Article

A simple affine-invariant spline interpolation over triangular meshes

László L. Stachó [1,†,‡]

[1] Bolyai Institute, University of Szeged, Aradi Vértanúk tere 1; stacho@math.u-szeged.hu

Abstract: Given a triangular mesh, we obtain an orthogonality free analogue of the classical local Zlámal-Ženišek spline procedure with simple explicit affine-invariant formulas in terms of the normalized barycentric coordinates of the mesh triangles. Our input involves first order data at mesh points and, instead of adjusting normal derivatives at side middle points, we construct the elementary splines by adjusting the Fréchet derivatives at three given directions along the edges with the result of bivariate polynomials of degree 5. By replacing the real line $\mathbb{R}$ with a generic field $\mathbb{K}$, our results admit a natural interpretation with possible independent interest and the proofs are short enough for graduate courses.

Keywords: polynomial $\mathcal{C}^1$-spline; triangular mesh; first order data; affine invariance over fields

1. Introduction

With the rapid increase in computing capacity, spline interpolation over triangular meshes became a popular issue in numerical mathematics: given the data of coordinates of points from some 2D surface, triangularization techniques and then $\mathcal{C}^1$-spline constructions are widely used for approximating the underlying surface with high accuracy. The related literature with large computational demands and spectacular outcome is enormous. Beautiful examples relatively close to our context are [1]Hahman(2000), [2]Cao(2019) and the references therein.

Our aim is somewhat the opposite direction. We investigate "minimalist" approaches: given a triangular mesh on the plane, find a method producing a $\mathcal{C}^1$ spline with *polynomials of law degree* on the mesh triangles which is "local" in the sense that the coefficients for any mesh triangle can be calculated with an explicit formula depending only on the location and the given data (as function values, differential requirements etc.) associated with the vertices of two adjacent triangles. Our paper originates from computer algebraic studies of the classical method by [3]Zlámal-Ženišek(1971) based upon the fact that the requirement of adjusting fifth-degree polynomials for function, gradient and Hessian values along with normal derivatives at edge middle points of a single mesh-triangle gives rise to a $\mathcal{C}^1$-spline. Originally they have only proved that the linear system of 21 equations for calculating the 21 coefficients for the adjustment admits a unique solution. Recently [4]Sergienko et al.(2014) published the rather sophisticated related explicit formulas, which motivated us to develop an axiomatic approach to locally generated polynomial spline methods [5][Stachó(2019)]. Our recent work is a non-straightforward application of the results there, though it is self-contained formally. We only use the principal shape functions Φ and Θ below provided by Theorem 2.3 there in the simplest form with no need to any hint of their provenience.

We are going to describe a family of local $\mathcal{C}^1$-spline procedures with really simple explicit affine-invariant 5-degree polynomials in terms of barycentric weights by adjusting first order data at vertices. Despite our results seem like a variant of the procedure by Zlámal-Ženisek (ZZ *for later use*), they cannot be deduced as a special case because of being free of the concept of orthogonality. The proofs, which may have independent interest, are basically different from the arguments in (ZZ).

Citation: Stachó, L.L. Affine-invariant splines for triangular meshes. *Journal Not Specified* **2022**, *1*, 0.
https://doi.org/

Received:
Accepted:
Published:

2. Main results

Throughout this work let

$$\Phi(t) := t^3(10 - 15t + 6t^2), \qquad \Theta(t) := t^3(4 - 3t).$$

Fix also any non-degenerate triangle $\boldsymbol{T}$ with vertices $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ on the plane $\mathbb{R}^2$ along with three affine functions $\boldsymbol{x} \mapsto f_i + A_i(\boldsymbol{x} - \boldsymbol{p}_i)$ $\big($that is $f_i \in \mathbb{R},\ A_i \in \mathcal{L}(\mathbb{R}^2, \mathbb{R})\big)$ and define

$$F_0(\boldsymbol{x}) := \sum_{i=1}^{3} \Big[\Phi\big(\lambda_i(\boldsymbol{x})\big) f_i + \Theta\big(\lambda_i(\boldsymbol{x})\big) A_i(\boldsymbol{x} - \boldsymbol{p}_i)\Big] \tag{1}$$

where $\lambda_1, \lambda_2, \lambda_3 : \mathbb{R}^2 \to \mathbb{R}$ are the *barycentric weights* determined unambiguously by the relations

$$\sum_{i=1}^{3} \lambda_i(\boldsymbol{x}) = 1, \quad \boldsymbol{x} = \sum_{i=1}^{3} \lambda_i(\boldsymbol{x}) \boldsymbol{p}_i \quad (\boldsymbol{x} \in \mathbb{R}^2).$$

Theorem 1. *Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3 \in \mathbb{R}^2$ be arbitrary vectors such that $\boldsymbol{u}_k \not\parallel (\boldsymbol{p}_j - \boldsymbol{p}_k)$ whenever $\{i, j, k\} = \{1, 2, 3\}$. Then there exist constants $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{R}$ which can be formulated explicitly in terms of $\lambda_1, \lambda_2, \lambda_3$ (see (18) later) such that the function*

$$F(\boldsymbol{x}) := F_0(\boldsymbol{x}) + \sum_{\ell=1}^{3} \zeta_\ell \lambda_\ell(\boldsymbol{x})^{-1} \prod_{m=1}^{3} \lambda_m(\boldsymbol{x})^2 \tag{2}$$

along with its Fréchet derivatives $F'(\boldsymbol{x})\boldsymbol{v} := \frac{d}{d\tau}\big|_{\tau=0} F\big(\boldsymbol{x} + \tau \boldsymbol{v}\big)$ behave on the edges of $\boldsymbol{T}$ for any triple (i, j, k) of different indices as follows:

