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Dedicated to the memory of Prof. I. Joó

On the level set method in minimax theory

L.L. STACHÓ [1,2]

ABSTRACT. *Our purpose is to give a unified treatment along with some new generalizations of a series of non-linear minimax theorems of Sion type involving one or two functions proved via intersection theorems applied to their level sets.*

1. Introduction. Since the appearance of von Neumann's minimax theorem [14] the phenomena of inequalities of the type $(*)$ $\inf_{x\in X}\sup_{y\in Y} f(x,y) = \sup_{y\in Y}\inf_{x\in X} f(x,y)$ attracted mathematicians continuously. The idea of attacking the problem through the structure of the level sets (L) $\{x \in X : f(x,y) \geq \alpha\}, \{y \in Y : f(x,y) \leq \alpha\}$ appeared first perhaps in a work of M. Sion [16]: to conclude $(*)$ in topological vector spaces, along with the continuity of the function f it suffices to require the convexity and compactness of all the level sets (L). In 1972, Brézis-Nirenberg-Stampacchia [1] established that $(*)$ holds if each member of (L) is only algebraically closed and some of them is compact. Later on, it was shown [7, 18] that this theorem implies Ky Fan's minimax theorem [4] and its extended version by König with $\frac{1}{2}$-convexity [13] via a bilinear lifting procedure.

Still in 1969 Wu [20] suggested that the notion of convexity should be extended toward non-linearity: in particular he observed that the system of straight line segments can be replaced by certain systems of Jordan arcs for the definition of convex sets in a minimax theorem of Sion type. In 1980 Joó [6] rediscovered Wu's trick in the context of classical convex-concave functions and his simple proof became popular. The author [17] recognized that modifications in this proof lead to minimax theorems far beyond the context of topological vector spaces: the latter can even be replaced by general topological spaces and a system (called interval space structure) of arbitrary connected sets joining all couples of points can play the role of straight line segments in a Sion type minimax theorem. In 1989 Kindler-Trost [10] deduced the natural analogue of the Brézis-Nirenberg-Stampacchia theorem in the setting of interval spaces and, by introducing the concept of pavements [11], recently Kindler gave an interesting axiomatic study of sufficient conditions on the level sets of f leading to minimax theorems. In 1981 Komiya [12] suggested that an abstract convex hull operation for the purposes of mathematical economy and minimax theory should be nothing else as a finitely generated monotone singleton preserving closure operation. In 1991 C. Horvath [5] found a theorem which enabled applications of continuous selection arguments, Brower's fixed point theorem and Kastner-Kuratowski-Mazurkiewicz type arguments analogous to the familiar ones used in the context of classical linear convexity in the case of any Komiya type convex hull operation with contractible values for finite sets (called now H-convexity in a widespread terminology).

Recently completely abstract minimax theorems without even topology [8,22] and two functions minimax problems of the type $\sup_x \inf_y f(x,y) \leq \inf_y \sup_x g(x,y)$ with $f \leq g$ [9,3,19 and references therein] turned to focus of attention. Concerning the level set method, besides positive results, in the pioneering work of Thompson and Yuan [19] an example appears indicating the limits of the use of successful arguments for one function minimax theorems in interval spaces. In contrast, very recently Wu and Zhang [21] found a far reaching two functions minimax theorem in terms of H-convexity of level sets.

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In this paper we present a lemma (called Metatheorem) which covers the schemes of the proofs of the above mentioned two functions minimax theorems in interval spaces and which leads to a new and natural one function minimax theorem. We show that some new results stated in terms of Chang's W-convexity [2] respectively in pure terms of topology involving seemingly no convexity are available by interval spaces. On the other hand, by a careful logical analysis, we point out that several minimax results concerning H-convexity can be extended beyond the setting of Komiya type convexity.

2. Generalized convexity

Definition. 2.1. Let X be any set and let $s : 2^X \to 2^X$ be a given map assigning some (possibly empty) subset to any subset of X. We say that a subset C of X is *s-convex* (or we write C s-convex $\subset X$ if $s(Z) \subset C$ $(Z \subset C)$. We refer to s as the *generator map* of a convexity or simply as a convexity on X.

A convexity s on X is *finitely generated* if $s(Z) = \emptyset$ for any Z infinite $\subset Z$.

A function $\phi : X \to \mathbb{R}$ is *s-quasiconvex* if $\{x : \ \phi(x) < \alpha\}, \{x : \ \phi(x) \le \alpha\}$ s-convex $\subset X$ $(\alpha \in \mathbb{R})$. Furthermore ϕ is *s-quasiconcave* if $-\phi$ is s-quasiconvex.

Remark. 2.2. The following fundamental properties of s-convexity are straightforward.

1) Intersections of families of s-convex sets are s-convex.
2) The union of an increasing net of s-convex sets is s-convex if s is finitely generated.
3) The map $\overline{s}(Z) := \bigcap\{C \ \ s\text{-convex} \subset X : \ \ C \supset Z\}$ $(Z \subset X)$ is a Komiya type convex hull operation whenever s is finitely generated and $s\{x\} = \{x\}$ $(x \in X)$. Conversely, a finitely generated convexity $s : 2^X \to 2^X$ is a Komiya type convex hull operation if and only if $s\{x\} = \{x\}$ $(x \in X)$ holds along with the monotonicity property $s(A) \supset s(B) \supset B$ $(X \supset A \supset B)$.

