LIPSCHITZIAN RETRACTS AND CURVES AS FRACTALS

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Abstract. We prove that, under not too restrictive conditions, the union of finitely many strong fractals, that is invariant sets of finite families of proper contractions, as defined in [1], is a strong fractal. Hence we establish collage theorems for non-affine strong fractals in terms of Lipschitzian retracts. We show that any rectifiable curve is a strong fractal though there is a simple arc which is not a strong fractal.

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1. Introduction, basic concepts

Let (X, d) be a metric space. By a *contraction* on X we mean a mapping $T: X \to X$ such that $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$. Given $\lambda \geq 0$, the mapping $T: X \to X$ is called a λ-*contraction* if $d(Tx, Ty) \leq \lambda d(x, y)$ for all $x, y \in X$. A set $K \subset X$ is a *weak Lipschitzian retract* of X if $K = R(K) = R(X)$ for some λ-contraction $R: X \to X$. (The usual terminology is the following: K is a *Lipschitzian retract* of X if $K = R(X)$ for some λ-contraction $R: X \to X$ such that $R(x) = x$ for all $x \in K$).

In accordance with [2], by a *fractal* in X we mean a non-empty compact set $K \subset X$ such that $K = \bigcup_{i=1}^{n} T_i(K)$ for some contractions $T_1, \ldots, T_n: X \to X$. We say that K is an ε-*fractal* if the above maps T_i are ε-contractions. If K is an ε-fractal for some $0 < \varepsilon < 1$, then we call it a *strong fractal*. We also say that the set K is the *invariant set* of $(T_1, \ldots, T_n)$, see [1]. It is known that for any finite set of λ-contractions, with $\lambda < 1$, there is a unique invariant set, see [1]. For the classical examples of fractals the contractions above are usually similarities or affine transformations. So here we use the term fractal in a wider sense. We proved in [2] that the set of strong fractals in $\mathbb{R}^N$ is a nowhere dense F_σ subset of the set of compact subsets of $\mathbb{R}^N$ with respect to the Hausdorff metric.

In this article we deal with two questions. The first one is whether the union of strong fractals is a strong fractal; the second question is whether continuous curves are strong fractals. We prove in Proposition 2.2 that the union of finitely many strong fractals is a strong fractal under the condition that the sets are weak Lipschitzian retracts of the space. It turns out that, in the Euclidean spaces, if the sets are compact weak Lipschitzian retracts of the space with at least one of them having non-empty interior, then their union is a strong fractal; the sets don't have to be strong fractals; see Theorem 2.3.

As for the second question, we prove in Theorem 3.6 that any rectifiable curve is a strong fractal. We also give an example of a simple arc, in fact the graph of a continuous function, that is not a strong fractal; see Theorem 4.3.

2. Unions of weak Lipschitzian retracts

As a preparation, we prove the following simple statement.

2.1. Lemma. *Let K be an ε_0-fractal in X for some $0 < \varepsilon_0 < 1$. Then K is an ε-fractal for any $0 < \varepsilon < 1$.*

Proof. Trivially, if $0 < \eta < \varepsilon < 1$ and K is an η-fractal, then K is an ε-fractal. On the other hand, K is an ε_0^m-fractal for any $m = 1, 2, \dots$ for the following reason. By assumption, $K = \bigcup_{i=1}^n T_i(K)$ for some ε_0-contractions $T_1, \dots, T_n \colon X \to X$. Then the composed mappings $T_{i_1} \circ \dots \circ T_{i_m}$ are all ε_0^m-contractions with $K = \bigcup_{i_1,\dots,i_m=1}^n T_{i_1} \circ \dots \circ T_{i_m}(K)$. $\square$

2.2. Proposition. *Let $K_1, \dots, K_r$ be strong fractals and weak Lipschitzian retracts of X at the same time. Then $\bigcup_{j=1}^r K_j$ is a strong fractal.*

