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Holomorphic automorphisms of continuous products of balls*,**

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Abstract. We study continuous product domains in the space $\mathcal{C}(\Omega, E)$ of all continuous E-valued functions on Ω, where Ω is a compact Hausdorff topological space and E is an arbitrary JB*-triple, and discuss the group of holomorphic automorphisms of domains of that type.

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0. Introduction

In [10] Vigué introduced the notion of a continuous product of bounded domains in complex Banach spaces and studied the group of biholomorphic automorphisms of domains of that type. Continuous product domains have a natural fibration and it is reasonable to look for fibre-preserving automorphisms and vector fields. Let $\mathrm{G}(\mathbb{D})$ be the connected component of the identity in the group of all holomorphic automorphisms of a continuous product $\mathbb{D}$, endowed with the topology of local uniform convergence. Under certain restrictions on $\mathbb{D}$ (see conditions [2a] and [2b] in [10] th. 1.8), the identity transformation has a neighbourhood that consists of fibre-preserving automorphisms. Condition [2b] is of geometrical nature and it is satisfied whenever $\mathbb{D}$ is convex; however to check that [2a] is satisfied for specific domains $\mathbb{D}$ may be a non trivial matter. That was a key point in [11], where continuous products in $\mathcal{C}(\Omega, \mathbb{C})$, the space of continuous complex valued

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functions, were discussed. On the other hand, by result of Dineen-Klimek-Timoney [3], if $\mathbb{D}$ is a continuous product of *balls* then every transformation in $\mathsf{G}(\mathbb{D})$ is fibre-preserving.

In this paper we are concerned with some Jordan theoretic generalizations of [11]. We replace the complex line $\mathbb{C}$ by an arbitrary Banach space E and study certain continuous product domains $\mathbb{D}$ in $\mathcal{C}(\Omega, E)$, the space of all continuous E-valued functions $f: \Omega \to E$. In Sect. 1 we consider a family $(D_\omega)_{\omega\in\Omega}$ of domains in E with some mild restrictions on the smoothness of the boundaries ∂D_ω and the following property: for every couple of points a, $b \in D_\omega$ there exists a complex geodesic which is a holomorphic retract of D_ω, passes through the points a, b and stretches to ∂D_ω. We establish that then all transformations in $\mathsf{G}(\mathbb{D})$ are fibre preserving. By a result of Lempert [7], the biholomorphic images of finite dimensional convex sets with smooth boundary have the above retract property. If a domain D in E is the image of a bounded circular symmetric domain in E by a biholomorphic mapping that extends bicontinuously to the boundary ∂D, then D has the above retract property.

In Sect. 2 we apply the previous results to the study of the holomorphic geometry of a continuous product of balls. In Sect. 3 we define continuous families of balls in partial JB*-triples by using bounded weights. We then characterize the complete vector fields in the product domain in terms of the triple product and the family of weights. In this manner we reobtain some of the results in [11] whereas others are no longer valid in the new context as it is shown by counterexamples. In particular, we show that if Z is any partial JB*-triple and Z_0 denotes its symmetric subspace, then $\mathcal{C}(\Omega, Z)_0 = \mathcal{C}(\Omega, Z_0)$ which generalizes a well known result about JB*-triples of continuous functions.

1. Continuous products and the fibre-preserving property

Throughout this section, Ω and E denote respectively a compact topological space and a complex Banach space with norm $\|\cdot\|$ and open unit ball $D \subset E$. By $\mathcal{C}(\Omega, E)$ we denote the space of all continuous E-valued functions $f: \Omega \to E$, endowed with the usual operations and the norm of the supremum. Let $U \subset E$ be a domain. A holomorphic vector field in U is an E-valued holomorphic function $X: U \to E$, and it is said to be complete in U if for every initial condition $z \in U$, the maximal solution $y_t(z) := \exp tX(z)$ of the initial value problem

$$\frac{d}{dt} y_t(z) = X(y_t(z)), \qquad y_0(z) = z,$$

called the exponential of X, is defined in the whole real line $\mathbb{R}$ and satisfies $y_z(t) \in U$ for all $t \in \mathbb{R}$. We let $\mathfrak{g}(U)$ denote the set of all complete holomorphic vector fields in U. We shall use that if U is a bounded circular domain containing the origin then every element X in $\mathfrak{g}(U)$ is a continuous polynomial $E \to E$ of degree ≤ 2, [6]. Finally, a bounded circular domain U such that $0 \in U$ is holomorphically symmetric if for every pair $a, b \in U$ there is a biholomorphic automorphism h of U such that $h(a) = b$. Recall ([10] def. 1.5)

1.1 Definition A domain $\mathbb{D} \subset \mathcal{C}(\Omega, E)$ is the continuous Ω-product of the family $(D_\omega)_{\omega\in\Omega}$ of bounded domains in E if the following two conditions hold

$$\begin{aligned} \mathbb{D} &= \{f \in \mathcal{C}(\Omega, E) : f(\omega) \in D_\omega, \ \ (\omega \in \Omega)\}, \\ D_\omega &= \{f(\omega) : f \in \mathbb{D}\}, \quad (\omega \in \Omega). \end{aligned}$$

In that case $\mathbb{D}$ consists of continuous sections of

$$\mathbb{D}_\star := \{(\omega, x) \in \Omega \times E : \omega \in \Omega, \ x \in D_\omega\}$$

with respect to the fibration $p: \mathbb{D}_\star \to \Omega$ given by $(\omega, x) \mapsto \omega$.