$$F(\boldsymbol{p}_i) = f_i, \quad F'_{\boldsymbol{v}}(\boldsymbol{p}_i) = A_i \boldsymbol{v} \quad (\boldsymbol{v} \in \mathbb{R}^2), \tag{3}$$

$$F\big(t\boldsymbol{p}_i + (1-t)\boldsymbol{p}_j\big) = \Phi(t) f_i + [1 - \Phi(t)] f_j + [\Theta(t) A_i - \Theta(1-t) A_j](\boldsymbol{p}_j - \boldsymbol{p}_i), \tag{4}$$

$$F'\big(t\boldsymbol{p}_i + (1-t)\boldsymbol{p}_j\big)\boldsymbol{u}_k = \big[\Theta(t) A_i + \Theta(1-t) A_j\big]\boldsymbol{u}_k. \tag{5}$$

As a consequence, given a triangular mesh, we can obtain modifications of the celebrated (ZZ) spline procedure [3]Zlámal-Ženišek(1971), [4]Sergienko(2014) regardless to second order data but with simple explicit scalar product free formulas in terms of affine functions. Notice that, due to their invariant affine invariance, our results cannot be deduced from (ZZ) e.g. by setting the input second derivatives at the vertices to 0.

Recall that by a *triangular mesh* we mean a family $\mathcal{T} = \big\{\boldsymbol{T}_1, \ldots \boldsymbol{T}_N\big\}$ of closed triangles in $\mathbb{R}^2$ such that the intersection $\boldsymbol{T}_m \cap \boldsymbol{T}_n$ is either a common edge or a common vertex or empty for different indices m, n. Given any triangle $\boldsymbol{T} \subset \mathbb{R}^2$, $\mathrm{Vert}(\boldsymbol{T})$ and $\mathrm{Edge}(\boldsymbol{T})$ will denote the set of its vertices resp. edges, and we write $\mathrm{Vert}(\mathcal{T}) := \bigcup_{n=1}^{N} \mathrm{Vert}(\boldsymbol{T}_n)$ resp. $\mathrm{Edge}(\mathcal{T}) := \bigcup_{n=1}^{N} \mathrm{Edge}(\boldsymbol{T}_n)$. By a *data set of first order* for the mesh $\mathcal{T}$ we mean a family

$$\mathcal{F} = \big\{(\boldsymbol{p}, f_{\boldsymbol{p}}, A_{\boldsymbol{p}}) : \boldsymbol{p} \in \mathrm{Vert}(\mathcal{T})\big\} \quad \text{with } f_{\boldsymbol{p}} \in \mathbb{R},\ A_p \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}). \tag{6}$$

We call a $\mathcal{C}^1$-smooth function $f : \cup\mathcal{T} := \bigcup_{n=1}^{N} \boldsymbol{T}_n \to \mathbb{R}$ a *polynomial $\mathcal{C}^1$-spline* for the data $\mathcal{F}$ over $\mathcal{T}$ if the restrictions $f|\boldsymbol{T}_n$ are polynomials $\mathbb{R}^2 \to \mathbb{R}$ with Taylor expansion $f_{\boldsymbol{p}} + A_{\boldsymbol{p}}(\boldsymbol{x} - \boldsymbol{p})$ around the points $\boldsymbol{p} \in \mathrm{Vert}(\boldsymbol{T}_n)$.[1]

For our later considerations, $\mathcal{T} = \{\boldsymbol{T}_1, \ldots, \boldsymbol{T}_N\}$ will stand a fixed triangular mesh. Given any mesh triangle $\boldsymbol{T}_n$, we shall write $\lambda_{n,\boldsymbol{p}}$ $\big(\boldsymbol{p} \in \mathrm{Vert}(\boldsymbol{T}_n)\big)$ for its barycentric weights

[1] I.e. f is continuously differentiable on $\mathrm{Interior}\big(\cup\mathcal{T}\big)$, furthermore there are polynomials $P_1, \ldots, P_N$ in 2-variables such that $f([\xi, \eta]) = P_n(\xi, \eta)$ whenever $[\xi, \eta] \in \boldsymbol{T}_n$ $(n = 1, \ldots, N)$ satisfying $f(\boldsymbol{p}) = f_{\boldsymbol{p}}$, $\partial P_n / \partial\xi\big|_{[\xi,\eta]=p} = A_{\boldsymbol{p}}[1, 0]$, $\partial P_n / \partial\eta\big|_{[\xi,\eta]=p} = A_{\boldsymbol{p}}[0, 1]$ at the points $\boldsymbol{p} \in \mathrm{Vert}(\boldsymbol{T}_n)$.

(i.e. $x = \sum_{p\in \mathrm{Vert}(T_n)}(x)p$ for any $p \in \mathbb{R}^2$), and $E_{n,p}$ will denote the edge opposite to the vertex p in T_n.

