrac{\log\rho_s}{\gamma_s},\rho_{s+1},\dots,\rho_{n-1}\Big)$ $\quad\forall(\rho_1,\dots,\rho_{n-1})\in K$ where $K\equiv$

$\equiv\{(\rho_1,\dots,\rho_{n-1})>0 : p(\rho_1,\dots,\rho_{n-1},0)<1\}$ and the function $v:K\to\mathbb{R}_+$ is defined implicitly by $p(\rho_1,\dots,\rho_{n-1},v(\rho_1,\dots,\rho_{n-1}))=1$. Next observe that any of the functions $\omega_{\lambda_1,\dots,\lambda_{n-2}}:I_{\lambda_1,\dots,\lambda_{n-2}}\ni$ $\ni\rho\mapsto\rho^{-1/\gamma_s}\cdot[1-v((\lambda_1\rho^{1/\gamma_s})^{\gamma_1},\dots,(\lambda_{s-1}\rho^{1/\gamma_s})^{\gamma_{s-1}},\rho,\lambda_s,\dots,\lambda_{n-2})^2]$, where $\lambda_1,\dots,\lambda_{n-2}\ \mathbb{R}_+$ are arbitrarily fixed and $I_{\lambda_1,\dots\lambda_{n-2}}$ denotes the interval $\{\rho>0 : p((\lambda_1\rho^{1/\gamma_s})^{\gamma_1},\dots,(\lambda_{s-1}\rho^{1/\gamma_s})^{\gamma_{s-1}},\rho,$ $\lambda_s,\dots,\lambda_{n-2},0)<1\}$, is constant. Indeed: $\dfrac{d^+}{d\rho}\Big|_{\rho_0}\omega_{\lambda_1,\dots\lambda_{n-2}}(\rho)=$

$$=\Big(-\frac{1}{\gamma_s}\Big)\rho_0^{-\frac{1}{\gamma_s}-1}\,[1-v((\lambda_1\rho_0^{1/\gamma_s})^{\gamma_1},\dots,\lambda_{n-2})^2]+\rho_0^{-\frac{1}{\gamma_s}}\,(-2v((\lambda_1\rho_0^{\frac{1}{\gamma_s}})^{\gamma_1},\dots,$$

$$\lambda_{n-2}))\cdot\Big[\sum_{j=1}^{s-1}\pi_j\lambda_j\rho_0^{\frac{\gamma_1}{\gamma_s}-1}\frac{\gamma_j}{\gamma_s}+\pi_s\Big]\text{ for some }(\pi_1,\dots\pi_{n-1})\in$$

$$\in\Big[-\text{subgrad}\Big|_{((\lambda_1\rho_0^{1/\gamma_s})^{\gamma_1},\dots,(\lambda_{s-1}\rho_0^{1/\gamma_s})^{\gamma_{s-1}},\rho_0,\lambda_s,\dots,\lambda_{n-2})}(-v)\Big].$$

That is, by setting $(\rho_1,\dots,\rho_{n-1})\equiv((\lambda_1\rho_0^{1/\gamma_s})^{\gamma_1},\dots,(\lambda_{s-1}\rho_0^{1/\gamma_s})^{\gamma_{s-1}},$ $\rho_0,\lambda_s,\dots,\lambda_{n-2})$, we have $-\gamma_s\rho_0^{\frac{1}{\gamma_s}+1}\dfrac{d^+}{d\rho}\Big|_{\rho_0}\omega_{\lambda_1,\dots,\lambda_{n-2}}(\rho)=$

$$=[1-v(\rho_1,\dots,\rho_{n-1})^2]+2\,v(\rho_1,\dots,\rho_{n-1})\sum_{j=1}^{n-1}\pi_j\rho_j\rho_s\gamma_j=\text{ by Corollary 10}=0.$$

On the other hand, concaveness of v establishes the absolute continuity of each $\omega_{\lambda_1,\dots,\lambda_{n-2}}$.

Now the definition (and clearly only this definition)
$\varphi(\lambda_1,\dots,\lambda_{n-2}) \equiv$ [the (unique) element of range $\omega_{\lambda_1,\dots,\lambda_{n-2}}$]
satisfies (30). The mentioned continuity properties of φ and the relation $\varphi(0,\dots,0)=1$ are obvious from the definition of $\omega_{\lambda_1,\dots,\lambda_{n-1}}(.)$.

Assume n=3 and s=2. Then Lemma 18 (25) entails $p(\rho_1,0,\rho_3)= =1 \Leftrightarrow 1-\rho_3^2=\rho_1^{1/\gamma_1} \quad \forall \rho_1,\rho_2 \in \mathbb{R}_+$. Hence $\sqrt{1-\rho_1^{1/\gamma_1}} = v(\rho_1,0) =$

$$= \lim_{\rho_2 \downarrow 0} v(\rho_1,\rho_2) = \lim_{\rho_2 \downarrow 0} \sqrt{1-\rho_2^{1/\gamma_2}\varphi(\rho_1^{1/\gamma_1}/\rho_2^{1/\gamma_2})} \quad \forall \rho_1 \in [0,1). \text{ Thus}$$

$$\rho_1^{1/\gamma_1} = \lim_{\rho_2 \downarrow 0} \frac{\rho_2^{1/\gamma_2}}{\rho_1^{1/\gamma_1}} \cdot \varphi\left(\frac{\rho_1^{1/\gamma_1}}{\rho_2^{1/\gamma_2}}\right) = \rho_1^{1/\gamma_1} \lim_{\lambda \uparrow \infty} \frac{\varphi(\lambda)}{\lambda} \quad \forall \rho_1 \in (0,1). \square$$

__Example.__ Let $\gamma_1,\gamma_2 \in [1,\infty)$ and φ be any convex increasing continuous $\mathbb{R}_+ \to \mathbb{R}_+$ function such that $\varphi(0)=1$, $\varphi'(\lambda)=1 \quad \forall \lambda > 1$. Then the set $K \equiv \{(\zeta_1,\zeta_2,\zeta_3) : 1 > |\zeta_3|^2 + |\zeta_2|^{\gamma_2} \varphi(|\zeta_1|^{\gamma_1}/|\zeta_2|^{\gamma_2})\}$ is a convex subset of $\mathbb{C}^3$. The Banach space E_φ supported by $\mathbb{C}^3$ and equipped with the norm $\|f\| \equiv \inf\{\rho>0 : f \in \rho K\}$ $(f \in \mathbb{C}^3)$ can be considered as an atomic vector lattice with $(E_\varphi)_+ = \mathbb{R}_+^3$ and $\mathbb{C}(0,0,1) \subset (E_\varphi)_0$ (i.e. $\Delta \cdot (0,0,1) \subset \mathrm{Aut}\, B(E_\varphi)\{0\}$).

__Proof.__ Observe that for the Borel measure μ on $[0,1]$ defined by $\mu([0,\xi]) \equiv \varphi^{(+)}(\xi) \quad \forall \xi \ [0,1]$ we have $\varphi = \int_0^1 \varphi_\xi d\mu(\xi)$ where $\varphi_\xi \equiv [\mathbb{R}_+ \ni \lambda \mapsto \max\{1, \lambda + (1-\xi)\}] \ \forall \xi \in [0,1]$. Hence the function

$\tilde{p}(\rho_1,\rho_2,\rho_3)\equiv\rho_3^2+\rho_2^{\alpha_2}\varphi(\rho_1^{\gamma_1}/\rho_2^{\gamma_2})=\int_0^1(\rho_3^2+\rho_2^{\alpha_2}\max\{1,\frac{\rho_1^{\gamma_1}}{\rho_2^{\gamma_2}}+(1-\xi)\})d\mu(\xi)=$

$=\int_0^1(\rho_3^2+\max\{\rho_2^{\gamma_2},\rho_1^{\gamma_1}+(1-\xi)\rho_2^{\gamma_2}\})d\mu(\xi)$ is a convex increasing $\mathbb{R}_+^3\to\mathbb{R}_+$ function. Since $K=\{(\zeta_1,\zeta_2,\zeta_3):\tilde{p}(|\zeta_1|,|\zeta_2|,|\zeta_3|)<1\}$, this fact establishes that the norm and positive cone defined above render $\mathbb{C}^3$ a complex atomic vector lattice (with minimal ideals $\mathbb{C}(1,0,0)$, $\mathbb{C}(0,1,0)$, $\mathbb{C}(0,0,1)$). On the other hand, from Remark 5 we see that the convex increasing $\mathbb{R}^3_+\to\mathbb{R}_+$ function $p(\rho_1,\rho_2,\rho_3)\equiv\|(\rho_1,\rho_2,\rho_3)\|$ satisfies (27) for $\alpha_j\equiv 1/\gamma_j$ $(j=1,2)$. But then Lemma 18 ensures that for the bilinear map $q:E_\varphi\times E_\varphi\to E$ defined by $q(e_j,e_k)\equiv$ $\equiv-\gamma_{jk}(\delta_{3j}e_k+\delta_{3k}e_j)$ $(j,k=1,2,3)$ where $\gamma_{jj}=\frac{1}{2}$ $(j=1,2,3)$, $\gamma_{3k}\equiv$ $\equiv\gamma_{k3}\equiv 1/\gamma_k$ $(k=1,2)$ and $\gamma_{12}\equiv\gamma_{21}\equiv 0$ (further $\delta_3\equiv 1_{\{3\}}(.)$), respectively, the vector field $f\mapsto(0,0,1)+q(f,f)$ on $\mathbb{C}^3$ belongs to log*Aut B (E). □

<u>The automorphisms $\exp[B(E)\ni f\mapsto c+q_c(f,f)]$</u>

<u>Lemma 22.</u> Let H denote a Hilbert space with scalar product $\langle\,|\,\rangle$, further let $c\in H\setminus\{0\}$ and $q:H\times H\to H$ be the bilinear map $(f,g)\mapsto-\frac{1}{2}\langle f|c\rangle g-\frac{1}{2}\langle g|c\rangle f$. Set $D\equiv\{f\in H:\exists v:\mathbb{R}\to H$ function with $v(0)=f$ and $v'=c+q(v,v)\}$, $c^o\equiv c/\|c\|$ and $F^t\equiv\exp[D\ni f\mapsto t\cdot(c+q(f,f))]$ $(t\in R)$. Then

$$(31)\qquad F^tf=M_{\|c\|t}(\langle f|c^o\rangle)\cdot c^o+M^{\perp}_{\|c\|t}(\langle f|c^o\rangle)\cdot(f-\langle f|c^o\rangle c^o)$$

holds for every $f\in D$ and $t\in\mathbb{R}$ where M_t and $M_t^{\perp}$ denote the Möbius and co-Möbius transformations

(32′) $\quad M_t: \mathbb{C} \ni \zeta \mapsto \frac{\zeta + \text{th}\, t}{1+\zeta\, \text{th}\, t} \quad (= \text{th}(t + \text{area th}\, \zeta))$

(32″) $\quad M_t^{\perp}: \mathbb{C} \;\; \zeta \mapsto \frac{\sqrt{1-(\text{th}\, t)^2}}{1+\zeta\, \text{th}\, t} \quad (= \exp \int_0^t (-M_\tau(\zeta))\, d\tau)$,

respectively, for all $t \in \mathbb{R}$. (Here areath (.) means the multivalued function $\zeta \mapsto \{\alpha \in \mathbb{C}\, (\text{th}\, \alpha \zeta\})$.

<u>Proof.</u> By definition of the exponential map, we have $\frac{d}{dt} F^t f = c - q(F^t f, F^t f) = c - \langle F^t f | c \rangle F^t f \quad \forall t \in R, f \in D$. Let us fix f D arbitrarily. Then the function $[\mathbb{R} \ni t \mapsto F^t f]$ is the solution of the initial value problem $\left. \begin{array}{l} \frac{d}{dt} x = \|c\| (c^o - \langle x | c^o \rangle x) \\ x(0) = f \end{array} \right\}$. Thus the function $[\mathbb{R} \ni t \mapsto \langle F^t f | c^o \rangle]$ is the solution of the initial value problem

$\left. \begin{array}{l} \frac{d}{dt} \xi = \|c\| (1 - \xi^2) \\ \xi(0) = \langle f | c^o \rangle \end{array} \right\}$ which is easy to calculate: Let $\alpha \in \mathbb{C}$ be such that $\text{th}\, \alpha = \langle f | c^o \rangle$. Observe that $\frac{d}{dt} \text{th}(\|c\| t + \alpha) = \frac{d}{dt} \frac{\text{sh}(\|c\| t + \alpha)}{\text{ch}(\|c\| t + \alpha)} = \frac{\|c\| \text{ch}^2(\|c\| t + \alpha) - \|c\| \text{sh}^2(\|c\| t + \alpha)}{\text{ch}^2(\|c\| t + \alpha)} = \|c\| (1 - \text{th}^2(\|c\| t + \alpha))$. Hence, using the additional laws,

$$\langle F^t f | c^o \rangle = \text{th}(\|c\| t + \alpha) = \frac{\text{th}(\|c\| t) + \text{th}\, \alpha}{1 + \text{th}(\|c\| t)\, \text{th}\, \alpha} = \frac{\text{th}(\|c\| t) + \langle f | c^o \rangle}{1 + \langle f | c^o \rangle\, \text{th}(\|c\| t)} =$$