Proof. There is a finite constant $\lambda > 1$ such that $K_j = R_j(K_j) = R_j(X)$, $j = 1, \dots, r$ with some λ-contractions $R_1, \dots, R_r \colon X \to X$. By Lemma 2.1, there are finitely many λ^{-2}-contractions $T_{k,1}, \dots, T_{k,n_k} \colon X \to X$, such that $K_k = \bigcup_{i=1}^{n_k} T_{k,i}(K_k)$, $k = 1, \dots, r$. Then the mappings $R_j \circ T_{k,i}$, $1 \le j, k \le r$, $1 \le i \le n_k$, are λ^{-1}-contractions such that

$$\bigcup_{j=1}^r K_j = \bigcup_{j=1}^r R_j(K_j) = \bigcup_{j=1}^r R_j\Big(\bigcup_{k=1}^r K_k\Big) = \bigcup_{j,k=1}^r R_j(K_k) = \bigcup_{j,k=1}^r R_j\Big(\bigcup_{i=1}^{n_k} T_{k,i}(K_k)\Big)$$
$$= \bigcup_{j,k=1}^r \bigcup_{i=1}^{n_k} R_j \circ T_{k,i}(K_k) \, . \quad \square$$

2.3. Theorem. *Let $K_1, \dots, K_r$ be compact weak Lipschitzian retracts of $\mathbb{R}^N$ with at least one of them having non-empty interior. Then $K = \bigcup_{j=1}^r K_j$ is a strong fractal in $\mathbb{R}^N$.*

Proof. Assume $\mathrm{Int}(K_1) \ne \emptyset$, that is, the interior of the set K_1 is not empty. We may also assume that $0 \in \mathrm{Int}(K_1)$. Since each set K_j is a weak Lipschitzian retract of $\mathbb{R}^N$, there exist a constant $\lambda > 1$ and λ-contractions $R_1, \dots, R_r \colon \mathbb{R}^N \to \mathbb{R}^N$ so that $K_j = R_j(K_j) = R_j(\mathbb{R}^N)$, $1 \le j \le r$. For any point $p \in \mathbb{R}^N$, consider the λ^{-2}-homothetic map $T_p(x) := p + \lambda^{-2}(x - p)$, $x \in \mathbb{R}^N$. Since $0 \in \mathrm{Int}(K)$, for any $p \in \mathbb{R}^N$ the image $T_p(K_1)$ is a neighborhood of p. Hence, by the compactness of K, there are finitely many points

$p_1,\ldots,p_n \in K$ such that $\bigcup_{i=1}^n T_{p_i}(K) \supset K$. We complete the proof with the observation that the mappings $R_j \circ T_{p_i}$ are λ^{-1}-contractions with

$$K = \bigcup_{j=1}^N R_j(K_j) = \bigcup_{j=1}^N R_j(K) = \bigcup_{j=1}^N R_j(\mathbb{R}^N) = \bigcup_{j=1}^N R_j\Big(\bigcup_{i=1}^n T_{p_i}(K)\Big)$$
$$= \bigcup_{i=1}^n \bigcup_{j=1}^N R_j \circ T_{p_i}(K)\ . \quad \square$$

2.4. Remark. It is well known that any closed convex subset of $\mathbb{R}^N$ is a contractive retract of $\mathbb{R}^N$. On the other hand, using an argument similar to the one used in the proof of Theorem 2.3, we can see that *any non-empty compact convex subset of* $\mathbb{R}^N$ *is a strong fractal.* Indeed, let $0<\varepsilon<1$ and a non-empty compact convex set $K \subset \mathbb{R}^N$ be given. Let U be the smallest affine subspace of $\mathbb{R}^N$ containing K. The case of $\dim(U)=0$ (K consisting of one point) is trivial. Assume $\dim(U)\geq 1$. Then the interior of K with respect to the relative topology of U is non-empty. Since K is compact, there is a finite family $\{v_1,\ldots,v_n\}\subset U$ such that $\bigcup_{i=1}^n \big(\varepsilon K + (1-\varepsilon)v_i\big) \supset K$. Thus, by Proposition 2.2, we can also conclude that *any finite union of non-empty compact convex subsets of* $\mathbb{R}^N$ *is a strong fractal.*

