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# LOCALLY GENERATED POLYNOMIAL $\mathcal{C}^1$-SPLINES OVER TRIANGULAR MESHES

L.L. STACHÓ

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*Abstract.* We classify all possible local linear procedures over triangular meshes resulting in polynomial $C^1$-spline functions with affinely uniform shape for the basic functions at the edges, and fitting the 9 value- and gradient data at the vertices of the mesh members. There is a unique procedure among them with shape functions and basic polynomials of degree 5 and all other admissible procedures are its perturbations with higher degree.

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## 1. INTRODUCTION

By a *triangular mesh* we mean a finite family of closed non-degenerate triangles on the plane $\mathbb{R}^2$ with pairwise non-intersecting interiors and admitting only common vertices or edges. As usually, we regard $\mathbb{R}^2$ as the set of all real couples $[\xi,\eta]$ considered also as $1\times 2$ (row) matrices. We shall use the standard notations $x^{[1]} = x : [\xi,\eta] \mapsto \xi$, $x^{[2]} = y : [\xi,\eta] \mapsto \eta$ and $\langle \mathbf{u}|\mathbf{v}\rangle := \sum_{j=1}^2 x^{[j]}(\mathbf{u})x^{[j]}(\mathbf{v})$ for the Cartesian coordinates and scalar product, respectively. We write $\|\mathbf{u}\| = \langle \mathbf{u}|\mathbf{u}\rangle^{1/2}$ for the norm of $\mathbf{u}\in\mathbb{R}^2$ and $\mathrm{Co}(\mathbf{S})$ for the convex hull of $\mathbf{S}\subset\mathbb{R}^2$ resp. $\det(\mathbf{u},\mathbf{v}) = x^{[1]}(\mathbf{u})x^{[2]}(\mathbf{v}) - x^{[1]}(\mathbf{v})x^{[2]}(\mathbf{u})$ for $2\times 2$-determinants. Given a triangular mesh $\mathcal{T} = \{\mathbf{T}_1,\dots,\mathbf{T}_M\}$, in the sequel $\mathrm{Vert}(\mathbf{T}_k)$ and $\mathrm{Edge}(\mathbf{T}_k)$ will denote the sets of vertices resp. closed edges of the mesh members, $\mathrm{Dom}(\mathcal{T}) := \bigcup_{k=1}^M \mathbf{T}_k$, $\mathrm{Edge}(\mathcal{T}) := \bigcup_{k=1}^M \partial\mathbf{T}_k$, $\mathrm{Vert}(\mathcal{T}) := \bigcup_{k=1}^M \mathrm{Vert}(\mathbf{T}_k)$ will stand for the domain covered by $\mathcal{T}$, the line figure covered by all edges and the collection of all vertices, respectively. Recall that, given a *gradient data*

$$F = \left\{ \left(\mathbf{p}, f_{\mathbf{p}}, [f'_{x,\mathbf{p}}, f'_{y,\mathbf{p}}]\right) : \mathbf{p}\in\mathrm{Vert}\mathcal{T}) \right\} \subset \mathrm{Vert}(\mathcal{T})\times\mathbb{R}\times\mathbb{R}^2$$

on the set of the vertices in $\mathcal{T}$, a function $f:\mathbf{D}\to\mathbb{R}$ is a $\mathcal{C}^1$-*extension* of $F$ on $\mathbf{D} := \mathrm{Dom}(\mathcal{T})$ if $f$ has a continuous gradient $\mathbf{p}\mapsto \nabla f(\mathbf{p}) = \left[\frac{\partial}{\partial x} f(\mathbf{p}), \frac{\partial}{\partial y} f(\mathbf{p})\right]$ on $\mathrm{Interior}(\mathbf{D})$ which admits a continuous extension to $\mathbf{D}$ as well (denoted also by $\nabla f$) such that

$$f(\mathbf{p}) = f_{\mathbf{p}}, \quad \nabla f(\mathbf{p}) = \left[f'_{x,\mathbf{p}}, f'_{y,\mathbf{p}}\right] \qquad \left(\mathbf{p}\in\mathrm{Vert}(\mathcal{T})\right). \tag{1.1}$$

A $\mathcal{C}^1$-extension $f:\mathbf{D}\to\mathbb{R}$ of $F$ is said to be a *$\mathcal{C}^1$-spline interpolation* of $F$ with respect to the mesh $\mathcal{T}$ if the restrictions $f|\mathbf{T}_k$ are polynomials of the coordinates $x,y$.

There exists a large variety of $\mathcal{C}^1$-splines for any admissible $\mathcal{T}$ and $F$ which can be obtained e.g. as global polynomial extensions with Hermite type interpolation [4]. Obviously global polynomial fitting may primarily be interesting only from a pure theretical view point due to too large polynomial degree and hence high numerical instability. A better alternative could be an imitation of tensor product splines (e.g. with Catmull-Rom type hermition curves on edges developed for rectangular meshes [6,5]). This consists the construction of $\mathcal{C}^1$-splines as linear combinations on the rectangular mesh members from affine images of tensor products from only two special polynomials $\Phi,\Psi:[0,1]\to[0,1]$ (actually $\Phi(t)=t^2(3-2t)$, $\Psi(t)=t^2(1-t)$). Postulates A,B below exhibit two main features of most known tensor product spline procedures which can naturally be generalized even to procedures

$$\mathfrak{S}:(\mathcal{T},F)\mapsto f_{\mathcal{T},F}\quad\big(\mathcal{T}\text{ triang. mesh, }F\text{ grad. data on Vert}(\mathcal{T})\big)\tag{1.2}$$

furnishing $\mathcal{C}^1$-spline interpolation functions from gradient data at the vertices over triangular meshes.

**Postulate A.** Linearity and being locally generated: *There are polynomial functions*

$$\varphi_{\mathbf{T}}^{\mathbf{p}},\ \psi_{\mathbf{T}}^{1;\mathbf{p}},\ \psi_{\mathbf{T}}^{2;\mathbf{p}}:\mathbf{T}\to\mathbb{R}\quad\big(\mathbf{T}\in\{\textit{non-deg. triangles}\},\ \mathbf{p}\in\mathrm{Vert}(\mathbf{T})\big)$$

*depending only on the couple of the triangle* $\mathbf{T}$ *with a distingvished vertex such that the restriction of* $\mathfrak{S}$ *to any mesh triangle* $\mathbf{T}\in\mathcal{T}$ *has the form*

$$f_{\mathcal{T},F}|\mathbf{T}=\sum_{\mathbf{p}\in\mathrm{Vert}(\mathbf{T})}\Big[f_{\mathbf{p}}\varphi_{\mathbf{T}}^{\mathbf{p}}+f'_{x,\mathbf{p}}\psi_{\mathbf{T}}^{1;\mathbf{p}}+f'_{y,\mathbf{p}}\psi_{\mathbf{T}}^{2;\mathbf{p}}\Big].\tag{1.3}$$

If Postulate A holds and $\mathrm{Vert}(\mathbf{T})=\{\mathbf{a},\mathbf{b},\mathbf{p}\}$, in terms of the canonical frame vectors

$$\mathbf{e}^{[0]}:=\mathbf{0}=[0,0],\ \mathbf{e}^{[1]}:=[1,0],\ \mathbf{e}^{[2]}:=[0,1]$$

we necessarily have

$$\begin{aligned}&\varphi_{\mathbf{T}}^{\mathbf{p}}(\mathbf{p})=1,\ \nabla\varphi_{\mathbf{T}}^{\mathbf{p}}(\mathbf{p})=\mathbf{0},\qquad \psi_{\mathbf{T}}^{j;\mathbf{p}}(\mathbf{p})=0,\ \nabla\psi_{\mathbf{T}}^{j;\mathbf{p}}(\mathbf{p})=\mathbf{e}^{[j]};\\&\varphi_{\mathbf{T}}^{\mathbf{p}}(\mathbf{x})=\psi_{\mathbf{T}}^{j;\mathbf{p}}(\mathbf{x})=0,\ \nabla\varphi_{\mathbf{T}}^{\mathbf{p}}(\mathbf{x})=\nabla\psi_{\mathbf{T}}^{j;\mathbf{p}}(\mathbf{x})=\mathbf{0}\quad\big(\mathbf{x}\in\mathrm{Co}\{\mathbf{a},\mathbf{b}\}\big).\end{aligned}\tag{1.4}$$

The first statement in (1.4) is immediate from (1.3), The second one is a consequence of the fact that given any point $\widetilde{\mathbf{p}}$ forming an adjacent triangle $\widetilde{\mathbf{T}}:=\mathrm{Co}\{\mathbf{a},\mathbf{b},\widetilde{\mathbf{p}}\}$, considering the mesh $\mathcal{T}:=\{\mathbf{T},\widetilde{\mathbf{T}}\}$ with gradient data $F(\mathbf{q})=(0,\mathbf{0})$ where $\mathbf{q}=\mathbf{a},\mathbf{b},\widetilde{\mathbf{p}}$, we must have $f_{\mathcal{T},F}\equiv 0$ on $\widetilde{\mathbf{T}}$ and hence also $\nabla f_{\mathcal{T},F}\equiv 0$ on the common edge $\mathrm{Co}\{\mathbf{a},\mathbf{b}\}$ of the triangles $\mathbf{T},\widetilde{\mathbf{T}}$.

Locally generated linear spline procedures have the computational advantage that the resulting functions can be calculated on any mesh triangle regardless to what happens at vertices outside. A practical disadvantage is that in most cases only function values are available (mostly from scanned data) and convenient gradient values must be guessed or found by optimizing procedres.

**Postulate B.** Uniform shape on edges: (1.3) *holds and there are polynomial functions* $\Phi,\Psi : [0,1]\to\mathbb{R}$ *such that*

$$\Phi(0)=\Psi(0)=\Phi'(0)=\Psi'(0)=\Psi(1)=0, \quad \Phi(1)=\Psi'(1)=1 \tag{1.5}$$

*and the graps of the basic functions* $\varphi_{\mathbf{T}}^{\mathbf{p}}$ *on the edges of the triangle* $\mathbf{T}$ *are affine images of the graph of* $\Phi$*, those of* $\psi_{\mathbf{T}}^{j,\mathbf{p}}$ $(j=1,2)$ *are affine images of the graph of* $\Psi$.