Theorem 2. *Let* (6) *be a first order data set for $\mathcal{T}$ and let $\{u_E :\ E \in \mathrm{Edge}(\mathcal{T})\} \subset \mathbb{R}^2$ a family of vectors with $u_E \not\parallel E$. Then we can find constants*

$$\{\zeta_{p,E} :\ E \in \mathrm{Edge}(T),\ p \in \mathrm{Vert}(T)\setminus E \text{ for some } T \in \mathcal{T}\} \subset \mathbb{R}$$

such that the union $F : \cup\mathcal{T} \to \mathbb{R}$ of the polynomial functions $F_n : T_n \to \mathbb{R}$ obtained by replacing the terms λ_ℓ $(\ell = 1,2,3)$ in (1), (2) *with $\lambda_{n,p}$ $(p \in \mathrm{Vert}(T_n))$ as*

$$F_n(x) := \sum_{p\in \mathrm{Vert}(T_n)} \Big[\Phi\big(\lambda_{n,p}(x)\big) f_p + \Theta\big(\lambda_{n,p}(x)\big) A_p(x-p) + \zeta_{p,E_{n,p}} \lambda_{n,p}(x)^{-1} \prod_{q\in \mathrm{Vert}(T_n)} \lambda_{n,q}(x)^2\Big]$$

is a polynomial $\mathcal{C}^1$-spline for the data $\mathcal{F}$ over $\mathcal{T}$ such that

$$F'\big(tp + (1-t)q\big)u_E = \big(\Phi(t)A_p + [1-\Phi(t)]A_q\big)u_E \quad \text{whenever } E = [p,q] \in \mathrm{Vert}(\mathcal{T}),\ 0 < t < 1.$$

Remark 1. In course of the proof, with a straightforward adaptation of Theorem 1, we get an explicit expression for $\zeta_{p,E}$ in terms of the barycentric weights of the triangle $T := [\text{Convex hull of } \{p\} \cup E] \in \mathcal{T}$.

3. Preliminaries, straightforward observations

Our basic polynomials Φ, Θ have the Hermitian interpolation properties

$$\Phi(0) = \Phi'(0) = \Phi'(1) = 0, \quad \Phi(1) = 1, \quad \Phi'(t) = 30\,t^2(1-t)^2; \tag{7}$$

$$\Theta(0) = \Theta'(0) = \Theta(1) = 0, \quad \Theta(1) = 1, \quad \Theta'(t) = 12\,t^2(1-t). \tag{8}$$

Given any indices i, j, k with $\{i, j, k\} = \{1, 2, 3\}$,

$$\lambda_i(p_i) = 1, \quad \lambda_i(x) = 0 \ \text{ for } x \in [p_j, p_k] = \{(1-t)p_j + tp_k :\ t \in [0,1]\}; \tag{9}$$

$$\lambda_i'(z)v \equiv \lambda_i(z+p_j) = \lambda_i(z+p_k) = \lambda_i\big(z + (1-t)p_j + tp_k\big) \quad \text{independently of } z, t. \tag{10}$$

Remark 2. It is customary to express the weight λ_i in terms of the natural inner product $\big\langle([\xi_1, \xi_2]\big|[\eta_1, \eta_2]\big\rangle = \sum_{\ell=1}^2 \xi_\ell \eta_\ell$ of $\mathbb{R}^2$ as $\lambda_i(x) = \langle x - p_j | \langle m_i | m_i\rangle^{-1} m_i \rangle$ where $m_i = p_i - r_i$ is the height vector of the triangle T with the closest point r_i to p_i on the line connecting p_j with p_k. Formulas obtained by means of this inner product (like the explicit form of the (ZZ) basic functions published recently [4]Sergienko(2014)) are only invariant with respect to the isometries of $\mathbb{R}^2$ while our approach is free of metric considerations and can be generalized to *purely algebraic settings* by replacing $\mathbb{R}$ with an arbitrary field $\mathbb{K}$. In the sequel we write

$$G_i := [v \mapsto \lambda_\ell'(p_i)v] \tag{11}$$

for the (constant) Fréchet derivative of λ_i regarded as a linear functional $\mathbb{R} \to \mathbb{R}$ but avoiding to identify it with the gradient vector $\langle m_i | m_i \rangle^{-1} m_i$.

Notice that, as being formulated in terms of polynomials $\mathbb{R}^2 \to \mathbb{R}$, the functions $\lambda_1, \lambda_2, \lambda_3$ and F in Theorem 1 extend to to $\mathbb{R}^2$ by means of the same algebraic expressions, furthermore the identities (4),(5) hold on the whole line $\{tp_i + (1-t)p_j : t \in \mathbb{R}\}$ By (9) we have $\lambda_m(p_n) = \delta_{m,n}$ in terms of the Kronecker symbol $\delta_{m,n} = [1 \text{ if } m = n,\ 0 \text{ else}]$, furthermore the monomials

$$C_{\nu_1,\nu_2,\nu_3}(x) = \lambda_1(x)^{\nu_1} \lambda_2(x)^{\nu_2} \lambda_3(x)^{\nu_3}$$

satisfy

$$C_{\nu_1,\nu_2,\nu_3}(\boldsymbol{x}) = 0 \quad \text{for } \boldsymbol{x} \in \partial\boldsymbol{T} = \bigcup[\text{Edges of } \boldsymbol{T}] \quad \text{if } \min\{\nu_1,\nu_2,\nu_3\} \geq 1,$$
$$\big[C_{\nu_1,\nu_2,\nu_3}\big]'(\boldsymbol{x})\boldsymbol{v} = \sum_{\ell=1}^{3} \nu_\ell \lambda_\ell(\boldsymbol{x})^{-1} C_{\nu_1,\nu_2,\nu_3}(\boldsymbol{x}) G_\ell \boldsymbol{v}.$$