The postulate $s\{x\} = \{x\}$ $(x \in X)$ seems to be superfluous even in several typical applications of Komiya type convexities as we shall see in the context of generalized H-convexities. One of the key observations in [20,6,17] giving rise to several KKM-type arguments can immediately be generalized to s-convexity as follows.

Lemma. 2.3. *Let* $f : X \times Y \to \mathbb{R}$, $s : 2^Y \to 2^Y$. *Given* $\alpha \in \mathbb{R}$, *equivalent statements are*

(i) $\{y : \ f_1(x,y) \le \alpha\}$ s-quasiconvex $\subset Y$ *for all* $x \in X$,

(ii) $K^{(\alpha)}(y) \subset \bigcup_{z \in Z} K_1^{(\alpha)}(z)$ *whenever* $y \in s(Z)$ *and* $Z \subset X$.

In the sequel we shall be interested in the following two extreme types of convexities.

Example. 2.4. *1) Convexities of interval spaces.* Recall [17] that an interval space is a topological space X equipped with a map $[\ ,\ ] : \ X \times X \to \{\text{connected subsets of } X\}$ such that $a, b \in [a,b] = [b,a]$ $(a, b \in X)$. For $a, b \in X$, the figure $[a,b]$ is usually called the interval between the points a, b and a set $C \subset X$ is convex (with respect to interval structure $[\ ,\ ]$) if $[a,b] \subset C$ $(a, b \in C)$. Clearly, the set C is $[\ ,\ ]$-convex iff it is s-convex for the finitely (actually binarily) generated convexity

$$s\{a,b\} := [a,b] \quad \text{for } a, b \in X, \quad s(Z) := \emptyset \quad \text{for } \#Z \ne 2 \text{ or } \#Z \ne 1 \ .$$

In [10] even the axiom of symmetry $[a,b] = [b,a]$ is relaxed, however, the symmetric interval structure $\langle a,b\rangle := [a,b] \cup [b,a]$ $(a, b \in X)$ leads to the same convex sets as $[\ ,\ ]$.

Convex sets in Chang's W-spaces [2] can completely be characterized as s-convex sets for some finitely generated $s : 2^X \to 2^X$ with connected values with respect to some topology on X such that $s(F) \supset F$ $(F \text{ finite} \subset X)$ and $s(\emptyset) = \emptyset$. By passing to the truncated generator function $s_2(Z) := [s(Z) \text{ if } \#Z \le 2, \ \ \emptyset \text{ if } \#Z > 2]$ we get the convexity of an interval space such that $\{\,s\text{-convex sets}\} \subset \{\,s_2\text{-convex sets}\}$. Thus, since interval spaces can also be regarded as W-spaces, interval structures are order dense among W-structures on a given set with respect to the ordering by inclusion of the families of convex sets. Therefore a statement is valid for all W-spaces if and only if it holds for all interval spaces. In particular the setting of interval spaces is sufficient to deduce Sion type minimax theorems with W-convexity.

2) Generalized H-convexity. By a generalized H-convexity on a topological space X we mean a finitely generated convexity $s : 2^X \to 2^X$ such that

$$s(F) := \bigcup_{F \subset G \text{ finite} \subset X} \varphi_G(\Delta_F) \qquad (F \text{ finite} \subset X)$$

for some family $\Phi := \{\varphi_F : \ F \text{ finite} \subset X\}$ of continuous mappings $\varphi_F : \Delta_F \to X$ where Δ_F is the standard notation for the abstract simplex

$$\Delta_F := \{(\lambda_x : \ x \in F) : \ 0 \le \lambda_x \ \ (x \in F), \ \textstyle\sum_{x \in F} \lambda_x = 1\}$$

with vertices in F. According to a celebrated theorem of Ch. Horvath [5], in our terminology, given any finitely generated convexity $S : 2^X \to \{\text{connected subsets of } X\}$ with $F \subset S(F) \subset S(G)$ $(F \subset G \text{ finite} \subset X)$, there exits a generalized H-convexity $S^* : 2^X \to 2^X$ such that $\{\,S\text{-convex sets}\} \subset \{\,S^*\text{-convex sets}\}$. Obviously, pseudoconvex spaces in the sense of [9] are generalized H-spaces.

3. Two functions minimax theorems on interval spaces

In general, let X, Y be any non-empty sets and let $(A, >)$ be an index net directed upward. For any index $\alpha \in A$, let $K^{(\alpha)}, L^{(\alpha)}$ be setvalued functions $Y \to 2^X$ such that

1) $K^{(\alpha)}(y) \subset K^{(\beta)}(y), \ L^{(\alpha)}(y) \subset L^{(\beta)}(y), \ \emptyset \ne K^{(\alpha)}(y) \subset L^{(\alpha)}(y)$ $(\alpha > \beta, \ y \in Y)$.