2.5. Example. *Any finite union of bilipschitzian images of the unit ball in* $\mathbb{R}^N$ *is a strong fractal.* For a proof, observe that any bilipschitzian image of the closed unit ball B of $\mathbb{R}^N$ is a Lipschitzian retract of $\mathbb{R}^N$. Indeed, if $K=F(B)$ with some mapping $F\colon \mathbb{R}^N \to \mathbb{R}^N$ such that $\lambda^{-1}d(x,y) \leq d\big(F(x),F(y)\big) \leq \lambda d(x,y)$ and $\lambda>1$, then each component $\phi_k\colon F\to\mathbb{R}$, $1\leq k\leq N$ of the mapping $F^{-1}\colon K\to B$ extends to a Lipscitzian function $\widehat{\phi}_k\colon \mathbb{R}^N \to \mathbb{R}$ with Lipschitz constant λ. Hence F^{-1} admits an extension $\Phi\colon \mathbb{R}^N\to\mathbb{R}^N$ with Lipschitz constant at most $\sqrt{N}\lambda$. The metric projection P which maps every point $x\in\mathbb{R}^N$ to the nearest point in B is a well-defined 1-contraction. Then we have $K=R(K)=R(\mathbb{R}^N)$ with the retraction $R := F\circ P\circ\Phi$ having Lipschitz constant at most $\sqrt{N}\lambda^2$. As a consequence of the N-dimensional analog of Jordan's curve theorem, any homeomorphic image of B in $\mathbb{R}^N$ is a compact set with a non-empty simply connected interior. Thus we can apply Theorem 2.3 to conclude the proof of the statement.

3. Curves as fractals

3.1. Proposition. *Assume C is a compact Lipschitzian retract of the metric space X (with distance function d) such that C itself is a strong fractal in the space $(C,d|C)$. Then C is a strong fractal also in (X,d).*

Proof. By assumption, there exists a mapping $F\colon X\to X$ such that $F(X)=F(C)=C$ and $d\big(F(x),F(y)\big)\leq\lambda d(x,y)$, $x,y\in X$, for some constant $\lambda>1$. Since C is a strong fractal in $(X,d|X)$, by Lemma 1, there are λ^{-2}-contractions $L_1,\ldots,L_r\colon C\to C$ (with respect to the metric d) such that $C = \bigcup_{i=1}^r L_i(C)$. Then, with the λ^{-1}-contractions $\widetilde{L}_i := L_i \circ F\colon X \to C$, we have $C = \bigcup_{i=1}^r L_i(C) = \bigcup_{i=1}^r \widetilde{L}(X) = \bigcup_{i=1}^r \widetilde{L}(C)$. $\square$

3.2 Definition. By a *rectifiable curve* in a metric space (X,d) we mean a mapping $\Gamma\colon [\alpha,\beta] \to X$ from a finite interval to X such that

$$\mathrm{Length}(\Gamma) := \inf\Big\{ \sum_{i=1}^n d\big(\Gamma(t_{i-1}),\Gamma(t_i)\big) :\ \alpha = t_0 < t_1 < \cdots < t_n = \beta,\ n = 1,2,\ldots\Big\} < \infty.$$

3.3. Remark. Every rectifiable curve admits an arc-length proportional reparametrization. That is, if $\Gamma\colon [\alpha,\beta] \to X$ is a rectifiable curve, then there is a rectifiable curve $\widetilde{\Gamma}\colon [0,\mathrm{Length}(\Gamma)] \to X$ with the following properties. We have $\widetilde{\Gamma} = \Gamma \circ \tau$ with some monotone increasing function $\tau\colon [0,\mathrm{Length}(\Gamma)] \to [\alpha,\beta]$ such that

$$\mathrm{Length}(\widetilde{\Gamma}|[\xi,\eta]) = \mathrm{Length}(\Gamma)|[\tau(\xi),\tau(\eta)]) = \eta - \xi$$

for all $0 \le \xi < \eta \le \mathrm{Length}(\Gamma)$.

3.4. Lemma. *Let $\Gamma\colon [0,\ell] \to X$ be an arclength parametrized rectifiable curve. Then its range $C := \Gamma[0,\ell]$ is a Lipschitzian retract of X.*

Proof. Notice that Γ is a 1-contraction. Define $\delta := \max\big\{d\big(\Gamma(0),\Gamma(t)\big) :\ t \in [0,\ell]\big\}$ and let

$$F(x) := \Gamma\big(\ell\delta^{-1} d(\Gamma(0),x)\big), \qquad x \in X.$$

Then F is an $\ell\delta^{-1}$-contraction of X onto C with $F(C) = \Gamma\{\ell\delta^{-1}d(\Gamma(0),\Gamma(t) :\ t \in [0,\ell]\} = \Gamma[0,\ell] = C$. $\square$