In particular, independently of the choice of the coefficients ζ_ℓ, 64

$$F(\boldsymbol{x}) = F_0(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \partial\boldsymbol{T}, \tag{12}$$

$$\Big[\sum_{\ell=1}^{3} \zeta_\ell \lambda_\ell^{-1} \prod_{m=1} \lambda_m^2\Big]_{\boldsymbol{v}}'(\boldsymbol{x}) = \zeta_k \lambda_i(\boldsymbol{x})^2 \lambda_j(\boldsymbol{x})^2 G_k \boldsymbol{v} \quad \text{for } \boldsymbol{x} \in [\boldsymbol{p}_i, \boldsymbol{p}_j] \text{ if } \{i,j,k\} = \{1,2,3\}. \tag{13}$$

Proof of Theorem 1 65

Fix the indices i, j, k arbitrarily such that $\{i,j,k\} = \{1,2,3\}$. Consider a generic point

$$\boldsymbol{x}_t := t\boldsymbol{p}_i + (1-t)\boldsymbol{p}_j \tag{14}$$

on the edge $[\boldsymbol{p}_i, \boldsymbol{p}_j]$ of the triangle $\boldsymbol{T}$. Since the weights λ_ℓ are affine functions,

$$\lambda_i(\boldsymbol{x}_t) = t, \ \lambda_j(\boldsymbol{x}_t) = 1-t, \ \lambda_k(\boldsymbol{x}_t) = 0 \quad (0 \leq t \leq 1). \tag{15}$$

Since $\Phi(1-t) = 1 - \Phi(t)$, in view of (15) we get

$$\begin{aligned}F_0(\boldsymbol{x}_t) &= \sum_{\ell=i,j,k} \Big[\Phi\big(\lambda_\ell(\boldsymbol{x}_t)\big) f_\ell + \Theta\big(\lambda_\ell(\boldsymbol{x}_t)\big) A_\ell\big(\boldsymbol{x}_t - \boldsymbol{p}_\ell\big)\Big] = \\ &= \Big[\Phi(t) f_i + (1-t)\Theta(t) A_i(\boldsymbol{p}_j - \boldsymbol{p}_i)\Big] + \Big[\big(1-\Phi(t)\big) f_j + t\Theta(1-t) A_j(\boldsymbol{p}_i - \boldsymbol{p}_j)\Big] + \\ &\quad + \Big[\Phi(0) f_k + \Theta(0) A_k\big(t\boldsymbol{p}_i + (1-t)\boldsymbol{p}_j - \boldsymbol{p}_k\big)\Big].\end{aligned}$$

That is, by (7) and (8),

$$\begin{aligned}F\big(t\boldsymbol{p}_i + (1-t)\boldsymbol{p}_j\big) &= t^3(10 - 15t + 6t^2) f_i + (1-t)^3(1 + 3t + 6t^2) f_j + \\ &\quad + t^3(1-t)(4-3t) A_i(p_j - p_i) + (1-t)^3 t(1+3t) A_j(\boldsymbol{p}_i - \boldsymbol{p}_j).\end{aligned} \tag{16}$$

As for the Fréchet derivatives along the edge $[\boldsymbol{p}_i, \boldsymbol{p}_j]$, in view of (12) and (13) we get

$$F'\big(t\boldsymbol{p}_i + (1-t)\boldsymbol{p}_j\big)\boldsymbol{v} = F_0'\big(t\boldsymbol{p}_i + (1-t)\boldsymbol{p}_j\big)\boldsymbol{v} + \zeta_k t^2(1-t)^2 G_k \boldsymbol{v}.$$

Notice that in general we have

$$F_0'(\boldsymbol{x})\boldsymbol{v} = \sum_{\ell=i,j,k} \Big[\Phi'\big(\lambda_\ell(\boldsymbol{x})\big)[G_\ell \boldsymbol{v}] f_\ell + \Theta'\big(\lambda_\ell(\boldsymbol{x})\big)[G_\ell \boldsymbol{v}] A_\ell(\boldsymbol{x} - \boldsymbol{p}_\ell) + \Theta\big(\lambda_\ell(\boldsymbol{x})\big) A_\ell \boldsymbol{v}\Big].$$

In particular, since at the generic point (14) on $[p_i, p_j]$, we have $\boldsymbol{x}_t - p_i = (1-t)(p_j - p_i)$, resp. $\boldsymbol{x}_t - p_j = t(p_i - p_j)$,

$$\begin{aligned}F_0'(\boldsymbol{x}_t)\boldsymbol{v} &= F_0'\big(t\boldsymbol{p}_i + (1-t)\boldsymbol{p}_j\big)\boldsymbol{v} = \\ &= \Big[\Phi'(t)[G_i \boldsymbol{v}] f_i + \Theta'(t)[G_i \boldsymbol{v}](1-t) A_i(\boldsymbol{p}_j - \boldsymbol{p}_i) + \Theta(t) A_i \boldsymbol{v}\Big] + \\ &\quad + \Big[\Phi'(1-t)[G_j \boldsymbol{v}] f_j + \Theta'(1-t)[G_j \boldsymbol{v}] t\, A_j(\boldsymbol{p}_i - \boldsymbol{p}_j) + \Theta(1-t) A_j \boldsymbol{v}\Big] + \\ &\quad + \Big[\Phi'(0)[G_k v] f_k + \Theta'(0)[G_k \boldsymbol{v}] A_k(\boldsymbol{x}_t - \boldsymbol{p}_k) + \Theta(0) A_k \boldsymbol{v}\Big].\end{aligned}$$

