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Minimax theorems beyond topological vector spaces

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1. Introduction

The numerous applications and generalizations of von Neumann's classical minimax theorem constitute an important chapter of modern convex analysis. However, all proofs make essential use of some variant of Brouwer's fixed point theorem, a result that has seemingly nothing to do with convexity but closely connected with the vector space structure of $\mathbf{R}^n$.

In his recent paper [3], I. Joó submitted a completely new and elementary proof of Ky Fan's minimax principle, based on a simple fixed point theorem that can be easily proved by the usual methods of convex analysis. Now the converse question arises: Is it possible to give an extension of the concept of convexity that allows us to find a proof of Brouwer's fixed point theorem proceeding along the lines of the fixed point theorem in [3].

Unfortunately, we cannot furnish yet a definitive answer to this problem. However, by an examination of the proofs in [1] and [3] we can find a deep argument that may provide some hope in an affirmative answer. Namely, these proofs do not touch the algebraic structure of the underlying vector spaces and the only property arising from convexity which is actually used is the trivial topological fact that the interval $[0, 1]$ is connected.

The main purpose of the present article is to show how these remarks yield new generalizations of the Ky Fan and Brézis—Nirenberg—Stampacchia minimax principles, respectively, for topological spaces that are richer but axiomatically simpler than the familiar topological vector spaces.

Our goals will be the following three observations:

a) The most suitable concept in describing the topological situation that occurs in the minimax principles is perhaps the interval space defined (in Section 2) as a topological space equipped with a system of connected subsets that play the role

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of closed segments in vector spaces. In such spaces the convexity of sets and quasiconvexity of functions have a natural interpretation and Joó's method (even with some simplifications) can be applied to establish an extension of Ky Fan's minimax theorem.

b) On the other hand, by shifting the emphasis from the topology on the order structure of one of the underlying spaces, a little change in the crucial steps of [1] (summarized there in formulae (3), (4), (5)) leads to a new elementary proof and generalization for certain interval spaces of the Brézis—Nirenberg—Stampacchia minimax theorem [4, p. 289] that provides a deeper explanation of the asymmetry noted in [4, Remark p. 290].

c) We can answer by a counterexample a question of L. Nirenberg [5, p. 144] concerning the conjectured most general form of minimax theorems in topological vector spaces.

I am indebted to I. Joó for the stimulating discussions and for having called my attention to Nirenberg's question.

2. A Joó type minimax theorem in interval spaces

Definition. By an *interval space* we mean a topological space X endowed with a mapping $[.,.]$: $X\times X\to\{$connected subsets of $X\}$ such that $x_1, x_2\in[x_1, x_2]=[x_2, x_1]$ for all $x_1, x_2\in X$.

In interval spaces it makes sense to speak of convexity in a natural way:

Definition. A subset K of an interval space X is *convex* if for every $x_1, x_2\in K$ we have $[x_1, x_2]\subset K$. Obviously, this concept preserves the following fundamental properties of convexity in vector spaces:

Proposition 1. *In any interval space X, convex sets are connected or empty. The intersection of any family of convex sets is convex. The union of any increasing (with respect to inclusion) net is convex.*

For our purposes it is of more importance that, although convex functions cannot be defined on interval spaces in a reasonable manner, the concept of quasiconvexity of functions can be extended to interval spaces.

Definition. A function f mapping an interval space X into $\mathbf{R}$ is *quasiconvex* or *quasiconcave* if $f(z)\leqq\max\{f(x_1), f(x_2)\}$ or $f(z)\geqq\min\{f(x_1), f(x_2)\}$ whenever $x_1, x_2\in X$ and $z\subset[x_1, x_2]$. Thus f is quasiconvex [quasiconcave] iff the sets $\{x: f(x)\leqq\gamma\}$ $[\{x: f(x)\geqq\gamma\}]$ are convex for all $\gamma\in\mathbf{R}$.

To extend the proof in [1] for interval spaces, we need the following generalization of the fixed point theorem in [3]:

Proposition 2. *Let Y be an interval space, X a topological space and $K(\cdot)$ a mapping of Y into the family of compact subsets of X, such that*

(1) $K(y)\neq\emptyset$ *for all* $y\in Y$,

(2) $K(z)\subset K(y_1)\cup K(y_2)$ *whenever* $z\in[y_1, y_2]$ *and* $y_1, y_2\in Y$,

(3) $\bigcap_{i=1}^{n} K(y_i)$ *is connected or empty for every* $y_1, \dots, y_n\in Y$ $(n=1, 2, \dots)$,

(4) $x\in K(y)$ *whenever* $y=\lim_{i\in\mathscr{I}} y_i$, $x=\lim_{i\in\mathscr{I}} x_i$ *and* $x_i\in K(y_i)$ *for all* $i\in\mathscr{I}$. *Then we have* $\bigcap_{y\in Y} K(y)\neq\emptyset$.