Let $\mathcal{K} := \{\bigcap_{(\alpha,y) \in F} K^{(\alpha)}(y) : \ F \text{ finite} \subset A \times Y\}$, $\mathcal{L} := \{\bigcap_{(\alpha,y) \in F} L^{(\alpha)}(y) : \ F \text{ finite} \subset A \times Y\}$.

Lemma. 3.1. (Metatheorem). *Suppose* $I : Y \times Y \to 2^Y$ *is a mapping such that*

2) $L^{(\alpha)}(z) \subset L^{(\alpha)}(y_1) \cup L^{(\alpha)}(y_2)$ $\qquad (z \in I(y_1, y_2), \ \alpha \in A)$;
3) $K \subset L_1 \cup^{\bullet} L_2$, $\Rightarrow K \subset L_1$ or $K \subset L_2$ $\qquad (K \in \mathcal{K}; \ L_1, L_2 \in \mathcal{L}, \ L_1 \cap L_2 = \emptyset)$;
4) *if* $K \in \mathcal{K}$ $L_1, L_2 \in \mathcal{L}$, $\alpha \in A$, $y_1, y_2 \in Y$ *and* $i : I(y_1, y_2) \to \{1, 2\}$ *then* $L_1 \cap L_2 = \emptyset$ and $K \cap K^{(\alpha)}(y) \subset L_{i(y)}$ $(y \in Y)$ imply $i(y) \equiv 1$ or $i(y) \equiv 2$;
5) $\bigcap_{(\alpha,y) \in F} L^{(\alpha)}(y) \ne \emptyset$ implies $\bigcap_{(\alpha,y) \in F} K^{(\alpha)}(y) \ne \emptyset$ *for* F finite $\subset A \times Y$.

Then $\emptyset \notin \mathcal{K}$, *that is the family* $\mathcal{K}$ *has finite intersection property.*

Proof. Let $n := \inf\{\#F : \bigcap_{(\alpha,y) \in F} L^{(\alpha)}(y) \ne \emptyset, \ F \text{ finite} \subset A \times Y\}$. By 1), $n > 1$. By 5) also $n = \inf\{\#F : \ \bigcap_{(\alpha,y) \in F} K^{(\alpha)}(y) \ne \emptyset, \ F \text{ finite} \subset A \times Y\}$.

Suppose indirectly $n < \infty$. Choose $(\alpha_1, y_1), \dots, (\alpha_n, y_n)$ such that $\bigcap_{i=1}^n L^{(\alpha_i)}(y_i) = \bigcap_{i=1}^n K^{(\alpha_i)}(y_i) = \emptyset$. Fix $\alpha \in A$ with $\alpha > \alpha_1, \alpha_2$ and let

$$K := \bigcap_{i > 2} K^{(\alpha_i)}(y_i) \ , \quad L_k := L^{(\alpha_k)}(y_k) \cap \bigcap_{i > 2} K^{(\alpha_i)}(y_i) \qquad (k = 1, 2) \ .$$

Then $L_1 \cap L_2 = \emptyset$ and, by the definition of n, $\emptyset \neq K \cap K^{(\alpha)} \subset^{\text{by } 2)1)} \subset L_1 \cup^{\bullet} L_2$ $(y \in [y_1, y_2])$. Then 3) implies either $K \cap K^{(\alpha)}(y) \subset L_1$ or $K \cap K^{(\alpha)}(y) \subset L_2$ for any $y \in [y_1, y_2]$. Thus by 4) we even have the stronger alternatives of either $K \cap K^{(\alpha)}(y) \subset L_1$ for all $y \in [y_1, y_2]$ or $K \cap K^{(\alpha)}(y) \subset L_2$ for all $y \in [y_1, y_2]$. This contradicts the facts $K \cap K^{(\alpha)}(y_1) \subset L_1$, $K \cap K^{(\alpha)}(y_2) \subset L_2$ with $L_1 \cap L_2 = \emptyset$.

Remark. 3.2. The minimax theorems in [16,1,20,6, 17,10,19] can be deduced by an immediate application of this Metatheorem with $A := \mathbb{R}$ to the level sets $K^{(\alpha)}(y) := \{x \in X : f_1(x,y) > \alpha\}$ and $L^{(\beta)}(x) := \{y \in Y : f_2(x,y) > \beta\}$ $(\alpha, \beta \in \mathbb{R};\ x \in X,\ y \in Y)$ (and $f_1 \equiv f_2$ in case of one function minimax theorems).

Theorem. 3.3. *Let X, Y be interval spaces, $f_1, f_2 : X \times Y \to \mathbb{R}$ with $f_1 \le f_2$ and*

(i) $f_1(\bullet, y)$ *is* quasiconcave *and* lower *semicontinuous on any interval of X* $(y \in Y)$,
(ii) $f_2(x, \bullet)$ *is* quasiconvex *and* upper or lower *semicont. on any interval of Y* $(x \in X)$,
(iii) $\{x : f_2(x,y_1), \dots, f_2(x,y_n) \ge \beta\} \neq \emptyset \Rightarrow \{x : f_1(x,y_1), \dots, f_2(x,y_n) \ge \beta\} \neq \emptyset$ *for any* $\{y_1, \dots, y_n\}$ finite $\subset Y$ *and* $\beta \in \mathbb{R}$.