$= M_{\|c\| t}(\langle f | c^o \rangle)$. Therefore, to complete the proof of (31), it suffices to show that

$$(33)\qquad \langle F^t f|g\rangle = \langle f|g\rangle\cdot M^{\perp}_{\|c\|t}(\langle f|c^o\rangle)\quad \forall t\in\mathbb{R}\ \text{ whenever }\ g\perp c.$$

Let $g\in H$ be any such fixed vector that $g\perp c$. Now we have

$$\frac{d}{dt}\langle F^t f|g\rangle = -\|c\|\langle F^t f|c^o\rangle\langle F^t f|c^o\rangle = -\|c\|\cdot M_{\|c\|t}(\langle f|c^o\rangle)\cdot$$

$\cdot\langle F^t f|g\rangle \quad \forall t\in\mathbb{R}$ and $\langle F^o f|g\rangle=\langle f|g\rangle$. Hence we immediately obtain

$$\langle F^t f|g\rangle=\langle f|g\rangle \exp\int_0^t(-M_{\|c\|\tau}(\langle f|c^o\rangle))\|c\|d\tau=\langle f|g\rangle\exp\int_0^{\|c\|t}(-M_\tau\cdot$$

$\cdot(\langle f|c^o\rangle)d\tau$. Again, by letting $\alpha\in\text{area th}\,\langle f|c^o\rangle$, we see

$$\exp\int_0^{\|c\|t}(-M_\tau(\langle f|c^o\rangle))d\tau=\exp\int_0^{\|c\|t}[-\text{th}(\tau+\alpha)]d\tau=\exp[\log^*\text{ch}\alpha -$$

$-\log^*\text{ch}(\|c\|t+\alpha)] = \dfrac{\text{ch}\,\alpha}{\text{ch}(\|c\|t+\alpha)}$ for sufficiently small values of t (in $\mathbb{R}$) where the function $\log^*$ is any continuous branch cut of the multivalued function $\log:\zeta\mapsto\{\eta\ \mathbb{C}:\exp\eta=\zeta\}$ in a neighbourhood of α. Since the function $t\mapsto\langle F^t f|g\rangle$ is analytic, hence $\langle F^t f|g\rangle=$

$$=\langle f|g\rangle\cdot\frac{\text{ch}\,\alpha}{\text{ch}(\|c\|t+\alpha)}=\langle f|g\rangle\frac{\text{ch}\,\alpha}{\text{ch}(\|c\|t)\text{ch}\alpha+\text{sh}(\|c\|t)\text{sh}\alpha}=\langle f|g\rangle\cdot$$

$$\cdot\frac{1/\text{ch}(\|c\|t)}{1+\text{th}(\|c\|t)\text{th}\alpha}=\frac{\sqrt{1+\text{th}^2(\|c\|t)}}{1+\langle f|c^o\rangle\text{th}(\|c\|t)}\cdot\langle f|g\rangle\quad\forall t\in\mathbb{R}$$ which proves (33). □

<u>Corollary 14.</u> $D=\{f\in H:\langle f|c^o\rangle\notin(-\infty,-1)\cup(1,\infty)\}$. In particular, $D\supset\overline{B}(H)$.

Proof. It is easy to see from the proof of Lemma 22 that $f \in D$ whenever the right hand side of the formula (31) is well-defined for every $t \in \mathbb{R}$. This latter requieres $0 \neq 1 + \langle f | c^o \rangle \, th(\|c\| t) \quad \forall t \in \mathbb{R}$ whence $f \in D$ if and only if $-\langle f | c^o \rangle \notin$ $\notin \text{range } cth(\|c\| \cdot)\big|_{\mathbb{R}} = (-\infty, -1) \cup (1, \infty)$. □

Proposition 9. Assume $\mathcal{M}_o \neq \emptyset$, and for each $n \in \mathcal{M}_o$ let $\rho_n \in \mathbb{R}_+$ and c_n^o denote an arbitrarily fixed unit vector in E_{S_n}. Set $c \equiv \sum_{n \in \mathcal{M}_o} \rho_n c_n^o$, $D \equiv \{f \in E : \exists v : \mathbb{R} \to E \quad v(0) = f, v' = c + q_c(v,v)\}$ and $F^t \equiv$ $\equiv \exp[D \ni f \mapsto t \cdot (c + q_c(f,f))]$ $(t \in \mathbb{R})$. Then we have

$$(34') \qquad 1_{S_n} \cdot (F^t f) = M_{\rho_n t}(\langle f | c_n^o \rangle) \cdot c_n^o + M^{\perp}_{\rho_n t}(\langle f | c_n^o \rangle) \cdot \cdot (1_{S_n} f - \langle f | c_n^o \rangle c_n^o)$$

$$(34'') \qquad (F^t f)(x) = f(x) \cdot \prod_{n \in \mathcal{M}_o} [M_{\rho_n t}(\langle f | c_n^o \rangle)]^{2\gamma_{nm}}$$

whenever $x \in S_m$ and $m \notin \mathcal{M}_o$

for all $f \in D$ and $t \in \mathbb{R}$.[17]

Proof. Let $X = \{x_1, \ldots, x_N\}$ where $x_j \in S_{m_j}$ $(j = 1, \ldots, N)$.

---

[17] Here $M_\tau, M_\tau^\perp$ are the transformations defined by (32'),(32"). For $f, g \in E$, $\langle f | g \rangle$ means their scalar product in $L^2(X)$ (i.e. $\langle f | g \rangle \equiv \sum_{x \in X} f(x) \cdot \overline{g(x)}$).

From Proposition 7 (24*) we directly deduce $q_c(f,g) =$

$$= \sum_{x,y\in X} f(x)g(y)\; q_c(1_x,1_y) = \sum_{k=1}^{N} [-\sum_{j=1}^{N} \gamma_{m_j m_k} (f(x_j)g(x_k) + $$

$$+ f(x_k)g(x_j)) \cdot \overline{c(x_j)}] \cdot 1_{x_k} = \sum_{k=1}^{N} [-\sum_{n\in \mathcal{M}_o} \gamma_{nm_k} (\langle f|\rho_n c_n^o\rangle g(x_k) + $$

$$+ \langle g|\rho_n c_n^o\rangle f(x_k)] \cdot 1_{x_k} \qquad \forall\, f,g\in E.$$ Hence

$$q_c(f,g)\Big|_{S_n} = -\tfrac{1}{2}[\langle f|\rho_n c_n^o\rangle\cdot (g|_{S_n}) + \langle g|\rho_n c_n^o\rangle\cdot (f|_{S_n})] \quad \forall n\in \mathcal{M}_o,$$

$$q_c(f,g)(x_k) = -\sum_{n\in\mathcal{M}_o} \gamma_{nm_k} [\langle f|\rho_n c_n^o\rangle g(x_k) + \langle g|\rho_n c_n^o\rangle f(x_k)]$$

whenever $x_k \in X\setminus X_o$.

By definition of the exponential map, given $f\in D$ the mapping $t\mapsto F^t f$ is the solution of the initial value problem

$$\left.\begin{aligned} \frac{d}{dt}x &= c + q_c(x,x) \\ x(0) &= f \end{aligned}\right\}.$$ Therefore we have

$$(35')\qquad \frac{d}{dt}(F^t f)\Big|_{S_n} = c\Big|_{S_n} - \langle F^t f|\rho_n c_n^o\rangle\cdot (F^t f)\Big|_{S_n} \quad \text{and}$$

$$(F^o f)\Big|_{S_n} = f\Big|_{S_n}$$

$$(35'')\qquad \frac{d}{dt}(F^t f)(x_k) = -\sum_{n\in\mathcal{M}_o} 2\gamma_{nm_k} \langle F^t f|\rho_n c_n^o\rangle\cdot (F^t f)(x_k),$$

$$(F^o f)(x_k) = f(x_k)$$

whenever $f\in D$ and $x_k\in X\setminus X_o$. Now let us fix any $n\in\mathcal{M}_o$ and

set $D_n \equiv \{h \in L^2(S_n) : \exists v : \mathbb{R} \to L^2(S_n) \quad v(0)=h, v' = c|_{S_n} - \langle v | \rho_n c_n^o \rangle v\}$.
From (35′) it readily follows $D_n \supset \{f|_{S_n} : f \in D\}$ and

$$(F^t f)_{S_n} = F_n^t(f|_{S_n}) \quad \forall t \in \mathbb{R}, f \in D \quad \text{where} \quad F_n^t \equiv \exp[D_n \ni h \mapsto t\cdot(c|_{S_n} - \langle h | (c|_{S_n}) \rangle_{L^2(S_n)} h)].$$

Hence we see, by applying Lemma 22 to the space $H \equiv L^2(S_n)$ and the vector $c|_{S_n}$, that (34′) holds. Since (34′) implies $\langle F^t f | \rho_n c_n^o \rangle = \langle ((F^t f)|_{S_n}) | (c|_{S_n}) \rangle_{L^2(S_n)} = M_{\rho_n t}(\langle f | c_n^o \rangle)\cdot \rho_n$, given $x_k \in X \setminus X_o$, the solution of the initial value problem (35") is $(F^t f)(x_k) = f(x_k)\cdot \exp \sum_{n \in \mathcal{M}_o} [-2\gamma_{nm_k} \int_0^t M_{\rho_n \tau}(\langle f | c_n^o \rangle)\cdot \rho_n \, d\tau] =$

$$= f(x_k) \prod_{n \in \mathcal{M}_o} [\exp \int_0^{\rho_n t} (-M_\tau(\langle f | c_n^o \rangle))\, d\tau]^{2\gamma_{nm_k}}.$$

Taking (32") into consideration, (34") is immediate. □

Corollary 15. $\{f \in E : \forall c \in E_o \; \exists v : \mathbb{R} \to E \quad v(0)=f, v' = c + q_c(v,v)\} =$
$= \{f \in E : \|(f|_{S_n})\|_{L^2(S_n)} < 1 \quad \forall n \in \mathcal{M}_o\}$.

Proof. From the proof of (35") we see that for any given $c \in E_o$ we have $\{f \in E : \exists v : \mathbb{R} \to E \; v(0)=f, v'=c+q_c(v,v)\} = \{f \in E : f|_{S_n} \in D_n \; \forall n \in \mathcal{M}_o\}$ where the sets D_n (depending on c) are de fined as in the proof of Proposition 9. Thus, using Corollary 14 to express D_n, we obtain $\{f \in E : \forall c \in E_o \exists v : \mathbb{R} \to E \; v(0) = f, v' = c + q_c(v,v)\} =$

$$= \bigcap_{c \in E_o} \{f \in E : \langle (f|_{S_n}) | (c|_{S_n}) \rangle_{L^2(S_n)} \notin (-\infty, -\|c|_{S_n}\|_{L^2(S_n)}) \cup (\|c|_{S_n}\|_{L^2(S_n)}, \infty) \quad \forall n \in \mathcal{M}_o\} \subset$$

$\subset$ |since $\forall f\in E\ f\cdot 1_{X_o}\in E_o| \subset \{f\in E : \|f|_{S_n}\|^2_{L^2(S_n)} \notin (-\infty, \|f|_{S_n}\|_{L^2(S_n)}) \cup$ $\cup (\|f|_{S_n}\|_{L^2(S_n)}, \infty)\ \forall n\in\mathcal{M}_o\} = \{f\in E : \|f|_{S_n}\|_{L^2(S_n)} < 1\ \ \forall n\in\mathcal{M}_o\}$. The converse inclusion $\{f\in E : \|f|_{S_n}\| < 1\ \ \forall n\in\mathcal{M}_o\} \subset \bigcap_{c\in E_o} \{f\in E :$ $\langle (f|_{S_n}) | (c|_{S_n}) \rangle_{L^2(S_n)} \notin (-\infty, -\|c|_{S_n}\|_{L^2(S_n)}) \cup (\|c|_{S_n}\|_{L^2(S_n)}, \infty)\ \forall n\in\mathcal{M}_o\}$ is trivial. □

<u>Proposition 10.</u> $\overline{B}(E)\cap E_o = \{f\in E : f|_{X\setminus X_o} = 0, \|f|_{S_n}\|_{L^2(S_n)} \leq 1$ $\forall m\in\mathcal{M}_o\}$.

<u>Proof.</u> Since for every $c\in E_o$ $(= \sum_{x\in X_o} \mathbb{C}\, 1_x$ by Corollary 14) the vector field $f \mapsto c + q_c(f,f)$ is tangent to $\partial B(E)$, the classical existence (and uniqueness) theorem concerning the solution of initial value problems establishes $\overline{B}(E) = \{f\in E : \forall c\in E_o\ \exists v:\mathbb{R}\to\overline{B}(E)$ $v(0)=f,\ v' = c + q_c(v,v)\}$. Hence (by Corollary 15) $\overline{B}(E) \subset \{f\in E :$ $\|f|_{S_n}\|_{L^2(S_n)} \leq 1\ \ \forall n\in\mathcal{M}_o\}$. Therefore it suffices to show that $B(E)\cap E_o \supset \{f\in E_o : \|f|_{S_n}\|_{L^2(S_n)} < 1\ \ \forall n\in\mathcal{M}_o\}$.

Let $f\in E_o$ be arbitrarily fixed and suppose $\|f|_{S_n}\|_{L^2(S_n)} < 1$ $\forall n\in\mathcal{M}_o$. Define the functions c_n $(n\in\mathcal{M}_o)$ by $c_n \equiv \begin{cases} \|f|_{S_n}\|^{-1}_{L^2(S_n)} \cdot 1_{S_n} \cdot f & \text{if } f|_{S_n} \neq 0 \\ 0 & \text{if } f|_{S_n} = 0 \end{cases}$.