That is, under Postulate B, for the generic points $\mathbf{y}_t := t\mathbf{p}+(1-t)\mathbf{a}$ on the edge $\mathrm{Co}\{\mathbf{a},\mathbf{p}\}$, resp. $\mathbf{z}_t := t\mathbf{p}+(1-t)\mathbf{b}$ on $\mathrm{Co}\{\mathbf{b},\mathbf{p}\}$ we have

$$\begin{aligned} \varphi^{\mathbf{p}}_{\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}}(\mathbf{y}_t) &= \mathrm{const}_{\mathbf{a},\mathbf{p}}\Phi(t), \quad \psi^{j,\mathbf{p}}_{\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}}(\mathbf{y}_t) = \mathrm{const}^{(j)}_{\mathbf{a},\mathbf{p}}\Psi(t), \\ \varphi^{\mathbf{p}}_{\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}}(\mathbf{z}_t) &= \mathrm{const}_{\mathbf{b},\mathbf{p}}\Phi(t), \quad \psi^{j,\mathbf{p}}_{\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}}(\mathbf{z}_t) = \mathrm{const}^{(j)}_{\mathbf{b},\mathbf{p}}\Psi(t) \end{aligned} \tag{1.6}$$

while for the points $\mathbf{x}_t := (1-t)\mathbf{a}+t\mathbf{b}$ on the edge $\mathrm{Co}\{\mathbf{a},\mathbf{b}\}$ we simply have

$$\begin{aligned} \varphi^{\mathbf{p}}_{\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}}(\mathbf{x}_t) &= \psi^{j,\mathbf{p}}_{\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}},(\mathbf{x}_t) = 0, \\ \nabla\varphi^{\mathbf{p}}_{\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}}(\mathbf{x}_t) &= \nabla\psi^{j,\mathbf{p}}_{\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}}(\mathbf{x}_t) = \mathbf{0}. \end{aligned} \tag{1.7}$$

In the sequel we call $\Phi,\Psi$ the *shape functions* of the spline procedure (1.2) satisfying Postulate B. Notice that the requirements $\Phi(0)=\Phi'(0)=\Psi(0)=\Psi'(0)$ follow automatically from the order condition (1.4) on the edge $\mathrm{Co}\{\mathbf{a},\mathbf{b}\}$.

Our aim in this paper is a parametric classification of all procedures satisfying Postulates A,B which produce $\mathcal{C}^1$-smooth functions.

It is remarkable that there is a unique one among them with lawest degree (degree 5) which turns out to be homothetically invariant. From the view point of applications, the results provide the complete list of hermitian $\mathcal{C}^1$-splines with shape uniformity over edges from which one can choose the best fit one with respect to various aspects. It is worth to relate the latter fact to the local linear polynomial spline interpolation procedures on the basis of the Zlámal-Ženišek (ZZ) equations [8,9]. These methods rely upon the fact that, given a triangular mesh with gradient and Hessian data at the vertices and normal derivative values at edge middle points, there is a unique fitting spline with 5th degree polynomials. The 21 polynomial coefficients over any mesh triangle can be obtained as the unique solution of a system of 21 straightforward linear equations. Explicit formulas along with estimates concerning the approximation of smooth functions with this method were published recently [7]. Recently there are computer algebras (as MAPLE 14 or Wolfram Mathematica 12) being powerful enough for a completely symbolic solution of the ZZ-equations and it is easy to give examples where the ZZ-approximaton produces only a $\mathcal{C}^1$-smooth function[1]. It seems that our first order approach with the shape conditions of Postulate B provides a geometrically motivated alternative to several problems discussed in [7]. As remarked also in [1], first order approches with a few (actually 9 in [1]) free parameters may have practical advantages versus higher order methods due to the fact that data sampling can rarely support Hessian data or even adequate guesses for them. The usual way to overcome such problems is minimizing curvature functionals with given function values at mesh vertices but varying the data of first and higher derivatives. This can be carried out with relatively low computational costs for our splines in Theorem 2.3, namely about at least 5 times less computational effort than with ZZ-baed procedures.

## 2. MAIN RESULTS

Recall that given a non-degenerate triangle $\mathbf{T} \subset \mathbb{R}^2$ with $\{\mathbf{a},\mathbf{b},\mathbf{c}\} = \mathrm{Vert}(\mathbf{T})$, the *normalized baricentric coordinates* of a point $\mathbf{x}$ are the terms of the necessarily unique triple $\big[\lambda_{\mathbf{T}}^{\mathbf{a}}(\mathbf{x}),\lambda_{\mathbf{T}}^{\mathbf{b}}(\mathbf{x}),\lambda_{\mathbf{T}}^{\mathbf{c}}(\mathbf{x})\big] \in \mathbb{R}^3$ such that

$$\mathbf{x} = \lambda_{\mathbf{T}}^{\mathbf{a}}(\mathbf{x})\mathbf{a} + \lambda_{\mathbf{T}}^{\mathbf{b}}(\mathbf{x})\mathbf{b} + \lambda_{\mathbf{T}}^{\mathbf{c}}(\mathbf{x})\mathbf{c}, \quad \lambda_{\mathbf{T}}^{\mathbf{a}}(\mathbf{x}) + \lambda_{\mathbf{T}}^{\mathbf{b}}(\mathbf{x}) + \lambda_{\mathbf{T}}^{\mathbf{c}}(\mathbf{x}) = 1.$$

We reserve the symbols $\lambda_{\mathbf{T}}^{\mathbf{p}}$ as standard notation. It is well-known from elementary analytic plain geomertry [2] that

$$\lambda_{\mathbf{T}}^{\mathbf{p}}(\mathbf{x}) = \mathrm{area}(\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{x}\})/\mathrm{area}(\mathbf{T}) \qquad \big(\mathbf{x} \in \mathbf{T}\big)$$

thus normalized baricentric coordinates can easily be calculated by means of determinants or inner products with a $(\pi/2)$-rotation:

$$\lambda_{\mathbf{T}}^{\mathbf{p}}(\mathbf{x}) = \frac{\det(\mathbf{x}-\mathbf{a},\mathbf{x}-\mathbf{b})}{\det(\mathbf{p}-\mathbf{a},\mathbf{p}-\mathbf{b})} = \frac{\big\langle(\mathbf{b}-\mathbf{a})\mathbf{R}\big|\mathbf{x}-\mathbf{a}\big\rangle}{\big\langle(\mathbf{b}-\mathbf{a})\mathbf{R}\big|\mathbf{p}-\mathbf{a}\big\rangle} \quad \text{where } \mathbf{R} := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{2.1}$$

For later use we also introduce the abbreviating notations

$$x_{\mathbf{p}}^{[j]} := x^{[j]} - x^{[j]}(\mathbf{p}), \ \xi_{\mathbf{p},\mathbf{a}}^{\mathbf{v}} := \frac{\langle \mathbf{v}-\mathbf{a}|\mathbf{p}-\mathbf{a}\rangle}{\|\mathbf{p}-\mathbf{a}\|^2}, \ \overline{\xi}_{\mathbf{p},\mathbf{a}}^{\mathbf{v}} := \frac{\langle \mathbf{v}-\mathbf{a}|(\mathbf{p}-\mathbf{a})\mathbf{R}\rangle}{\|\mathbf{p}-\mathbf{a}\|^2}. \tag{2.2}$$

As for geometric interpretation, $\xi_{\mathbf{p},\mathbf{a}}^{\mathbf{v}}$ resp. $\overline{\xi}_{\mathbf{p},\mathbf{a}}^{\mathbf{v}}$ are the affine coordinates of the point $\mathbf{v}$ with respect to the orthogonal frame $\big[\mathbf{a},\mathbf{p},\mathbf{a}+(\mathbf{p}-\mathbf{a})\mathbf{R}\big]$ with origin $\mathbf{a}$ so that $\mathbf{v} = \mathbf{a} + \xi_{\mathbf{p},\mathbf{a}}^{\mathbf{v}}(\mathbf{p}-\mathbf{a}) + \overline{\xi}_{\mathbf{p},\mathbf{a}}^{\mathbf{v}}(\mathbf{p}-\mathbf{a})\mathbf{R}$.

**Theorem 2.3.** *There is a unique local linear polynomial $\mathcal{C}^1$-spline procedure acting on triagular meshes with the property of uniform shape on vertices*[2] *and having shape functions with minimal computational complexity. Its shape functions are*

$$^*\Phi(t) = t^3(10-15t+6t^2), \qquad ^*\Psi(t) = t^3(t-1)(4-3t).$$

[1] Example: $f(x,y) = x_+^2 y^2(1-x-y)$ on the mesh $\{\mathbf{T}_k : k = 1,2\}$, $\mathbf{T}_k = \mathrm{Co}\{\mathbf{0},(-1)^k\mathbf{e}^{[1]},\mathbf{e}^{[2]}\}$ with vanishing ZZ data except for the edge middle point $\frac{1}{2}\mathbf{e}^{[1]} + \frac{1}{2}\mathbf{e}^{[2]}$ of $\mathbf{T}_2$.

[2] That is satisfying Postulates A,B with $f_{\mathcal{T},F} \in \mathcal{C}^1\big(\mathrm{Dom}(\mathcal{T})\big)$.

*The corresponding shape functions* (*for a non-degenerate triangle* $\mathbf{T} = \mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}$ *with distinguished vertex* $\mathbf{p}$) *have the form*

$$
\begin{aligned}
{}^*\varphi_{\mathbf{T}}^{\mathbf{p}} &= {}^*\Phi\big(\lambda_{\mathbf{T}}^{\mathbf{p}}\big) + 30\,[\lambda_{\mathbf{T}}^{\mathbf{p}}]^2\lambda_{\mathbf{T}}^{\mathbf{a}}\lambda_{\mathbf{T}}^{\mathbf{b}}\Big[\xi_{\mathbf{p},\mathbf{a}}^{\mathbf{b}}\lambda_{\mathbf{T}}^{\mathbf{a}} + \xi_{\mathbf{p},\mathbf{b}}^{\mathbf{a}}\lambda_{\mathbf{T}}^{\mathbf{b}}\Big],\\
{}^*\psi_{\mathbf{T}}^{j,\mathbf{p}} & \frac{{}^*\Psi\big(\lambda_{\mathbf{T}}^{\mathbf{p}}\big)}{\lambda_{\mathbf{T}}^{\mathbf{p}}-1}x_{\mathbf{p}}^{[j]} + 12[\lambda_{\mathbf{T}}^{\mathbf{p}}]^2\lambda_{\mathbf{T}}^{\mathbf{a}}\lambda_{\mathbf{T}}^{\mathbf{b}}\Big[\xi_{\mathbf{p},\mathbf{a}}^{\mathbf{b}}x_{\mathbf{p}}^{[j]}(\mathbf{a})\lambda_{\mathbf{T}}^{\mathbf{a}} + \xi_{\mathbf{p},\mathbf{b}}^{\mathbf{a}}x_{\mathbf{p}}^{[j]}(\mathbf{b})\lambda_{\mathbf{T}}^{\mathbf{b}}\Big].
\end{aligned}
$$