Thus in view of (7) and (8) it follows

$$F_0'(\boldsymbol{x}_t)\boldsymbol{v} = \Big[30\,t^2(1-t)^2[G_i\boldsymbol{v}]f_i + 12\,t^2(1-t)^2[G_i\boldsymbol{v}]A_i(\boldsymbol{p}_j-\boldsymbol{p}_i) + t^3(4-3t)A_i\boldsymbol{v}\Big] + \\ + \Big[30\,t^2(1-t)^2[G_j\boldsymbol{v}]f_j + 12\,t^2(1-t)^2[G_j\boldsymbol{v}]A_j(\boldsymbol{p}_i-\boldsymbol{p}_j) + (1-t)^3(1+3t)A_j\boldsymbol{v}\Big].$$

Hence we conclude that

$$\begin{aligned} &F'(\boldsymbol{x}_t)\boldsymbol{v} = t^2(1-t)^2\big[\zeta_k[G_k\boldsymbol{v}] + M_{i,j}\boldsymbol{v}\big] + t^3(4-3t)A_i\boldsymbol{v} + (1-t)^3(1+3t)A_j\boldsymbol{v} \\ &\text{where}\quad M_{i,j}\boldsymbol{v} := 30\big([G_i\boldsymbol{v}]f_i + [G_j\boldsymbol{v}]f_j\big) + 12\big([G_i\boldsymbol{v}]A_i - [G_j\boldsymbol{v}]A_j\big)(\boldsymbol{p}_j-\boldsymbol{p}_i). \end{aligned} \tag{17}$$

Proof of (3) . This follows from (16) and (17) by setting $t := 1$.

Proof of (4). Equivalent form of (17).

Proof of (5). Consider (17) with $\boldsymbol{v} := \boldsymbol{u}_k$. Observe that $G_k\boldsymbol{u}_k \neq 0$ since $\boldsymbol{u}_k \not\parallel (\boldsymbol{p}_j-\boldsymbol{p}_i)$. Thus the coefficient

$$\zeta_k := -\frac{M_{i,j}\boldsymbol{u}_k}{G_k\boldsymbol{u}_k} = -\frac{1}{G_k\boldsymbol{u}_k}\Big[30\big([G_i\boldsymbol{u}_k]f_i + [G_j\boldsymbol{u}_k]f_j\big) + 12\big([G_i\boldsymbol{u}_k]A_i - [G_j\boldsymbol{u}_k]A_j\big)(\boldsymbol{p}_j-\boldsymbol{p}_i)\Big] \tag{18}$$

is well-defined. Applying it, for the generic point (14) on the edge $[\boldsymbol{p}_i, \boldsymbol{p}_j]$, we get

$$F'(\boldsymbol{x}_t)\boldsymbol{u}_k = t^3(4-3t)A_i\boldsymbol{u}_k + (1-t)^3(1+3t)A_j\boldsymbol{u}_k$$

independently of the location of the third vertex $\boldsymbol{p}_k$ of the triangle $\boldsymbol{T}$. The proof is complete.

Corollary 1. *By writing $\boldsymbol{v} = \alpha\boldsymbol{u}_k + \beta(\boldsymbol{p}_i - \boldsymbol{p}_j)$, we have*

$$\begin{aligned} F'(\boldsymbol{x}_t)\boldsymbol{v} &= \alpha F'(\boldsymbol{x}_t)\boldsymbol{u}_k + \beta\big[F'(\boldsymbol{x}_t)\big](\boldsymbol{p}_i-\boldsymbol{p}_j) = \\ &= \alpha\Big[\Phi(t)A_i + [1-\Phi(t)]A_j\Big]\boldsymbol{u}_k + \beta\frac{d}{dt}F(\boldsymbol{x}_t). \end{aligned} \tag{19}$$

Proof of Theorem 2

It suffices to verify the following two statements:

(i) *Given $\boldsymbol{p} \in \mathrm{Vert}(\mathcal{T})$ and $\boldsymbol{v} \in \mathbb{R}^2$, we have $F(\boldsymbol{p}) = f_{\boldsymbol{p}}$ and $F'(\boldsymbol{p})\boldsymbol{v} = A_{\boldsymbol{p}}\boldsymbol{v}$.*

(ii) *Given two adjacent mesh triangles $\boldsymbol{T}_m, \boldsymbol{T}_n \in \mathcal{T}$, with common edge $[\boldsymbol{p}, \boldsymbol{q}]$, for the points $\boldsymbol{x}_t = t\boldsymbol{p} + (1-t)\boldsymbol{q}$ on the line connecting $\boldsymbol{p}, \boldsymbol{q}$ we have $F(\boldsymbol{x}_t) = F_m(\boldsymbol{x}_t) = F_n(\boldsymbol{x}_t)$ and $F_m'(\boldsymbol{x}_t)\boldsymbol{v} = F_n'(\boldsymbol{x}_t)\boldsymbol{v}$ for any $\boldsymbol{v} \in \mathbb{R}^2$.*

Proof of (i): Choose any mesh triangle $\boldsymbol{T}_n \in \mathcal{T}$ with $\boldsymbol{p} \in \mathrm{Vert}(\boldsymbol{T}_n)$. By writing $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ with $\boldsymbol{p}_1 = \boldsymbol{p}$ for the vertices of $\boldsymbol{T}_n$, an application of (3) in Theorem 1 with $F := F_n$ shows that $F_n(\boldsymbol{p}) = F_n(\boldsymbol{p}_1) = f_{\boldsymbol{p}_1} = f_{\boldsymbol{p}}$ and $F_n'(\boldsymbol{p})\boldsymbol{v} = F_n'(\boldsymbol{p}_1)\boldsymbol{v} = A_1\boldsymbol{v} = A_{\boldsymbol{p}}\boldsymbol{v}$ independently of which mesh triangle $\boldsymbol{T}_n$ with vertex $\boldsymbol{p}$ is considered.