Then the families $\{\{x : f_2(x,y) \ge \beta\} : \beta < \beta_*,\ y \in Y\}$ *and* $\{\{x : f_1(x,y) \ge \beta\} : \beta < \beta_*.\ y \in Y\}$ *have finite intersection property where* $\beta_* := \sup_x \inf_y f_2(x,y)$.

Proof. We can apply a modified version of the Metatheorem with the net

$$A := \{(\zeta, \infty), [\zeta, \infty) : \zeta \in \mathbb{R}\}, \qquad \alpha < \beta \overset{\text{def}}{\Leftrightarrow} \alpha \supset \beta \neq \alpha \quad (\alpha, \beta \in A)$$

and the families of upper level sets

$$K^{(\alpha)}(y) := \{x : f_1(x,y) \in \alpha\}, \quad L^{(\alpha)}(y) := \{x : f_2(x,y) \in \alpha\} \qquad (\alpha \in A,\ y \in Y).$$

Indeed, 1) holds automatically since $\mathcal{K}, \mathcal{L}$ are defined in terms of upper level sets of functions. By (i), K convex $\subset X$ for $K \in \mathcal{K}$ and, by (ii), for all $x_1, x_2 \in X$ we have the alternatives either $L \cap [x_2, x_2]$ closed $\subset [x_1, x_2]$ $(L \in \mathcal{L})$ or $L \cap [x_2, x_2]$ open $\subset [x_1, x_2]$ $(L \in \mathcal{L})$. Hence condition 3) is immediate. By (ii) we also have the local intersection property

$$\forall\, y_1, y_2 \in Y,\ K \in \mathcal{K},\ z \in [y_1, y_2],\ \alpha \in A \quad \exists\, x \in K \cap K^{(\alpha)}(z)$$
$$\exists\, V \in \mathcal{V}_x \qquad x \in K \cap K^{(\alpha)}(v) \qquad (v \in V \cap [y_1, y_2]).$$

This property with 3) implies 4). By (ii) we have the following refinement of 3).

3*) $\forall x_1, x_2 \in X\ \ \forall\, y_1, y_2 \in Y\ \ \forall\, \alpha < \gamma \in A\ \ \exists\, \beta \in A$ with $\alpha < \beta < \gamma$ and $L^{(\beta)}(y_k) \cap [x_1, x_2]$ open $\subset [x_1, x_2]$ or $L^{(\beta)}(y_k) \cap [x_1, x_2]$ closed $\subset [x_1, x_2]$ for $k = 1, 2$.

Therefore we can slightly modify the argument of the Metatheorem to complete the proof as follows. Defining n as before, by assumption (iii) we have again

$$n = \inf\{\#F : \textstyle\bigcap_{(\alpha,y) \in F} K^{(\alpha)}(y) \neq \emptyset,\ F \text{ finite} \subset A \times Y\} > 1.$$

By choosing $(\alpha_1, y_1), \dots, (\alpha_n, y_n)$, K and L_1, L_2 in the same way as in course of the previous proof, we get

$$\emptyset \neq K^{(\alpha)}(z) \subset L^{(\alpha)}(z) \cap \textstyle\bigcap_{i>2} L^{(\alpha_i)}(y_i) \subset L_1 \cup^{\bullet} L_2 \qquad \text{for any } \alpha > \alpha_1, \alpha_2.$$

Fix $\alpha, \gamma \in A$ with $\gamma > \alpha > \alpha_1, \alpha_2$ arbitrarily. To complete the proof with an analogous argument as before, it suffices to show

3') either $K^{(\gamma)}(z) \subset L_1$ or $K^{(\gamma)}(z) \subset L_2$ $\qquad (z \in [y_1, y_2])$.

Suppose that, in the contrary, we can find $x_1 \in K^{(\gamma)}(z) \cap K \cap L_1$, $x_2 \in K^{(\gamma)}(z) \cap K \cap L_2$. Choose β in accordance with $3^*_A)$. By assumption $1_A)$, $[x_1, x_2] \subset K \cap K^{(\gamma)}(z)$. Therefore

$$[x_1, x_2] \subset L^{(\gamma)}(z) \cap K \cap [x_1, x_2] = L^{(\gamma)}(z) \cap [x_1, x_2] \subset$$
$$\subset (L^{(\alpha)}(y_1) \cup L^{(\alpha)}(y_2)) \cap [x_1, x_2] \subset \textstyle\bigcap_{k=1}^{2} (L^{(\beta)}(y_k) \cap [x_1, x_2]).$$

Since $L^{(\beta)}(y_k) \cap [x_1, x_2] \subset L_k$ $(k = 1, 2)$ and $L_1 \cap L_2 = \emptyset$, for $k = 1, 2$ we have $L^{(\beta)}(y_k) \cap [x_1, x_2] = L_k \cap [x_1, x_2]$. This is impossible by the connectedness of $[x_1, x_2]$ and the choice of $\beta \in A$ ensuring that both $L^{(\beta)}(y_1) \cap [x_1, x_2]$ and $L^{(\beta)}(y_2) \cap [x_1, x_2]$ are closed or open in $[x_1, x_2]$ simultaneously.