Proof. We must show that the family $K(Y)$ has the finite intersection property, i.e.

$$\bigcap_{i=1}^{n} K(y_i)\neq\emptyset \quad \text{for every} \quad y_1, \dots, y_n\in Y \tag{3'}$$

for all $n\in\mathbf{N}$. We prove (3′) by induction on n. For $n=1$, (3′) follows from (1). Suppose that (3′) holds for $n=1, \dots, k$ but there are $y_1^*, \dots, y_{k+1}^*$ such that $\bigcap_{i=1}^{k+1} K(y_i^*)=\emptyset$. Consider now the mapping $y\mapsto K^*(y)\equiv K(y)\cap\bigcap_{i=3}^{k+1} K(y_i^*)$. It readily follows from our induction hypothesis that $K^*(y)\neq\emptyset$ for all $y\in Y$. Moreover, (2) and (3) ensure that

(5) $K^*(z)$ is a connected subset of $K^*(y_1^*)\cup K^*(y_2^*)$ for any $z\in[y_1^*, y_2^*]$.

By definition, $K^*(y_1^*)\cap K^*(y_2^*)=\emptyset$. (5) implies that for every $z\in[y_1^*, y_2^*]$, the connected set $K^*(z)$ is the disjoint union of the compact sets $K^*(z)\cap K^*(y_j^*)$ $(j=1, 2)$. Hence

(5′) either $K^*(z)\subset K^*(y_1^*)$ or $K^*(z)\subset K^*(y_2^*)$ for any $z\in[y_1^*, y_2^*]$.

Thus the sets $S_j\equiv\{z\in[y_1^*, y_2^*]:\ K^*(z)\subset K^*(y_j^*)\}$ $(j=1, 2)$ are disjoint non-empty and $S_1\cup S_2=[y_1^*, y_2^*]$. But from (4) we see that both S_1 and S_2 must be closed in $[y_1^*, y_2^*]$. (In fact, let $j=1$ or 2 be fixed and let $(y_i\colon i\in\mathscr{I})$ be a net in S_j with $y_i\to y\in[y_1^*, y_2^*]$. For any index $i\in\mathscr{I}$, pick a point $x_i\in K^*(y_i)$ arbitrarily. Since by the definition of S_j, the sets $K^*(y_i)$ are contained in the compact $K^*(y_j^*)$, for a suitable subnet $(x_{i_m}\colon m\in\mathscr{M})$ we have $x_{i_m}\to x$ for some $x\in K^*(y_j^*)$. Now (4) ensures that $x\in K^*(y)$ whence $K^*(y)\subset K^*(y_j^*)$.) However this contradicts our axiomatic assumption that intervals are connected.

Theorem 1. *Let X, Y be compact interval spaces and let $f\colon X\times Y\to\mathbf{R}$ be a continuous function such that*

(6^x) *the subfunctions* $x\mapsto f(x, y)$ *are quasiconcave for any fixed* $y\in Y$,

(6^y) *the subfunctions* $y\mapsto f(x, y)$ *are quasiconvex for any fixed* $x\in X$.

Then $\gamma_*\equiv\max_x\min_y f(x, y)=\min_y\max_x f(x, y)\equiv\gamma^*$.

Proof. A standard compactness argument establishes that both γ_* and γ^* are attained (thus the statement of Theorem 1 makes sense). Then obviously we have $\gamma_* \leqq \gamma^*$. The converse inequality $\gamma_* = \max_x \min_y f(x, y) \geqq \gamma^*$ is equivalent to the existence of some $x_0 \in X$ such that for all $y \in Y$ we have $f(x_0, y) \geqq \gamma^*$.

For each $y \in Y$, let $K(y)$ be defined by $K(y) \equiv \{x : f(x, y) \geqq \gamma^*\}$. Thus to $\gamma_* \geqq \gamma^*$ we have to show $\bigcap_{y \in Y} K(y) \neq \emptyset$.