Since the domain $B(E)$ is absorbing in E, exists $t_o>0$ such that, there by setting $\delta_n \equiv \dfrac{\text{real area th}\, \| f|_{S_n} \|_{L^2(S_n)}}{t_o}$ $(n \in \mathcal{M}_o)$, the vector $c \equiv \equiv \sum_{n\in\mathcal{M}_o} \delta_n$ belongs to $B(E)$. Then consider the mapping $F \equiv \exp[B(E) \ni g \mapsto \mapsto t_o \cdot (c + q_c(g,g))]$. By (34′) and we have $F(0) = \sum_{n\in\mathcal{M}_o} M_{\delta_n t_o}(0) \cdot c_n = = \sum_{n\in\mathcal{M}_o} \text{th}(\delta_n t_o) \cdot c_n = \sum_{n\in\mathcal{M}_o} \| f|_{S_n} \|_{L^2(S_n)} \cdot c_n = f$. But F is an automorphism of $B(E)$ whence $f = F(0) \in B(E)$. □

Corollary 16. $B(E) \cap E_o = \{\exp[f \mapsto c + q_c(f,f)](0) : c \in E_o\}$. □

Complete description of $\text{Aut}_o B(E)$

Lemma 23. Given $c \in E_o$ and $n \in \mathcal{M}$, we have $q_c(h_1,h_2) \in E_{S_n}$ whenever a) $h_1,h_2 \in E_o$ and $h_1 \in E_{S_n}$, b) $h_1 \in E_{S_n}$ and $n \notin \mathcal{M}_o$.

Proof. a) For any $x \in X$, let $m(x) \equiv$ [the (unique element of $\{m \in \mathcal{M} : x \in S_m\}$]. Then $q_c(h_1,h_2) = q_c\left(\sum_{x\in S_n} h_1(x) 1_x, \sum_{y \in X_o} h_2(y) 1_y\right) =$

$= |\text{by Proposition 7 (24*)}| = -\sum_{x\in S_n} \sum_{y\in X_o} \gamma_{nm(y)} \cdot h_1(x) h_2(y) [\overline{c(x)} 1_y + + \overline{c(y)} 1_x] = \left|\text{since } \gamma_{nm} = \begin{cases} 1/2 & \text{if } m=n \\ 0 & \text{if } m \in \mathcal{M}_o \setminus \{n\} \end{cases}\right| = -\frac{1}{2} \sum_{x,y\in S_n} h_1(x) h_2(y) \cdot \cdot [\overline{c(x)} 1_y + \overline{c(y)} 1_x] \in \sum_{z\in S_n} \mathbb{C}\, 1_z = E_{S_n}.$

b) $q_c(h_1,h_2) = -\sum_{x\in S_n}\sum_{y\in X} h_1(x)h_2(y)\gamma_{nm(y)}\,[\overline{c(x)}\,1_y + \overline{c(y)}\,1_x] =$

$= |$since $\gamma_{nm} = 0$ if $m \notin \mathcal{M}_o$ and since $c(x) = 0$ for $x \notin X_o| =$

$= -\sum_{x\in S_n}\sum_{y\in X_o} h_1(x)h_2(y)\ {}_{nm(y)}\overline{c(y)}\,1_x \quad E_{S_n}. \|$

<u>Lemma 24.</u> Let $n\in\mathcal{M}$, $f\in E_{S_n}$ with $\|f\|<1$ and $v\in\log^*\mathrm{Aut}\,B(E)$. Set $F^t \equiv \exp(t\cdot v|_{B(E)})$ $f_t \equiv F^t(f)$, $g_t \equiv F^t(0)$ $(t\in R)$. Then $f_t - g_t \in E_{S_n}$ $\forall t\in\mathbb{R}$.

<u>Proof.</u> Let us write c and $\ell(.)$ for the constant and linear part of $v(.)$, respectively. Now

$$(36)\qquad \frac{d}{dt}(f_t-g_t) = v(f_t)-v(g_t) = [c+\ell(f_t)+q_c(f_t,f_t)] - [c+\ell(g_t)+ + q_c(g_t,g_t)] = \ell(f_t-g_t)+q_c(f_t-g_t,f_t+g_t).$$

Introduce the mapping $A:\mathbb{R}\times E_{S_n}\to E$ defined by $A(t,h)\equiv\ell(h) + q_c(h,f_t+g_t)$. Since $\ell\in\log^*\mathrm{Aut}\,B(E)$ (cf. Theorem 6 d)), from Theorem 10 we obtain $\ell(E_{S_n}) = E_{S_n}$. Therefore Lemma 23 b) establishes $A(t,h)\in E_{S_n}$ $(\forall t\in\mathbb{R},\ h\in E_{S_n})$ whenever $n\notin\mathcal{M}_o$. Moreover, since any vector field belonging to $\log^*\mathrm{Aut}\,B(E)$ is tangent to E_o (see [KU1]), we have $f_t, g_t\in E_o$ $\forall t\in\mathbb{R}$ whenever $f\in E_o$. Thus, by Lemma 23 a), $A(t,h)\in E_{S_n}$ $\forall t\in\mathbb{R}, h\in E_{S_n}$ also if $n\in\mathcal{M}_o$. That is, in any case $A(\mathbb{R},E_{S_n})\subset E_{S_n}$. Consider the initial value problem

$$\begin{cases}\frac{d}{dt}x = A(t,x) \\ x(0) = f,\ x(.)\in C^1(\mathbb{R},E_{S_n})\end{cases}$$[18]. Since $A(\mathbb{R},E_{S_n})\subset E_{S_n}$, it has a unique solution $\varphi_f:\mathbb{R}\to E_{S_n}$. Hence (also by uniquess) φ_f is the unique solution of $\left.\begin{matrix}\frac{d}{dt}x = A(t,x) \\ x(0)=f, x(.)\in C^1(\mathbb{R},E)\end{matrix}\right\}$. Hence (36) yields $f_t - g_t = \varphi_f(t) \quad \forall t\in\mathbb{R}$. □

Corollary 17. If L is a linear member of $Aut_o B(E)$ then $L(B(E_{S_n}))\subset B(E_{S_n}) \quad \forall n\in\mathcal{M}$, i.e. {linear elements of $Aut_o B(E)$}= $=\{U|_{B(E)} : U\in\{L^2(X)\text{-isometries}\},\ U(E_{S_n})\subset E_{S_n} \quad \forall n\in\mathcal{M}\}$.

Proof. For some $v\in\log^*Aut\,B(E)$ we have $L=\exp(v|_{B(E)})$. Thus, by Lemma 24, for any $n\in\mathcal{M}$, $L(f)=L(f)-L(0)\in E_{S_n}$ whenever $f\in E_{S_n}$. The second statement is immediate from Theorem 10 now. □

At this point we can summarize our results concerning Aut B of finite dimensional atomic Banach lattices as follows:

Theorem 11.[19)20)] A mapping $F:B(E)\to E$ belongs to $Aut_o B(E)$

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18) $C^k(\mathbb{R},V)\equiv\{k$ times continuously differentiable $\mathbb{R}\to V$ functions$\}$ for every $k\in\mathbb{N}$ and topological vector space V.

19) Special case of the main theorem (stated without proof) in [Sun1] for convex complete finite dimensional Reinhardt domains.

20) For the notations see p.107,[15)], (32′) and (32"); E denoting a finite dimensional Banach lattice on $\mathbb{C}^X$ ($\equiv$\{X→C functions\} where X is a given (finite) set) such that $\|1_x\| = 1 \quad \forall x\in X$.

if and only if for each $m\in\mathcal{M}$ there can be found an $L^2(S_m)$-isometry U_m and for all $n\in\mathcal{M}_o$ there exist unit vectors $e_n\in L^2(S_n)$ and constants $\rho_n\in\mathbb{R}_+$, respectively, such that for any $f\in E$ and $m\in\mathcal{M}$, by setting $f_m\equiv f|_{S_m}$, we have

$$(34^*)\qquad (Ff)|_{S_m} = U_m[M_{\rho_m}(\langle f_m|e_m\rangle)\cdot e_m + M^{\perp}_{\rho_m}(\langle f_m|e_m\rangle)\cdot (f_m - \langle f_m|e_m\rangle\cdot e_m)] \quad \text{whenever} \quad m\in\mathcal{M}_o$$

$$(34^{**})\qquad (Ff)|_{S_m} = U_m[(\prod_{n\in\mathcal{M}_o} M^{\perp}_{\rho_n}(\langle f_n|e_n\rangle)^{2\gamma_{nm}})\cdot f_m] \quad \text{whenever} \quad m\in\mathcal{M}\setminus\mathcal{M}_o.$$

Proof. In view of Corollary 17 and Proposition 9, the only thing we have to see is that any $F\in \mathrm{Aut}_o B(E)$ can be factorized as $F = U\tilde{F}$ where U is a linear member of $\mathrm{Aut}_o B(E)$ and $\tilde{F}$ is of the form $\tilde{F} = \exp[B(E)\ni g\mapsto c + q_c(g,g)]$ for some $c\in E_o$. Observe that by Corollary 16 we can find $c_1\in E_o$ such that the mapping $F_1\equiv\exp[B(E)\ni g\mapsto c_1+q_{c_1}(g,g)]$ satisfies $F_1(0)=F(0)$. Now we have $F(F_1^{-1}(0))=0$. Then a classical theorem of Carathéodory establishes that the mapping $F\circ F_1^{-1}$ is linear, i.e. the choices $\tilde{F}\equiv F_1$ and $U\equiv F\circ F_1^{-1}$ satisfy our requirements. □

In order to generalize this theorem, it is more convenient to use the following equivalent results concerning $\log^*\mathrm{Aut}\,B(E)$ that are contained in Proposition 6, Theorem 10 and Propositions 7,8,10 but which can be deduced also directly from Theorem 11:

Corollary 18. $E_o=\{f\in E : f(x)=0 \quad \forall x\in \bigcup_{n\in\mathcal{M}_o} S_n\}$. For all $f\in E_o$ we have $\|f\|=\max\{\|f|_{S_n}\|_{L^2(S_n)} : n\in\mathcal{M}_o\}$. The Lie algebra $\log^*\mathrm{Aut}\,B(E)$ consists of those vector fields $v(.)$ on E for which there can be found $c\in E_o$, $\ell\in\{\text{linear } E\to E \text{ maps}\}$ and $q\in\{\text{bilinear } E\times E\to E \text{ maps}\}$ such that $v=[f\mapsto c+\ell(f)+q(f,f)]$, $\ell(E_{S_m})\subset \subset E_{S_m}$ $(\forall m\in\mathcal{M})$, $\langle \ell(1_x),1_y^*\rangle+\overline{\langle \ell(1_y),1_x^*\rangle}=0$ and $q(1_x,1_y)=-\gamma_{mn}\cdot \cdot[\overline{c(x)}1_y+\overline{c(y)}1_x]$ whenever $x\in S_m, y\in S_n$ $(\forall m,n\in\mathcal{M})$. The matrix Γ has the properties $0\le\gamma_{mn}\le 1$ $\forall m,n\in\mathcal{M}$, $\gamma_{mn}=\begin{cases}1/2 & \text{if } m=n\\ 0 & \text{if } m\neq n\end{cases}$ $\forall m,n\in\mathcal{M}_o$, $\gamma_{mn}=0$ $\forall m,n\notin\mathcal{M}_o$. □

Chapter 7

On Aut B in some infinite dimensional atomic Banach lattices

As we have seen in the previous chapter, the Projection Principle and the Kaup-Upmeier Theorem enables us to reduce a good deal of the algebraic description of Aut B(E) (for Banach lattices E specified at the beginning of Chapter 6) to merely three dimensional analogous problems. In a similar way as we have done it in finite dimensions, it is not hard to calculate the exact values of $q_c(1_x,1_y)$ and to show $\langle \ell(1_x),1_y^*\rangle + \overline{+\langle \ell(1_y),1_x^*\rangle} = 0$ for all $x,y \in X$, $c \in E_o$ and linear members ℓ of log*Aut B(E). It can be expected that, from these observations, the complete characterization of Aut B(E) is available by some limiting process if the functions 1_x ($x \in X$) are dense in some suitable sense in E.

To illustrate the power of the Projection Principle, we shall derive our further results directly from Corollary 18 (or which is the same from the main theorem of [Sun1]) and Theorems 6,7 without touching the details (even in a generalized form) of the proof of Corollary 18.

Throughout this chapter X denotes a (fixed) set and E is a Banach lattice on such a sublattice of $\mathbb{C}^X$ ($\equiv\{X \to C$ functions$\}$) that contains 1_x for all $x \in X$. We shall write $\mathfrak{J}$ for the upward directed net of the finite subsets of X and we assume $\|f\| = \sup\{\|1_Y f\| : Y \in \mathfrak{F}\}$ $(= \lim_{Y\in\mathfrak{F}} \|1_Y f\|)$ $\forall f \in E$ and $\|1_x\| = 1$ $\forall x \in X$.