Corollary. 3.4. *The above hypothesis and* (iv) $\forall\, L \in \mathcal{L}\ \ \exists K \in \mathcal{K}\ \ \overline{K}$ compact $\subset L$ *imply* $\inf_y \sup_x f(x,y) \le \sup_x \inf_y g(x,y)$.

Remark. 3.5. *Purely topological minimax theorems via interval spaces.*
Let X, Y be topological spaces and assume

$1_B)$ $K^{(\alpha)}(y)$ connected $\subset X$ $\qquad (\alpha \in A,\ y \in Y)$;
$3_B)$ $L^{(\alpha)}(y)$ open or closed $\subset X$ $\qquad (\alpha \in A,\ y \in Y)$;
$3^*_B)$ $\bigcap_{(\alpha,x) \in H} \{y \in Y : x \notin L^{(\alpha)}(y)\}$ connected $\subset Y$ $\qquad (H \subset A \times X)$.

Notice that assumption $3^*_B)$ gives rise to the interval structure

$$[y_1, y_2] := \bigcap \{\{y : x \notin L^{(\alpha)}(y)\} : x \notin L^{(\alpha)}(y_1),\ x \notin L^{(\alpha)}(y_2),\ x \in X,\ \alpha \in A\} =$$
$$= \{y \in Y : \forall\, \alpha \in A\ \ L^{(\alpha)}(y) \subset L^{(\alpha)}(y_1) \cup L^{(\alpha)}(y_2)\}.$$

$4_B)$ $\forall\, y_1, y_2 \in Y$, $K \in \mathcal{K}$, $z \in [y_1, y_2]$, $\alpha \in A$ $\ \exists\, x \in K^{(\alpha)}(z) \cap K$
$\exists\, V \in \{\text{neighborhoods of } z\}$ $\ x \in K^{(\alpha)}(v) \cap K$ $\quad (v \in V)$.

Then we have the implications $1_B) + 3_B) \Rightarrow 3)$, $\quad 3) + 3^*_B) + 4_B) \Rightarrow 4)$.

Corollary. 3.6. *Let X, Y be topological spaces and let $f_1, f_2 : X \times Y \to \mathbb{R}$ with $f_1 \le f_2$ be functions such that*

(a) $[y_1, y_2] := \{y : f_2(x,y) \le f_2(x,y_1) \vee f_2(x,y_2)\ (x \in X)\}$ connected $\subset Y$ *for all* $y_1, y_2 \in Y$;
(b) $f_2(\bullet, y)$ *is* upper /lower/ *semicontinuous on X for any $y \in Y$;*
(c) *the sets* $\bigcap_{(\alpha,y) \in F} \{x : f_1(x,y) \ge \alpha\}$ *are connected for F* finite $\subset A \times Y$;
(d) $f_1(x, \bullet)$ *is* lower *semicontinuous on Y for any $x \in X$;*
(e) $\{x : f_2(x,y_1), \dots, f_2(x,y_n) \ge \beta\} \neq \emptyset \Rightarrow \{x : f_1(x,y_1), \dots, f_1(x,y_n) \ge \beta\} \neq \emptyset$ *for any* $\{y_1, \dots, y_n\}$ finite $\subset Y$ *and* $\beta \in \mathbb{R}$.

Then the families $\mathcal{K}, \mathcal{L}$ have finite intersection property for $K^{(\alpha)}(y) := \{x : f_1(x,y) \ge \alpha\}$, $L^{(\alpha)}(y) := \{x : f_2(x,y) \ge \alpha\}$.

4. Two functions minimax theorems on generalized H-spaces

Theorem. 4.1. (Tarafdar-type selection theorem). *Let X, Y be topological spaces, X being compact; $s : 2^Y \to 2^Y$ a generalized H-convexity with* $s(G) = \bigcup_{F \subset G} \varphi_G(\Delta_F)$ *for*

G finite $\subset Y$ *and let* $K : X \to \{\text{nonempty } s\text{-convex subsets of } Y\}$ *be a multifunction with the local intersection property*

$$\forall\, x \in X \quad \exists\, U_x \text{ neighborhood of } x \quad \exists\, y_x \in K(x) \qquad y_x \in K(u) \quad (u \in U_x)\ .$$

Then for some G finite $\subset Y$ *there exists a continuous function* $f : X \to \Delta_G$ *such that* $\varphi_G(f(x)) \in K(x)$ *for all* $x \in X$.