**Theorem 2.4.** *A spline procedure acting on triangular meshes and satisfying Postulates A,B produces* $\mathcal{C}^1$*-smooth splines if and only if its shape functions are of the form*

$$\Phi(t) = {}^*\Phi(t) + t^3(1-t)^3\Phi_1(t), \qquad \Psi(t) = {}^*\Psi(t) + t^3(1-t)^3\Psi_1(t) \tag{2.5}$$

*and the shape functions* (*for a non-degenerate triangle* $\mathbf{T} = \mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}$ *with distinguished vertex* $\mathbf{p}$) *can be written in terms of the modified shape function*

$$\Theta(t) := \Psi(t)/(t-1) \tag{2.5$'$}$$

*and the rotation matrix* $\mathbf{R}$ *in* (2.1) *as*

$$
\begin{aligned}
\varphi_{\mathbf{T}}^{\mathbf{p}} &= \Phi\big(\lambda_{\mathbf{T}}^{\mathbf{p}}\big) + [\lambda_{\mathbf{T}}^{\mathbf{p}}]^2\lambda_{\mathbf{T}}^{\mathbf{a}}\lambda_{\mathbf{T}}^{\mathbf{b}}P_{\mathbf{a},\mathbf{b}}^{\mathbf{p}}\big(\lambda_{\mathbf{T}}^{\mathbf{b}},\lambda_{\mathbf{T}}^{\mathbf{a}}\big),\\
\psi_{\mathbf{T}}^{j,\mathbf{p}} &= \Theta\big(\lambda_{\mathbf{T}}^{\mathbf{p}}\big)x_{\mathbf{p}}^{[j]} + [\lambda_{\mathbf{T}}^{\mathbf{p}}]^2\lambda_{\mathbf{T}}^{\mathbf{a}}\lambda_{\mathbf{T}}^{\mathbf{b}}Q_{\mathbf{a},\mathbf{b}}^{j,\mathbf{p}}\big(\lambda_{\mathbf{T}}^{\mathbf{b}},\lambda_{\mathbf{T}}^{\mathbf{a}}\big)
\end{aligned}
\tag{2.6}
$$

*where*

$$
\begin{aligned}
P_{\mathbf{a},\mathbf{b}}^{\mathbf{p}}(s,t) &= s\Big\{\xi_{\mathbf{p},\mathbf{b}}^{\mathbf{a}}\frac{\Phi'(1-s)}{(1-s)^2s^2} + \overline{\xi}_{\mathbf{p},\mathbf{b}}^{\mathbf{a}}k_{\mathbf{b}}^{0,\mathbf{p}}(s)\Big\} + \\
&\quad + t\Big\{\xi_{\mathbf{p},\mathbf{a}}^{\mathbf{b}}\frac{\Phi'(1-t)}{(1-t)^2t^2} + \overline{\xi}_{\mathbf{p},\mathbf{a}}^{\mathbf{b}}k_{\mathbf{a}}^{0,\mathbf{p}}(t)\Big\} + stR_{\mathbf{a},\mathbf{b}}^{0,\mathbf{p}}(s,t),\\
Q_{\mathbf{a},\mathbf{b}}^{j,\mathbf{p}}(s,t) &= s\Big\{\xi_{\mathbf{p},\mathbf{b}}^{\mathbf{a}}x_{\mathbf{p}}^{[j]}(\mathbf{b})\frac{\Theta'(1-s)}{s(1-s)^2} + \overline{\xi}_{\mathbf{p},\mathbf{b}}^{\mathbf{a}}k_{\mathbf{b}}^{j,\mathbf{p}}(s)\Big\} + \\
&\quad + t\Big\{\xi_{\mathbf{p},\mathbf{a}}^{\mathbf{b}}x_{\mathbf{p}}^{[j]}(\mathbf{a})\frac{\Theta'(1-t)}{t(1-t)^2} + \overline{\xi}_{\mathbf{p},\mathbf{a}}^{\mathbf{b}}k_{\mathbf{a}}^{j,\mathbf{p}}(t)\Big\} + stR_{\mathbf{a},\mathbf{b}}^{j,\mathbf{p}}(s,t)
\end{aligned}
\tag{2.7}
$$

*with the following free options in* (2.6) *resp.* (2,7)*:*

(i) $\Phi_1, \Psi_1 : [0,1] \to \mathbb{R}$ *are arbitrary polynomial functions,*

(ii) $(\mathbf{p},\mathbf{q}) \mapsto k_{\mathbf{q}}^{i,\mathbf{p}}$ $(i = 0,1,2)$ *are arbitrary maps assigning polynomial functions* $\mathbb{R} \to \mathbb{R}$ *to pairs of distinct points,*

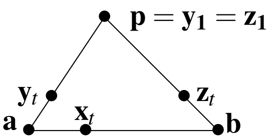
(iii) $(\mathbf{p},\mathbf{q},\mathbf{r}) \mapsto R_{\mathbf{q},\mathbf{r}}^{i,\mathbf{p}}$ $(i = 0,1,2)$ *are arbitrary maps assigning polynomials* $\mathbb{R}^2 \to \mathbb{R}$ *to triples of non-collinear points with the symmetry* $R_{\mathbf{q},\mathbf{r}}^{i,\mathbf{p}}(s,t) \equiv R_{\mathbf{r},\mathbf{q}}^{i,\mathbf{p}}(t,s)$.

**Remark 2.8.** (i) Actually, Theorem 2.3 is simply a corollary of Theorem 2.4 by setting the options (i)$-$(iv) to 0. We emphasize it for its potential practical and educational use.

(ii) The formally rational expressions in $(2.6-2.7)$ are polynomials. Indeed, $\Phi'(1-t)/\left[t^2(1-t)^2\right] = 30-3(1-2t)\Phi_1(1-t)+t(1-t)\Phi_1'(1-t)$, resp. $\Psi(t)/(t-1) = t^3[(4-3t)-(1-t)^2\Psi_1(t)]$, $\Theta'(1-t)/[t(1-t)^2] = 12+(2-5t)\Psi_1(1-t)-t(1-t)\Psi_1'(1-t)$.

(iii) $\lambda_{\mathbf{T}}^{\mathbf{p}},\lambda_{\mathbf{T}}^{\mathbf{a}},\lambda_{\mathbf{T}}^{\mathbf{b}}$ are the affine functions determined by the properties $\mathrm{Line}\{\mathbf{a},\mathbf{b}\}=\left(\lambda_{\mathbf{T}}^{\mathbf{p}}=0\right)$, $\mathrm{Line}\{\mathbf{b},\mathbf{p}\}=\left(\lambda_{\mathbf{T}}^{\mathbf{a}}=0\right)$, $\mathrm{Line}\{\mathbf{a},\mathbf{p}\}=\left(\lambda_{\mathbf{T}}^{\mathbf{b}}=0\right)$,$\lambda_{\mathbf{T}}^{\mathbf{p}}(\mathbf{p})=\lambda_{\mathbf{T}}^{\mathbf{a}}(\mathbf{a})=\lambda_{\mathbf{T}}^{\mathbf{b}}(\mathbf{b})=1$. For the parametrized edge points in $(1.6)$ we have

$$\begin{aligned}
&\lambda_{\mathbf{T}}^{\mathbf{p}}(\mathbf{x}_t)=\lambda_{\mathbf{T}}^{\mathbf{a}}(\mathbf{z}_t)=\lambda_{\mathbf{T}}^{\mathbf{b}}(\mathbf{y}_t)\equiv 0,\\
&\lambda_{\mathbf{T}}^{\mathbf{p}}(\mathbf{y}_t)=\lambda_{\mathbf{T}}^{\mathbf{p}}(\mathbf{z}_t)=\lambda_{\mathbf{T}}^{\mathbf{b}}(\mathbf{x}_t)\equiv t,\\
&\lambda_{\mathbf{T}}^{\mathbf{a}}(\mathbf{x}_t)=\lambda_{\mathbf{T}}^{\mathbf{a}}(\mathbf{y}_t)=\lambda_{\mathbf{T}}^{\mathbf{b}}(\mathbf{z}_t)\equiv 1-t.
\end{aligned}$$




On the other hand $x^{[j]}(\mathbf{y}_t)=(1-t)x^{[j]}(\mathbf{a}-\mathbf{p})=(1-t)x_{\mathbf{p}}^{[j]}(\mathbf{a})$ resp. $x^{[j]}(\mathbf{z}_t)=(1-t)x^{[j]}(\mathbf{b}-\mathbf{p})=(1-t)x_{\mathbf{p}}^{[j]}(\mathbf{b})$. Hence, with the formulas $(2.6)$, the shape conditions $(1.6)$ hold automatically with $\mathrm{const}_{\mathbf{a},\mathbf{p}}=\mathrm{const}_{\mathbf{b},\mathbf{p}}=1$ and $\mathrm{const}_{\mathbf{a},\mathbf{p}}^{(j)}=x^{[j]}(\mathbf{p}-\mathbf{a})$ resp. $\mathrm{const}_{\mathbf{b},\mathbf{p}}^{(j)}=x^{[j]}(\mathbf{p}-\mathbf{b})$, furthermore also $(1.7)$ is fulfilled.

(iv) One can check with symbolic computer algebra that all the spline procedures described in Theorem 2.4 produce $\mathcal{C}^1$-functions. It suffices to establish only that, given any two adjacent non-degenerate triangles $\mathbf{T}:=\mathrm{Co}\{\mathbf{p},\mathbf{a},\mathbf{b}\}$ resp. $\widetilde{\mathbf{T}}:=\mathrm{Co}\{\mathbf{p},\mathbf{a},\widetilde{\mathbf{p}}\}$ with common edge $\mathrm{Co}\{\mathbf{p},\mathbf{a}\}$ and distinguished point $\mathbf{p}$, the gradient vectors of the shape functions $\varphi_{\mathbf{T}}^{\mathbf{p}},\psi_{\mathbf{T}}^{j,\mathbf{p}}$ coincide with those of $\varphi_{\widetilde{\mathbf{T}}}^{\mathbf{p}},\psi_{\widetilde{\mathbf{T}}}^{j,\mathbf{p}}$ at the points $\mathbf{y}_t=(1-t)\mathbf{a}+t\mathbf{b}$. Indeed, hence it follows that the unit spline functions $f_{\mathcal{T},F_{\mathbf{p}}^i}$ $\left(\mathbf{p}\in\mathrm{Vert}(\mathcal{T}),\, i=0,1,2\right)$ corresponding to the gradient data $F_{\mathbf{p}}^0:=\left\{[\mathbf{p},1,\mathbf{0}],[\mathbf{q},0,\mathbf{0}]:\mathbf{q}\in\mathrm{Vert}(\mathcal{T})\setminus\{\mathbf{p}\}\right\}$ resp. $F_{\mathbf{p}}^j:=\left\{[\mathbf{p},0,\mathbf{e}^{[j]}],[\mathbf{q},0,\mathbf{0}]:\mathbf{q}\in\mathrm{Vert}(\mathcal{T})\setminus\{\mathbf{p}\}\right\}$ $(j=1,2)$ are continuously differentiable.