For any $Y \subset X$, we set $E_Y \equiv 1_Y E (= \{f \in E : f|_{X \setminus Y} = 0\})$ and $Y_o \equiv \{y \in Y : \exists F \in \mathrm{Aut}\, B(E_Y)\ \ F(0)(y) \neq 0\}$. If $Y \in \mathfrak{F}$, Corollary 18 establishes the existence of a (unique) finest partition $\{S_m^Y : m \in \mathcal{M}(Y)\}$ of Y and a (unique) symmetric real matrix $\Gamma^Y \equiv (\gamma_{m,n}^Y)_{m,n\in\mathcal{M}(Y)}$, respectively, such that $\ell(E_{S_m^Y}) \subset E_{S_m^Y}$ for each linear $\ell \in \log^*\mathrm{Aut}\, B(E_Y)$ and $\gamma_{mn}^Y = 0$ whenever $S_m^Y, S_n^Y \not\subset Y_o$, $\gamma_{mn}^Y = \begin{cases} 1/2 & \text{for } m=n \\ 0 & \text{for } m \neq n \end{cases}$ whenever $S_m^Y, S_n^Y \subset Y_o$ and $[E_Y \ni f \mapsto 1_{S_m^Y} + 2\gamma_{mn}^Y (\sum_{x\in S_m^Y} f(x)) \cdot 1_{S_m^Y} f] \in \log^*\mathrm{Aut}\, B(E)$ whenever $S_m^Y \subset Y_o$ and $S_n^Y \not\subset Y_o$. We shall reserve the notations $\mathcal{M}(Y), S_m^Y, \gamma_{mn}^Y, \Gamma^Y$ to indicate the above partition of $Y(\in\mathfrak{F})$ and matrix on $\mathcal{M}(Y)$, respectively. Finally we shall write $\mathcal{M}_o(Y) \equiv \{m \in \mathcal{M}(Y) : S_m^Y \subset Y_o\}$ (for $Y \in \mathfrak{F}$).

Lemma 25. If $Y \subset Z \in \mathfrak{F}$ then the partition $\{S_n^Z \cap Y : n \in \mathcal{M}(Z)\}$ on Y is finer than $\{S_m^Y : m \in \mathcal{M}(Y)\}$ (i.e. $\forall m \in \mathcal{M}(Z)\ \exists \mathcal{N} \subset \mathcal{M}(Z)$ $S_m^Y = \bigcup_{n\in\mathcal{N}} S_n^Z$).

Proof. Let $\emptyset \neq Y \subset Z \in \mathfrak{F}$, $n \in \mathcal{M}(Z)$ and $y \neq x, y\ S_n^Z \cap Y$ be given. We have to see that $x, y \in S_m^Y$ for some $m \in \mathcal{M}(Y)$.

By Corollary 18, the vector field $\ell \equiv [E_Z \ni f \mapsto f(y)1_x - f(x)1_y]$

is a linear member of $\log^*\mathrm{Aut}\,B(E\ )$. Since the map $P \equiv$ $\equiv [E \ni f \mapsto 1_Y f]$ is a band projection of E_Z onto E_Y, Theorem 7′ establishes that the field $\ell_Y \equiv P\ell|_{E_Y}$ is a linear member of $\log^*\mathrm{Aut}\,B(E_Y)$. Observe that $\exp(t\ell_Y) = [E_Y\ f \mapsto (f(x)\cos t +$ $+ f(y)\sin t)1_x + (-f(x)\sin t + f(y)\cos t)1_y]$ $\forall t \in \mathbb{R}$. Thus $\exp(\ell_Y)$ maps the subspace $L^2(\{x,y\})$ isometrically onto itself and it vanishes on $L^2(Y \setminus \{x,y\})$. Hence Corollary 18 entails that x and y belong to the same member of the partition $\{S_m^Y : m \in \mathcal{M}(Y)\}$.□

Proposition 11. There exists a (unique) coarsest partition $\{S_m : m \in \mathcal{M}\}$ of X such that $\{S_m \cap Y : m \in \mathcal{M}\}$ is finer than $\{S_m^Y :$ $m \in \mathcal{M}(Y)\}$ for each $Y \in \mathfrak{F}$. Namely, this partition $\{S_m : m \in \mathcal{M}\}$ consists of the equivalence classes of the (equivalence) relation $\approx$ defined by

$$(37) \qquad x \approx y \overset{\text{def}}{\Longleftrightarrow} \forall Y \in \mathfrak{F}\ \ x,y \in Y \Longrightarrow \exists m \in \mathcal{M}(Y)\ \ x,y \in S_m^Y \ \text{(for } x,y \in X).$$

Proof. Clearly, if the relation $\approx$ is an equivalence then the partition of X formed by the equivalence classes of $\approx$ is coarser than any other partition $\{S'_m : m \in \mathcal{M}'\}$ of X with the property that $\forall Y \in \mathfrak{F}$ $\{S'_m \cap Y : m \in \mathcal{M}'\}$ is finer than $\{S_m^Y : m \in \mathcal{M}(Y)\}$. Hence it suffices to show that $\approx$ is an equivalence.

Reflexivity and symmetry of $\approx$ are trivial. To prove its transitivity, let $x,y,z \in X$ be arbitrarily chosen so that we have $x \approx y \approx z$ and let Y be any finite subset of X containing x and z. Now, by (37), we have $x,y \in S_m^{Y \cup \{y\}}$ and $y,z \in S_n^{Y \cup \{y\}}$ for some

$m,n\in\mathcal{M}(Y\cup\{y\})$ respectively. That is, $\exists n\in\mathcal{M}(Y\cup\{y\})$ $x,y,z\in S_n^{Y\cup\{y\}}$. But then Lemma 25 establishes that $\exists m\in\mathcal{M}(Y)$ $x,z\in S_m^Y$. □

In the sequel we shall keep fixed the notation $\{S_m : m\in\mathcal{M}\}$ for the partition of X described in Proposition 11.

Lemma 26. For all $m\in\mathcal{M}$ and $f\in E_{S_m}$ we have $\|f\|=\|f\|_{L^2(X)}$ $(<\infty)$.

Proof. Let $m\in\mathcal{M}$ and $f\in E_{S_m}$ be given. Now, by assumption, $\|f\|=\lim_{Y\in\mathfrak{F}}\|1_Y f\|$ holds. Furthermore, from Proposition 11 we deduce the existence of a (unique) mapping $n:\{Y\in\mathfrak{F}:S_m\cap Y\neq\emptyset\}\to\bigcup_{Y\in\mathfrak{F}}\mathcal{M}(Y)$ such that $n(Y)\in\mathcal{M}(Y)$ and $S_m\cap Y\subset S^Y_{n(Y)}$ whenever $Y\in\mathfrak{F}$ and $S_m\cap Y\neq\emptyset$. Using this map $n(.)$, we can write $1_Y f=1_{S^Y_{n(Y)}}\cdot f$ $(\forall Y\in\mathrm{dom}(n))$. Since $\{U$ linear $E_Y\to E_Y$ map$:U(E_{S^Y_r})=E_{S^Y_r},\ \|(Ug)|_{S^Y_r}\|_{L^2(S^Y_r)}=\|g|_{S^Y_r}\|_{L^2(S^Y_r)}$ $\forall g\in E_Y$ $\forall r\in\mathcal{M}(Y)\}=\exp\{\ell$ linear field on $E_Y:\ell(E_{S^Y_r})\subset E_{S^Y_r},\langle\ell(1_x),1_y^*\rangle+$ $+\overline{\langle\ell(1_y),1_x^*\rangle}=0$ $\forall x,y\in S^Y_r$ $\forall r\in\mathcal{M}(Y)\}\subset$ |by Corollary 18| $\subset\{$linear members of $\mathrm{Aut}_0B(E)\}=\{$surjective E_Y-isometries$\}$, we have $\|1_{S^Y_{n(Y)}}f\|=$ $=\|U_{f,Y}(1_{S^Y_{n(Y)}}f)\|=\|\,\|f|_{S^Y_{n(Y)}}\|_{L^2(S^Y_{n(Y)})}\cdot 1_{x_Y}\|=\|f|_{S^Y_{n(Y)}}\|_{L^2(S^Y_{n(Y)})}=$ $=\|1_{S^Y_{n(Y)}}f\|_{L^2(X)}=\|1_Y f\|_{L^2(X)}$ $\forall Y\in\mathrm{dom}(n)$ where x_Y denotes a (fixed) element of $S^Y_{n(Y)}$ and $U_{f,Y}$ is such a linear $E_Y\to E_Y$ map that for some $V\in\{L^2(Y)$-unitary operators$\}$ we have $(U_g)|_Y=$ $=V(g|_Y)$ $\forall g\in E_Y$ and $U_{f,Y}f=\|f|_{S^Y_{n(Y)}}\|_{L^2(S^Y_{n(Y)})}\cdot 1_{x_Y}$ and $U(E_{S^Y_r})=$

$= E_{S_r^Y} \quad \forall r \in \mathcal{M}(Y)$. Thus $\|f\| = \sup_{Y \in \mathfrak{F}} \|1_Y f\| = \sup_{Y \in \mathrm{dom}(n)} \|1_Y f\| =$

$= \sup_{Y \in \mathrm{dom}(n)} \|1_Y f\|_{L^2(X)} = \sup_{Y \in \mathfrak{F}} \|1_Y f\|_{L^2(X)} = \|f\|_{L^2(X)}$. □

Corollary 19. For any $Y \in \mathfrak{F}$ and $V \in \{L^2(Y)\text{-unitary operators}\}$, the map $U \equiv [E_Y \ni f \mapsto V(f|_Y) \cup [X \setminus Y \ni x \mapsto 0]]$ is an isometry of E_Y whenever $U(E_{S_r^Y}) = E_{S_r^Y} \quad \forall r \in \mathcal{M}(Y)$. Thus if $f, g \in E$ vanish outside of a finite subset of X and $\|f|_{S_m}\|_{L^2(S_m)} = \|g|_{S_m}\|_{L^2(S_m)} \quad \forall m \in \mathcal{M}$ then $\|f\| = \|g\|$. □

Lemma 27. Let $f, g \in E$ and assume $\|f|_{S_m}\|_{L^2(S_m)} \le \|g|_{S_m}\|_{L^2(S_m)}$ $\forall m \in \mathcal{M}$ [21]. Then $\|f\| \le \|g\|$.

Proof. Consider any $Y \in \mathfrak{F}$ and $\varepsilon \in [0,1)$. Since the family $\{m \in \mathcal{M} : Y \cap S_m \ne \emptyset\}$ is finite and since $\|g|_{S_m}\|_{L^2(S_m)} = \sup_{Z \in \mathfrak{F}} \|1_Z g|_{S_m}\|_{L^2(S_m)}$ $\forall m \in \mathcal{M}$, we can fix $Z_{Y,\varepsilon} \in \mathfrak{F}$ such that $Z_{Y,\varepsilon} \supset Y$ and $\|\varepsilon 1_Y f|_{S_m}\|_{L^2(S_m)} \le$ $\le \|1_{Z_{Y,\varepsilon}} g|_{S_m}\|_{L^2(S_m)} \quad \forall m \in \mathcal{M}$. For any $m \in \mathcal{M}$, let us pick a point $x_m \in S_m$. Then (from Lemma 26) we obtain $\|\varepsilon 1_Y f\| = \|\sum_{m \in \mathcal{M}} \|\varepsilon 1_Y f|_{S_m}\|_{L^2(S_m)} \cdot 1_{x_m}\|$ and $\|1_{Z_{Y,\varepsilon}} g\| = \|\sum_{m \in \mathcal{M}} \|1_{Z_{Y,\varepsilon}} g|_{S_m}\|_{L^2(S_m)} \cdot 1_{x_m}\|$. Therefore (since $0 \le$ $\le \sum_{m \in \mathcal{M}} \|\varepsilon 1_Y f|_{S_m}\|_{L^2(S_m)} 1_{x_m} \le \sum_{m \in \mathcal{M}} \|1_{Z_{Y,\varepsilon}} g|_{S_m}\|_{L^2(S_m)} 1_{x_m}$) $\|\varepsilon 1_Y f\| \le \|1_{Z_{Y,\varepsilon}} g\| \le$ $\le \|g\|$. Hence $\|f\| = \sup\{\|\varepsilon 1_Y f\| : \varepsilon \in [0,1), Y \in \mathfrak{F}\} \le \|g\|$. □

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[21] Remark that $\|g|_{S_m}\|_{L^2(S_m)} = \|1_{S_m} g\|_{L^2(X)} =$ by Lemma 31 $= \|1_{S_m} g\| < \|g\| < \infty$.

Corollary 20. If $\|f|_{S_m}\|_{L^2(S_m)} = \|g|_{S_m}\|_{L^2(S_m)}$ $\forall m \in \mathcal{M}$ then $\|f\| = \|g\|$. □

Proposition 12. Let $f \in \overline{\bigcup_{Y \in \mathfrak{F}} E_Y}$ and $g \in \mathbb{C}^X$ be such functions that $\|f|_{S_m}\|_{L^2(S_m)} = \|g|_{S_m}\|_{L^2(S_m)}$ $\forall m \in \mathcal{M}$. Then $g \in E$ and $\|f\| = \|g\|$.