Proof. By the compactness of the space X, there exists a finite partition of unity on X subordinated to the covering $\{U_x :\ x \in X\}$. That is, we can choose a finite family $\{x_1, \dots, x_N\} \subset X$ along with a set $\{\psi_1, \dots, \phi_N\}$ of continuous functions $X \to [0,1]$ such that $\sum_{i=1}^{N} \psi_i = 1$ and $\{x \in X :\ \psi_i(x) > 0\} \subset U_{x_i}$ $(i = 1, \dots, N)$. Then, with the set $G := \{y_{x_1}, \dots, y_{x_N}\}$, the function $f(x) := \varphi_G(\sum_{i=1}^{N} \psi_i(x) 1_{y_{x_i}})$ $(x \in X)$ suits our requirements. Indeed, f is obviously continuous as a finite composition of continuous operations. On the other hand, given any point $x \in X$, with the index set $I_x := \{i :\ \psi_i(x) > 0\}$ we have $x \in U_{x_i}$ $(i \in I_x)$. Therefore, with the subset $F_x := \{y_{x_i} :\ i \in I_x\}$ of G we have also $F_x \subset K(x)$ and hence $f(x) = \varphi_G(\sum_{i \in I_x} \psi_i(x) 1_{y_{x_i}}) \in \varphi_G(\Delta_{F_x}) \subset s(F_x) \subset K(x)$ by the s-convexity of $K(x)$.

Lemma. 4.2. *Let* $X_1, \dots, X_n$ *compact generalized H-spaces and, for* $j = 1, \dots, n$, *let*

$$K_j : \widehat{X}_j \to \{C :\ \emptyset \neq C \text{ convex} \subset X_j\} \qquad \text{where} \quad \widehat{X}_j := \textstyle\prod_{i : i \neq j} X_i$$

be multifunctions with the local intersection property

$$\forall\, \widehat{x} \in \widehat{X}_j \quad \exists\, \widehat{U}\text{-neighborhood of } \widehat{x} \text{ in } \widehat{X}_j \quad \exists\, x \in X_j :\quad x \in K_j(\widehat{u}) \quad (\widehat{u} \in \widehat{U})\ .$$

Then there exist $x_1 \in X_1, \dots, x_n \in X_n$ *such that* $x_j \in K_j(x_1, \overset{\wedge j}{\dots}, x_n)$ $(j = 1, \dots, n)$.

Proof. By the selection theorem we can find natural numbers $N_1, \dots, N_n$ and, for $j = 1, \dots, n$, continuous functions

$$f_j : \widehat{X}_j \to \Delta_{N_j} := \Delta_{\{1, \dots, N_j\}}\ , \quad \varphi_j : \Delta_{N_j} \to X_j \quad \text{with} \quad \varphi_j(f_j(\widehat{x})) \in K_j(\widehat{x}) \text{ for } \widehat{x} \in \widehat{X}_j\ .$$

Then we can apply Brower's fixed point theorem to the function $\Psi := F \circ \varphi$ where

$$F(x_1, \dots, x_n) := \big(f_1(x_2, \dots, x_n), \dots, f_n(x_1, \dots, x_{n-1})\big) \quad (x_1 \in X_1, \dots, x_n \in X_n),$$
$$\varphi(\xi_1, \dots, \xi_n) := \big(\varphi_1(\xi_2, \dots, \xi_n), \dots, \varphi_n(\xi_1, \dots, \xi_{n-1})\big) \quad (\xi_1 \in \Delta_{N_1}, \dots, \xi_n \in \Delta_{N_n}).$$

Therefore there exists $\xi \in \Delta_{N_1} \times \cdots \times \Delta_{N_n}$ with $\Psi(\xi) = \xi$. Then for the point $x := \varphi(\xi)$ we have $\varphi(F(x)) = \varphi \circ F(\varphi(\xi)) = \varphi \circ F \circ \varphi(\xi) = \varphi \circ \Psi(\xi) = \varphi(\xi) = x$. The components $x_1, \dots, x_n$ of x suit $x_j \in K_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ $(j = 1, \dots, n)$.

Corollary. 4.3. *Let* $X_1, \dots, X_n$ *be compact generalized H-spaces and, for* $j = 1, \dots, n$, *let* $f_j : X_1 \times \cdots \times X_n \to \mathbb{R}$ *resp.* $a_j : \widehat{X}_j \to \mathbb{R}$ *be functions such that for any fixed* $(x_1, \overset{\wedge j}{\dots}, x_n) \in \widehat{X}_j$ *the subfunction* $f_j(x_1, \dots, x_{j-1}, \bullet, x_{j+1}, \dots, x_n)$ *is quasiconvex on* X_j *and for each* $\widehat{x} := (x_1, \overset{\wedge j}{\dots}, x_n) \in \widehat{X}_j$ *there exists a neighborhood* $\widehat{U}$ *of the point* $\widehat{x}$ *along with some* $y \in X_j$ *such that* $f_j(u_1, \dots, u_{j-1}, y, u_{j+1}, \dots, u_n) < a_j(u_1, \overset{\wedge j}{\dots}, u_n)$ *for* $(u_1, \overset{\wedge j}{\dots}, u_n) \in U$. *Then there exist* $x_1 \in X_1, \dots, x_n \in X_n$ *such that*

$$f_j(x_1, \dots, x_n) < a_j(x_1, \overset{\wedge j}{\dots}, x_n) \qquad (j = 1, \dots, n)\ .$$

Proof. We can apply Lemma 4.2 to the multifunctions

$$K_j(x_1, \overset{\wedge j}{\dots}, x_n) := \{y \in X_j :\ f_j(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_n) > a_j(x_1, \overset{\wedge j}{\dots}, x_n)\}\ .$$