Proof of (ii): Let $\boldsymbol{T}_m, \boldsymbol{T}_n \in \mathcal{T}$ be two adjacent triangles with common edge $[\boldsymbol{p}, \boldsymbol{q}]$. Necessarily $\mathrm{Vert}(\boldsymbol{T}_m) = \{\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}\}$ and $\mathrm{Vert}(\boldsymbol{T}_n) = \{\boldsymbol{p}, \boldsymbol{q}, \overline{\boldsymbol{r}}\}$ with suitable mesh points $\boldsymbol{r}, \overline{\boldsymbol{r}} \in \mathrm{Vert}(\mathcal{T})$. An application of (4) in Theorem 1 with $\boldsymbol{p}_1 := \boldsymbol{p}, \boldsymbol{p}_2 := \boldsymbol{q}$, $\boldsymbol{p}_3 := r$ and $F := F_m$ shows that

$$\begin{aligned} F_m(\boldsymbol{x}_t) &= \Phi(t)f_1 + [1-\Phi(t)]f_2 + \big[\Theta(t)A_1 + \Theta(1-t)A_2\big](\boldsymbol{p}_j - \boldsymbol{p}_i) = \\ &= \Phi(t)f_{\boldsymbol{p}} + [1-\Phi(t)]f_{\boldsymbol{q}} + \big[\Theta(t)A_{\boldsymbol{p}} + \Theta(1-t)A_{\boldsymbol{q}}\big](\boldsymbol{q} - \boldsymbol{p}). \end{aligned}$$

The same conclusion holds when replacing $(\boldsymbol{r}, F_m)$ with $(\overline{\boldsymbol{r}}, F_n)$. Thus we have $F_m(\boldsymbol{x}_t) = F_n(\boldsymbol{x}_t)$ along the edge $[\boldsymbol{p}, \boldsymbol{q}]$ (moreover along the whole straight line connecting $\boldsymbol{p}$ and

$\boldsymbol{q}$) independently of the location of the third vertices $\boldsymbol{r}$ resp. $\bar{\boldsymbol{r}}$. Since the functions F_n $(n = 1, \ldots, N)$ are $\mathbb{R}^2 \to \mathbb{R}$ polynomials, their union $F : \bigcup_{n=1}^{N} \boldsymbol{T}_n \to \mathbb{R}$ with $F(x) = F_n(\boldsymbol{x})$ whenever $\boldsymbol{x} \in \boldsymbol{T}_n$ is a well-defined continuous function.

From a similar application of (5) in Theorem 1 and (19) finishing its proof applied with $F := F_m$, $\boldsymbol{p}_1 := \boldsymbol{p}$, $\boldsymbol{p}_2 := \boldsymbol{q}$, $\boldsymbol{p}_3 := \boldsymbol{r}$, $u_3 := \boldsymbol{u}_{[p,q]}$, $\boldsymbol{v} := \alpha u_3 + \beta(\boldsymbol{p}_2 - \boldsymbol{p}_1) = \alpha \boldsymbol{u}_{[p,q]} + \beta(\boldsymbol{q} - \boldsymbol{p})$ and $\zeta_3 :=$ [obtained by (19)] we conclude that

$$F_m'(\boldsymbol{x}_t)\boldsymbol{v} = \alpha\Big[\Phi(t)A_{\boldsymbol{p}} + [1 - \Phi(t)]A_{\boldsymbol{q}}\Big]\boldsymbol{u}_{[\boldsymbol{p},\boldsymbol{q}]} + \beta\frac{d}{dt}F_m(\boldsymbol{x}_t).$$

We get the same when the index m is replaced with n, though the values ζ_3 may differ in (19) when calculating with $\boldsymbol{p}_3 := \boldsymbol{r}$ for m and $\boldsymbol{p}_3 := \bar{\boldsymbol{r}}$ for n, respectively. We know already that the functions F_m, F_n coincide along the common edge $[\boldsymbol{p}, \boldsymbol{q}]$. Therefore, with the joint function $F = \bigcup_{n=1}^{N} F_n | \boldsymbol{T}_n$ we indeed have $F'(\boldsymbol{x}_t)v = F_m'(\boldsymbol{x}_t)\boldsymbol{v} = F_n\prime(\boldsymbol{x}_t)\boldsymbol{v} = \alpha\Big[\Phi(t)A_{\boldsymbol{p}} + [1 - \Phi(t)]A_{\boldsymbol{q}}\Big]\boldsymbol{u}_{[\boldsymbol{p},\boldsymbol{q}]} + \beta\frac{d}{dt}F(\boldsymbol{x}_t)$. Q. e. d.

4. Algorithm

INPUT:
$K \in \mathbb{N} = \{1, 2, \ldots\}$ for the number of mesh points;
List $\mathbf{v}_k = [v_k^x, v_k^y] \in \mathbb{R}^2$ $(k = 1, \ldots, K)$ of mesh points;
List $\mathbf{f}_k \in \mathbb{R}$ $(k = 1, \ldots, K)$ for function data at mesh points;
List $\mathbf{A}_k(\xi, \eta) = \mathbf{A}_k^x \xi + \mathbf{A}_k^y \eta$, $\mathbf{A}_k^x, \mathbf{A}_k^y \in \mathbb{R}$ $(k = 1, \ldots, K)$
$N \in \mathbb{N}$ for the number of mesh triangles;
of linear forms for prescribed derivatives at mesh points;
List $[i_{n,1}, i_{n,2}, i_{n,3}] \in \mathbb{N}^3$ $(n = 1, \ldots, N)$ of indices with $1 \le i_{n,1} < i_{n,2} < i_{n,3} \le N$
such that $\mathrm{Vert}(\mathbf{T}_n) = \{\mathbf{v}_{i_{n,1}}, \mathbf{v}_{i_{n,2}}, \mathbf{v}_{i_{n,3}}\}$;
List $\mathbf{u}_{m,n} = [u_{m,n}^x, u_{m,n}^y] \in \mathbb{R}^2$ $(0 \le m, n \le N,\ m \ne n)$ of vectors
such that $\mathbf{u}_{m,n} = \mathbf{u}_{n,m} \not\parallel (\mathbf{v}_m - \mathbf{v}_n)$;