Proof. Since $f \in \overline{\bigcup_{Y \in \mathfrak{F}} E_Y}$, we can choose a sequence $Y_1 \subset Y_2 \subset \ldots$ in $\mathfrak{F}$ and functions $f_1, f_2, \ldots \in E$ such that $f_n \in E_{Y_n}$ and $\|f - f_n\| < \frac{1}{n}$ $\forall n \in \mathbb{N}$. Since E is a Banach lattice,

$$\|f - 1_{Y_n} f\| = \|1_{X \setminus Y_n} f\| \leq \|(1_{Y_n} f - f_n) + 1_{X \setminus Y_n} f\| = \|f - f_n\| < \frac{1}{n} \quad \forall n \in \mathbb{N}.$$

Then consider the index families $\mathcal{M}_n \equiv \{m \in \mathcal{M} : S_m \cap Y_n = \emptyset\}$ (n=1,2,... observe that each $\mathcal{M}_n$ is finite). Since by Lemma 26 we have $\|f|_{S_m}\|_{L^2(S_m)} = \|1_{S_m} f\| \leq \|f\| < \infty$ $\forall m \in \mathcal{M}$, we also can choose a sequence $Z_1 \subset Z_2 \subset \ldots$ in $\mathfrak{F}$ such that $Z_n \subset \bigcup_{m \in \mathcal{M}_n} S_m$ and $\|1_{Z_n \cap S_m} f\|_{L^2(x)} > (1 - \frac{1}{n|\mathcal{M}_n|^3}) \|1_{S_m} f\|$ and $\|1_{Z_n \cap S_m} g\|_{L^2(x)} > (1 - \frac{1}{n|\mathcal{M}_n|^3}) \|1_{S_m} f\|$ for all $n \in \mathbb{N}$ where $|\mathcal{M}_n| \equiv$ cardinality $\mathcal{M}_n$. Now, by setting $\lambda_{nm} \equiv \|1_{Z_n \cap S_m} g\| / \|1_{Z_n \cap S_m} f\|$, hence we obtain

$$(38) \qquad 1 - \frac{1}{n|\mathcal{M}_n|^3} < \lambda_{nm} < \frac{1}{1 - \frac{1}{n|\mathcal{M}_n|^3}} \quad \forall m \in \mathcal{M}_n \ \forall n \in \mathbb{N} \setminus \{1\}.$$

Define the following sequences of functions in E with finite supports:

$$f_n \equiv 1_{Z_n} f \ , \quad g_n \equiv 1_{Z_n} g \ , \quad \tilde{g}_n \equiv \sum_{m\in\mathcal{M}_n} \lambda_{nm} 1_{Z_n \cap S_m} f \qquad (n=1,2,\ldots).$$

From Corollary 20 it follows $\|g_n\| = \|\tilde{g}_n\|$ $\forall n\in\mathbb{N}$ (since $\|g_n|_{S_m}\|_{L^2(S_m)} = \|\tilde{g}_n|_{S_m}\|_{L^2(S_m)}$ $\forall m\in\mathcal{M}$). To complete the proof, we shall show that

(39′) $(g_n : n\in\mathbb{N})$ is a Cauchy sequence in E

(39″) $\lim_{n\to\infty} \|\tilde{g}_n\|$ $(= \lim_{n\to\infty} \|g_n\| = \|f\|$.

Indeed: $|f_n - \tilde{g}_n| = |\sum_{m\in\mathcal{M}_n} (1-\lambda_{nm}) 1_{Z_n\cap S_m} f| \le \sum_{m\in\mathcal{M}_n} |1-\lambda_{nm}| \cdot 1_{Z_n\cap S_m} \cdot |f| \le$

$\le$by (38) $\le \frac{1}{n|\mathcal{M}_n|^3 - 1} \cdot \sum_{m\in\mathcal{M}_n} 1_{Z_n\cap S_m} |f| \le \frac{1}{n|\mathcal{M}_n|^3-1} |f|$ $(n>1)$. Hence

$\|f_n - \tilde{g}_n\| \le \frac{1}{n-1}\|f\| \to 0$ $(n\to\infty)$ which proves (39″).

To the proof of (39′), let us fix any $n, n'\in\mathbb{N}$ with $n'>n>1$. Now we have $\mathcal{M}_{n'} \supset \mathcal{M}_n$ and $Z_{n'} \supset Z_n$ whence

$$(40) \qquad |g_{n'} - g_n| = \sum_{m\in\mathcal{M}_{n'}\setminus\mathcal{M}_n} 1_{Z_n\cap S_m} |g| + \sum_{m\in\mathcal{M}_n} 1_{(Z_{n'}\setminus Z_n)\cap S_m} |g| \le$$

$$\le \sum_{m\in\mathcal{M}_{n'}\setminus\mathcal{M}_n} 1_{Z_n\cap S_m} |g| + \sum_{m\in\mathcal{M}_n} 1_{S_m\setminus Z_n} |g| .$$

Since $\|\sum_{m'\in\mathcal{M}_{n'}\setminus\mathcal{M}_n} 1_{Z_{n'}\cap S_{m'}} |g| \,|_{S_m}\|_{L^2(S_m)} = \|1_{\bigcup_{m'\in\mathcal{M}_{n'}\setminus\mathcal{M}_n} S_m} \cdot \tilde{g}_n|_{S_m}\|_{L^2(S_m)}$ $\forall m\in\mathcal{M}$,

we have by Corollary 20

$$(40')\quad \Big\|\sum_{m\in\mathcal{M}_{n'}\setminus\mathcal{M}_n} 1_{Z_{n'}\cap S_m}|g|\Big\| = \Big\|1_{\bigcup_{m'\in\mathcal{M}_{n'}\setminus\mathcal{M}_n} S_{m'}}\cdot \tilde g_{n'}\Big\| = \Big\|\sum_{m\in\mathcal{M}_{n'}\setminus\mathcal{M}_n} \lambda_{n'm} 1_{S_m} f_{n'}\Big\| =$$

$$= \Big\|\sum_{m\in\mathcal{M}_{n'}\setminus\mathcal{M}_n} \lambda_{n'm} 1_{S_m} |f_{n'}|\Big\| \le \text{by } (38) \le \frac{n'|\mathcal{M}_{n'}|^3}{n'|\mathcal{M}_{n'}|^3-1}\Big\|1_{\bigcup_{m\in\mathcal{M}\setminus\mathcal{M}_n} S_m}|f_{n'}|\Big\| \le$$

$$\le 2\Big\|f_{n'}-1_{\bigcup_{m\in\mathcal{M}_n} S_m} f_{n'}\Big\| = 2\Big\|1_{Z_{n'}}(f-1_{\bigcup_{m\in\mathcal{M}_n} S_m} f)\Big\| \le 2\|f-1_{Z_n} f\|.$$

On the other hand,

$$(40'')\quad \Big\|1_{\bigcup_{m\in\mathcal{M}_n} S_m\setminus Z_n}|g|\Big\| < \sum_{m\in\mathcal{M}_n}\|1_{S_m\setminus Z_n}|g|\| = \text{by Lemma 26} =$$

$$= \sum_{m\in\mathcal{M}_n}\|1_{S_m\setminus Z_n} g\|_{L^2} =$$

$$= \sum_{m\in\mathcal{M}_n}\sqrt{\|g|_{S_m}\|_{L^2}^2-\|g|_{S_m\cap Z_n}\|_{L^2}^2} = \sum_{m\in\mathcal{M}_n}\sqrt{\|f|_{S_m}\|_{L^2}^2-\lambda_{nm}^2\|f|_{Z_n\cap S_m}\|_{L^2}^2} \le$$

$$\le \sum_{m\in\mathcal{M}_n}\sqrt{\|f|_{S_m}\|_{L^2}^2-\lambda_{nm}^2\Big(1-\frac{1}{n|\mathcal{M}_n|^3}\Big)^2\|f|_{S_m}\|_{L^2}^2} \le \text{by } (38) \le$$

$$\le \sum_{m\in\mathcal{M}_n}\|f|_{S_m}\|_{L^2}\sqrt{1-\Big(1-\frac{1}{n|\mathcal{M}_n|^3}\Big)^4} \le \|f\|\sum_{m\in\mathcal{M}_n}\sqrt{1-\Big(1-\frac{1}{n|\mathcal{M}_n|^3}\Big)^4} =$$

$$= \|f\|\,|\mathcal{M}_n|\sqrt{\frac{4}{n|\mathcal{M}_n|^3}-\frac{6}{n^2|\mathcal{M}_n|^6}+\frac{4}{n^3|\mathcal{M}_n|^9}-\frac{1}{n^4|\mathcal{M}|_n^{12}}} \le \sqrt{\frac{15}{n|\mathcal{M}_n|}}\,\|f\|.$$

Combining (40), (40') and (40"), we obtain

$$\|g_{n'}-g_n\| \le \Big\|1_{\bigcup_{m\in\mathcal{M}_{n'}\setminus\mathcal{M}_n} Z_{n'}\setminus S_m}|g|\Big\| + \Big\|1_{\bigcup_{m\in\mathcal{M}_n} S_m\setminus Z_n}|g|\Big\| \le$$

$$\le 2\|f-f_n\| + (15/n)^{1/2}\|f\| \to 0 \quad \text{if} \quad n', n\to\infty. \ \square$$

Corollary 21. If $E = \overline{\bigcup_{Y\in\mathcal{F}} E_Y}$ and for any $m\in\mathcal{M}$, U_m denotes an $L^2(S_m)$-unitary operator then the mapping $U: E\to\mathbb{C}^X$ defined by $Uf \equiv \bigcup_{m\in\mathcal{M}} U_m(f|_{S_m})$ (i.e. $(Uf)|_{S_m} = U_m(f|_{S_m})$ $\forall m\in\mathcal{M}$) ranges in E, morever $U|_{B(E)}\in \mathrm{Aut}_o B(E)$. □

Lemma 28. For any $m\in\mathcal{M}$, let A_m be a (linear) $L^2(S_m)\to L^2(S_m)$ operator and define the mapping $A: E\to\mathbb{C}^X$ by $Af \equiv \bigcup_{m\in\mathcal{M}} A_m(f|_{S_m})$. Then $A(E)\subset E$ entails $\sup_{m\in\mathcal{M}} \|A_m\|_{L^2(S_m)} < \infty$.

Proof. Set $M \equiv \sup_{m\in\mathcal{M}} \|A_m\|_{L^2(S_m)}$ and assume $M=\infty$. Then we can choose a sequence $m_1, m_2, \ldots \in\mathcal{M}$ such that $\|A_{m_n}\|_{L^2(S_{m_n})} > n^3$ $\forall n\in\mathbb{N}$. Lemma 26 and the hypothesis $E = \overline{\bigcup_{Y\in\mathcal{F}} E_Y}$ imply $E_{S_m} = \{f\in\mathbb{C}^X : f|_{X\setminus S_m}=0,\ f|_{S_m}\in L^2(S_m)\}$ $\forall m\in\mathcal{M}$. Hence, for every $n\in\mathbb{N}$, we can choose a function $f_n\in E_{S_{m_n}}$ such that $\|f|_{S_{m_n}}\|_{L^2} = 1$ and $\|A_{m_n}(f_n|_{S_{m_n}})\|_{L^2} > \frac{n^3}{2}$. Consider now the function $f \equiv \sum_{n=1}^{\infty} \frac{1}{n^2} f_n$. Clearly $f\in E$. On the other hand, $(Af)|_{S_{m_n}} = \frac{1}{n^2} A_{m_n}(f|_{S_{m_n}})$ whence we would have $\|Af\| \geq \|1_{S_{m_n}} Af\| = \|1_{S_{m_n}} Af|_{L^2} = \|(Af)|_{S_{m_n}}\|_{L^2} \geq \frac{1}{n^2}\cdot\frac{n^3}{2} = \frac{n}{2}$ $\forall n\in\mathbb{N}$ in case of $Af\in E$ which is impossible. □

Corollary 22. If range $A\subset E$ then $\|A\| = M (\equiv \sup_{m\in\mathcal{M}} \|A_m\|_{L^2(S_m)}) < \infty$.

Proof. For any $f\in E$ and $m\in\mathcal{M}$, $\|(Af)|_{S_m}\|_{L^2}=\|A_m(f|_{S_m})\|_{L^2}\leq$ $\leq\|A_m\|_{L^2}\cdot\|f|_{S_m}\|_{L^2}\leq M\cdot\|f|_{S_m}\|_{L^2}$. Thus, by Lemma 27, $\|Af\|\leq M\|f\|$ $\forall f\in E$ i.e. $\|A\|\leq M$. The converse inequality is trivial, since from Lemma 26 we obtain $\|A\|=\sup_{\|f\|\leq 1}\|Af\|\geq\sup_{f\in B(E_{S_m})}\|Af\|=\|A_m\|_{L^2(S_m)}$ $\forall m\in\mathcal{M}$. □

Lemma 29. Let ℓ denote a linear vector field on E. Then to $\ell\in\log^*\mathrm{Aut}\,B(E)$ it is necessary and sufficient that there esist a family $\{A_m:\ iA_m\in\{\text{self adjoint } L^2(S_m)\text{-operators}\},\ m\in\mathcal{M}\}$ such that $\sup_{m\in\mathcal{M}}\|A_m\|_{L^2(S_m)}<\infty$ and $\ell(f)=\bigcup_{m\in\mathcal{M}}A_m(f|_{S_m})$ $(\in E\,!)$ $\forall f\in E$.

Proof. The sufficiency part of the statement is trivial. Necessity: Suppose $\ell\in\log^*\mathrm{Aut}\,B(E)$, $m\in\mathcal{M}$.