3.5. Lemma. *Let $\Gamma\colon [0,\ell] \to X$ be an arclength parametrized rectifiable curve. Then its range $C := \Gamma[0,\ell]$ is a strong fractal in itself.*

Proof. By assumption, with the interval $I := [0,\ell]$, we have $\Gamma\colon I \to X$ with $\eta - \xi = \mathrm{Length}(\Gamma|[\xi,\eta])$ for all $0 \le \xi < \eta \le \ell$. Fix $\varepsilon \in (0,1)$ arbitrarily and, for each point $z \in X$, define the mappings

$$F_{z,s}^{(\sigma)}(x) := \Gamma\Big(P_I\big(s + \sigma\varepsilon\big[d\big(x,z\big) - d\big(x,\Gamma(s)\big)\big]\big)\Big), \qquad x \in X,\ \sigma = \pm 1,$$

where $P_I\colon \mathbb{R} \to I$ is the metric projection $P_I(\xi) := \max\{0,\min\{\ell,\xi\}\}$ onto the interval $I = [0,\ell]$. Each $F_{z,s}^{(\sigma)}$ is an ε-contraction into the range $C := \Gamma(I)$ of Γ. Since C cannot consist only of a single point, given any parameter $s \in I$, we can choose a point $z_s = \Gamma(t_s) \in C$ such that $z_s \ne \Gamma(s)$. Then the closed interval $I_s := \big\{d\big(\Gamma(t),z_s\big) - d\big(\Gamma(t),\Gamma(s)\big) :\ t \in [0,\ell]\big\}$ containing 0 has non-zero length. Therefore, the interval $\bigcup_{\sigma=\pm1}\big[s + \sigma\varepsilon I_s\big]$ is a neighborhood of s. Thus, by the compactness of the interval I, there are finitely many parameters $s_1, \dots, s_r \in I$ such that $I \subset \bigcup_{i=1}^r \bigcup_{\sigma=\pm 1} \big[s + \sigma\varepsilon I_s\big]$. It follows that

$$\begin{aligned} C = \Gamma(I) &= \Gamma\Big(P_I\big(\bigcup_{i=1}^r \bigcup_{\sigma=\pm 1} \big[s + \sigma\varepsilon I_s\big]\big)\Big) \\ &= \bigcup_{i=1}^r \bigcup_{\sigma=\pm 1} \Gamma\Big(P_I\big(\big[s + \sigma\varepsilon I_s\big]\big)\Big) \\ &= \bigcup_{i=1}^r \bigcup_{\sigma=\pm 1} F_{z_{s_i}, s_i}(C)\ . \quad \square \end{aligned}$$

3.6. Theorem. *The range $\Gamma[\alpha, \beta]$ of any rectifiable curve $\Gamma\colon [\alpha, \beta] \to X$ is a strong fractal.*

Proof. This is immediate from Proposition 3.1 and Lemmas 3.4 and 3.5. $\square$

4. A counter-example

4.1. Lemma. *Let I be an interval in $\mathbb{R}$ and let $f\colon I \to \mathbb{R}$ be a continuous function. Assume $y_1, y_2 \in f(I)$. Then there is a closed subinterval J of I such that $f(J) = [y_1, y_2]$.*

Proof. Let $t_1, t_2 \in I$ be any couple of points with $f(t_i) = y_i$, $i = 1, 2$. We may assume $t_1 < t_2$ and $y_1 < y_2$. Let $x_1 := \sup\{t :\in [t_1, t_2] :\ f(t) = y_1\}$. Clearly $f(x_1) = y_1$, $x_1 < t_2$ and, by Bolzano's theorem, $f(t) > y_1$ for $x_1 < t \le t_2$. Then let $x_2 := \inf\{t :\in [x_1, t_2] : f(t) = y_2\}$. In this case $f(x_2) = y_2$, $x_1 < x_2$ and $f(t) < y_1$ for $x_1 < t \le x_2$. Thus the choice $J := [x_1, x_2]$ suits the requirements of the lemma. $\square$