1.2 Lemma *Let Ω, E and $\|\cdot\|_\omega$, $(\omega \in \Omega)$, respectively be a compact topological space, a complex Banach space and a family of norms in E with open unit balls D_ω. Assume that $\mathbb{D} := \{f \in \mathcal{C}(\Omega, E) : f(\omega) \in D_\omega, \ (\omega \in \Omega)\}$ is a bounded domain in $\mathcal{C}(\Omega, E)$. Then $\mathbb{D}$ is the continuous Ω-product of the family $(D_\omega)_{\omega\in\Omega}$ if and only if there are constants $0 < m \leq M < \infty$ such that*

$$m\|\cdot\| \leq \|\cdot\|_\omega \leq M\|\cdot\|, \qquad (\omega \in \Omega),$$

and the function $N(\omega, x) := \|x\|_\omega$ is upper semicontinuous on $\Omega \times E$.

Proof. It is elementary that $\mathbb{D}_\star = \{(\omega, x) \in \Omega \times E : \omega \in \Omega, \ x \in D_\omega\}$ is an open set in $\Omega \times E$. Clearly $\mathbb{D}_\star = \{(\omega, x) \in \Omega \times E : N(\omega, x) < 1\}$. Since $\|\cdot\|_\omega$ is homogeneous, for $\varrho > 0$ the sets $\{(\omega, x) \in \Omega \times E : N(\omega, x) < \varrho\}$ are open in $\Omega \times E$ and this means that N is upper semicontinuous. Since the origin is an interior point of $\mathbb{D}$, there exists $\delta > 0$ such that $\{(\omega, x) \in \Omega \times E : \|x\| < \delta\} \subset \mathbb{D}_\star$. Therefore

$$\|x\|_\omega < \frac{1}{\delta}\|x\|, \qquad (\omega \in \Omega, \ x \in E).$$

Since $\mathbb{D}$ is bounded, there is a number $\varrho > 0$ such that $\mathbb{D}_\star \subset \{(\omega, x) \in \Omega \times E : \|x\| < \varrho\}$, hence

$$\|x\|_\omega > \frac{1}{\varrho}\|x\|, \qquad (\omega \in \Omega, \ x \in E).$$

Conversely, let $\omega \in \Omega$ and $0 \neq x \in D_\omega$ be given and assume that $\alpha \mapsto \|x\|_\alpha$ is upper semicontinuous. Recall that then there exists a continuous function $\phi: \Omega \to \mathbb{R}$ such that

$$\phi(\omega) = \|x\|_\omega, \qquad \phi(\alpha) \geq \|x\|_\alpha, \qquad (\alpha \in \Omega).$$

Then the function

$$f_{\omega,x} : \alpha \mapsto \frac{\|x\|_\alpha}{\phi(x)} x$$

satisfies $f_{\omega,x} \in \mathbb{D}$ and so $D_\omega = \{f_{\omega,x}(\omega) : x \in D_\omega\}$ which means that $\mathbb{D}$ is the continuous product over Ω of the family $(D_\alpha)_{\alpha\in\Omega}$. □

1.3 Definition A holomorphic vector field $X: \mathbb{D} \to C(\Omega, E)$ on the continuous Ω-product of the family $(D_\omega)_{\omega\in\Omega}$ of domains in E is said to have the *fibre preserving property* (the FPP for short) if there exists a family of holomorphic vector fields $v_\omega: D_\omega \to E$ such that

$$[Xf](\omega) = v_\omega(f(\omega)), \qquad (\omega \in \Omega,\ f \in \mathbb{D}).$$

Let $\mathbb{D}$ be the continuous Ω-product of a family $(D_\alpha)_{\alpha\in\Omega}$ of open balls in E. We shall establish a sufficient condition to ensure the following property: For every $\omega \in \Omega$ and every pair a, $b \in D_\omega$ there exists a holomorphic function $F: D_\omega \to \mathbb{D}$ such that

$$[F(a)](\omega) = a, \qquad [F(b)](\omega) = b, \qquad (1).$$

To use (1) we introduce Lempert's retract property in the following definition, where as usual Δ and $\mathbb{T}$ denote respectively the open unit disk and its boundary.

1.4 Definition A domain C in a complex Banach space E is said to have *Lempert's retract property* (LRP for short) if for every pair $a, b \in C$, $a \neq b$, there exists an injective continuous mapping $\psi: \overline{\Delta} \to \overline{C}$ such that

$$a, b \in \psi(\Delta)\,, \quad \psi|_\Delta \text{ is holomorphic}\,, \quad \psi(\Delta) \subset C\,, \quad \psi(\mathbb{T}) \subset \partial C$$

and there exists a surjective holomorphic projection $P : C \to \psi(\Delta)$, $P \circ P = P$.

1.5 Lemma *Let E and D respectively be a complex Banach space and the image of a bounded circular symmetric domain in E by a biholomorphic map which extends bicontinuously to ∂D. Then D has the LRP.*

Proof. Since circular bounded symmetric domains are convex [5], it suffices to prove the statement for the special case of D being the unit ball of E. Assume $a, b \in D = \{x \in E : \|x\| < 1\}$. Consider first the case $a = 0$. By the Hahn-Banach theorem we can find a continuous linear functional $\lambda : E \to \mathbb{C}$ such that $\|\lambda\| = 1$ and $\lambda(b) = \|b\|$. Then the choice $\psi(\zeta) := \zeta\, b/\|b\|$, $(\zeta \in \overline{\Delta})$, and $P(x) := \lambda(x)\, b/\lambda(b)$, $(x \in D)$, suits our requirements. In the case $a \neq 0$ we proceed as follows. Since D is symmetric, by a result of Kaup [6] (see also [4] prop. 3.2), we can choose a biholomorphic mapping $A : W \to E$ of some neighbourhood W of $\overline{D}$ such that $A|_D$ is a biholomorphic automorphism, $A(\partial D) = \partial D$ and $A(a) = 0$.