OUTPUT:
List $\mathbf{F}_n(\xi, \eta)$ $(n = 1, \ldots, N)$ of polynomials with coefficients in $\mathbb{R}$.

CALCULATION: Consecutively, for each index $n = 1, 2, \ldots, N$, we compute the polynomial $F_n(\xi, \eta)$ by applying Theorem 1 and (18) as follows:

For $\ell = 1, 2, 3$ let
$\boldsymbol{p}_\ell := \mathbf{v}_{i_{n,\ell}}$, $f_\ell := \mathbf{f}_{i_{n,\ell}}$, $A_\ell(\xi, \eta) := \mathbf{A}_{i_{n,\ell}}(\xi, \eta)$,
$\boldsymbol{u}_1 := \mathbf{u}_{i_{n,2}, i_{n,3}}$, $\boldsymbol{u}_2 := \mathbf{u}_{i_{n,3}, i_{n,1}}$, $\boldsymbol{u}_3 := \mathbf{u}_{i_{n,1}, i_{n,2}}$;
For technical reasons, for $m = 1, 2, 3$ we set also
$\boldsymbol{p}_{m+3} := p_m$, $f_{m+3} := f_m$, $A_{m+3}(\xi, \eta) := A_m(\xi, \eta)$, $\boldsymbol{u}_{ell+3} := \boldsymbol{u}_m$;
After setting the actual values for using the formulas in Theorem, for $\ell = 1, 2, 3$,
establish the barycentric weights and their derivatives as affine resp. linear forms
in terms of the outer product $[\alpha, \beta] \wedge [\gamma, \delta] := \det\left[\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right] = \alpha\delta - \beta\gamma$ (see [6]Berger(1987)):

$$D := \boldsymbol{p}_1 \wedge \boldsymbol{p}_2 + \boldsymbol{p}_2 \wedge \boldsymbol{p}_3 + \boldsymbol{p}_3 \wedge \boldsymbol{p}_1,$$
$$\lambda_\ell(\xi, \eta) := \Big[[\xi, \eta] \wedge (\boldsymbol{p}_{\ell+1} - \boldsymbol{p}_{\ell+2}) + \boldsymbol{p}_{\ell+1} \wedge \boldsymbol{p}_{\ell+2}\Big] / D,$$
$$G_\ell(\xi, \eta) := \Big[[\xi, \eta] \wedge (\boldsymbol{p}_{\ell+1} - \boldsymbol{p}_{\ell+2})\Big] / D;$$

For cyclic indices, we set also
$\lambda_{m+3}(\xi, \eta) := \lambda_m(\xi, \eta)$, $G_{m+3}(\xi, \eta) := G_m(\xi, \eta)$ $(m = 1, 2, 3)$.
Then, for $k = 1, 2, 3$, we compute the correction coefficients by means of (18):

$$\zeta_k := -\frac{1}{G_k u_k} \sum_{d=1}^{2} \Big[30[G_{k+d} u_k] f_{k+d} - (-1)^d 12 [G_{k+d} u_k]\, A_{k+d}(\boldsymbol{p}_{k+2} - \boldsymbol{p}_{k+1})\Big];$$

Finally we let

$$\mathbf{F}_n(\xi,\eta) := \sum_{\ell=1}^{3}\Big[\Phi\big(\lambda_\ell(\xi,\eta)\big)f_\ell + \Theta\big(\lambda_\ell(\xi,\eta)\big)A_\ell\big([\xi,\eta]-p_\ell\big)+ \\ +\zeta_\ell\lambda_\ell(\xi,\eta)\lambda_{\ell+1}(\xi,\eta)^2\lambda_{\ell+2}ell(\xi,\eta)^2\Big].$$

5. Version in pure algebraic setting

We consider the possibility of replacing the real line $\mathbb{R}$ with an arbitrary (possibly finite) field $\mathbb{K}$. Though ordering is no longer available, in particular we cannot speak of edges $[\boldsymbol{p},\boldsymbol{q}] = \{t\boldsymbol{p}+(1-t)\boldsymbol{q} : 0 \le t \le 1\}$ or triangles $\boldsymbol{T} = \{t\boldsymbol{p}+s\boldsymbol{q}+(1-s-t)\boldsymbol{r} : 0 \le s,t,s+t \le 1\}$ in $\mathbb{K}$ any longer, the concept of *lines* $\mathrm{Line}(\boldsymbol{p},\boldsymbol{q}] := \{t\boldsymbol{p}+(1-t)\boldsymbol{q} : t \in \mathbb{K}\}$ connecting distict points $\boldsymbol{p},\boldsymbol{q} \in \mathbb{K}$ makes sense and is widely used in algebraic geometry. From classical geometry we can also save the concept of *non-degenerate point triples* $\{\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3\} \subset \mathbb{K}^2$ by requiring that the expression $\boldsymbol{p}_1\wedge\boldsymbol{p}_2+\boldsymbol{p}_2\wedge\boldsymbol{p}_3+\boldsymbol{p}_3\wedge\boldsymbol{p}_1$ (which corresponds to a non-zero multiple of the area of the triangle with vertices $\boldsymbol{p}_j$ in the case $\mathbb{K}=\mathbb{R}$) should not vanish. Parallellity of two vectors $\boldsymbol{u},\boldsymbol{v} \in \mathbb{K}^2$ can also be well-defined with the property $\boldsymbol{u}\wedge\boldsymbol{v} \neq 0$.