From the definition of γ^* we see that $K(y) \neq \emptyset$ for any $y \in Y$. The continuity of f implies that $K(y)$ is compact and from (6^x) we obtain that $K(y)$ is convex for all $y \in Y$. From (6^y) it follows $K(z) = \{x : f(x, z) \geqq \gamma^*\} \subset \{x : \max\{f(x, y_j) : j = 1, 2\} \geqq \gamma^*\} = \bigcup_{j=1}^{2} \{x : f(x, y_j) \geqq \gamma^*\} = \bigcup_{j=1}^{2} K(y_j)$ whenever $z \in [y_1, y_2]$. Finally, also from the continuity of f we deduce (4). Since convex sets are connected or empty, Proposition 2 can be applied, whose conclusion is $\bigcap_{y \in Y} K(y) \neq \emptyset$.

We close this section with the following question:

Question. Is there a choice of X, Y and K in Proposition 2 such that the conclusion $\bigcap_{y \in Y} K(y) \neq \emptyset$ be a known equivalent of Brouwer's fixed point theorem?

3. A generalization of the Brézis—Nirenberg—Stampacchia minimax theorem

Definition. We shall say that an interval space Y is *Dedekind complete* if for every pair of points $y_1, y_2 \in Y$ and convex subsets $H_1, H_2 \subset Y$ with $y_j \in H_j$ ($j = 1, 2$) and $[y_1, y_2] \subset H_1 \cup H_2$ there exists $z \in H_1$ such that $[y_2, z] \setminus \{z\} \subset H_2$ or there exists $z \in H_2$ such that $[y_1, z] \setminus \{z\} \subset H_2$.

Lemma 1. *Let Y be a convex subset of some real Hausdorff topological vector space with its natural interval structure $[y_1, y_2] \equiv \{(1-\lambda) y_1 + \lambda y_2 : \lambda \in [0, 1]\}$ (for each $y_1, y_2 \in Y$). Then Y is a Dedekind complete interval space.*

Proof. Given y_1, y_2 and H_1, H_2 as above, set $z \equiv (1-\lambda^*) y_1 + \lambda^* y_2$ where $\lambda^* \equiv \sup\{\lambda \in [0, 1] : (1-\lambda) y_1 + \lambda y_2 \in H_1\}$. Then $z \in [y_1, y_2]$ and $[y_j, z] \setminus \{z\} \subset H_j$ ($j = 1, 2$).

Proposition 3. *Let X be an interval space, Y a Dedekind complete Hausdorff interval space and $f \colon X \times Y \to \mathbf{R}$ a function such that*

(7^x) *the subfunctions $x \mapsto f(x, y)$ are quasiconcave on X and upper semicontinuous on any interval of X (for all fixed $y \in Y$).*

(7^y) *the subfunctions $y \mapsto f(x, y)$ are quasiconvex on Y and lower semicontinuous on any interval of Y (for all fixed $x \in X$). Then the family $\mathcal{F}$ of X-subsets defined by*

$$\mathcal{F} \equiv \{\{x : f(x, y) \geqq \gamma\} : y \in Y, \gamma < \gamma^*\}, \quad \text{where} \quad \gamma^* \equiv \inf_y \sup_x f(x, y), \tag{8}$$

has the finite intersection property whenever $\gamma^* > -\infty$.

Proof. The definition of γ^* ensures that $F\neq\emptyset$ for any $F\in\mathscr{F}$ (and $\mathscr{F}\neq\emptyset$ if $\gamma^*>-\infty$). Assume now that we have

$$\bigcap_{i=1}^{n} F_i \neq \emptyset \quad \text{for every choice of} \quad F_1, \dots, F_n \in \mathscr{F}, \tag{9}$$

but $\bigcap_{i=1}^{n+1} F_i^* = \emptyset$ where $F_1^*, \dots, F_{n+1}^*$ are some given elements of $\mathscr{F}$. To complete the

11 proof, we show that this is impossible.