We construct the linear geodesic ψ' and the linear projection P' to the couple $a' := A(a) = 0$ and $b' := A(b) \in D$ according to the previous instructions. Then the composite mappings $\psi := A^{-1} \circ \psi'$ and $P := A^{-1} \circ P' \circ A$ meet the requirements. $\square$

1.6 Remark In [2] Dineen, Timoney and Vigué have established the existence of complex geodesics in bounded convex domains of complex Banach spaces but no holomorphic projection with the properties of P above was ever considered as far as we know.

1.7 Proposition *Let Ω, E and $(D_\alpha)_{\alpha\in\Omega}$, respectively be a compact topological space, a complex Banach space and a family of bounded starlike domains in E with the LRP. Assume that*

$$D_\alpha = \{x \in E : \ \nu_\alpha(x) < 1\}, \qquad (\alpha \in \Omega)$$

for a family of continuous positive homogeneous functions $(\nu_\alpha)_{\alpha\in\Omega}$ such that set

$$\mathbb{D} := \{f \in C(\Omega, E) : \ f(\omega) \in D_\omega, \quad (\omega \in \Omega)\}$$

is a bounded domain and $\{f(\omega) : \ f \in \mathbb{D}\} = D_\omega$ for all $\omega \in \Omega$. Then for every $\omega \in \Omega$ and $a, b \in D_\omega$ there exists a holomorphic map $F : D_\omega \to \mathbb{D}$ such that

$$[F(a)](\omega) = a, \qquad [F(b)](\omega) = b.$$

In particular, every complete holomorphic vector field in $\mathbb{D}$ has the FPP.

Proof. Fix $\omega \in \Omega$ and $a, b \in D_\omega$ arbitrarily. With the notation established before, let us construct the complex geodesic ψ and the projection P associated to the couple $a, b \in D_\omega$. We show that for some continuous function $g : \Omega \to [1, \infty)$ with $g(\omega) = 1$ the holomorphic map $F : D_\omega \to C(\Omega, E)$ defined by

$$F(x) := [\alpha \mapsto g(\alpha)^{-1} P(x)], \qquad (x \in D_\omega)$$

ranges in $\mathbb{D}$. To this aim it suffices to establish the existence of a continuous function g such that $g(\alpha)^{-1}z \in D_\alpha$ for any $\alpha \in \Omega$ and $z \in \text{range}(P)$. Since $D_\alpha = \{x \in E : \ \nu_\alpha(x) < 1\}$ and $\text{range}(P) = \psi(\Delta)$, this condition means

$$g(\alpha) \geq \phi(\alpha) := \sup_{0\neq z\in\psi(\Delta)} \nu_\alpha(z)/\nu_\omega(z), \qquad g(\omega) = 1, \qquad (\alpha \in \Omega)\,.$$

By the compactness of Ω and the fact that $\phi(\omega) = 1$, such a continuous function g exists if the function ϕ is upper semicontinuous.

By composing ψ with a suitable Möbius transformation of Δ, we may assume that $\psi(0) = 0$ if we have $0 \in \psi(\Delta \cup \mathbb{T})$ since, by the maximum principle, the possibility $0 \in \psi(\mathbb{T})$ is excluded. Thus, with a suitable integer $K \geq 0$, we can write

$$\psi(\zeta) = \zeta^K \widetilde{\psi}(\zeta)\,, \quad \widetilde{\psi}(\zeta) \neq 0 \qquad (|\zeta| < 1 + \varepsilon)$$

for some continuous map $\widetilde{\psi} : \Delta \cup \mathbb{T} \to E$ which is holomorphic on Δ. Then we have

$$\phi(\alpha) = \sup_{\substack{0<r<1 \\ 0\leq\theta\leq 2\pi}} \frac{\nu_\alpha(e^{iK\theta}\widetilde{\psi}(re^{i\theta})}{\nu_\omega(e^{iK\theta}\widetilde{\psi}(re^{i\theta})} = \max_{\substack{0\leq r\leq 1 \\ 0\leq\theta\leq 2\pi}} \frac{\nu_\alpha(e^{iK\theta}\widetilde{\psi}(re^{i\theta})}{\nu_\omega(e^{iK\theta}\widetilde{\psi}(re^{i\theta})}\,.$$

A standard compactness argument shows that

$$\phi(\alpha) = \frac{\nu_\alpha(e^{iK\theta(\alpha)}\widetilde{\psi}(r(\alpha)e^{i\theta(\alpha)})}{\nu_\omega(e^{iK\theta(\alpha)}\widetilde{\psi}(r(\alpha)e^{i\theta(\alpha)})}, \qquad (\alpha \in \Omega)$$

for suitable functions $r : \Omega \to [0,1]$ and $\theta : \Omega \to [0,2\pi]$. Observe that the function $N(\alpha,x) := \nu_\alpha(x)$ is upper semicontinuous on $\Omega \times E$. Indeed, since $\mathbb{D}$ is the continuous product over Ω of the domains $D_\alpha = \{x \in E : \nu_\alpha(x) < 1\}$, it is well-known [10] that the set $\mathbb{D}_\star := \{(\omega,\, x) \in \Omega \times E{:}\, \omega \in \Omega,\, x \in D_\omega\}$ is open in $\Omega \times E$. Clearly $\mathbb{D}_\star = \{(\omega,\, x) \in \Omega \times E{:}\, N(\omega,\, x) < 1\}$. Since ν_ω is positive homogeneous, for $\varrho > 0$ the level sets $\{(\omega,\, x) \in \Omega \times E{:}\, N(\omega,\, x) < \varrho\}$ are open in $\Omega \times E$.