## 3. Proof of Theorem 2.4

Henceforth we consider an arbitrarily fixed procedure $\mathfrak{S}:(\mathcal{T},F)\mapsto f_{\mathcal{T},F}$ which satisfies Postulates A,B and produces continuous but not necessarily continuously differentiable functions. We reserve the notations $\varphi_{\mathbf{T}}^{\mathbf{p}}$, $\psi_{\mathbf{T}}^{j,\mathbf{p}}$ resp. $\Phi,\Psi,\Theta(t):=\Psi(t)/(t-1)$ for the basic functions resp. shape functions as established in Section 1. In accordance with $(1.5)$ we can write

$$\Phi(t)=t^2(3-2t)+t^2(1-t)^2\Phi_0(t), \qquad \Psi(t)=t^2(t-1)+t^2(1-t)^2\Psi_0(t) \quad (3.1)$$

and $\Theta(t)=t^2+t^2(t-1)\Psi_0(t)$ with suitable polynomials $\Phi_0,\Psi_0$.

Next we are going to express the constraints (1.4), (1.6), (1.7) on the basic functions in terms of $\Phi, \Psi$ and the baricentric coordinates. To this aim, we recall the following folklore fact from elementary algebraic geometry relating the root curves with a product decomposition of multivariate polynomials which is an easy consequence of Bézout's Theorem [3].

**Remark 3.2**(i) If $\mathbf{L}_0, \mathbf{L}_1, \ldots, \mathbf{L}_m$ are distinct straight lines such that $\mathbf{L}_k = \big(\ell_k = 0\big)$ with the affine functions (i.e. polynomials of first degree) $\ell_k : \mathbb{R}^2 \to \mathbb{R}$ $(k = 1, \ldots, m)$ then a polynomial $\mathbb{R}^2 \to \mathbb{R}$ is divisable with $\prod_{k=0}^m \ell_k^{\nu_k}$ if and only if, for any index $k$, it vanishes in order $\nu_k$ at the points of $\mathbf{L}_k$. In particular, given a non-degenerate triangle $\mathbf{T} := \mathrm{Co}\{\mathbf{a}, \mathbf{b}, \mathbf{p}\}$, a polynomial $Q : \mathbb{R}^2 \to \mathbb{R}$ of two variables has the form

$$Q = [\lambda_{\mathbf{T}}^{\mathbf{p}}]^{\nu_0} [\lambda_{\mathbf{T}}^{\mathbf{a}}]^{\nu_1} [\lambda_{\mathbf{T}}^{\mathbf{b}}]^{\nu_2}$$

for some polynomial $q : \mathbb{R}^2 \to \mathbb{R}$ if and only if it vanishes in order $\nu_0$ at the points of $\mathrm{Line}\{\mathbf{a}, \mathbf{b}\}$, order $\nu_1$ at $\mathrm{Line}\{\mathbf{p}, \mathbf{b}\}$ and order $\nu_2$ at $\mathrm{Line}\{\mathbf{p}, \mathbf{a}\}$, respectively.[3]

(ii) If $Q : \mathbb{R}^2 \to \mathbb{R}$ is a polynomial of two variables, we can write

$$\begin{aligned} & Q(x,y) = Q(0,0) + xq_1(x) + yq_2(y) + xyq_3(x,y) \quad \text{where} \\ & \quad q_1(x) := [Q(x,0)] - Q(0,0)/x, \quad q_2(y) := [Q(0,y) - Q(0,0)]/y, \\ & \quad q_3(x,y) := \big[Q(x,y) - [Q(0,0) + xq_1(x) + yq_2(y)]\big]/(xy) \end{aligned}$$

are well-defined polynomials in one resp. two variables.

We shall call the $\mathbb{R}^2$-polynomial $Q_0(x,y) := Q(0,0) + xq_1(x) + yq_2(y)$ of first degree the *principal part* of $Q$.

**Lemma 3.4.** *The basic functions* $\varphi_{\mathbf{T}}^{\mathbf{p}}, \psi_{\mathbf{T}}^{j,\mathbf{p}}$ *for* $\mathbf{T} = \mathrm{Co}\{\mathbf{a}, \mathbf{b}, \mathbf{p}\}$ *have the form*

$$\begin{aligned} \varphi_{\mathbf{T}}^{\mathbf{p}} &= \Phi(\lambda_{\mathbf{T}}^{\mathbf{p}}) + [\lambda_{\mathbf{T}}^{\mathbf{p}}]^2 \lambda_{\mathbf{T}}^{\mathbf{a}} \lambda_{\mathbf{T}}^{\mathbf{b}} \,\mathrm{Pol}(\lambda_{\mathbf{T}}^{\mathbf{b}}, \lambda_{\mathbf{T}}^{\mathbf{a}}), \\ \psi_{\mathbf{T}}^{j,\mathbf{p}} &= \Theta(\lambda_{\mathbf{T}}^{\mathbf{p}}) x_{\mathbf{p}}^{[j]} + [\lambda_{\mathbf{T}}^{\mathbf{p}}]^2 \lambda_{\mathbf{T}}^{\mathbf{a}} \lambda_{\mathbf{T}}^{\mathbf{b}} \,\mathrm{Pol}(\lambda_{\mathbf{T}}^{\mathbf{b}}, \lambda_{\mathbf{T}}^{\mathbf{a}}) \end{aligned}$$

*in terms of the baricentric coordinates* (2.1)*,, the shape functions* 3.1*, the modified shape function* (2.5′) *and with suitable polynomials of two variables.*

**Proof.** Fix any triangle $\mathbf{T} := \mathrm{Co}\{\mathbf{a}, \mathbf{b}, \mathbf{p}\}$. As mentioned, necessarily (3.1) holds and $\Theta$ is a polynomial. Consider the functions

$$f := \Phi(\lambda_{\mathbf{T}}^{\mathbf{p}}), \qquad g^{(j)} := \Theta(\lambda_{\mathbf{T}}^{\mathbf{p}}) \cdot x_{\mathbf{p}}^{[j]}.$$

Along the edge $\mathrm{Co}\{\mathbf{a}, \mathbf{p}\}$, at the points $\mathbf{y}_t := (1-t)\mathbf{a} + t\mathbf{p}$ we have $\lambda_{\mathbf{T}}^{\mathbf{p}}(\mathbf{y}_t) = t$, $\lambda_{\mathbf{T}}^{\mathbf{b}}(\mathbf{y}_t) = 0$, $\lambda_{\mathbf{T}}^{\mathbf{a}}(\mathbf{y}_t) = [1 - \lambda_{\mathbf{T}}^{\mathbf{a}}(\mathbf{y}_t) - \lambda_{\mathbf{T}}^{\mathbf{b}}(\mathbf{y}_t) = 1 - t$. Observe that the functions $f, g^{(j)}$

[3] $Q$ vanishes in order $\nu$ at the point $[x_0, y_0]$ if $\frac{\partial^{k+m}}{\partial x^k \partial y^m} Q(x_0, y_0) = 0$ whenever $k + m < \nu$.

suit the shape uniformity conditions because

$$f(\mathbf{y}_t)=\Phi(t),\quad g^{(j)}(\mathbf{y}_t)=\Theta(t)\langle\mathbf{e}^{[j]}|\mathbf{y}_t-\mathbf{p}\rangle=\Theta(t)(1-t)\langle\mathbf{e}^{[j]}|\mathbf{a}-\mathbf{p}\rangle= \\ =\langle\mathbf{e}^{[j]}|\mathbf{p}-\mathbf{a}\rangle\Psi(t)$$

and since $f,g^{(j)}$ are polynomial multiples of $[\lambda_\mathbf{T}^\mathbf{p}]^2$. Also, since $\mathbf{y}_1=\mathbf{p}$, $f(\mathbf{p})=\Phi(\lambda_\mathbf{T}^\mathbf{p}(\mathbf{y}_1))=\Phi(1)=1$ and

$$\begin{aligned}\nabla f(\mathbf{p})&=\Phi'\big(\lambda_\mathbf{T}^\mathbf{p}(\mathbf{y}_1)\big)\nabla\lambda_\mathbf{T}^\mathbf{p}(\mathbf{y}_1)=0\cdot\nabla\lambda_\mathbf{T}^\mathbf{p}(\mathbf{y}_1)=\mathbf{0},\\ \nabla g^{(j)}(\mathbf{p})&=\nabla_{\mathbf{x}=\mathbf{y}_1}\Big[\Theta\big(\lambda_\mathbf{T}^\mathbf{p}(\mathbf{x})\big)x_\mathbf{p}^{[j]}(\mathbf{x})\Big]=\\ &=x_\mathbf{p}^{[j]}(\mathbf{p})\nabla_{\mathbf{x}=\mathbf{y}_1}\Theta\big(\lambda_\mathbf{T}^\mathbf{p}(\mathbf{x})\big)+\Theta\big(\lambda_\mathbf{T}^\mathbf{p}(\mathbf{p})\big)\nabla_{\mathbf{x}=\mathbf{y}_1}x^{[j]}(\mathbf{p})=\\ &=0\cdot\Theta'(1)\nabla\lambda_\mathbf{T}^\mathbf{p}(\mathbf{y}_1)+\Theta(1)\mathbf{e}^{[j]}=\mathbf{e}^{[j]}.\end{aligned}$$