By (8) we may suppose that $F_i^*=\{x\colon f(x, y_i^*)\geqq\gamma_i^*\}$ $(i=1, \dots, n+1)$ with $y_1^*, \dots, y_{n+1}^*\in Y$ and $\gamma^*>\gamma_1^*\geqq\dots\geqq\gamma_{n+1}^*$. Set

$$G \equiv \bigcap_{i=3}^{n+1} \{x\colon f(x, y_i^*) > \gamma_i^*\} \quad \text{and} \quad K(y) \equiv \{x\in G\colon f(x, y) > \gamma_1^*\} \quad \text{for all} \quad y\in Y. \tag{10}$$

Now (7^x) implies that each set $K(y)$ is convex in X and from (10) and (9) we see that

$$K(y) \supset \left\{x\colon f(x, y) \geqq \frac{\gamma_1^*+\gamma^*}{2}\right\} \cap \bigcap_{i=3}^{n+1} \left\{x\colon f(x, y_i^*) \geqq \frac{\gamma_i^*+\gamma^*}{2}\right\} \neq \emptyset \quad (\text{for all} \quad y\in Y).$$

Also in this proof, the key property of the mapping $y\mapsto K(y)$ is that

(2^*) $K(z)\subset K(y_1)\cup K(y_2)$ whenever $z\in[y_1, y_2]$ (for all $y_1, y_2\in Y$) which can be deduced from (10) and (7) as follows: $K(z)=\{x\in G\colon f(x, z)>\gamma_1^*\}\subset \subset\{x\in G\colon \{\max f(x, y_j)\colon j=1, 2\}>\gamma_1^*\}=\bigcup_{j=1}^{2}\{x\in G\colon f(x, y_j)>\gamma_1^*\}=K(y_1)\cup K(y_2)$.
Hence it follows that

(5^*) either $K(z)\subset K(y_1^*)$ or $K(z)\subset K(y_2^*)$ for any $z\in[y_1^*, y_2^*]$.

Indeed, $x_1\in K(z)\cap K(y_1^*)$ and $x_2\in K(z)\cap K(y_2^*)$ implies that for the sets $T_j\equiv \equiv[x_1, x_2]\cap F_j^*\cap\bigcap_{i=3}^{n+1} F_i^*$ $(j=1, 2)$ we have $T_1\cap T_2\subset\bigcap_{i=1}^{n+1} F_i^*=\emptyset$ and $[x_1, x_2]\supset T_1\cup T_2\supset \supset[x_1, x_2]\cap(F_1^*\cup F_2^*)\cap G\supset[x_1, x_2]\cap\bigcup_{j=1}^{2} K(y_j^*)\supset$ by (2) $\supset[x_1, x_2]\cap K(z)=[x_1, x_2]$. By ($7^x$) the sets F_i^* are closed in X $(i=1, \dots, n+1)$ whence T_1 and T_2 are closed in $[x_1, x_2]$. But this contradicts the connectedness of $[x_1, x_2]$. Thus (5^*) holds.

(2^*) and (5^*) show that the sets

$$H_j^* \equiv \{z\colon K(z) \subset K(y_j^*)\} \quad (j = 1, 2) \tag{11}$$

are convex in Y, $H_1^*\cup H_2^*\supset[y_1, y_2]$ and $y_j^*\in H_j^*$ $(j=1, 2)$. Since the interval space Y was assumed to be Dedekind complete, there exist $j\in\{1, 2\}$ and $z^*\in H_j^*$ such that

$$[y_k^*, z^*]\setminus\{z^*\} \subset H_k^* \quad \text{where} \quad k\in\{1, 2\}\setminus\{j\}. \tag{12}$$

From (10) and (11) we have

$$f(x^*, z^*) > \gamma_1^* \quad \text{for all} \quad x^*\in K(z^*). \tag{13}$$

On the other hand, if $x^* \in K(z^*)$ then $x^* \notin K(y_k^*)$. From (12) and (11) it follows $K(y_k^*) \supset K(z)$ for all $z \in [y_k^*, z^*] \setminus \{z^*\}$ whence we obtain by (10) that

$$(13') \qquad f(x^*, z) \leqq \gamma_1^* \quad \text{for all} \quad z \in [y_k^*, z^*] \setminus \{z^*\} \quad \text{and} \quad x^* \in K(z^*).$$

Since the topology of Y was supposed to be Hausdorff and since the interval $[y_k^*, z^*]$ is connected, the point z^* belongs to the closure of $[y_k^*, z^*] \setminus \{z^*\}$. But then (7^y) and (13′) imply $f(x^*, z^*) \leqq \gamma_1^*$ for all $x^* \in K(z^*)$ $(\neq \emptyset)$ which contradicts (13).