Finally we establish the upper semicontinuity of ϕ as follows. We fix any $\alpha \in \Omega$ along with a net $(\alpha_j)_{j\in J}$ converging to α in Ω and we verify that $\limsup_j \phi(\alpha_j) \leq \phi(\alpha)$. By passing to a suitable subnet, we may assume without loss of generality that

$$\begin{aligned}&\phi(\alpha_j) \to \limsup_j \phi(\alpha_j) = \lim_j \phi(\alpha_j), \quad r(\alpha_j) \to \varrho, \quad \theta(\alpha_j) \to \vartheta,\\ &\nu_{\alpha_j}(e^{Ki\theta(\alpha_j)}\widetilde{\psi}(r(\alpha_j)e^{i\theta(\alpha_j)}) \to \mu\end{aligned}$$

for some $\varrho \in [0,1]$, $\vartheta \in [0,2\pi]$ and $\mu \in \mathbb{R}_+$. Then, taking into account the continuity of $\widetilde{\psi}$ and ν_ω, we have

$$\begin{aligned}e^{Ki\theta(\alpha_j)}\widetilde{\psi}(r(\alpha_j)e^{i\theta(\alpha_j)}) &\to e^{Ki\vartheta}\widetilde{\psi}(\varrho e^{i\vartheta}) \neq 0\,,\\ \nu_\omega(e^{Ki\theta(\alpha_j)}\widetilde{\psi}(r(\alpha_j)e^{i\theta(\alpha_j)})) &\to \nu_\omega(e^{Ki\vartheta}\widetilde{\psi}(\varrho e^{i\vartheta})) > 0\,.\end{aligned}$$

By the upper semicontinuity of the function N, also

$$\begin{aligned}\mu &= \lim_j N(\alpha_j, e^{Ki\theta(\alpha_j)}\widetilde{\psi}(r(\alpha_j)e^{i\theta(\alpha_j)}))N(\alpha, e^{Ki\vartheta}\widetilde{\psi}(\varrho e^{i\vartheta}))\\ &= \nu_\alpha(e^{Ki\vartheta}\widetilde{\psi}(\varrho e^{i\vartheta}))\,.\end{aligned}$$

It follows

$$\begin{aligned}\limsup_j \phi(\alpha_j) = \mu &\le \frac{\nu_\alpha(e^{iK\vartheta}\widetilde{\psi}(\varrho e^{i\vartheta}))}{\nu_\omega(e^{iK\vartheta}\widetilde{\psi}(\varrho e^{i\vartheta}))} \le \max_{\substack{0\le\overline{\varrho}\le 1\\ 0\le\overline{\vartheta}\le 2\pi}} \frac{\nu_\alpha(e^{iK\overline{\vartheta}}\widetilde{\psi}(\overline{\varrho}e^{i\overline{\vartheta}}))}{\nu_\omega(e^{iK\overline{\vartheta}}\widetilde{\psi}(\overline{\varrho}e^{i\overline{\vartheta}}))}\\ &= \frac{\nu_\alpha(e^{iK\theta(\alpha)}\widetilde{\psi}(r(\alpha)e^{i\theta(\alpha)}))}{\nu_\omega(e^{iK\theta(\alpha)}\widetilde{\psi}(r(\alpha)e^{i\theta(\alpha)}))} = \phi(\alpha)\,.\end{aligned}$$

This shows that $\mathbb{D}$ satisfies the conditions of Theorem 2.8 in [10]. Hence any automorphism in $\mathsf{G}(\mathbb{D})$, the identity connected component in the group of all holomorphic automorphism of $\mathbb{D}$, has the FPP. □

1.8 Remark If we restrict ourselves to continuous products of *convex* domains, then by a result of Dineen-Klimek-Timoney [3] no auxiliary conditions are necessary to guarantee that the transformations in $\mathsf{G}(\mathbb{D})$ have the FPP.

1.9 Theorem *Let Ω, E and $(D_\omega)_{\omega\in\Omega}$, respectively be a compact topological space, a complex Banach space and a family of bounded circular domains in E. Let $\mathbb{F} \subset C(\Omega, E)$ be a subspace for which the following three properties hold:* (1) *$\mathbb{F}$ contains the constant functions $\mathbf{1}_\Omega e := [\omega \mapsto e]$ $(e \in E)$,* (2) *the set $\mathbb{D} := \{f \in \mathbb{F} : \ f(\omega) \in D_\omega \ \ (\omega \in \Omega)\}$ is a bounded domain in $\mathbb{F}$,* (3) *$D_\omega = \{f(\omega) : \ f \in \mathbb{D}\}$ for all $\omega \in \Omega$. Assume furthermore that every transformation in $\mathsf{G}(\mathbb{D})$ has the FPP. Then for every vector field $X{:}\,\mathbb{F} \to \mathbb{F}$ the following statements are equivalent:*

(i) $X \in \mathfrak{g}(\mathbb{D})$

(ii) *There is a family of vector fields $v_\omega{:}\, E \to E$ $(\omega \in \Omega)$, with the following properties:*
$v_\omega \in \mathfrak{g}(D_\omega)$ $(\omega \in \Omega)$, the function $\omega \mapsto v_\omega(f(\omega))$ belongs to $\mathbb{F}$, and $[X(f)](\omega) = v_\omega(f(\omega))$ holds for every $f \in \mathbb{F}$ and every $\omega \in \Omega$.