Therefore the difference functions $\varphi_\mathbf{T}^\mathbf{p}-f$ and $\psi_\mathbf{T}^{j,\mathbf{p}}-g^{(j)}$ vanish on the edge $\mathrm{Co}\{\mathbf{a},\mathbf{p}\}$ of the triangle $\mathbf{T}$. Similar arguments with the points $\mathbf{z}_t:=(1-t)\mathbf{b}+t\mathbf{p}$ show that $\varphi_\mathbf{T}^\mathbf{p}-g$ and $\psi_\mathbf{T}^{j,\mathbf{p}}-g^{(j)}$ vanish on $\mathrm{Co}\{\mathbf{b},\mathbf{p}\}$. By (1.7) their gradients also vanish on the edge $\mathrm{Co}\{\mathbf{a},\mathbf{b}\}=\big(\lambda_\mathbf{T}^\mathbf{p}=0\big)$. Hence (cf. Remark 3.2) they are polynomial multiples of $[\lambda_\mathbf{T}^\mathbf{p}]^2\lambda_\mathbf{T}^\mathbf{a}\lambda_\mathbf{T}^\mathbf{b}$, say $\varphi_\mathbf{T}^\mathbf{p}=f+[\lambda_\mathbf{T}^\mathbf{p}]^2\lambda_\mathbf{T}^\mathbf{a}\lambda_\mathbf{T}^\mathbf{b}\Pi_{\mathbf{p},\mathbf{T}}^{(0)}$ and $\psi_\mathbf{T}^{j,\mathbf{p}}=g^{(j)}+[\lambda_\mathbf{T}^\mathbf{p}]^2\lambda_\mathbf{T}^\mathbf{a}\lambda_\mathbf{T}^\mathbf{b}\Pi_{\mathbf{p},\mathbf{T}}^{(j)}$, respectively. Since $\lambda_\mathbf{T}^\mathbf{a},\lambda_\mathbf{T}^\mathbf{b}$ are linearly independent affine functionals, the mapping $\Lambda_{\mathbf{a},\mathbf{b}}^\mathbf{p}:\mathbf{x}\mapsto\big[\lambda_\mathbf{T}^\mathbf{a}(\mathbf{x}),\lambda_\mathbf{T}^\mathbf{b}(\mathbf{x})\big]$ is an affine coordinatization on the plain $\mathbb{R}^2$. Thus we can express each term $\varphi_\mathbf{T}^\mathbf{p},\psi_\mathbf{T}^{j,\mathbf{p}}$ as a polynomial of the coordinates $\Lambda_{\mathbf{a},\mathbf{b}}^\mathbf{p}$ which completes the proof.

**Notation 3.5.** For later convenience, without danger of confusion, we introduce the unifying context-free notations

$$\lambda_{\mathbf{a},\mathbf{b}}^\mathbf{p}:=\lambda_{\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}}^\mathbf{p},\quad f_{\mathbf{a},\mathbf{b}}^{0,\mathbf{p}}:=\varphi_{\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}}^\mathbf{p},\quad f_{\mathbf{a},\mathbf{b}}^{j,\mathbf{p}}:=\psi_{\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}}^{j,\mathbf{p}}\quad(j=1,2).$$

Furthermore, in view of Lemma 3.4, we shall write

$$f_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}=\Phi^{[i]}(\lambda_{\mathbf{a},\mathbf{b}}^\mathbf{p})x_\mathbf{p}^{[i]}+[\lambda_{\mathbf{a},\mathbf{b}}^\mathbf{p}]^2\lambda_{\mathbf{a},\mathbf{p}}^\mathbf{b}\lambda_{\mathbf{b},\mathbf{p}}^\mathbf{a}P_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(\lambda_{\mathbf{a},\mathbf{p}}^\mathbf{b},\lambda_{\mathbf{b},\mathbf{p}}^\mathbf{a})\quad(i=0,1,2)\tag{3.6}$$

where

$$\Phi^{[0]}:=\Phi,\quad\Phi^{[1]}:=\Phi^{[2]}:=\Theta,\quad x_\mathbf{p}^{[0]}:\mathbf{x}\mapsto1\ \text{ with }\ \mathbf{e}^{[0]}:=\nabla x_\mathbf{p}^{[0]}=\mathbf{0}$$

and the terms $P_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}$ $(i=0,1,2)$ are polynomials with coefficients depending on the ordered tuple $(i,\mathbf{p},\mathbf{a},\mathbf{b})$. Notice that necessarily

$$P_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(s,t)\equiv P_{\mathbf{b},\mathbf{a}}^{i,\mathbf{p}}(t,s)\tag{3.7}$$

due to the trivial index symmetries $\lambda_{\mathbf{u},\mathbf{v}}^\mathbf{w}\equiv\lambda_{\mathbf{v},\mathbf{u}}^\mathbf{w}$ and $f_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}\equiv f_{\mathbf{b},\mathbf{a}}^{i,\mathbf{p}}$.

**Lemma 3.8.** *We have* $f_{\mathcal{T},F} \in \mathcal{C}^1\big(\mathrm{Dom}(\mathcal{T})\big)$ *for every triangular mesh with arbitrary gradient data if and only if*

$$\mathbf{b} \mapsto \nabla f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(\mathbf{y}) \equiv \mathrm{const}_{\mathbf{p},\mathbf{a},\mathbf{y}} \quad \text{for fixed } \mathbf{p} \neq \mathbf{a} \text{ and } \mathbf{y} \in \mathrm{Co}\{\mathbf{p},\mathbf{a}\}. \tag{3.9}$$

**Proof.** Given any non-degenerate triangle $\mathbf{T} = \mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}$, by construction, for the points $\mathbf{x}_t := (1-t)\mathbf{a} + t\mathbf{b}$, $\mathbf{y}_t := (1-t)\mathbf{a} + t\mathbf{p}$ and $\mathbf{z}_t := (1-t)\mathbf{b} + t\mathbf{p}$ on the edges of the triangle $\mathbf{T}$ we have $f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(\mathbf{x}_t) = 0$ independently of $\mathbf{p}$, $f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(\mathbf{y}_t) = \Phi^{[i]}(t)$ independently of $\mathbf{b}$ and $f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(\mathbf{z}_t) = \Phi^{[i]}(t)$ independently of $\mathbf{a}$. Thus the shape conditions are automatic from (3.6). Moreover, given any triangle $\widetilde{\mathbf{T}}$ with a common edge but disjoint interior to $\mathbf{T}$, the function pairs $\varphi^{\mathbf{p}}_{\mathbf{T}}, \varphi^{\mathbf{p}}_{\widetilde{\mathbf{T}}}$ resp. $\psi^{j,\mathbf{p}}_{\mathbf{T}}, \psi^{j,\mathbf{p}}_{\widetilde{\mathbf{T}}}$ touch continuosly. The analogous necessary and sufficent condition for a $\mathcal{C}^1$-smooth touching is that the gradient pairs $\nabla\varphi^{\mathbf{p}}_{\mathbf{T}}, \nabla\varphi^{\mathbf{p}}_{\widetilde{\mathbf{T}}}$ resp. $\nabla\psi^{j,\mathbf{p}}_{\mathbf{T}}, \nabla\psi^{j,\mathbf{p}}_{\widetilde{\mathbf{T}}}$ coincide on the common edge:

$$\begin{aligned}
&\text{(i)} \ \ \nabla f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(\mathbf{x}) = \nabla f^{i,\widetilde{\mathbf{p}}}_{\mathbf{a},\mathbf{b}}(\mathbf{x}) && \text{if } \mathbf{x} \in \mathrm{Co}\{\mathbf{a},\mathbf{b}\} = \mathbf{T} \cap \mathrm{Co}\{\mathbf{a},\mathbf{b},\widetilde{\mathbf{p}}\},\\
&\text{(ii)} \ \ \nabla f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(\mathbf{y}) = \nabla f^{i,\widetilde{\mathbf{b}}}_{\mathbf{a},\mathbf{p}}(\mathbf{y}) && \text{if } \mathbf{y} \in \mathrm{Co}\{\mathbf{a},\mathbf{p}\} = \mathbf{T} \cap \mathrm{Co}\{\mathbf{a},\widetilde{\mathbf{b}},\mathbf{p}\},\\
&\text{(iii)} \ \ \nabla f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(\mathbf{z}) = \nabla f^{i,\widetilde{\mathbf{a}}}_{\mathbf{b},\mathbf{p}}(\mathbf{z}) && \text{if } \mathbf{z} \in \mathrm{Co}\{\mathbf{b},\mathbf{p}\} = \mathbf{T} \cap \mathrm{Co}\{\widetilde{\mathbf{a}},\mathbf{b},\mathbf{p}\}.
\end{aligned} \tag{3.10}$$

Observe that (3.10(i)) holds automatically with the trivial value $\mathbf{0}$. Furthermore conditions (3.10(ii)) and (3.10(iii)) are analogous (by changing the roles of $\mathbf{a}$ and $\mathbf{b}$ resp. $\widetilde{\mathbf{a}}$ and $\widetilde{\mathbf{b}}$). Finally we observe that, in (3.10(ii)), for fixed $\mathbf{a},\mathbf{p}$ and $\mathbf{y} \in \mathrm{Co}\{\mathbf{a},\mathbf{p}\}$ we can choose the points $\mathbf{b}$ and $\widetilde{\mathbf{b}}$ on different half plain components of $\mathbb{R}^2 \setminus \mathrm{Line}\{\mathbf{a},\mathbf{p}\}$ arbitrarily. This implies that all the vectors $\nabla f^{i,\mathbf{b}}_{\mathbf{a},\mathbf{p}}(\mathbf{y})$, $\nabla f^{i,\widetilde{\mathbf{b}}}_{\mathbf{a},\mathbf{p}}(\mathbf{y})$ with $\mathbf{b},\widetilde{\mathbf{b}} \in \mathbb{R}^2 \setminus \mathrm{Line}\{\mathbf{a},\mathbf{p}\}$ must be the same. Due to the construction (1.3), the fact that all the pairs $\varphi^{\mathbf{p}}_{\mathbf{T}}, \varphi^{\mathbf{p}}_{\widetilde{T}}$ resp. $\psi^{j,\mathbf{p}}_{\mathbf{T}}, \psi^{j,\mathbf{p}}_{\widetilde{\mathbf{T}}}$ of basic functions touch $\mathcal{C}^1$-smoothly in case of adjacent triangles $\mathbf{T},\widetilde{\mathbf{T}}$, ensures that the splines $f_{\mathcal{T},F}$ are all $\mathcal{C}^1$-smooth as well.