Proposition. 4.4. *Let* $X_1, \dots, X_n$ *be compact generalized H-spaces and let* $f_1, \dots, f_n : X_1 \times \cdots \times X_n \to \mathbb{R}$ *be continuous functions such that for any fixed* $(x_1, \overset{\wedge j}{\dots}, x_n) \in \widehat{X}_j$ *the subfunction* $f_j(x_1, \dots, x_{j-1}, \bullet, x_{j+1}, \dots, x_n)$ *is quasiconvex on* X_j $(j = 1, \dots, n)$. *Then there exist* $x_1 \in X_1, \dots, x_n \in X_n$ *such that*

$$f_j(x_1, \dots, x_n) = \min_{y \in X_j} f_j(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_n) \qquad (j = 1, \dots, n)\ .$$

Proof. The functions $c_j(x_1, \overset{\wedge j}{\dots}, x_n) := \min_{y \in X_j} f_j(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_n)$ are continuous. To prove this statement, we may assume $j = 1$ without loss of generality. As being the infimum of the continuous functions $f_1(\bullet, \widehat{x})$ $(\widehat{x} \in \widehat{X}_1)$, the function c_1 is upper semicontinuous. Let $\widehat{x} := (x_2, \dots, x_n) \in \widehat{X}_1$ be arbitrarily given and suppose indirectly that $\widehat{x}^i := (x_2^i, \dots, x_n^i)$ $(i \in I)$ is a net converging to $\widehat{x}$ in $\widehat{X}_1$ but $\liminf_{i \in I} c_1(\widehat{x}^i) < c_1(\widehat{x})$. By passing to a suitable subnet, we may assume that the limit of $(c_1(\widehat{x}^i) :\ i \in I)$ exists. Choose the points $y^i \in X_1$ $(i \in I)$ in a manner such that $f_1(y^i, \widehat{x}^i) = c_1(\widehat{x}^i)$ $(i \in I)$ By passing to a suitable subnet again, we may also assume that the net $(y^i :\ i \in I)$ converges to some point $y \in X_1$. Then we get the contradiction $c_1(\widehat{x}) > \lim_{i \in I} c_1(\widehat{x}^i) = \lim_{i \in I} f_1(y^i, \widehat{x}^i) = f_1(y, \widehat{x}) \geq \min_{z \in X_1} f_1(z, \widehat{x}) = c_1(\widehat{x})$.

For any $\varepsilon > 0$ let $a_j^\varepsilon := c_j - \varepsilon$ $\quad (j = 1, \dots, n)$. By the continuity of the functions f_j, we can apply the previous corollary (with a_j^ε instead of a_j) to conclude that for each $\varepsilon > 0$ there exists $\mathbf{x}^\varepsilon := (x_1^\varepsilon, \dots, x_n^\varepsilon) \in X_1 \times \cdots \times X_n$ such that $f_j(x_1, \dots, x_n) < c_j(x_1, \overset{\wedge j}{\dots}, x_n) + \varepsilon$ $(j = 1, \dots, n)$. Then any cluster point of the net $(\mathbf{x}^\varepsilon :\ \varepsilon > 0)$ suits our requirements.

Corollary. 4.5. *Let* X_1, X_2 *be compact generalized H-spaces,* $f_1, f_2 : X_1 \times X_2 \to \mathbb{R}$ *with* $f_1 \leq f_2$ *and let* $a_k : X_k \to \mathbb{R}$ $(k = 1, 2)$ *be functions such that*

(i) $f_1(\bullet, x_2)$ *is quasiconcave for any* $x_2 \in X_2$;
(ii) $\forall\, x_2 \in Y \ \ \exists\, x_1 \in X_1 \ \ \exists\, V$ neighborhood of x_2 : $\quad f_1(x_1, v) > a_2(v)$ $(v \in V)$;
(iii) $f_2(x_1, \bullet)$ *is quasiconvex for any* $x_1 \in X_1$;
(iv) $\forall\, x_1 \in X_1 \ \ \exists\, x_2 \in Y \ \ \exists\, U$ neighborhood of x_1 : $\quad f_2(u, x_2) < a_1(u)$ $(u \in U)$.

Then there exist $x_1 \in X_1$ *and* $x_2 \in Y$ *with* $f_1(x_1, x_2) > a_2(x_2)$ *and* $f_2(x_1, x_2) < a_1(x_1)$.

Proof. By assumptions (i)+(ii), $K_1(x_2) := \{x_1 \in X_1 :\ f_1(x_1, x_2) > a_2(x_2)\}$ $(x_2 \in X_2)$ is a non-empty convex valued multifunction $Y \to 2^{X_1}$ with local intersection property. Similarly, by (iii)+(iv), $K_2(x_1) := \{x_2 \in X_2 :\ f_2(x_1, x_2) < a_1(x_1)\}$ $(x_1 \in X_1)$ is also a non-empty convex valued multifunction $X_1 \to 2^{X_2}$ with local intersection property. By Lemma 4.2 there exist $x_1 \in X_1$ and $x_2 \in Y$ with $x_1 \in K_2(x_1)$ and $x_2 \in K_1(x_2)$.