Proof. (i)⇒ (ii). This is clear by the fibre preserving property (1) and the fact that $D_\omega = \{f(\omega){:}\, f \in \mathbb{F})\}$ for any $\omega \in \Omega$.

(ii)⇒ (i). By assumption v_ω is a complete holomorphic vector field in D_ω. Hence every v_ω is a continuous polynomial $E \to E$ of degree ≤ 2. For every $e \in E$ the function $\omega \mapsto v_\omega(\mathbf{1}_\Omega \cdot e)$ belongs to $\mathbb{F}$ and hence is continuous by assumption. Thus by the compactness of Ω, for each fixed $e \in E$ the set $\{v_\omega(e){:}\, \omega \in \Omega\}$ is bounded in E. That is, for every $e \in E$ there is a constant $M_e < \infty$ such that

$$\|v_\omega(e)\| \le M_e, \qquad (e \in E,\ \omega \in \Omega) \qquad (2).$$

For $n \in \mathbb{N}$ let $H_n{:}= \{x \in E{:}\, \|v_\omega(x)\| \le n, \ (\omega \in \Omega)\}$. By (2) we have $E = \bigcup_{n\in\mathbb{N}} H_n$, and as E is a Banach space, by Baire category theorem there is some $x_0 \in E$ and some $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subset H_n$ for some $n \in \mathbb{N}$. Then

$$\|v_\omega(x)\| \leq n \quad \text{for all} \quad x \in B_\varepsilon(x_0).$$

But for every fixed $\omega \in \Omega$ the function $x \mapsto v_\omega(x)$ is a polynomial, hence the mapping $X: f \mapsto X(f)$ is a polynomial $\mathbb{F} \to \mathbb{F}$ which is bounded on the open ball $B_\varepsilon(\mathbf{1}_\Omega \cdot x_0) := \{f \in \mathbb{F} : \ \|f(\omega) - x_0\| < \varepsilon\}$. In particular X is a continuous polynomial $\mathbb{F} \to \mathbb{F}$. Since we have assumed that X is fibre-preserving, this entails that X is complete in $\mathbb{D}$ by the proof of ([10] th. 1.8). $\square$

2. The symmetric part of a continuous product

In this section we apply the previous results to the study of the holomorphic geometry of the domain $\mathbb{D}$, the continuous product of the family of balls $D_\omega = \{x \in E: \|x\|_\omega < 1\}$, $(\omega \in \Omega)$. Notice that $\mathbb{D}$ can also be regarded as the unit ball of the Banach space $\mathbb{E}: = \mathcal{C}(\Omega, E)$ with a suitable norm.

The appropriate algebraic tool for the description of the Lie algebra of all complete holomorphic vector fields in a bounded circular domain is the category of partial JB*-triples. By a partial Jordan triple ([1], [8]) we mean a structure $(E, E_0, \{\cdot,\cdot,\cdot\})$ where E is a complex Banach space, E_0 is a complex linear subspace of E, and $\{\cdot,\ \cdot,\ \cdot\}: E \times E_0 \times E \to E$ (called the triple product) is a continuous real trilinear map satisfying the axioms for all $a \in E_0$ and all $x, y \in E$

(i) $\{E_0\, E_0\, E_0\} \subset E_0$
(ii) $\{x\, a\, y\}$ is symmetric complex linear in x, y and conjugate linear in a.
(iii) $\{a\, b\, \{x\, c\, y\}\} = \{\{a\, b\, x\}\, c\, y\} - \{x\, \{b\, a\, c\}\, y\} + \{x\, c\, \{a\, b\, y\}\}$
(iv) $\{x\, a\, \{x\, b\, x\}\} = \{\{x\, a\, x\} b\, x\}$
(v) $a \Box a$ is a hermitian positive element of $\mathcal{L}(E)$
(vi) $\|\{aaa\}\| = \|a\|^3$.

To be concise we say that E is a partial JB*-triple and refer to E_0 as the symmetric subspace of E. In case $E_0 = E$ we speak of a JB*-triple. Given a complex Banach space, its unit ball D gives rise to a natural partial JB*-triple structure on E as follows: The manifold $E_0: = \mathfrak{g}(D)0 = \{X(0): X \in \mathfrak{g}(D)\}$ is a closed complex linear subspace of E and there is a unique partial JB*-triple product on $E \times E_0 \times E$ with

$$\mathfrak{g}(D) = \mathfrak{g}^+(D) \oplus \mathfrak{g}^-(D)$$

where

$$\mathfrak{g}^+(D) = \{X'(0): X \in \mathfrak{g}(D)\}, \quad \mathfrak{g}^-(D) = \{[z \mapsto v - \{z\, v\, z\}]: v \in E_0\}.$$

In the sequel we shall use the fact that each surjective linear isometry of the space E is an automorphism of the triple product associated with the unit ball D.