Theorem 2. *Suppose that X is an interval space, Y is a Dedekind complete Hausdorff interval space and that the function $f\colon X \times Y \to \mathbf{R}$ has the properties* (7^y),

(7^x_*) *the subfunctions $x \mapsto f(x, y)$ are upper semicontinuous and quasiconcave on the whole X (for all fixed $y \in Y$),*

(7^c) *for some $\gamma < \inf_y \sup_x f(x, y)$ and $y \in Y$, the set $\{x\colon f(x, y) \geqq \gamma\}$ is compact.*

Then we have $\max_x \inf_y f(x, y) = \inf_y \sup_x f(x, y)$.

Proof. From the definition of the operations inf and sup it follows immediately that $\sup_x \inf_y f(x, y) \leqq \inf_y \sup_x f(x, y)$. Therefore again it suffices to prove that $\inf_y f(x_0, y) \geqq \gamma^* (\equiv \inf_y \sup_x f(x, y))$ for some $x_0 \in X$, or equivalently that the family $\mathcal{F}$ defined by (8) admits a common point.

Now (7^c) ensures that $\gamma^* > -\infty$ and that some member of $\mathcal{F}$ is a non-empty compact set. By (7^x_*), each member of $\mathcal{F}$ is a closed subset of X. Hence $\bigcap \mathcal{F} \neq \emptyset$ if and only if $\mathcal{F}$ has the finite intersection property. But this is a directe consequence of Proposition 3.

Corollary. (Brézis—Nirenberg—Stampacchia) *If X is a convex subset of a real Hausdorff topological vector space, Y is a convex subset in a real vector space and $f\colon X \times Y \to \mathbf{R}$ is a function satisfying (7^x_*), (7^y) and (7^c) then we have* $\max_x \inf_y f(x, y) = \inf_y \sup_x f(x, y)$.

Proof. Let us endove Y with any locally convex Hausdorff vector space topology. (It is always possible e.g. by taking the convex core topology on the supporting vector space of Y, cf. [6, p. 110, (2.10)].) Then by Lemma 1 we can apply Theorem 2.

4. A counterexample concerning the extendibility of Theorem 2

In the light of the proof of Proposition 3, we can answer (negatively) the question raised by L. Nirenberg [5, p. 144] whether condition (7^x_*) can be replaced by the weaker condition (7^x) in the Brézis—Nirenberg—Stampacchia minimax theorem.

Theorem 3. *There exist locally convex Hausdorff topological vector spaces F, G and compact convex subsets $X\subset F$ and $Y\subset G$, further a function $f\colon X\times Y\to\{0,1\}$ satisfying* (7^x), (7^y), *and such that* $0=\max_x \min_y f(x,y)$ *and* $1=\max_y \min_x f(x,y)$.

Remark. It is well-known from elementary convex analysis that a convex subset K of a finite dimensional real Hausdorff topological vector space V is closed if and only if it is algebraically closed (i.e. if the sets $\{\lambda\in\mathbf{R}\colon u+\lambda\cdot v\in K\}$ are closed for all $u, v\in V$) [6, p. 59, p. 9]. Hence (7^x) [respectively (7^y)] implies that the subfunctions $x\mapsto f(x,y)$ [$y\mapsto f(x,y)$] restricted to the intersection of $X[Y]$ with any finite dimensional linear submanifold of F [G] are all upper [lower] semicontinuous.

Proof. Let G be the space of the functions mapping $\mathbf{N}(\equiv\{1,2,\dots\})$ into $\mathbf{R}$ endowed with the pointwise convergence topology and let $Y\equiv\{y\in G\colon \text{range}\,(y)\subset[0,1]\}$. Thus Y is homeomorphic to the compact product space $[0,1]^{\mathbf{N}}$. For $i=1,2,\dots$ let e_i denote the function $e_i\colon n\mapsto\delta_{in}(=1$ if $i=n$, 0 if $i\neq n)$. Set $H_n\equiv\text{co}\,\{e_i\colon i>n\}$ (the symbol *co* standing for the algebraic convex hull operation; $n=1,2,\dots$). Clearly, the sets H_n are algebraically closed (because the vectors $e_1, e_2, \dots$ are linearly independent). Further we have $\bigcap_{n=1}^{\infty} H_n=\emptyset$. Therefore the function