Consider any point $x\in S_m$. If $n\neq m$ and $y\in S_n$ then, by definition of the partition $\{S_n: n\in\mathcal{M}\}$, we can find $Z_y\in\mathfrak{F}$ such that $x,y\in Z$ and $\langle\tilde{\ell}(1_x),1_y^*\rangle=0$ $\forall\tilde{\ell}\in\log^*\mathrm{Aut}\,B(E)$. By the Projection Principle (Theorem 8′) $[E_{Z_y}\ni f\mapsto 1_{Z_y}\ell(f)]\in\log^*\mathrm{Aut}\,B(E_{Z_y})$, thus $\langle\ell(1_x),1_y^*\rangle=0$ $\forall y\in X\setminus S_m$. That is $\ell(1_x)\in E_{S_m}$. Since the functions with finite supports are dense in E_{S_m} $(\simeq L^2(S_m)$ by Lemma 26), we have $\ell(E_{S_m})\subset E_{S_m}$.

Consider any $y\in S_m$ and set $Y\equiv\{x,y\}$, $\ell_Y\equiv[E_Y\ni f\mapsto 1_Y\ell(f)]$. The Projection Principle establishes $\ell_Y\in\log^*\mathrm{Aut}\,B(E_Y)$. Since the partition $\{S_n\cap Y: n\in\mathcal{M}\}$ is finer than $\{S_n^Y: n\in\mathcal{M}(Y)\}$, we have $\{Y\}=$

$=\{S_n^Y : n\in\mathcal{M}(Y)\}$. Hence Corollary 18 implies $\langle \ell(1_x),1_y^*\rangle = \langle \ell_Y(1_x),1_y^*\rangle =$
$= -\overline{\langle \ell_Y(1_y),1_x^*\rangle} = -\overline{\langle \ell(1_y),1_x^*\rangle}$ $\forall x,y\in S_m$. Thus (by the classical Hellinger-Toeplitz Theorem) for some selfadjoint $L^2(S_m)$-operator B, $\ell(f)|_{S_m} = iB(f|_{S_m})$ $\forall f\in E$. Then Lemma 28 completes the proof. □

Next we turn to examine the quadratic part of log*Aut B(E). Henceforth we reserve the symbols $X_o, \mathcal{M}_o$ to denote the sets $X_o \equiv \{x\in X : \exists f\in E_o (\equiv \mathbb{C}\ \mathrm{Aut}\, B(E)\ \{0\})\ \ f(x)\neq 0\}$ and $\mathcal{M}_o \equiv \{m\in\mathcal{M} : \exists x\in X_o\ \ x\in S_m\}$, respectively.

Proposition 13. $X_o = \bigcup_{n\in\mathcal{M}_o} S_n$ and $\|f\| = \sup_{n\in\mathcal{M}_o} \|f|_{S_n}\|_{L^2(S_n)}$ $\forall f\in E_{X_o}$.

Proof. To $X_o = \bigcup_{n\in\mathcal{M}_o} S_n$, it is enough to show $S_n\setminus X_o \neq \emptyset$ $\forall n\in\mathcal{M}_o$. Assume $n\in\mathcal{M}_o$ and $x\in S_n\setminus X_o$. By definition of X_o, there exist $x_o\in S_n$ and $F_o\in \mathrm{Aut}\, B(E)$ with $F_o(0)(x_o)\neq 0$. Let $U : E\to\mathbb{C}^X$ be the mapping $Uf \equiv 1_{X\setminus\{x,x_o\}} f + f(x)1_{x_o} + f(x_o)1_x$. Since $(Uf)-f\in\mathbb{C}1_x + \mathbb{C}1_{x_o}$ $\forall f\in E$, range $U\subset E$. Then from Corollary 20 we see that U is an E-isometry. Thus, since $U^2 = \mathrm{id}_E$, we have $U|_{B(E)} = \mathrm{Aut}\, B(E)$. Hence $F\equiv U\circ F_o\in \mathrm{Aut}\, B(E)$ and $F(0)(x) = F_o(0)(x_o)\neq 0$ contradicting $x\notin X_o$.

Consider any $Y,Z\in\mathfrak{F}$ with $\emptyset\neq Y\subset Z\subset X_o$. Then the Projection Principle entails $\mathcal{M}_o(Y)=Y$ and $\mathcal{M}_o(Z)=Z$. By Proposition 10 (or if we want to use only Corollary 18 from Chapter 6 then by repeating the proof of Proposition 10) we have $\overline{B}(E_Y)$ $(=\overline{B}(E_{Y_o})) = \{g\in E_Y :$ $\|g|_{S_m^Y}\|_{L^2}<1$ $\forall m\in\mathcal{M}(Y)\ (=\mathcal{M}_o(Y)\}$ and $\overline{B}(E_Z) = \{g\in E_Z : \|g|_{S_m^Z}\|_{L^2}<1$ $\forall m\in$

$\in \mathcal{M}(Z)\}$. Therefore $\|g\| =$ gauge $\bar{B}(E_Y) = \inf\{\rho > 0 : \|g|_{S_m^Y}\|_{L^2} \le \rho \ \forall m \in \mathcal{M}(Y)\} =$

$= \max_{m \in \mathcal{M}(Y)} \|g|_{S_m^Y}\|_{L^2}$ whenever $g \in E_Y$ and similarly $\|g\| = \max_{m \in \mathcal{M}(Z)} \|g|_{S_m^Z}\|_{L^2}$ $\forall g \in E_Z$. Thus $\forall g \in E_Y$ $\|g\| = \max_{m \in \mathcal{M}(Y)} \|g|_{S_m^Y}\|_{L^2} = \max_{m \in \mathcal{M}(Z)} \|g|_{S_m^Z}\|_{L^2}$. This is possible only if

$$(41) \qquad \{S_m^Y : m \in \mathcal{M}(Y)\} = \{Y \cap S_m^Z : m \in \mathcal{M}(Z)\} \setminus \{\emptyset\} \Longleftarrow \emptyset \ne Y \subset Z \text{ finite} \subset X_o .$$

Since $\{S_m : m \in \mathcal{M}\}$ is the coarsest partition of X with the property $\{S_m \cap Y : m \in \mathcal{M}\}$ is finer than $\{S_m^Y : m \in \mathcal{M}(Y)\}$ $\forall Y \in \mathfrak{F}$, we must have by (41)

$$(41') \qquad \{S_m^Y : m \in \mathcal{M}(Y)\} = \{S_m \cap Y : m \in \mathcal{M}\} \setminus \{\emptyset\} \quad \forall Y \text{ finite} \subset X_o,$$

$$(42) \qquad \|g\| = \sup_{m \in \mathcal{M}_o} \|g|_{S_m}\|_{L^2} \quad \text{if } g \in E \text{ has finite support contained in } X_o.$$

Now let $f \in E_{X_o}$. By our basic assumption, $\|f\| = \sup_{Y \in \mathfrak{F}} \|1_Y f\| =$ $=$ by (42) $= \sup_{Y \in \mathfrak{F}} \sup_{n \in \mathcal{M}_o} \|f|_{Y \ S_n}\|_{L^2} = \sup_{n \in \mathcal{M}_o} \sup_{Y \in \mathfrak{F}} \|f|_{Y \cap S_n}\| = \sup_{n \in \mathcal{M}_o} \|f|_{S_n}\|_{L^2}$. □

Proposition 14. There exists a unique matrix $\Gamma \equiv (\gamma_{mn})_{m,n \in \mathcal{M}}$ such that $\gamma_{mn} = 0$ if $m,n \in \mathcal{M}_o$ and

$$(43) \qquad q_c(1_x, 1_y) = -\gamma_{mn}(\overline{c(x)}\, 1_y + \overline{c(y)}\, 1_x) \text{ whenever } c \in E_o, x \in S_m, y \in S_n .$$

This matrix Γ necessarily satisfies $0 \le \gamma_{mn} = \gamma_{nm} \le 1$ $\forall m,n \in \mathcal{M}$ and

$\gamma_{mm} = \frac{1}{2}, \gamma_{mn} = 0 \Leftarrow m \neq n \quad \forall m,n \in \mathcal{M}_o$.

Proof. Let $c \in E_o$, $x,y,z \in X$ be arbitrarily given and set $Y \equiv \{x,y,z\}$, $v \equiv [E_Y \ni f \mapsto 1_Y \cdot (c + q_c(f,f))]$. By definition of q_c (cf. Theorem 6 d)) and by the Projection Principle, we have $v \in$ $\in \log^*\mathrm{Aut}\,B(E_Y)$. Thus, since the map $q_c^Y \equiv [(f,g) \mapsto 1_Y q_c(f,g)]$ is the unique symmetric bilinear $E_Y \times E_Y \to E_Y$ transformation with $[f \mapsto 1_Y c + q(f,f)] \in \log^*\mathrm{Aut}\,B(E_Y)$, we have $1_Y q_c(1_x,1_y) = q_c^Y(1_x,1_y) =$ $= -\gamma_{mn}^Y(\overline{1_Y c(x)}1_y + \overline{1_Y c(y)}1_x) = -\gamma_{mn}^Y(\overline{c(x)}1_y + \overline{c(y)}1_x)$ if $x \in S_m^Y$ and $y \in S_n^Y$. Therefore $\langle q_c(1_x,1_y), 1_z^* \rangle = 0$ whenever $z \notin \{x,y\}$ and $\langle q_c(1_x,1_y), 1_x^* \rangle = -\gamma_{mn}^{\{x,y\}}\overline{c(x)}$, $\langle q_c(1_x,1_y), 1_y^* \rangle = -\gamma_{mn}^{\{x,y\}}\overline{c(y)}$ if $x \in S_m^{\{x,y\}}$ and $y \in S_n^{\{x,y\}}$. Hence existence and uniqueness of a symmetric matrix $\Gamma : \mathcal{M}^2 \to [0,1]$ satisfying (43) and $\gamma_{mn} = 0$ for $m,n \notin \mathcal{M}_o$ is immediate. Then suppose $m,n \in \mathcal{M}_o$, $x \in S_m, y \in S_n$. Write $Y \equiv \{x,y\}$ and let $x \in S_{m'}^Y$, $,y \in S_{n'}^Y$. From (41') we deduce $m=n \Leftrightarrow m'=n'$. The Projection Principle establishes $Y_o = Y$. Thus, by Corollary 18, $\gamma_{mn} = \gamma_{m'n'}^Y = \frac{1}{2}\delta_{m'n'} = \frac{1}{2}\delta_{mn}$. □

Lemma 30. Given $f,g \in \overline{\bigcup_{Y \in \mathcal{F}} E_Y}$, $m \in \mathcal{M}$, $z \in S_m$ and $c \in E_o$, we have $\sum_{n \in \mathcal{M}_o} \gamma_{mn} \sum_{x \in S_n} |f(x)||c(x)| < \infty$ and

$$\text{(44)} \qquad \langle q_c(f,g), 1_z^* \rangle = -f(z) \sum_{n \in \mathcal{M}_o} \gamma_{mn} \langle g|_{S_n} | c|_{S_n} \rangle_{L^2(S_n)} - g(z) \sum_{n \in \mathcal{M}_o} \gamma_{mn} \langle f|_{S_n} | c|_{S_n} \rangle_{L^2(S_n)}.$$

Proof. If f and g have finite support then $\langle q_c(f,g), 1_z^* \rangle =$
$= \sum_{x,y\in X} f(x)\, g(y) \langle q_c(1_x,1_y), 1_z^* \rangle = \sum_{n_1,n_2\in\mathcal{M}} \sum_{x\in S_{n_1}} \sum_{y\in S_{n_2}} (-\gamma_{n_1 n_2}) \cdot$
$\cdot f(x) g(y) \langle q_c(1_x,1_y), 1_z^* \rangle$. Hence we readily obtain (44) in this special case.

Then let f denote an arbitrary element of $\overline{\bigcup_{Y\in\mathfrak{F}} E_Y}$. From the order increasing property of the lattice norms, it is not hard to deduce that any $X\to\mathbb{C}$ function $\tilde{f}$ with $|\tilde{f}|<|f|$ belongs to $\overline{\bigcup_{Y\in\mathfrak{F}} E_Y}$, moreover $\tilde{f}$ is the norm limit of the net $(1_Y\tilde{f} : Y\in\mathfrak{F})$. Thus for the function $h : X\to\mathbb{C}$ defined by $h(x) \equiv$

$$\equiv \begin{cases} |f(x)|c(x)/|c(x)| & \text{if } x\neq z \text{ and } c(x)\neq 0 \\ -1/2 & \text{if } x=z \\ |f(x)| & \text{else} \end{cases}$$

we have $h\in \overline{\bigcup_{Y\in\mathfrak{F}} E_Y}$ and

$$\langle q_c(h,h), 1_z^* \rangle = \lim_{Y\in\mathfrak{F}} \langle q_c(1_Y h, 1_Y h), 1_z^* \rangle = \lim_{Y\in\mathfrak{F}} \sum_{n\in\mathcal{M}_o} \gamma_{mn} \langle 1_Y h|_{S_n} | c|_{S_n} \rangle_{L^2} =$$
$$= \lim_{Y\in\mathfrak{F}} \Big(\sum_{n\in\mathcal{M}_o} \sum_{x\in S_n\setminus\{z\}} \gamma_{mn} |f(x)||c(x)| - \overline{c(z)} \Big).$$

Therefore $\sum_{n\in\mathcal{M}_o} \sum_{x\in S_n} |f(x)||c(x)|<\infty$ $(\forall f\in \overline{\bigcup_{Y\in\mathfrak{F}} E_Y})$ whence (44) is immediate now. □

As matter of fact, only the restriction to $\overline{\bigcup_{Y\in\mathfrak{F}} E_Y}$ can be calculated from the values $\langle q_c(1_x,1_y), 1_z^* \rangle$ $(x,y,z\in X)$. Therefore it is convenient to restrict our attention only to that case when E coincides with $\overline{\bigcup_{Y\in\mathfrak{F}} E_Y}$.