Remark that JB*-triples characterize algebraically the bounded symmetric domains in the sense that the Harris-Chandra realization of a bounded symmetric domain is circular, and symmetric circular domains are exactly the linear copies of the unit balls of JB*-triples. Partial JB*-triples characterize bounded circular domains in a weaker sense: Bounded circular domains are the linear copies of bounded domains that are invariant under the exponential of all vector fields of the form $[z \mapsto v - \{z\, v\, z\}]$, $(v \in E_0)$, and under the linear automorphisms of the triple product. A complete axiomatization is not yet known.

We refer to [1], [5], [8] [9], [12] for proofs and background material on JB*-triples and partial JB*-triples.

2.1 Theorem *Let Ω, E and $\|\cdot\|_\omega$, $(\omega \in \Omega)$, respectively be a compact topological space, a complex Banach space and a family of norms in E whose open unit balls D_ω have the LRP. Let $(E,\ E_{\omega 0},\ \{\cdot,\ \cdot,\ \cdot\}_\omega)$ be the partial JB*-triple associated with D_ω and denote by $\mathbb{D}$ the continuous Ω-product of the family $(D_\omega)_{\omega\in\Omega}$. Then for every vector field $X: \mathcal{C}(\Omega, E) \to \mathcal{C}(\Omega, E)$ the following statements are equivalent:*

(i) $X \in \mathfrak{g}^-(\mathbb{D})$

(ii) *There is continuous function a: $\Omega \to E$ with the following properties: $a(\omega) \in E_{\omega 0}$ for all $\omega \in \Omega$, and $\omega \mapsto \{f(\omega)\, a(\omega)\, f(\omega)\}_\omega$ is continuous whenever $f \in \mathcal{C}(\Omega, E)$.*

Proof. Let $(\mathbb{E}, \mathbb{E}_0, \{\cdot,\ \cdot,\ \cdot\})$ be the partial JB*-triple associated with $\mathbb{D}$. By (1), for each $a \in \mathbb{E}_0$ the vector field $X_a := [f \mapsto a - \{f\, a\, f\}]$ has the fibre form $X_a(f) = [\omega \mapsto v_{a\,\omega}(f(\omega))]$ with a suitable family $v_{a\,\omega} \in \mathfrak{g}(D_\omega)$, $(\omega \in \Omega)$. Since $X_a \in \mathfrak{g}^-(\mathbb{D})$ and $X_a'(0) = 0$, we actually have $v_{a\,\omega} \in \mathfrak{g}^-(D_\omega)$ for all $\omega \in \Omega$. Therefore $v_{a\,\omega}(z) = v_{a\,\omega}(0) - \{z\, v_{a\,\omega}(0)\, z\}$ for $z \in D_\omega$ and we even have $a(\omega) = v_{a\,\omega}(0)$. That is, the triple product associated with $\mathbb{E}$ has the fibration

$$\{f\, a\, f\} = [\omega \mapsto \{f(\omega)\, a(\omega)\, f(\omega)\}_\omega] \qquad (f \in \mathbb{E},\ a \in \mathbb{E}_0).$$

In particular, the fact $X \in \mathfrak{g}^-(\mathbb{D})$ is equivalent to having $X(f) = [\omega \mapsto a(\omega) - \{f(\omega)\, a(\omega)\, f(\omega)\}_\omega]$ for some function $a \in \mathbb{E}_0$. In this case (ii) holds since $X(f) \in \mathbb{E}$ should be a continuous function.

Conversely, if we assume (ii) it is straightforward to see that every vector field of the form X_a is complete in $\mathbb{D}$. □

2.2 Example 1. Let E be a complex Banach space with norm $\|\cdot\|$, open unit ball D and symmetric subspace E_0. Consider the family of norms $\|\cdot\|_\omega = \|\cdot\|$

for all $\omega \in \Omega$. In this case the domains $(D_\omega)_{\omega\in\Omega}$ coincide with D and the partial triple product of each factor coincides with the triple product in E. The product (which is now a power) domain $\mathbb{D}$ is the unit ball in $\mathcal{C}(\Omega, E)$ relative to the norm of the supremum. Thus we can apply (2), but now the function

$$\omega \mapsto \{f(\omega)\, a_\omega\, f(\omega)\}_\omega = \{f(\omega)\, a_\omega\, f(\omega)\}$$

is continuous whenever $f \in \mathcal{C}(\Omega, E)$ due to the continuity of the product in E. Thus we get

$$\mathfrak{g}^-(\mathbb{D}) \approx \mathcal{C}(\Omega, E_0), \quad \text{and} \quad \mathcal{C}(\Omega, E)_0 = \mathcal{C}(\Omega, E_0).$$

In case $E = E_0$, that is, if E is a JB*-triple, then we reobtain a classical result

$$\mathfrak{g}^-(\mathbb{D}) \approx \mathcal{C}(\Omega, E), \quad \text{and} \quad \mathcal{C}(\Omega, E)_0 = \mathcal{C}(\Omega, E).$$

3. Partial JB*-triples and weighted powers

Let E be a complex Banach space and denote by $\mathsf{GL}(E)$ the group of all bounded invertible linear operators on E. Let $\phi\colon \Omega \to \mathsf{GL}(E)$ be a function for which the following two conditions hold:

(a) Boundedness: There are constants $0 < m \leq M < \infty$ such that

$$m\,\|x\| \leq \|\phi(\omega)^{-1}x\| \leq M\,\|x\|, \qquad (x \in E,\ \omega \in \Omega).$$

(b) Upper semicontinuity: The function $(x, \omega) \mapsto \|\phi(\omega)^{-1}x\|$ is upper semicontinuous in $E \times \Omega$.