$$m(y)\equiv\min\{n\in\mathbf{N}\colon y\notin H_4\}$$

is well-defined for all $y\in G$. Now we define the space F as the set of the functions mapping Y into $\mathbf{R}$, also with the pointwise convergence topology, and we set $X\equiv\{x\in F\colon \text{range}\,(x)\subset[0,1]\}$. Again, X is homeomorphic to the compact product $[0,1]^Y$. To define the function f, first we introduce the following X-subset valued function $K(\cdot)$ on Y:

$$K(y)\equiv\text{co}\,\{1_{H_n}\colon n\geqq m(y)\}\quad(\text{for all } y\in Y)$$

where 1_{H_n} denotes the characteristic function of the set H_n (i.e. $1_{H_n}(y)=1$ if $y\in H_n$ and 0 else). Since the functions 1_{H_n} ($n\in\mathbf{N}$) are linearly independent, the sets $K(y)$ are algebraically closed (for all $y\in Y$). Then let

$$f(x,y)\equiv 1_{K(y)}(y)\quad(=1 \text{ if } x\in K(y),\ 0 \text{ if } x\notin K(y))\quad\text{for all}\quad x\in X,\ y\in Y.$$

To show (7^x), we have to check that for all $\gamma\in\mathbf{R}$, the sets $\{x\colon f(x,y)\geqq\gamma\}$ are algebraically closed for any $y\in Y$. But $\{x\colon f(x,y)\geqq\gamma\}=X$ if $\gamma\leqq 0$, $K(y)$ if $0<\gamma\leqq 1$, $\emptyset$ if $\gamma>1$.

In particular, $\{x\colon f(x,y)\geqq 1\}=K(y)\neq\emptyset$ for each $y\in Y$, whence $1=\max_{x,y} f(x,y)=$ $=\min_y \max_x f(x,y)$.

For (7^y) we must show that $\{y\colon f(x,y)\leqq\gamma\}$ is algebraically closed for all $\gamma\in\mathbf{R}$ and $x\in X$. Now we have $\{y\colon f(x,y)\leqq\gamma\}=\emptyset$ if $\gamma<0$, Y if $\gamma\geqq 1$, and if $0\leqq\gamma<1$ then

$\{y: f(x,y)\leqq\gamma\}=\{y: f(x,y)=0\}=\{y: x\notin K(y)\}=\{y: x_{H_n}\mathrm{co}\,\{1_{H_n}: n\geqq m(y)\}\}$. In case of $x\notin\mathrm{co}\,\{1_{H_n}: n\in\mathbf{N}\}$ we obviously have $\{y: x\notin\mathrm{co}\,\{1_{H_n}: n\geqq m(y)\}\}=Y$. If $x\in\mathrm{co}\,\{1_{H_n}: n\in\mathbf{N}\}$ then there exist finite sets $\mathscr{I}_x\subset\mathbf{N}$ and $\{\lambda_i^x: i\in\mathscr{I}_x\}\subset(0,\infty)$ such that $\sum_{i\in\mathscr{I}_x}\lambda_i^x=1$ and $x=\sum_{i\in\mathscr{I}_x}\lambda_i^x\cdot 1_{H_i}$, thus in this case we have $\{y: x\notin\mathrm{co}\,\{1_{H_n}: n\geqq m(y)\}\}=$ $=\{y:\ \min\mathscr{I}_x<m(y)\}=\{y:\ \min\mathscr{I}_x<\min\{n:\ y\notin H_n\}\}=\{y:\ \exists n\leqq\min\mathscr{I}_x\ y\notin H_n\}=$ $=\{y:\ \forall n\leqq\min\mathscr{I}_x\ y\in H_n\}=\bigcap_{n=1}^{\min\mathscr{I}}H_n=H_{\min\mathscr{I}_x}$ which is also convex and algebraically closed.

Since for any $x\in X$ we have seen that $\{y: f(x,y)=0\}=Y$ or H_n for some $n\in\mathbf{N}$, i.e. $\{y: f(x,y)=0\}\neq\emptyset$, we can conclude $0=\min_{x,y}f(x,y)=\max_x\min_y f(x,y)$.

Question. Does $\sup_x\inf_y f(x,y)=\inf_y\sup_x f(x,y)$ hold if the function $f\colon X\times Y\to\mathbf{R}$ is such that X and Y are convex compact subsets of some locally convex Hausdorff topological vector spaces and every restriction to any straight line segment contained in X [in Y] of the subfunctions $x\mapsto f(x,y)$ [$y\mapsto f(x,y)$] is continuous and *concave [convex]*?

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