4.2. Lemma. *Let $F\colon [\alpha, \beta] \to \mathbb{R}^2$ be an injective continuous mapping. Then any contractive image of the arc $C := F[\alpha, \beta]$ which is contained in C is also a simple arc of the form $F[\alpha', \beta']$ for some subinterval $[\alpha', \beta'] \subsetneq [\alpha, \beta]$.*

Proof. The arc C is compact and connected and the inverse mapping $F^{-1} : C \to [\alpha, \beta]$ is continuous. Consider any contraction $T : C \to C$. Since T is necessarily continuous, the set $T(C)$ is also compact. Observe that $\mathrm{diam}(C) > \mathrm{diam}(T(C))$. Indeed, for some points $a', b' \in T(C)$ we have $d(a', b') = \mathrm{diam}(T(C))$. Now there are points $a, b \in C$ with $T(a) = a'$ and $T(b) = b'$. Since T is a contraction, we have $\mathrm{diam}(C) \ge d(a, b) > d(a', b') = \mathrm{diam}(T(C))$. It follows that $T(C) \subsetneq C$. The set $F^{-1} \circ T(C)$ is a compact connected proper subset of $[\alpha, \beta] = F^{-1}(C)$. Hence $F^{-1} \circ T(C) = [\alpha', \beta'] \subsetneq [\alpha, \beta]$ with suitable $\alpha', \beta' \in [\alpha, \beta]$. $\square$

4.3. Theorem. *There is a simple arc in $\mathbb{R}^2$ which is not a strong fractal.*

Proof. Given any index $n = 1, 2, \ldots$, for $i = 0, 1, \ldots, 8^n$, let $\alpha_{n,i} := 2^{-n} + 16^{-n} i$ denote the endpoints of the equidistant partition of the interval $I_n := [2^{-n}, 2^{1-n}]$ into 8^n pieces. Furthermore, for $i = 1, \ldots, 8^n$, let $\beta_{n,i}$ be the middle point of the interval $I_{n,i} := [\alpha_{n,i-1}, \alpha_{n,i}]$, that is, $\beta_{n,i} = 2^{-n} + 16^{-n}(i - 1/2)$. In the sequel we shall write $I_{n,i}^{(-)} := [\alpha_{n,i-1}, \beta_{n,i}]$ and $I_{n,i}^{(+)} := [\beta_{n,i}, \alpha_{n,i}]$ for the left and right half of the interval $I_{n,i}$, respectively. In terms of these sequences, introduce the following points on the plane $\mathbb{R}^2$. Let $a_{n,i} := (\alpha_{n,i}, 0)$ $(n = 1, 2, \ldots,\ i = 0, \ldots, 8^n)$ and let $b_{n,i}$ denote the point in the positive half-plane $\mathbb{R} \times \mathbb{R}_+$ lying in distace 2^{1-n} from both $a_{n,i-1}$ and $a_{n,i}$. Notice that $b_n = \big(2^{-n} + (i - \frac{1}{2})16^{-n}, \delta_n\big)$ where $\delta_n = \sqrt{2^{2-2n} - 2^{-8n-2}}$. Consider the straight line segments $C_{n,i}^{(-)} := \big[a_{n,i-1}, b_{n,i}\big]$ and $C_{n,i}^{(+)} := \big[a_{n,i}, b_{n,i}\big]$, respectively. Observe that their union with the origin

$$C := \{(0,0)\} \cup \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{8^n} C_{n,i}^{(-)} \cup C_{n,i}^{(+)}$$

coincides with the graph in $\mathbb{R}^2$ of the continuous function $f\colon [0,1] \to \mathbb{R}^2$ defined by

$$f(0) := 0, \quad f(t) := \delta_n \arcsin\big|\sin(8^n \pi t)\big|\big) \ \text{ for } \ t \in I_n := [2^{-n}, 2^{1-n}] \ \ (n = 1, 2, \ldots).$$

Introduce the subpolygons $C_n := \operatorname{graph}(f|I_n) = \bigcup_{i=1}^{8^n} C_{n,i}^{(-)} \cup C_{n,i}^{(+)}$. It follows that

$$\operatorname{Length}(C_n) = 8^n \cdot 2 \cdot 2^{1-n} = 4^{n+1} > 3 \sum_{k=1}^{n-1} \operatorname{Length}(C_k), \qquad n = 2, 3, \ldots.$$