**Notation 3.11.** Given any ordered triple $(\mathbf{u},\mathbf{v},\mathbf{w})$ of non-collinear points, we shall write $\mathbf{g}^{\mathbf{w}}_{\mathbf{u},\mathbf{v}} := \nabla\lambda^{\mathbf{w}}_{\mathbf{u},\mathbf{v}}$ for the constant gradient vectors of the baricentric coordinate functions. Notice that, by (2.1),

$$\mathbf{g}^{\mathbf{w}}_{\mathbf{u},\mathbf{v}} := \frac{(\mathbf{u}-\mathbf{v})\mathbf{R}}{\big\langle (\mathbf{u}-\mathbf{v})\mathbf{R} \big| \mathbf{w}-\mathbf{u} \big\rangle} = \frac{\sigma^{\mathbf{w}}_{\mathbf{u},\mathbf{v}}(\mathbf{u}-\mathbf{v})\mathbf{R}}{\mathrm{area}(\mathrm{Co}\{\mathbf{u},\mathbf{v},\mathbf{w}\})}. \tag{3.12}$$

where $\sigma^{\mathbf{w}}_{\mathbf{u},\mathbf{v}} = \pm 1$ according as $(\mathbf{u},\mathbf{v},\mathbf{w})$ are oriented anticlockwise or clockwise. In particular, if $\mathbf{T} = \mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}$ is a non-degenerate triangle, we have

$$\begin{aligned}
&\mathbf{g}^{\mathbf{p}}_{\mathbf{a},\mathbf{b}} + \mathbf{g}^{\mathbf{a}}_{\mathbf{b},\mathbf{p}} + \mathbf{g}^{\mathbf{b}}_{\mathbf{a},\mathbf{p}} = \nabla\big[\lambda^{\mathbf{p}}_{\mathbf{a},\mathbf{b}} + \lambda^{\mathbf{a}}_{\mathbf{b},\mathbf{p}} + \lambda^{\mathbf{b}}_{\mathbf{a},\mathbf{p}}\big] = \nabla 1 = \mathbf{0},\\
&\mathbf{g}^{\mathbf{p}}_{\mathbf{a},\mathbf{b}} \perp \mathbf{b}-\mathbf{a}, \quad \mathbf{g}^{\mathbf{b}}_{\mathbf{a},\mathbf{p}} \perp \mathbf{a}-\mathbf{p}, \quad \mathbf{g}^{\mathbf{a}}_{\mathbf{b},\mathbf{p}} \perp \mathbf{b}-\mathbf{p}.
\end{aligned}$$

**Lemma 3.13.** *If* $\mathbf{T} = \mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}$ *is a non-degenerate triangle, at the points* $\mathbf{y}_t := (1-t)\mathbf{a}+t\mathbf{p}$ *of the edge* $\mathrm{Co}\{\mathbf{a},\mathbf{p}\}$ *we have*

$$\begin{aligned}\nabla f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(\mathbf{y}_t) =& x^{[i]}\big((1-t)(\mathbf{a}-\mathbf{p})\big)[\Phi^{[i]}]'(t)\,\mathbf{g}^{\mathbf{p}}_{\mathbf{a},\mathbf{b}}+\\ &+\Phi^{[i]}(t)\mathbf{e}^{[i]}+t^2(1-t)P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(0,1-t)\,\mathbf{g}^{\mathbf{b}}_{\mathbf{a},\mathbf{p}}.\end{aligned}\tag{3.14}$$

**Proof.** With the abbreviations

$$\ell_0 := \lambda^{\mathbf{p}}_{\mathbf{a},\mathbf{b}},\ \ell_1 := \lambda^{\mathbf{b}}_{\mathbf{a},\mathbf{p}},\ \ell_2 := \lambda^{\mathbf{a}}_{\mathbf{b},\mathbf{p}},\quad P^{[i]} := P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}},\quad G^{[i]} := \ell_0^2\ell_2 P^{[i]}(\ell_1,\ell_2)$$

we can write

$$\begin{aligned}\nabla f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}} &= \nabla\Big[x^{[i]}_{\mathbf{p}}\Phi^{[i]}(\ell_0)+\ell_1 G^{[i]}\Big] =\\ &= x^{[i]}_{\mathbf{p}}\nabla\big[\Phi^{[i]}(\ell_0)\big]+\Phi^{[i]}(\ell_0)\nabla x^{[i]}_{\mathbf{p}}+\ell_1\nabla G^{[i]}+G^{[i]}\nabla\ell_1 =\\ &= x^{[i]}_{\mathbf{p}}\Phi^{[i]\prime}(\ell_0)\nabla\ell_0+\Phi^{[i]}(\ell_0)\mathbf{e}^{[i]}+\ell_1\nabla G^{[i]}+G^{[i]}\nabla\ell_1.\end{aligned}$$

We complete the proof with the observations that

$$\begin{aligned}&\ell_0(\mathbf{y}_t)=t,\quad \ell_1(\mathbf{y}_t)=0,\quad \ell_2(\mathbf{y}_t)=1-t,\\ &x^{[i]}_{\mathbf{p}}(\mathbf{y}_t)=x^{[i]}\big((1-t)(\mathbf{a}-\mathbf{p})\big),\quad \nabla x^{[i]}_{\mathbf{p}}\equiv\mathbf{e}^{[i]}.\end{aligned}$$

**Remark 3.15.** To prove Theorem 2.4, we need a precise description for the coefficients of the polynomials $P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}$ in terms of the variables $\mathbf{a},\mathbf{b},\mathbf{p}$ such that (3.9) should hold.

According to Lemma 3.8, the procedure $\mathfrak{S} : (\mathcal{T},F)\mapsto f_{\mathcal{T},F}$ produces $\mathcal{C}^1$-splines for every admissible data if and only if, for any $t\in[0,1]$ and for any fixed pair $\mathbf{a},\mathbf{p}$ of distinct points, the gradient expressions (3.14) are independent of the variable $\mathbf{b}$ ranging in $\mathbb{R}^2\backslash\mathrm{Line}\{\mathbf{a},\mathbf{p}\}$. This latter condition can be formulated in terms of the $\mathbf{b}$-independent affine coordinates (2.2) as follows. By (3.12) we have

$$\begin{aligned}\mathbf{g}^{\mathbf{b}}_{\mathbf{a},\mathbf{p}} &= \frac{(\mathbf{a}-\mathbf{p})\mathbf{R}}{\langle(\mathbf{a}-\mathbf{p})\mathbf{R}|\mathbf{b}-\mathbf{a}\rangle} = \|\mathbf{p}-\mathbf{a}\|^{-2}(1/\overline{\xi}^{\,\mathbf{b}}_{\mathbf{p},\mathbf{a}})(\mathbf{p}-\mathbf{a})\mathbf{R},\\ \mathbf{g}^{\mathbf{p}}_{\mathbf{a},\mathbf{b}} &= \frac{(\mathbf{a}-\mathbf{b})\mathbf{R}}{\langle(\mathbf{a}-\mathbf{b})\mathbf{R}|\mathbf{p}-\mathbf{a}\rangle} = \frac{\xi^{(\mathbf{a}-\mathbf{b})\mathbf{R}}_{\mathbf{p},\mathbf{a}}(\mathbf{p}-\mathbf{a})+\overline{\xi}^{(\mathbf{a}-\mathbf{b})\mathbf{R}}_{\mathbf{p},\mathbf{a}}(\mathbf{p}-\mathbf{a})\mathbf{R}}{\|\mathbf{p}-\mathbf{a}\|^2\overline{\xi}^{\,\mathbf{b}}_{\mathbf{p},\mathbf{a}}} =\\ &= \|\mathbf{p}-\mathbf{a}\|^{-2}\Big[(\mathbf{p}-\mathbf{a})+(\xi^{\mathbf{b}}_{\mathbf{p},\mathbf{a}}/\overline{\xi}^{\,\mathbf{b}}_{\mathbf{p},\mathbf{a}})(\mathbf{p}-\mathbf{a})\mathbf{R}\Big].\end{aligned}$$

Thus we can rewrite (3.14) in the form

$$\begin{aligned}\nabla f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(\mathbf{y}_t) &= \Big[\mathbf{b}\text{-independent terms}\Big]+\\ &+\frac{x^{[i]}\big((1-t)(\mathbf{a}-\mathbf{p})\big)[\Phi^{[i]}]'(t)\xi^{\mathbf{b}}_{\mathbf{p},\mathbf{a}}+t^2(1-t)P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(0,1-t)}{\|\mathbf{p}-\mathbf{a}\|^2\overline{\xi}^{\,\mathbf{b}}_{\mathbf{p},\mathbf{a}}}(\mathbf{p}-\mathbf{a})\mathbf{R}.\end{aligned}\tag{3.16}$$

Hence we conclude immediately the following.

**Lemma 3.17.** *We have* (3.9) *if and only if for every pair* $\mathbf{p},\mathbf{a}$ *of distinct points there exist polynomials* $K_{\mathbf{a}}^{i,\mathbf{p}}$ $(i=0,1,2)$ *of one variable such that*

$$K_{\mathbf{a}}^{i,\mathbf{p}}(t) = x^{[i]}\big((1-t)(\mathbf{a}-\mathbf{p})\big)[\Phi^{[i]}]'(t)\frac{\xi_{\mathbf{p},\mathbf{a}}^{\mathbf{b}}}{\overline{\xi}_{\mathbf{p},\mathbf{a}}^{\mathbf{b}}} + \frac{t^2(1-t)}{\overline{\xi}_{\mathbf{p},\mathbf{a}}^{\mathbf{b}}}P_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(0,1-t) \tag{3.18}$$

*independently of the choice of* $\mathbf{b}$ *outside* Line$\{\mathbf{a},\mathbf{p}\}$.

We can regard (3.18) as a partial algebraic condition on the polynomials $P_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}$ of two variables, namely the below identity for $0<t<1$:

$$P_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(0,1-t) = \overline{\xi}_{\mathbf{p},\mathbf{a}}^{\mathbf{b}}\frac{K_{\mathbf{a}}^{i,\mathbf{p}}(t)}{t^2(1-t)} - \xi_{\mathbf{p},\mathbf{a}}^{\mathbf{b}}\frac{x^{[i]}\big((1-t)(\mathbf{a}-\mathbf{p})\big)[\Phi^{[i]}]'(t)}{t^2(1-t)}. \tag{3.19}$$