From now on, we always assume, in addition,

$$(**) \qquad E = \overline{\bigcup_{Y\in\mathfrak{F}} E_Y};$$

and for each $x\in X$, $m(x)$ shall denote the (unique) element of satisfying $x\in S_{m(x)}$.

Remark 6. If $\tilde{E}$ is any Banach lattice of $X \to \mathbb{C}$ functions such that $1_x \in \tilde{E}$ $\forall x \in X$ and $\tilde{E} = \overline{\bigcup_{Y \in \mathfrak{F}} \tilde{E}_Y}$ then we automatically have $\|\tilde{f}\| = \sup_{Y \in \mathfrak{F}} \|1_Y \tilde{f}\|$ for all $\tilde{f} \in \tilde{E}$ since $f \in \mathbb{C}^X$ and $|f| < |\tilde{f}|$ imply $f \in \tilde{E}$ and $\lim_{Y \in \mathfrak{F}} \|f - 1_Y f\| = 0$. (Proof: For any $\varepsilon > 0$ choose $Y_\varepsilon \in \mathfrak{F}$ and $\tilde{f}_\varepsilon \in E_{Y_\varepsilon}$ so that $\|\tilde{f} - \tilde{f}_\varepsilon\| < \varepsilon$. Then $Y, Z \in \mathfrak{F}$ and $Y, Z \subset X \setminus Y_{\varepsilon/2}$ entail $|1_Y f - 1_Z f| \leq$
$\leq |1_Y f| + |1_Z f| \leq 2|1_{Y \cup Z} \tilde{f}| \leq 2|\tilde{f} - 1_{Y_{\varepsilon/2}} \tilde{f}| = 2\,|1_{X \setminus Y_{\varepsilon/2}} (\tilde{f} - \tilde{f}_{\varepsilon/2})| \leq 2|\tilde{f} - \tilde{f}_{\varepsilon/2}|$ whence $\|1_Y f - 1_Z f\| < \varepsilon$.)

Proposition 15. $\mathrm{Aut}\, B(E)\, \{0\} = B(E) \cap E_{X_0}$.

Proof. It suffices to prove that $1_x \in E_0$ for all $x \in X_0$. From (44) we have already a (unique) candidate to be q_{1_x}. Namely, given $x \in X_0$, this is the mapping $q \equiv [(f,g) \mapsto [z \mapsto -\gamma_{m(x)m(z)} (f(x)g(z) - g(x)f(z))]]$. Remark 6 immediately establishes that q is a continuous bilinear $E \times E \to E$ transformation. Consider the vector field $v \equiv [E \ni f \mapsto 1_x + q(f,f)]$ and for any $Y \in \mathfrak{F}$ with $x \in Y$, set $v_Y \equiv$ $\equiv 1_Y v|_{E_Y}$. Observe that we have $v_Y(f) = 1_x - 2f(x)\gamma_{m(x)m(.)} f$ $\forall f \in E_Y$. Therefore $v(1_Y f) = 1_Y v(f)$ $\forall f \in E$. Hence $E_Y \cap (\mathrm{dom}\, \exp(v)) = \mathrm{dom}\, \exp(v_Y)$ and $\exp(v_Y)(f) = \exp(v)(f)$ $\forall f \in \mathrm{dom}\, \exp(v_Y)$. On the other hand, by writing $m_Y(y)$ for that element of $\mathcal{M}(Y)$ which fullfills $y \in S^Y_{m_Y(y)}$ (for any y Y), from the proof of Proposition 14 we see $\gamma_{m(x)m(y)} = \gamma^Y_{m_Y(x)m_Y(y)}$ $\forall y \in Y$. Thus $v_Y = [E_Y \ni f \mapsto 1_x - 2f(x)\gamma^Y_{m_Y(x)m_Y(.)} f]$ i.e., by definition of the matrix Γ^Y, $v_Y \in \log^* \mathrm{Aut}\, B(E_Y)$. Consequently $\overline{B}(E_Y) \subset \mathrm{dom}\, \exp(v_Y) \subset \mathrm{dom}\, \exp(v)$.

Let f be an arbitrarily fixed element of $\overline{B}(E)$, $t^* \equiv \equiv \sup\{t\in\mathbb{R} : f\, \mathrm{dom}\, \exp(tv)\}$, $t_* \equiv \inf\{t\in\mathbb{R} : f\in \mathrm{dom}\, \exp(tv)\}$ and for any $t\in(t_*,t^*)$, set $f_t \equiv \exp(tv)(f)$. Then, for any $Y\in\mathfrak{F}$ with $x\in Y$, we have $\frac{d}{dt}1_Y f_t = 1_Y \frac{d}{dt} f_t = 1_Y v(f_t) = v(1_Y f_t) = v_Y(1_Y f_t)$ on (t_*,t^*). Hence $1_Y f_t = \exp(tv_Y)(1_Y f)$ $\forall t\in(t_*,t^*)$. Thus, since $1_Y f\in 1_Y\overline{B}(E) = = \overline{B}(E_Y)$, $1_Y f^t = \exp(tv_Y)(1_Y f)\in \exp(tv_Y)(\overline{B}(E_Y)) = \overline{B}(E_Y) \subset \overline{B}(E)$ $\forall t\in(t_*,t^*)$ $\forall Y\in\{Z\in\mathfrak{F} : x\in Z\}$. Therefore $\|f_t\| = \sup_{Y\in\mathfrak{F}} \|1_Y f_t\| \le 1$ $\forall t\in(t_*,t^*)$. Since the field v is Lipschitzian on every bounded set, $t^*<\infty$ would imply $\overline{\lim_{t\uparrow t^*}} \|f_t\| = \infty$. That is $t^*=\infty$, and similarly $t_* = -\infty$. This fact means that (by arbitrariness of $f\in\overline{B}(E)$) the vector field v is complete in $\overline{B}(E)$.

Then assume $f\in\partial B(E)$ and set $T\equiv\{t\in\mathbb{R} : f_t\in\partial B(E)\}$. Consider any $t\in T$. By the Piccard-Lindelöf Theorem, there exists $\delta_o>0$ such that $f_t+\delta_o B(E)\subset \mathrm{dom}\,\exp(\tau v)$ for all $\tau\in(-\delta_o,\delta_o)$. It is well-known (cf. [KU1]) that the exponential map of a holomorphic vector field restricted to any open subset of its domain is holomorphic. Hence $f_{t+\tau} = \exp(\tau v)(f_t) = \lim_{\rho\downarrow 1} \exp(\tau v)(\rho f_t)$ $\forall\tau\in(-\delta_o,\delta_o)$. Since v is complete in $\overline{D}$ and since $\rho f_t\notin\overline{D}$ for $\rho>1$, we have $\exp(\tau v)(\rho f_t)\notin \notin\overline{D}$ $\forall\rho>1$ $\forall\tau\in(-\delta_o,\delta_o)$. Consequently, $f_{t+\tau}\in\partial\overline{B}(E)=\partial B(E)$ $\forall\tau\in(-\delta_o,\delta_o)$, i.e. the set T is open in $\mathbb{R}$. On the other hand, T is obviously closed and non-empty. Therefore $T=\mathbb{R}$, i.e. the field v is complete in $\partial B(E)$ whence (by Lemma 13) we obtain $v\in\log^*\mathrm{Aut}\, B(E)$. Thus $1_x = v(0)\in E_o$. □

We can discover from the proof the following

Corollary 23. If $\tilde{E}$ is a Banach space, D a bounded balanced domain in $\tilde{E}$ and v denotes a polynomial vector field on $\tilde{E}$ then $v \in \log^* \mathrm{Aut}\, D$ if and only if v is complete in $\overline{D}$. □

Lemma 31. $\sup_{k \in \mathcal{M}} \sum_{j \in \mathcal{M}_o} \gamma_{jk} < \infty$.

Proof. For $k \in \mathcal{M}_o$, $\sum_{j \in \mathcal{M}_o} \gamma_{jk} = \sum_{j \in \mathcal{M}_o} \frac{1}{2} \delta_{jk} = \frac{1}{2}$. Thus if

$\sup_{k \in \mathcal{M}} \sum_{j \in \mathcal{M}_o} \gamma_{jk} = \infty$ then there exists a sequence of distinct points $z_1, z_2, \ldots \in X \setminus X_o$ and a sequence $J_1 \subset J_2 \subset \ldots$ of finite subsets of $\mathcal{M}_o$ such that $\sum_{j \in J_n} \gamma_{jm(2_n)} > n^5$ $(n=1,2,\ldots)$. From every set S_j $(j \in \mathcal{M})$, let us pick an element x_j and define the function $c : X \to \mathbb{C}$ as follows

$$c(x) \equiv \begin{cases} 0 \text{ if } x \notin \bigcup_{n=1}^{\infty} \{x_j : j \in J_n\} \\ \sup \{\frac{1}{m} : x = x_j \text{ and } j \in J_m\} \text{ if } x \in \bigcup_{n=1}^{\infty} \{x_j : j \in J_n\} \end{cases}.$$

Since $1_{S_j} c = 0$ if $j \notin \bigcup_{n=1}^{\infty} J_n$ and $1_{S_j} c = \sup\{\frac{1}{m} : j \in J_m\} \cdot 1_{x_j}$ if $j \in \bigcup_{n=1}^{\infty} J_n$, we have $|1_{\bigcup\{S_j : j \in J_{n_1}\}} c - 1_{\bigcup\{S_j : j \in J_{n_2}\}} c| =$

$$= \sum_{j \in J_{n_1} \setminus J_{n_2}} |1_{S_j} c| = \sum_{j \in J_{n_1} \setminus J_{n_2}} \sup\{\frac{1}{m} : j \in J_m\} 1_{x_j} \leq \sum_{j \in J_{n_1} \setminus J_{n_2}} \frac{1}{n_2+1} 1_{x_j}$$

whenever $n_1 > n_2$. Hence $\| 1_{\bigcup\{S_j : j\in J_{n_1}\}} c - 1_{\{S_j : j\in J_{n_2}\}} c \| \le$

$\le \| \sum_{j\in J_{n_1}\setminus J_{n_2}} \frac{1}{n_2+1} 1_{x_j} \| =$ by Proposition 13 $= \frac{1}{n_2+1} \to 0$ if $n_1 > n_2 \to \infty$.

Thus $(1_{\bigcup\{S_j : j\in J_n\}} c : n\in\mathbb{N})$ is a Cauchy sequence whence $c\in E_o$

(since supp(c) $\bigcup_{n=1}^{\infty} \bigcup_{j\in J_n} S_j$).

Then consider the function $f \equiv c + \sum_{n=1}^{\infty} \frac{1}{n^2} 1_{z_n}$. Clearly $f\in E$.

However, we would have $\infty > \| q_c(f,f) \| \ge | \langle q_c(f,f), 1^*_{z_n} \rangle | =$ by (44) $=$

$= | -2f(z_n) \sum_{j\in \mathcal{M}_o} \gamma_{jm(z_n)} \langle f|_{S_j} | c|_{S_j} \rangle_{L^2(S_j)} | = \frac{2}{n^2} \sum_{j\in\mathcal{M}_o} \gamma_{jm(z_n)} \| c|_{S_j} \|^2_{L^2(S_j)} =$

$= \frac{2}{n^2} \sum_{j\in\mathcal{M}_o} \gamma_{jm(z_n)} \sup \{ \frac{1}{m^2} : j\in J_m \} \ge \frac{2}{n^2} \sum_{j\in J_n} \gamma_{jm(z)} \frac{1}{n^2} >$ by hypothesis $>$

$> \frac{n^5}{n^4} = n \quad \forall n\in\mathbb{N}$. □

Proposition 16. Let $c\in E_o$ and $f\in \overline{B}(E)$ be given. Suppose $c|_{S_j} = \rho_j c_j^o$ where $\rho_j \in \mathbb{R}_+$ and c_j^o is a unit vector in $L^2(S_j)$ $(j\in\mathcal{M}_o)$. Set $f^t \equiv \exp[g \mapsto t\cdot(c+q_c(g,g))]$ and $f_j^t \equiv f^t|_{S_j}$ $(j\in\mathcal{M})$ and denote by P_j the orthogonal projection $h \mapsto h - \langle h | c_j^o \rangle_{L^2(S_j)} c_j^o$ in $L^2(S_j)$ $(j\in\mathcal{M})$. Then

$$(45') \quad f_j^t = M_{\rho_j t}(\langle f_j^o | c_j^o \rangle_{L^2(S_j)}) c_j^o + M^{\perp}_{\rho_j t}(\langle f_j^o | c_j^o \rangle_{L^2(S_j)}) P_j f_j^o \quad \text{if } j\in\mathcal{M}_o$$

$$(45'') \quad f_k^t = \exp\left[-2 \sum_{j\in\mathcal{M}_o} \gamma_{jk}\rho_j \int_0^1 M_{\rho_j\tau}(\langle f_j^o | c_j^o\rangle_{L^2(S_j)})\,d\tau\right] f_k^o$$

$$\text{for } k\in\mathcal{M}\setminus\mathcal{M}_o$$

where the functions $M_\tau, M_\tau^\perp$ are defined by (32′) and (32″).