Corollary. 4.6. *Let* X_1, X_2 *be compact generalized H-spaces,* $f_1 \leq f_2 : X_1 \times X_2 \to \mathbb{R}$ *with properties* (i),(iii) *and*

(ii') $\forall\, y \in Y,\ x \in X_1,\ \varepsilon > 0\ \exists\, x' \in X_1\ \exists\, V$ neighborhood of y : $f(x', V) > f(x, y) - \varepsilon$;
(iv') $\forall\, x \in X_1,\ y \in X_2,\ \varepsilon > 0\ \exists\, y' \in X_2\ \exists\, U$ neighborhood of x : $f(U, y') < f(x, y) + \varepsilon$.

Then $\quad \inf_{x_2 \in X_2} \sup_{x_1 \in X_1} f_1(x_1, x_2) \leq \sup_{x_1 \in X_1} \inf_{x_2 \in Y} f_2(x_1, x_2)$.

Proof. The contrary $\sup_{x_1 \in X_1} \inf_{x_2 \in Y} f_2(x_1, x_2) < \alpha_2 < \alpha_1 < \inf_{x_2 \in Y} \sup_{x_1 \in X_1} f_1(x_1, x_2)$ (for suitable $\alpha_1, \alpha_2 \in \mathbb{R}$) would imply (with $a_1(x_1) \equiv \alpha_1$ and $a_2(x_2) \equiv \alpha_2$) the existence of points $x_k \in X_k$ $(k = 1, 2)$ such that $x_1 \in K_1(x_2)$ and $x_2 \in K_2(x_1)$ for $K_1(y) := \{x \in X_1 :\ f_1(x, y) > \alpha_1\}$ and $K_2(x) := \{y \in X_2 :\ f_2(x, y) < \alpha_2\}$. This contradicts $f_1 \leq f_2$.

Added in proof. Recently S. Park [15] introduced a concept called *G-convexity* closely related to our generalized H-convexity. It can be shown that given any G-convexity Γ on a topological space X, there exists a generalized H-convexity $s : 2^X \to 2^X$ such that $\{\Gamma\text{-convex sets}\} \subset \{s\text{-convex sets}\}$. Hence, in particular all the results of Section 4 hold when generalized H-convexity is replaced by G-convexity.

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ON WEAK CONVERGENCE TO FIXED POINTS OF NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we prove the following weak convergence theorem: Let C be a nonempty closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition or whose norm is Fréchet differentiable. Let T be a nonexpansive mapping from C into itself with a fixed point. Suppose that $\{x_n\}$ is given by $x_1 \in C$ and $x_{n+1} = \alpha_n T[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n$ for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and $\limsup_{n\to\infty} \beta_n < 1$, or $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$ and $\limsup_{n\to\infty} \beta_n < 1$. Then $\{x_n\}$ converges weakly to a fixed point of T. This is a generalization of the results of Tan and Xu, and Takahashi and Kim.

1. Introduction

Let E be a real Banach space and let C be a nonempty closed convex subset of E. Then a mapping T from C into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. For a mapping T from C into itself, we denote by $F(T)$ the set of fixed points of T. Now, we consider the following iteration scheme: $x_1 \in C$ and

$$x_{n+1} = \alpha_n T[\beta_n T x_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n \quad \text{for all } n \geq 1, \tag{1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. Such an iteration scheme was introduced by Ishikawa [3]; see also Mann [4]. Recently Tan and Xu [8] proved the following interesting result (Corollary 1): Let C be a nonempty closed convex subset of a uniformly convex Banach space E which satisfies Opial's condition or whose norm is Fréchet differentiable and let T be a nonexpansive mapping from C into itself with a fixed point. Then for any initial data x_1 in C, the iterates $\{x_n\}$ defined by (1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, $\sum_{n=1}^{\infty} \beta_n(1-\alpha_n) < \infty$ and $\limsup_{n\to\infty} \beta_n < 1$, converge weakly to a fixed point of T. On the other hand, Takahashi and Kim [7] proved the following (Corollary 2): Let C, E and T be as above and suppose $\alpha_n \in [a,b]$ and $\beta_n \in [0,b]$, or $\alpha_n \in [a,1]$ and $\beta_n \in [a,b]$ for some a, b with $0 < a \leq b < 1$. Then for any initial data x_1 in C, the iterates $\{x_n\}$ defined by (1) converge weakly to a fixed point of T. Note that Tan and Xu's result is applicable to the case of $\alpha_n = 1 - 1/n$ and $\beta_n = 1/n$ for all $n \geq 1$, while Takahashi and Kim's result is applicable to the case of $\alpha_n = \beta_n = 1/2$ for all $n \geq 1$.

In this paper, motivated by these two results, we prove the following weak convergence theorem: Let C, E and T be as above and suppose $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and $\limsup_{n\to\infty} \beta_n < 1$, or $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$ and $\limsup_{n\to\infty} \beta_n < 1$. Then for any initial data x_1 in C, the iterates $\{x_n\}$ defined by (1) converge weakly to a fixed point of T. Compare this with Tan and Xu's result [8] and Takahashi and Kim's result [7].

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