We refer to such a ϕ as an *admissible weight*. If $f \in \mathcal{C}(\Omega, E)$ then we have $m\,\|f(\omega)\| \leq \|\phi(\omega)^{-1}f(\omega)\| \leq M\,\|f(\omega)\|$, $(\omega \in \Omega)$, and so

$$m\,\|f\| \leq \|f\|_\phi \leq M\,\|f\| \quad \text{where} \quad \|f\|_\phi{:}= \sup_{\omega\in\Omega} \|\phi(\omega)^{-1}f(\omega)\|$$

is attainable by condition (b). Clearly $\|x\|_\omega{:}= \|\phi(\omega)^{-1}x\|$, $(x \in E)$, is a family of uniformly equivalent norms in E and the relation $\|f\|_\phi < 1$ is equivalent to $\|f(\omega)\| < 1$ for all $\omega \in \Omega$. This means that

$$\mathbb{D}{:}= \{f \in \mathcal{C}(\Omega, E)\colon \|f\|_\phi < 1\}$$

is the continuous product of the domains $D_\omega{:}= \{x \in E\colon \|x\|_\omega < 1\}$. Even if E is a JB*-triple, in general $\|\cdot\|_\phi$ does not coincide with the spectral norm of $\mathcal{C}(\Omega, E)$, hence $\mathbb{D}$ may fail to be homogeneous and it is reasonable to characterize the symmetric subspace of $(\mathcal{C}(\Omega, E),\ \|\cdot\|_\phi)$, that is, the symmetric subspace of the partial JB*-triple associated with $\mathbb{D}$. To that purpose let E_ω denote the space $(E,\ \|\cdot\|_\omega)$ and notice that $\phi(\omega)\colon E \to E_\omega$ is a surjective linear isometry for all $\omega \in \Omega$, hence in this case the factors

D_ω, $(\omega \in \Omega)$, are all isomorphic even if they may not coincide. The adjoint map $\phi(\omega)_\#: \mathfrak{g}(D) \to \mathfrak{g}(D_\omega)$ is an isomorphism of the corresponding Lie algebras of complete holomorphic vector fields which induces an isomorphism of the quadratic summands $\phi(\omega)_\#: \mathfrak{g}^-(D) \to \mathfrak{g}^-(D_\omega)$. Hence the triple product in the factor E_ω is given by

$$\{x\,y\,z\}_\omega = \phi(\omega)\{\phi(\omega)^{-1}x,\ \phi(\omega)^{-1}y,\ \phi(\omega)^{-1}z\},$$
$$(x,\ y,\ z \in E_\omega,\ \omega \in \Omega).$$

Moreover, if the ball $D \subset E$ has the LRP so do all factors $D_\omega \subset E_\omega$.

To simplify the notation we set $\tilde{f}(\omega) := \phi(\omega)^{-1}f(\omega)$ whenever ϕ is an admissible weight, f is continuous and $\omega \in \Omega$.

3.1 Theorem *Let Ω and E respectively be a compact topological space and a complex Banach space whose ball D has the LRP. Let E_0 denote the symmetric subspace of E and let $\phi: \Omega \to \mathsf{GL}(E)$ be an admissible weight. Then for every $v \in \mathcal{C}(\Omega, E)$ the following conditions are equivalent:*

(i) *The function v belongs to the symmetric subspace of $(\mathcal{C}(\Omega, E),\ \|\cdot\|_\phi)$.*
(ii) *$v(\omega) \in \phi(E_0)$ for all $\omega \in \Omega$ and the function $\phi\{\phi^{-1}f,\ \phi^{-1}v,\ \phi^{-1}f\}$ is continuous whenever so is f.*

Proof. It follows from (2.1) and the preceding discussion. □

Let σ denote the strong operator topology on $\mathsf{GL}(E)$. If a weight $\phi: \Omega \to \mathsf{GL}(E)$ satisfies the boundedness condition, so does its inverse $\phi^{-1}: \omega \mapsto \phi(\omega)^{-1}$, $(\omega \in \Omega)$. A bounded weight and its inverse have the same set of points of σ-continuity since the inversion operation $g \mapsto g^{-1}$ is σ-continuous on bounded subsets of $\mathsf{GL}(E)$. For any $f \in \mathcal{C}(\Omega, E)$ and any bounded weight, the function $\tilde{f} = \phi^{-1}f$ is continuous at every point of σ-continuity of ϕ. For every set $S \subset \Omega$ let $\mathcal{C}_S(\Omega, E)$ denote the ideal of the functions that vanish on S. As usual, we shall write $Q_u(v) := \{u\,v\,u\}$ for $u,\ v \in E$ to shorten the notation.