We are going to show that the curve C is not a fractal in $\mathbb{R}^2$ (moreover in C itself). Assume the contrary. Then there are finitely many contractions $T_1, \ldots, T_r\colon C \to C$ such that $C = \bigcup_{j=1}^{r} T_j(C)$. According to Lemma 4.2, each image set $T_j(C)$ is a simple arc. Hence

$$T_j(C) = F(K_j), \ \text{where}\, F(t) := \big(t, f(t)\big)\ (0 \le t \le 1) \text{ and } K_j \text{ is some closed interval} \subsetneq [0,1].$$

Clearly $\bigcup_{j=1}^{r} K_j = [0,1]$. Thus we may assume without loss of generality that $K_1 = [0, \alpha]$ for some $0 < \alpha < 1$. In particular, there is $n > 1$ with $K_1 \supset I_n = [2^{-n}, 2^{1-n}]$ and hence $C_n = F(I_n) \subset T_1(C)$. An application of Lemma 4.1 (with the mapping $S := F^{-1} \circ T_1 \circ F$ which sends $[0,1]$ continuously onto $[0,\alpha]$) shows that the inverse image by T_1 of each segment $C_{n,i}^{(\pm)}$ contains an arc (homeomorphic image of an interval) $D_{n,i}^{(\pm)} \subset C$ such that

$$T_1\big(D_{n,i}^{(\varepsilon)}\big) = C_{n,i}^{(\pm)}, \quad \varepsilon = \pm,\ i = 1, \ldots, 8^n.$$

Since T_1 is a contraction,

$$\operatorname{Length}\big(D_{n,i}^{(\varepsilon)}\big) = \operatorname{diam}\big(D_{n,i}^{(\varepsilon)}\big) > \operatorname{diam}\big(C_{n,i}^{(\varepsilon)}\big) = \operatorname{Length}\big(C_{n,i}^{(\varepsilon)}\big) = 2^{1-n}, \quad \varepsilon = \pm,\ i = 1, \ldots, 8^n.$$

Also there are closed intervals $K_i^{(\pm)} \subset [0,1]$, $i = 1, \ldots, 8^n$ such that

$$D_{n,i}^{(\varepsilon)} = F\big(K_i^{(\varepsilon)}\big), \quad \varepsilon = \pm,\ i = 1, \ldots, 8^n.$$

We have the following possibilities:

(1) $K_{n,i}^{(\varepsilon)} \subset [0, 2^{-n}]$;

(2) there is at most one of the intervals $K_{n,i}^{(\varepsilon)}$ containing the point 2^{-n} in its interior;

(3) $K_{n,i}^{(\varepsilon)} \subset [2^{-n}, 2^{1-n}]$;

(4) there is at most one of the intervals $K_{n,i}^{(\varepsilon)}$ containing the point 2^{1-n} in its interior;

(5) $K_{n,i}^{(\varepsilon)} \subset [2^{1-n}, 1]$.

Case (1) is impossible, because then we would have $\mathrm{diam}(D_{n,i}^{(\varepsilon)}) \leq \mathrm{diam}(F[0, 2^{-n}]) = \mathrm{diam}\big(\bigcup_{m=n+1}^{\infty} C_m) < 2^{1-n} = \mathrm{diam}(C_{n,i}^{(\varepsilon)})$.

In Case (3), the relation $\mathrm{diam}(D_{n,i}^{(\varepsilon)}) > \mathrm{diam}(C_{n,i}^{(\varepsilon)}) = 2^{1-n}$ is fulfilled only if $D_{n,i}^{(\varepsilon)}$ contains at least two segments of the form $C_{n,j}^{(\eta)}$. Hence there are at most $8^n/2$ such arcs $D_{n,i}^{(\varepsilon)}$.

Thus, taking into account also Cases (2) and (4) there are at least $8^n - 8^n/2 - 2 = 8^n/2 - 2$ arcs $D_{n,i}^{(\varepsilon)} = F(K_{n,i}^{(\varepsilon)})$ with $K_{n,i}^{(\varepsilon)}$ belonging to Case (5). Therefore, the total length of these arcs must be greater than $(8^n/2 - 2)2^{1-n} = 4^n - 2^{2-n}$. This exceeds the length of the arc $F[2^{1-n}, 1] = \bigcup_{m<n} C_m$, a contradiction. $\square$

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