Since, for fixed $\mathbf{a},\mathbf{p}$, the coordinates $\big(\xi_{\mathbf{p},\mathbf{a}}^{\mathbf{b}},\overline{\xi}_{\mathbf{p},\mathbf{a}}^{\mathbf{b}}\big)$ may assume arbitrary values $(r,s)$ with $s\neq 0$, from (3.19) we obtain the polynomial divisability relations $t^2(1-t)\big|K_{\mathbf{a}}^{i,\mathbf{p}}(t)$ and $t^2(1-t)\big|x^{[i]}\big((1-t)(\mathbf{a}-\mathbf{p})\big)[\Phi^{[i]}]'(t)$, respectively. Since $x^{[0]}\big((1-t)(\mathbf{a}-\mathbf{p})\big)\equiv 1$ and $x^{[0]}\big((1-t)(\mathbf{a}-\mathbf{p})\big)\equiv (1-t)x^{[j]}(\mathbf{a}-\mathbf{p})$ for $j=1,2$, with the aid of (3.22′) we can state (3.19) in the form

$$P_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}(0,1-t) = \frac{\big\langle \mathbf{b}-\mathbf{a}\big|\mathbf{p}-\mathbf{a}\big\rangle}{\|\mathbf{p}-\mathbf{a}\|^2}\, x_{\mathbf{p}}^{[i]}(\mathbf{a})\,\chi^{[i]}(t) + \frac{\big\langle \mathbf{b}-\mathbf{a}\big|(\mathbf{p}-\mathbf{a})\mathbf{R}\big\rangle}{\|\mathbf{p}-\mathbf{a}\|^2}\,\kappa_{\mathbf{a}}^{i,\mathbf{p}}(t) \tag{3.20}$$

with suitable polynomials $\chi^{[i]}$ and $\kappa_{\mathbf{a}}^{i,\mathbf{p}}$ $(i=0,1,2;\ \mathbf{a}\neq\mathbf{p}\in\mathbb{R}^2)$ of one variable. Actually

$$\kappa_{\mathbf{a}}^{i,\mathbf{p}}(t) = \frac{K_{\mathbf{a}}^{i,\mathbf{p}}(t)}{t^2(1-t)^2},\quad \chi^{[0]}(t) = \frac{[\Phi^{[0]}]'(t)}{t^2(1-t)} = \frac{\Phi'(t)}{t^2(1-t)},$$
$$x_{\mathbf{p}}^{[i]}(\mathbf{a})\chi^{[j]}(t) = \frac{x^{[j]}\big((1-t)(\mathbf{a}-\mathbf{p})\big)[\Phi^{[j]}]'(t)}{t^2(1-t)} = x_{\mathbf{p}}^{[j]}(\mathbf{a})\frac{[\Psi(t)/(t-1)]'}{t^2}$$

for $j{=}1,2$ on the basis of (3.19) In terms of the Kronecker-$\delta$, we can write even

$$\chi^{[i]}(t) = t^{-2}(1-t)^{-\delta_{i,0}}[\Phi^{[i]}]'(t)\qquad (i=0,1,2).$$

Clearly, the polynomials $K_{\mathbf{c}}^{i,\mathbf{p}}$ cannot be chosen arbitrarily. There is a unique obstacle: we obtained Lemma 3.13 and hence (3.18) by an inspection of $\nabla f_{\mathbf{a},\mathbf{b}}^{i,\mathbf{p}}$ on one of the edges of a triangle $\mathbf{T}=\mathrm{Co}\{\mathbf{a},\mathbf{b},\mathbf{p}\}$ at the distinguished point $\mathbf{p}$ (namely Co$\{\mathbf{a},\mathbf{p}\}$ with the parametrization $\mathbf{y}_t := (1-t)\mathbf{a}+t\mathbf{b}$) while also the analogous conclusion should also be taken simultaneously in to account with the second edge (namely Co$\{\mathbf{b},\mathbf{p}\}$

issued from $\mathbf{p}$. Applying a change $\mathbf{a}\leftrightarrow\mathbf{b}$ and taking into account the symmetry (3.7), we see that also

$$P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(1-t,0)=\frac{\langle\mathbf{a}-\mathbf{b}|\mathbf{p}-\mathbf{b}\rangle}{\|\mathbf{p}-\mathbf{b}\|^2}x^{[i]}_{\mathbf{p}}(\mathbf{b})\,\chi^{[i]}(t)+\frac{\langle\mathbf{a}-\mathbf{b}|(\mathbf{p}-\mathbf{b})\mathbf{R}\rangle}{\|\mathbf{p}-\mathbf{b}\|^2}\,\kappa^{i,\mathbf{p}}_{\mathbf{b}}(t). \tag{3.21}$$

We obtain the complete description for the families of polynomials $K^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}$ being admissible by Lemma 3.17 by the next obervation.

**Lemma 3.22.** *For any couple* $\mathbf{p}\neq\mathbf{c}\in\mathbb{R}^2$, *in* (3.20–21) *we have* $\chi^{[i]}(1)=\kappa^{i,\mathbf{p}}_{\mathbf{c}}(1)=0$.

**Proof**. Fix $i,\mathbf{p}\in\mathbb{R}^2$ and $\rho>0$ arbitrarily. Consider (3.20−21) for pairs $\mathbf{a},\mathbf{b}$ with $\|\mathbf{a}-\mathbf{p}\|=\|\mathbf{b}-\mathbf{p}\|=\rho$ written in the form

$$\mathbf{a}:=\mathbf{c}_\sigma,\ \ \mathbf{b}:=\mathbf{c}_\tau\quad\text{where}\quad \mathbf{c}_\tau:=\mathbf{p}+\rho\mathbf{u}_\tau,\ \mathbf{u}_\tau:=\cos\tau\mathbf{e}^{[1]}+\sin\tau\mathbf{e}^{[2]}.$$

Due to (3.7), with the abbreviations $\alpha:=\chi^{[i]}(1)$ and $\beta(\tau):=\kappa^{i,\mathbf{p}}_{\mathbf{c}_\tau}(1)$ we get

$$\begin{aligned}
0&=P^{i,\mathbf{p}}_{\mathbf{c}_\sigma,\mathbf{c}_\tau}(0,0)-P^{i,\mathbf{p}}_{\mathbf{c}_\tau,\mathbf{c}_\sigma}(0,0)=\\
&=\Big[\big(\langle\mathbf{u}_\tau|\mathbf{u}_\sigma\rangle-1\big)x^{[i]}(\mathbf{c}_\sigma)\alpha+\langle\mathbf{u}_\tau|\mathbf{u}_\sigma\mathbf{R}\rangle\beta(\sigma)\Big]-\\
&\quad-\Big[\big(\langle\mathbf{u}_\sigma|\mathbf{u}_\tau\rangle-1\big)x^{[i]}(\mathbf{c}_\tau)\alpha+\langle\mathbf{u}_\sigma|\mathbf{u}_\tau\mathbf{R}\rangle\beta(\tau)\Big]=\\
&=\big(\langle\mathbf{u}_\sigma|\mathbf{u}_\tau\rangle-1\big)\big[x^{[i]}(\mathbf{c}_\sigma)-x^{[i]}(\mathbf{c}_\tau)\big]\alpha+\langle\mathbf{u}_\tau|\mathbf{u}_\sigma\mathbf{R}\rangle\beta(\sigma)-\langle\mathbf{u}_\sigma|\mathbf{u}_\tau\mathbf{R}\rangle\beta(\tau)\big)=\\
&=\big[\cos(\sigma-\tau)-1\big]\big[x^{[i]}(\mathbf{c}_\sigma)-x^{[i]}(\mathbf{c}_\tau)\big]\alpha+\sin(\tau-\sigma)\big[\beta(\sigma)+\beta(\tau)\big].
\end{aligned}$$

Since $x^{[0]}\equiv1$, in any case we have $x^{[i]}_{\mathbf{p}}(\mathbf{c}_\sigma)-x^{[i]}_{\mathbf{p}}(\mathbf{c}_\tau)=\rho[x^{[i]}(\mathbf{u}_\sigma)-x^{[i]}(\mathbf{u}_\tau)]$. It follows

$$\begin{aligned}
\beta(\sigma)+\beta(\tau)&=\alpha\rho\frac{\cos(\tau-\sigma)-1}{\sin(\tau-\sigma)}\big[x^{[i]}(\mathbf{u}_\sigma)-x^{[i]}(\mathbf{u}_\tau)\big],\\
\big|\beta(\sigma)+\beta(\tau)\big|&\le\rho|\alpha|\frac{1-\cos(\tau-\sigma)}{\sin(|\tau-\sigma|)}\big\|\mathbf{u}_\sigma-\mathbf{u}_\tau\big\|\le\\
&\le2\rho|\alpha|[1-\cos(\tau-\sigma)].
\end{aligned}\tag{3.23}$$

Suppose indirectly $\beta(\tau)\neq0$ for some $\tau\in\mathbb{R}$. Let $\varepsilon:=|\beta(\tau)|$ and choose $\delta>0$ such that $2\rho|\alpha|(1-\cos\theta)<\varepsilon/4$ whenever $|\theta|\le\varepsilon$. Observe that then we have $|\beta(\tau)+\beta(\tau\pm\delta/2)|<\varepsilon/4$ that is $\beta(\tau\pm\delta/2)\in\big[-\varepsilon/4,\varepsilon/4\big]-\beta(\tau)$. Therefore $\beta(\tau+\delta/2)+\beta(\tau-\delta/2)\in\big[-\varepsilon/2,\varepsilon/2\big]-2\beta(\tau)\subset\big[-\varepsilon/2,\varepsilon/2\big]+\{-2\varepsilon,2\varepsilon\}=\big[-5\varepsilon/2,-3\varepsilon/2\big]\cup\big[3\varepsilon/2,5\varepsilon/2\big]$ i.e. $|\beta(\tau+\delta/2)+\beta(\tau-\delta/2)|\in\big[3\varepsilon/2,5\varepsilon/2\big]$. However, we also have $|\beta(\tau+\delta/2)+\beta(\tau-\delta/2)|<\varepsilon/4$ which leads to the contradiction $|\beta(\tau+\delta/2)+\beta(\tau-\delta/2)|\in\big[3\varepsilon/2,5\varepsilon/2\big]\cap\big[0,\varepsilon/4\big]=\varnothing$. By the arbitrariness of the radius $\rho$, the angle $\tau$ and the origin $\mathbf{p}$, we conclude that $\kappa^{i,\mathbf{p}}_{\mathbf{c}}(1)=0$ in any case.

For $i=1,2$ we get $\alpha=0$ i.e. $\chi^{[i]}(1)=0$ immediately by plugging $\beta(\tau)=\beta(\sigma)=0$ with $\sigma:=\tau+\pi/4$ in the first equation of (3.23). (Remark: $x^{[0]}(\mathbf{u}_\sigma)-x^{[0]}(\mathbf{u}_\tau)=1-1=0$, thus the argument does not work for $i=0$). In the case $i=0$ we conclude

$\alpha = 0$ as follows. Consider the difference of equations $(3.20-21)$ for $t = 1$ with $\mathbf{a} := \mathbf{p} + \mathbf{e}^{[1]}$ and $\mathbf{b} := \mathbf{p} + \mathbf{e}^{[1]} + \mathbf{e}^{[2]}$. Since $\kappa_{\mathbf{c}}^{i,\mathbf{p}} = 0$ ($\mathbf{c} = \mathbf{a}, \mathbf{b}$ is estabished already, we get simply $0 = -(1/2)\chi^{[0]}(1)$ which completes the proof.