On the other hand, it is also well-known that the formal derivation $\frac{d}{d\tau}\sum_{k=0}^{n}\alpha_k\tau^k := \sum_{k=1}^{n}k\alpha_k\tau^{k-1}$ $(\alpha_0,\ldots,\alpha_n \in \mathbb{K}$ gives rise to a calculus with multivariate polynomials with coefficients in $\mathbb{K}$ preserving the familiar identities as linearity, Leibniz rule, derivation formula of composite maps. Thus, since our computations in the section Algorithm involve only polynomial functions, we can conclude that the following theorem holds.

Theorem 3. *Let $[\boldsymbol{p}_1,\ldots,\boldsymbol{p}_K]$ be a sequence of distinct points in $\mathbb{K}^2$ and let $\big[[i_{n,1},i_{n,2},i_{n,3}] : n = 1,\ldots,N\big]$ be a sequence of distict triples of indices $1 \le i_{n,1} < i_{n,2} < i_{n,3} \le K$ such that the triples*

$$\mathrm{T}_n := \{\boldsymbol{p}_{i_{n,1}},\boldsymbol{p}_{i_{n,2}},\boldsymbol{p}_{i_{n,3}}\} \quad (n=1,\ldots,N)$$

of points are non-degenerate. Then given any sequence $[f_n : n = 1,\ldots K]$ constants in $\mathbb{K}$ along with a sequence $[A_n : n = 1,\ldots K]$ of linear forms $\mathbb{K}\to\mathbb{K}$ and any family $[\boldsymbol{u}_{m,n} : 1 \le m < n \le K]$ of vectors in $\mathbb{K}^2$ such that $\boldsymbol{u}_{m,n}\wedge(\boldsymbol{p}_m-\boldsymbol{p}_n) \neq 0$ $(1 \le m < n \le K)$, the sequence $[F_1,\ldots,F_N]$ of polynomial functions $\mathbb{K}\to\mathbb{K}$ obtained with the calculations in the section Algorithm*, has the following properties:* (i) $F_n(\boldsymbol{p}_k) = f_k,\ F_n'(\boldsymbol{p}_k)\boldsymbol{v} = A_k\boldsymbol{v}\ \ (\boldsymbol{v}\in\mathbb{K}^2)$ *whenever $\boldsymbol{p}_k \in \mathrm{T}_n$ for some n,*

(ii) $F_m|\mathrm{Line}(\boldsymbol{p}_i,\boldsymbol{p}_j) = F_n|\mathrm{Line}(\boldsymbol{p}_i,\boldsymbol{p}_j)$ *whenever $i \neq j$ and $\{\boldsymbol{p}_i,\boldsymbol{p}_j\} = \mathrm{T}_m \cap \mathrm{T}_n$,*

(iii) $F_n'\big(t\boldsymbol{p}_i+(1-t)\boldsymbol{p}_j\big)\boldsymbol{v} = \big[\Theta(t)A_i + \Theta(1-t)A_j\big)\boldsymbol{u}_k\ (t \in \mathbb{K})$ *whenever $\{\boldsymbol{p}_i,\boldsymbol{p}_j,\boldsymbol{p}_k\} = \mathrm{T}_n$.*

6. Conclusions

Our spline interpolation described above is a (ZZ) type procedure providing well-articulated explicit formulas of independent theoretical interest working even in abstract algebraic settings. From practical view points, for classical plane splines, the method is completely parallelizable, affine invariant and easy to optimize with respect to its free ζ-parameters. Applications on 3D triangular complexes even with non-trivial topology can also be expected, though this seems to be not longer a straightforward task.

Acknowledgments: This research was supported by the Ministry of Innovation and Technology of Hungary from the National Research, Development and Innovation Fund, project TKP2021-NVA-09.

References

1. S. Hahmann and G.-P. Bonneau, Triangular G^1 interpolation by 4-splitting domain triangles, *Computer Aided Geometric Design*, **2000**, *17*, 731-757.
2. J. Cao, Zh. Chen, X. Wei and Y.J. Zhang, A finite element framework based on bivariate simplex splines on triangle configurations, *Comput. Methods Appl. Mech. Engrg.*, **2019**, *357*, 112598.

3. M. Zlámal, A. and Ženišek; Mathematical aspects of the FEM, In *Technical Physical and Mathematical Principles of the FEM*, Editors V. Kolaž et al., Publishing House: Akademia, Praha, 1971, pp. 15-39.
4. I.V. Sergienko, O.M. Lytvyn, O.O. Lytvyn and O.I. Denisova, Explicit formulas for interpolating splines of degree 5 on the triangle, *Cybernetics and Systems Analysis*, **2014** *,50, No.5*, 670-678.
5. L.L. Stachó; Locally generated polynomial C1-splines over triangular meshes, arXiv, 2629120 Available online: https://arxiv.org/pdf/1903.11336.pdf (accessed on 27 March 2019).
6. M. Berger; *Geometry I*; Publisher: Springer Verlag, Berlin, Germany, 1987.