Proof. We have $\frac{d}{dt} f^t = c + q_c(f^t, f^t)$ and $f^o = f$. Thus if $j\in\mathcal{M}_o$ then

$$(46') \quad \frac{d}{dt} f_j^t = \text{by (44)} = \rho_j c_j^o + [S_j \ni z \mapsto -2f^t(z) \sum_{k\in\mathcal{M}_o} \gamma_{jk} \langle f_k^t | \rho_j c_j^o\rangle] =$$

$$= \text{by Proposition 14} = \rho_j(c_j^o - f_j^t \cdot \langle f_j^t | c_j^o\rangle).$$

Thus $f_j^t = f_j^o$ $(\forall t\in\mathbb{R})$ if $\rho_j = 0$. If $\rho_j \neq 0$ then we obtain (45′) from (46′) by applying Lemma 22 (31).

Then let $k\in\mathcal{M}\setminus\mathcal{M}_o$ and $z\in S_k$ be arbitrarily fixed. We have

$$(46'') \quad \frac{d}{dt} f^t(z) = \text{by (44)} = -2f^t(z) \sum_{j\in\mathcal{M}_o} \gamma_{jk}\langle f_j^t | \rho_j c_j^o\rangle = \text{by (45′)} =$$

$$= -2f^t(z) \sum_{j\in\mathcal{M}_o} \gamma_{jk}\rho_j M_{\rho_j t}(\langle f_j^o | c_j^o\rangle).$$

Since $\rho_j \le \|c\|$, $|\langle f_j^o | c_j^o\rangle_{L^2(S_j)}| < 1$ $\forall j\in\mathcal{M}_o$ and $M_\tau \in \mathrm{Aut}\bar{\Delta}$ $\forall \tau\in\mathbb{R}$, Lemma 31 establishes that the function $\tau \mapsto \sum_{j\in\mathcal{M}_o} \gamma_{jk}\rho_j M_{\rho_j\tau}(\langle f_j^o | c_j^o\rangle)$ is a bounded continuous function. Hence the initial value problem

$$\begin{cases} \frac{d}{dt}\varphi = -2\varphi(t) \sum_{j\in\mathcal{M}_o} \gamma_{jk}\rho_j M_{\rho_j\tau}(\langle f_j^o | c_j^o\rangle) \\ \varphi(0) = f(z) \end{cases}$$

admits a unique solution. One

readily verifies that this solution is $t \mapsto \exp[-2 \int_0^1 \sum_{j \in \mathcal{M}_0} \gamma_{jk} \rho_j \cdot$

$\cdot M_{\rho_j \tau}(\langle f_j^0 | c_j^0 \rangle) d\tau] f(z)$ whence (45") is immediate. □

<u>Corollary 24.</u> $\{\exp|f \to q_c(f,f)|(0) : c\ E_0\} = B(E_0)\ (=B(E) \cap E_{X_0})$.

<u>Proof.</u> Since, by (**), the $\mathbb{C}^X$-functions with finite support form a dens subset of E, from Proposition 13 it follows

$$(47) \quad E_0(=E_{X_0}) = \{f \in \mathbb{C}^X : \forall j \in \mathcal{M}_0 \quad f|_{S_j} \in L^2(S_j)$$
$$\text{and } \forall \varepsilon > 0 \ \{x \in X : |f(x)| > \varepsilon\} \quad \text{finite} \subset X_0\}.$$

Given $c \in E_0$, let $c|_{S_j} = \rho_j c_j^0$ where $\rho_j \in \mathbb{R}_+$ and $c_j^0 \in \partial B(L^2(S_j))$ $(j \in \mathcal{M})$. Set $\xi_j \equiv \text{area th}\,(\rho_j)$ $(j \in \mathcal{M})$. Since the function area th (.) is continuous at the point 0, from (47) we see that the function $c^* \equiv$ $\equiv \bigcup_{j \in \mathcal{M}} \xi_j c_j^0$ belongs to E_0. By Proposition 16, $\exp[f \mapsto c^* + q_{c^*}(f,f)](0) =$ $= (\bigcup_{j \in \mathcal{M}_0} M_{\xi_j}(0) c_j^0) \cup |X \setminus X_0 \ni X \mapsto 0|$ = by (32') $= \bigcup_{j \in \mathcal{M}} \text{th}(\xi_j) c_j^0 = \bigcup_{j \in \mathcal{M}} \rho_j c_j^0 = c$. □

<u>Corollary 25.</u> For every $F \in \text{Aut}\,B(E)$ there exist a unique $c \in E_0$ and a unique E-unitary operator L such that $F = L \exp[B(E) \ni$ $\ni f \mapsto c + q_c(f,f)]$. Moreover, if $F \in \text{Aut}_0 B(E)$ then $L|_{B(E)} \in \text{Aut}_0 B(E)$.

<u>Proof.</u> Given $F \in \text{Aut}\,B(E)$, by Corollary 24 there exists c E_0 such that the automorphism $Q \equiv \exp|B(E)\ f \to c + q_c(f,f)|$ satisfies $Q^{-1}(0) = F^{-1}(0)$. Furthermore, (45') establishes that such a choice of c is unique. Now we have $(F\ Q^{-1})(0) = 0$, thus (by Charathéodory's Theorem) the automorphism $F \circ Q^{-1}$ is linear. The second statement follows from the fact that $Q \in \text{Aut}_0 B(E)$. □

Hence the complete description of $\text{Aut}_0 B(E)$ is already immediate: Lemmas 29,30,31 ensure that the proofs of Lemma 23 and Lemma 24 can be carried out also in infinite dimensions without any modification. Thus we can conclude that the linear members of $\mathrm{Aut}_0 B(E)$ leave invariant each ball $B(E_{S_j})$ $(j\in\mathcal{M})$. Therefore, by Corollary 25 and Proposition 16, we can summarize our results in an abstract setting as follows.

Theorem 11*. If E denotes an atomic Banach lattice whose minimal ideals span (algebraically) a norm-dense submanifold of E then there exists a unique family $\{E_m : m\in\mathcal{M}\}$ of pairwise (lattice-) orthogonal Hilbertian projection bands (i.e., by the Neumann-Jordan Theorem [Ber1], projection bands with the property $\forall m\in\mathcal{M}\ \forall f,g\in E_m$ $\|f+g\|^2 = 2\|f\|^2 + 2\|g\|^2$) such that

1) any linear member of $\mathrm{Aut}_0 B(E)$ maps $B(E_m)$ onto itself $(m\in\mathcal{M})$

2) conversely, if for any $m\in\mathcal{M}$, U_m is an E_m-unitary operator then for some linear $L\in\mathrm{Aut}_0 B(E)$ we have $U|_{B(E_m)} = L|_{B(E_m)}$ $\forall m\in\mathcal{M}$.

Furthermore there exists a (unique) symmetric matrix $\Gamma \equiv (\gamma_{mn})_{m,n\in\mathcal{M}}$ of real numbers belonging to $[0,1]$ and a subset $\mathcal{M}_0$ of $\mathcal{M}$ that satisfy

3) $\forall m,n\in\mathcal{M}\setminus\mathcal{M}_0$ $\gamma_{mn}=0$, $\forall j\in\mathcal{M}_0$ $\gamma_{jj}=\frac{1}{2}$ and $\forall j,k\in\mathcal{M}_0$ $j\neq k \Rightarrow \gamma_{jk}=0$

4) $E_0 (\equiv \mathrm{C\,Aut}_0 B(E)\,\{0\}) = \text{norm-span}\,\{E_m : m\in\mathcal{M}_0\}$

5) $\|f\| = \sup\{\|P_m f\| : m\in\mathcal{M}_0\}$ $\forall f\in E_0$ where P_m denotes the band projection associated with E_m $(m\in\mathcal{M})$

6) a mapping $F : B(E)\to E$ belongs to $\mathrm{Aut}_0 B(E)$ if and only if

one can find E_m-unitary operators U_m $(m\in\mathcal{M})$ and unit vectors $c_j^o\in E_j$ with constants $\rho_j\in\mathbb{R}_+$ $(j\in\mathcal{M}_o)$, respectively, such that

$$\forall\varepsilon>0 \quad \{j\in\mathcal{M}_o:\ \rho_j>\varepsilon\} \text{ is finite,}$$

$$P_jF(f)=U_j[M_{\rho_j}(\langle P_jf|c_j^o\rangle)c_j^o+M_{\rho_j}(\langle P_jf|c_j^o\rangle)\cdot$$

$$\cdot(P_jf-\langle P_jf|c_j^o\rangle c_j^o)] \text{ if } j\in\mathcal{M}_o,$$

$$P_mF(f)=\exp\int_0^1\sum_{j\in\mathcal{M}_o}\gamma_{jm}\rho_jM_{\rho_j\tau}(\langle P_jf|c_j^o\rangle)d\tau\cdot U_mP_mf \text{ whenever } m\in\mathcal{M}\setminus\mathcal{M}_o$$

for all $f\in B(E)$ where M_t and $M_t^{\perp}$ stand for the $\bar{\Delta}\to\bar{\Delta}$ transformations (32'),(32"). □

Solution of the fixed point problem if $E=\overline{\bigcup_{\gamma\in\mathcal{F}}E_\gamma}$

In view of Proposition 16 and Corollary 25 we can give a definitive answer in a non-trivial special case of the question that motivated originally our investigations:

Theorem 12. Let E be an atomic Banach lattice which is norm-spanned by its minimal ideals. Then each biholomorphic automorphism of $\bar{B}(E)$ has a fixed point if and only if E_o $(\equiv\mathbb{C}\,\mathrm{Aut}\,B(E)\{0\})$ is a finite ℓ^∞-direct sum of Hilbert subspaces of E.

Proof. We may assume without loss of generality (cf. [Sch1]) that for some abstract set X, E is a sublattice of $\mathbb{C}^X$ containing the characteristic function of any finite subset of X endowed with such a complete lattice norm that $\|1_x\|=1$ $\forall x\in X$ and E =

$= \overline{\bigcup_{Y\in\mathfrak{F}} E_Y}$ (here $\mathfrak{F} \equiv \{$finite subsets of $X\}$, $E_Y \equiv 1_Y E$ as previously). Introducing the partition $\{S_m : m\in\mathcal{M}\}$ of X, the subfamily $\mathcal{M}_o$ of the index set $\mathcal{M}$ and the matrix Γ described in Propositions 11, 13 and 14, respectively, we have to show that $\mathcal{M}_o$ is finite if and only if $\forall F\in \mathrm{Aut}\,\overline{B}(E)\ \ \exists f\in\overline{B}(E)\ \ Ff = f$.

Suppose we can choose a sequence of distinct indexes $m_1, m_2, \ldots \in \mathcal{M}_o$. Then let us pick a point $x_j \in S_{m_j}$ for every $j\in\mathbb{N}$ and define the map $\tilde{F} : \overline{B}(E) \to \mathbb{C}^X$ by

$$(\tilde{F}f)(x_j) \equiv \frac{f(x_j) + \mathrm{th}(1/j)}{1 + f(x_j)\,\mathrm{th}(1/j)} \qquad (j = 1,2,\ldots)$$

$$(\tilde{F}f)(x) \equiv f(x)\ \exp \int_0^1 \sum_{j=1}^{\infty} \frac{1}{j}\gamma_{m_j n} \cdot \frac{f(x_j) + \mathrm{th}(\tau/j)}{1 + f(x_j)\,\mathrm{th}(\tau/j)}\, d\tau$$

whenever $x\in S_n \setminus \{x_1, x_2, \ldots\}$

Proposition 16 establishes $\tilde{F}\in \mathrm{Aut}_o\overline{B}(E)$ (cf. also [KU1,Corollary]). However, from $\tilde{F}f = f$ it would follow $f(x_j) = (+1 \text{ or } -1)\ \ \forall j\in\mathbb{N}$ which is impossible by (47).

Assume $\mathcal{M}_o$ is finite. Then any $F\ \mathrm{Aut}\,\overline{B}(E)$ is weakly continuous on E_o. (Indeed: By Corollary 25 we have $F = L\circ Q$ where L is a suitable E-operator and $Q = \exp[\overline{B}(E)\ni f\mapsto c+q_c(f,f)]$ for some $c\in E_o$. From (45'), finiteness of $\mathcal{M}_o$ and the weak continuity of the mappings $f\mapsto \langle f|_{S_j} | c_j^o\rangle$ we deduce that $\forall j\in\mathcal{M}_o\ \ f\mapsto 1_{S_j}\cdot Qf$ is weakly continuous. Thus, by (45"), $Q = \sum_{j\in\mathcal{M}_o} 1_{S_j} Q$ i.e. a finite sum of weakly continuous maps. Hence we conclude by remarking that Banach space operators are weakly continuous.) From the definition of E_o it follows $(\mathrm{Aut}\,\overline{B}(E))(\overline{B}(E_o)) = \overline{B}(E_o)$. Since the closed unit ball of any Hilbert space is weakly compact and since $\overline{B}(E_o)$ is homeomorphic to $\underset{j\in\mathcal{M}_o}{\times} \overline{B}(L^2(S_j))$, hence it follows by the Tychonoff-Schauder Fixed Point Theorem (see [DS1]) that every $F\in\mathrm{Aut}\,\overline{B}(E)$ admits fixed point. $\square$

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