**Corollary 3.24.** *The relations* (3.9) *hold if and only if we have* (3.18) *with the symmetry* (3.7) *where the polynomials* $K_{\mathbf{c}}^{i,\mathbf{p}}(t)$ *respectively* $x^{[i]}\big((1-t)(\mathbf{a}-\mathbf{p})\big)[\Phi^{[i]}]'(t)$ *are all divisable by* $t^2(1-t)^2$.

**Proof.** The relation $\kappa_{\mathbf{c}}^{i,\mathbf{p}}(1) = 0$ implies that there is a polynomial $\widetilde{\kappa}_{\mathbf{c}}^{i,\mathbf{p}}$ such that $\kappa_{\mathbf{c}}^{i,\mathbf{p}}(t) = (1-t)\widetilde{\kappa}_{\mathbf{c}}^{i,\mathbf{p}}(t)$ and $K_{\mathbf{c}}^{i,\mathbf{p}}(t) = t^2(1-t)\kappa_{\mathbf{c}}^{i,\mathbf{p}} = t^2(1-t)^2\widetilde{\kappa}_{\mathbf{c}}^{i,\mathbf{p}}(t)$ with some polynomial. Similarly, from $\chi^{[i]}(1) = 0$ we conclude that $\chi^{[i]}(t) = (1-t)\widetilde{\chi}^{[i]}(t)$ and $(1-t)x^{[i]}(\mathbf{a})[\Phi^{[i]}]'(t) = ((1-t)t^2\chi^{[i]}(t) = t^2(1-t)^2\widetilde{\chi}^{[i]}(t)$ with some polynomial $\widetilde{\chi}^{[i]}$.

**Corollary 3.25.** *We can write* $K_{\mathbf{c}}^{i,\mathbf{p}}(t) = t^2(1-t)^2 k_{\mathbf{c}}^{i,\mathbf{p}}(t)$ $(\mathbf{p} \neq \mathbf{c} \in \mathbb{R}^2)$ *and the admissible shape functions* $\Phi, \Psi$ *have the form*

$$\begin{aligned} &\text{(i)}\quad \Phi(t) = t^3(10-15t+6t^2) + t^3(1-t)^3\Phi_1(t), \\ &\text{(ii)}\quad \Psi(t) = t^3(t-1)(4-3t) + t^3(1-t)^3\Psi_1(t) \end{aligned} \tag{3.26}$$

*with suitable polynomials* $k_{\mathbf{c}}^{i,\mathbf{p}}, \Phi_2, \Psi_2$.

**Proof.** The stated form of $K_{\mathbf{c}}^{i,\mathbf{p}}$ is clear from (3.24). By definition $\Phi^{[0]}(t) = \Phi(t)$ and $x^{[0]}\big((1-t)(\mathbf{a}-\mathbf{p})\big) \equiv 1$. Furthermore, for $j = 1,2$ we have $\Phi^{[j]}(t) = \Psi(t)/(t-1)$ and $x^{[j]}\big((1-t)(\mathbf{a}-\mathbf{p})\big) = (1-t)x_{\mathbf{p}}^{[j]}(\mathbf{a})$. Thus, taking (3.1) into acount, the relation that $t^2(1-t)^2$ is a divisor of the polynomial $x^{[0]}\big((1-t)(\mathbf{a}-\mathbf{p})\big)[\Phi^{[0]}]'(t) = \Phi'(t) = 6t(1-t) + 2t(1-t)(1-2t)\Phi_0(t) + t^2(1-t)^2\Phi_0'(t)$ means simply that $t(1-t)$ is a divisor of $\big|6 + 2(1-2t)\Phi_0(t)$. Thus we have $6 + 2(1-2t)\Phi_0(t)|_{t=0,1} = 0$ implying that $\Phi_0(0) = -3$, $\Phi_0(1) = 3$. Therefore $\Phi_0(t) = -3 + 6t + t(1-t)\Phi_1(t)$ with some polynomial $\Phi_1$ and the generic form of $\Phi$ is $\big(3.1(\text{i})\big)$. Also according to (3.1), in the cases $j = 1,2$ we can write $\Psi(t) = -t^2(1-t) + t^2(1-t)^2\Psi_0(t)$ with some polynomial $\Psi_0$. Thus the relation that $t^2(1-t)^2$ is a divisor of the polynomial $x^{[j]}\big((1-t)(\mathbf{a}-\mathbf{p})\big)[\Phi^{[0]}]'(t) \equiv (1-t)x_{\mathbf{p}}^{[j]}(\mathbf{a})\big[\Psi(t)/(1-t)\big]'$ means that $t^2(1-t)\big|\big[\Psi(t)/(1-t)\big]' \equiv -2t + t(2-3t)\Psi_0(t) + t^2(1-t)^2\Psi_0'(t)$ is equivalent to saying $t(1-t)\big| -2 + (2-3t)\Psi_0(t)$ i.e. $-2 + (2-3t)\Psi_0(t)|_{t=0,1} = 0$ implying $\Psi_0(0) = 1$ and $\Psi_0(1) = -2$. Therefore $\Psi_0(t) = 1 - 3t + t^2(1-t)\Psi_1(t)$ with some polynomial $\Psi_1$ and the generic form of $\Psi$ is $\big(3.26(\text{i})\big)$.

**3.27. Finish of the proof of Theorem 2.4**

In view of $(3.21-21)$ and Remark 3.2(ii) we can write

$$\begin{aligned}
P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(s,t) &= P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(0,0) + s\big[\big(P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(s,0) - P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(0,0)\big)/s\big] + \\
&\quad + t\big[\big(P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(0,t) - P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(0,0)\big)/t\big] + st\mathrm{Pol}(s,t) = \\
&= s\big[\big(P^{i,\mathbf{p}}_{\mathbf{b},\mathbf{a}}(0,s)/s\big] + t\big[\big(P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(0,t)/t\big] + st\mathrm{Pol}(s,t) = \\
&= s\left[\overline{\xi}^{\,\mathbf{a}}_{\mathbf{p},\mathbf{b}} \frac{K^{i,\mathbf{p}}_{\mathbf{b}}(1-s)}{s^2(1-s)^2} - \xi^{\,\mathbf{a}}_{\mathbf{p},\mathbf{b}} \frac{x^{[i]}(\mathbf{b})\big[\Phi^{[i]}\big]'(1-s)}{s(1-s)^2}\right] + \\
&\quad + t\left[\overline{\xi}^{\,\mathbf{b}}_{\mathbf{p},\mathbf{a}} \frac{K^{i,\mathbf{p}}_{\mathbf{a}}(1-t)}{t^2(1-t)^2} - \xi^{\,\mathbf{b}}_{\mathbf{p},\mathbf{a}} \frac{x^{[i]}(\mathbf{a})\big[\Phi^{[i]}\big]'(1-t)}{t(1-t)^2}\right] + stR^{\mathbf{p}}_{\mathbf{a},\mathbf{b}}(s,t)
\end{aligned}$$

with suitable polynomials $K^{i,\mathbf{p}}_{\mathbf{c}}, \Phi^{[i]}, R^{\mathbf{p}}_{\mathbf{a},\mathbf{b}}$ of one- resp. two variables such that $t^2(1-t)^2\big|K^{i,\mathbf{p}}_{\mathbf{c}}(t)$ and $t^2(1-t)\big|x^{[i]}_{\mathbf{p}}(\mathbf{a})[\Phi^{[i]}]'(t)$. It is straightforward to check that the functions $f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}$ are polynomials in these cases and $P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(s,t) = P^{i,\mathbf{p}}_{\mathbf{b},\mathbf{a}}(t,s)$ if and only if $R^{\mathbf{p}}_{\mathbf{a},\mathbf{b}}(s,t) = R^{\mathbf{p}}_{\mathbf{b},\mathbf{a}}(t,s)$. It remains to show that the expressions

$$\begin{aligned}
&\nabla f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(\mathbf{y}_t) \quad \text{with} \quad \mathbf{y}_t := (1-t)\mathbf{a} + t\mathbf{p}, \\
&f^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}} = \Phi^{[i]}(\lambda^{\mathbf{p}}_{\mathbf{a},\mathbf{b}}) + [\lambda^{\mathbf{p}}_{\mathbf{a},\mathbf{b}}]^2 \lambda^{\mathbf{b}}_{\mathbf{a},\mathbf{p}} \lambda^{\mathbf{a}}_{\mathbf{b},\mathbf{p}} P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(\lambda^{\mathbf{b}}_{\mathbf{a},\mathbf{b}}, \lambda^{\mathbf{a}}_{\mathbf{b},\mathbf{p}})
\end{aligned}$$

are independent of the term $\mathbf{b}$ whenever

$$\begin{aligned}
&K^{i,\mathbf{p}}_{\mathbf{c}}(t) = t^2(1-t)^2 k^{i,\mathbf{p}}_{\mathbf{c}}(t), \\
&\Phi^{[0]}(t) = \Phi(t), \quad \Phi^{[1]}(t) = \Phi^{[2]}(t) \equiv \big[\Psi(t)/(t-1)\big]'
\end{aligned}$$

with arbitrary polynomials $k^{i,\mathbf{p}}_{\mathbf{c}}$ and the polynomials $\Phi, \Psi$ have the form $(3.26)$ with arbitrarily fixed polynomials $\Phi_1, \Psi_1$ of one variable.

Repeating the calculations of Lemma 3.13, we see that $(3.16)$ holds independently of the choice of $k^{i,\mathbf{p}}_{\mathbf{c}}, \Phi_1, \Psi_1, R^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}$. Notice that we have constructed the polynomials $P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(0,1-t) = P^{i,\mathbf{p}}_{\mathbf{b},\mathbf{a}}(1-t,0)$ in terms of $K^{i,\mathbf{p}}_{\mathbf{a}}$ in a manner such that $(3.18)$ should be fulfilled. Thus the expression

$$\big[\overline{\xi}^{\,\mathbf{b}}_{\mathbf{p},\mathbf{a}}\big]^{-1}\Big[x^{[i]}\big((1-t)(\mathbf{a}-\mathbf{p})\big)[\Phi^{[i]}]'(t)\xi^{\mathbf{b}}_{\mathbf{p},\mathbf{a}} + t^2(1-t)P^{i,\mathbf{p}}_{\mathbf{a},\mathbf{b}}(0,1-t)\Big]\ \big(=K^{i,\mathbf{p}}_{\mathbf{a}}(t)\big)$$

is independent of $\mathbf{b}$ automatically. This completes the proof in view of Lemma 3.17.

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*Author's address*

**L.L. Stachó**
University of Szeged, Bolyai Institute, Aradi véranúk tere 1, 6720 Szeged, Hungary
*E-mail address:* stacho@math.u-szeged.hu