3.2 Proposition *Let E be a JB*-triple and let $\phi: \Omega \to \mathsf{GL}(E)$, be an admissible weight whose set of points of σ-discontinuity is $S \subset \Omega$. Then the ideal $\mathcal{C}_S(\Omega, E)$ is contained in the symmetric subspace of $(\mathcal{C}(\Omega, E),\ \|\cdot\|_\phi)$.*

Proof. Let $c \in \mathcal{C}_S(\Omega, E)$. It suffices to show that $\omega \mapsto \phi(\omega)\,Q_{\tilde{f}(\omega)}(\tilde{c}(\omega))$ is continuous for all $f \in \mathcal{C}(\Omega, E)$, and for that it suffices to establish its continuity at every point $\omega \in S$. Fix $\omega_0 \in S$. Then we have

$$c(\omega_0) = 0, \qquad \tilde{c}(\omega_0) = 0, \qquad \phi(\omega_0)\,Q_{\tilde{f}(\omega_0)}(\tilde{c}(\omega_0)) = 0.$$

Let $\varepsilon > 0$. Since ϕ is bounded and c is continuous at ω_0, there is a neighbourhood U of ω_0 in Ω such that for $\omega \in U$

$$\|c(\omega)\| < \varepsilon, \qquad \text{and} \qquad \|\tilde{c}(\omega)\| \le M\,\|c(\omega)\|.$$

Therefore,

$$\begin{aligned}\|Q_{\tilde{f}(\omega)}(\tilde{c}(\omega))\| &\leq \|\tilde{f}(\omega)\|^2 \ \|\tilde{c}(\omega)\| \\ &\leq M^3 \|f\|^2 \varepsilon \quad (\omega \in U,\ f \in \mathcal{C}(\Omega, E)).\end{aligned}$$

Finally since $Q_{\tilde{f}(\omega_0)}(\tilde{c}(\omega_0)) = 0$,

$$\begin{aligned}&\|\phi(\omega)\, Q_{\tilde{f}(\omega)}(\tilde{c}(\omega)) - \phi(\omega_0)\, Q_{\tilde{f}(\omega_0)}(\tilde{c}(\omega_0))\| \\ &\leq \|\phi\| \cdot \|Q_{\tilde{f}(\omega)}(\tilde{c}(\omega))\| \\ &\leq \frac{1}{m} M^3 \|f\|^2 \varepsilon, \quad (\omega \in U,\ f \in \mathcal{C}(\Omega, E)).\end{aligned}$$

which shows that $\omega \mapsto \phi(\omega)\, Q_{\tilde{f}(\omega)}(\tilde{c}(\omega))$ is continuous (and null) at ω_0. □

3.3 Corollary ([11] Th. 7.1) *Let $\varrho\colon \Omega \to \mathbb{R}^+$ be a strictly positive lower semicontinuous function. Then the symmetric space of the partial JB*-triple associated with the domain*

$$\mathbb{D} := \{f \in \mathcal{C}(\Omega, \mathbb{C}) : |f(\omega)| < \varrho(\omega),\ (\omega \in \Omega)\}$$

is $\mathcal{C}_S(\Omega, \mathbb{C})$ where S stands for the set of discontinuity points of ϱ.

Proof. $\mathbb{D}$ is the weighted power of the unit disc Δ for the weight of multiplication by ϱ^{-1}. Therefore $a \in \mathbb{E}_0$ if and only if $\phi^{-1}\{(\phi f)\,(\phi a)\,(\phi f)\} = \varrho^{-2}\overline{a}f^2$ is continuous for every $f \in \mathcal{C}(\Omega, \mathbb{C})$. For $f = a$, the function $\varrho^{-2}\overline{a}a^2$ is continuous if and only if it vanishes on S. Conversely, if $a \in \mathcal{C}_S(\Omega, \mathbb{C})$ trivially $\varrho^{-2}\overline{a}f^2$ is continuous whenever so is f. □

The inclusion in proposition 3.2 may be strict even if we assume that the weight ϕ is strictly positive-valued as shown by the following counterexample (see [11] Th. 7.1)

3.4 Example Let E be the Hilbert space $E := \mathbb{C}^2$ with the usual scalar product and the triple product $\{xyx\} := (x \mid y)x$. Take $\Omega := [0, 1] \subset \mathbb{R}$ and let $S := \{a, b\} \subset \Omega$ with $a \neq b$. Let U, V be open disjoint neighbourhoods of a, b in Ω, respectively. Let $r_1, r_2\colon \Omega \to \mathbb{R}$ be two *strictly positive* real valued upper semicontinuous functions with exactly one discontinuity point at a, b respectively, and define a bounded weight $\phi\colon \Omega \to \mathsf{GL}(E)$ by $f \mapsto [\phi f](\omega) := (r_1(\omega) f_1(\omega),\ r_2(\omega) f_2(\omega))$, $(\omega \in \Omega)$, for $f = (f_1, f_2) \in \mathcal{C}(\Omega, \mathbb{C}^2)$. Clearly ϕ is a positive operator-valued weight for which S is exactly the set of discontinuities. Choose a function $c = (c_1, c_2) \in \mathcal{C}(\Omega, \mathbb{C}^2)$ such that $c_1, c_2 \geq 0$ on Ω with $c_1 + c_2 = 1$ and $c_1(U) = c_2(V) = \{0\}$. Now c vanishes at no point in S; however we have $\phi(\omega)\, Q_{\tilde{f}(\omega)}(\tilde{c}(\omega)) = \mu(\omega) f(\omega)$, $(\omega \in \Omega)$, where $\mu = f_1\overline{c_1}r_1^2 + f_2\overline{c_2}r_2^2$ is easily seen to be continuous on Ω, and so c belongs to the symmetric subspace of $(\mathcal{C}(\Omega, E),\ \|\cdot\|_\phi)$ even though c vanishes at no point in S.

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