PARTIAL JORDAN* TRIPLES WITH GRID BASE

L.L. STACHÓ

1. Introduction, basic concepts

Throughrout this work $\mathbb{K}$ denotes a commutative field of characteristic 0 with unit 1 and involution $\bar{\ }$ (called *conjugation*). Given a vector space E over $\mathbb{K}$, a subspace F of E and a 3-variable operation $(x,a,y) \mapsto \{xay\}$ mapping $E \times F \times E$ to E such that $\{FFF\} \subset F$, the algebraic structure $(E, \{\ \})$ is called the *partial Jordan* triple* E with base space F (over $\mathbb{K}$) if $\{xay\}$ is symmetric bilinear for $x,y \in E$, conjugate linear for $a \in F$ and satisfies the *Jordan identity*

$$(J) \qquad [a \,\square\, b, c \,\square\, d] = \{abc\} \,\square\, d - c \,\square\, \{dab\} \qquad (a,b,c,d \in F)$$

where, for any $x \in E$ and $f \in F$, the symbol $x \,\square\, f$ denotes the linear operator

$$x \,\square\, f \in \mathcal{L}(E), \qquad (x \,\square\, f)y := \{xfy\} \qquad (y \in E).$$

In particular, the base space F with the restricted triple product $\{\ \}|_{F\times F\times F}$ is a Jordan triple in the sense of Meyberg [5]. Notice that the vector fields

$$(V) \qquad [a + (b \,\square\, c)z + \{zdz\}]\frac{\partial}{\partial z} \qquad (a,b,c,d \in F)$$

form a 3-graded Lie algebra if and only if the axiom

$$\{\{zaz\}bz\} = \{za\{zbz\}\} \qquad (a,b \in F, z \in E)$$

of weak associativity holds. Partial Jordan* triples have a natural geometric background in the complex case ($\mathbb{K} = \mathbb{C}$). In [9] it is shown that the category of weakly associative partial JB*-triples can canonically be identified via the construction in [1] with the category of Lie algebras of complete holomorphic vector fields in bounded circular domains in Banach spaces. It seems that the key tool in the holomorphic classification of finite dimensional bounded bicircular domains by Panou [8] was the following observation: the 3-graded Lie algebra of vector fields of type (V) associated with a finite dimensional weakly associative complex partial Jordan* triple is unambiguously determined by its restriction to the base space, because the operation $R : L \mapsto L|_F$ for $L \in \mathrm{Span}_{\mathbb{C}} F \,\square\, F$ is injective. The proof in [8] relied heavily upon the assumptions $\dim(E) < \infty$ and $\mathbb{K} = \mathbb{C}$. In [10] we have established the injectivity of the restriction R by an analytic argument for infinite dimensional complex (or real) partial JB*-triples with finite dimensional base space without the assumption of weak associativity.

The aim of this note is to give a pure algebraic approach to the injectivity of the restriction R in an elegant algebraic context free of topology. The key tool to this approach will be the concept of weighted grids introduced in [11]. By a *weighted grid* in a Jordan* triple F we mean a linearly independent subset $G := \{g_w : w \in W\}$ of F indexed with a figure W contained in some $\mathbb{K}$-real vector space (vector space over the field $\mathrm{Re}(\mathbb{K}) := \{\xi \in \mathbb{K} : \xi = \overline{\xi}\}$) such that

$$\{g_u g_v g_w\} \in \mathbb{K} g_{u-v+w} \qquad (u, v, w \in W)$$

with the convention $g_z := 0$ when z is not in W. Grids in Neher's sense [6] can be regarded as weighted grids; on the other hand, the description of finite complex weighted grids can be deduced from classical grid theory [11].

After a self-contained exposition of weighted grids in Section 2, we shall prove the following

Theorem 1. *Let E be a partial Jordan* triple whose base space F is spanned by a finite weighted grid $G := \{g_w : w \in W\}$ of non-nil elements. Then the restriction $\sum_{u,v\in W} \gamma_{uv} g_u \,\square\, g_v \mapsto \sum_{u,v\in W} \gamma_{uv} g_u \,\square\, g_v|_F$ is injective.*

2. Weighted grids

Henceforth, throughout the whole work, E denotes an arbitrarily fixed partial Jordan* triple over $\mathbb{K}$ with base space F and triple product $\{\ \}$.

Definition 1. A nonzero element $e \in F$ is a tripotent with *sign* $\lambda \in \mathbb{K}$ if $\{eee\} = \lambda e$. Remark that the sign of a tripotent is unambiguously determined. We shall write $\mathrm{sgn}(e) := [\lambda \in \mathbb{K} : e^3 = \lambda e]$. Clearly, weighted grids consist of (signed) tripotents.

Lemma 1. *Suppose e is a tripotent in F with $\lambda := sgn(e)$ nonzero. Then $\lambda \in Re(\mathbb{K})$ and $F = \oplus_{k=0}^{2} \{x \in F : (e \,\square\, e)x = (k\lambda/2)x\}$.*

Proof. It is well-known [2] that for any fixed $c \in F$ the Jordan* triple F becomes a commutative Jordan algebra when equipped with the c-product $x \bullet^c y := \{xcy\}$. Hence, for any $a \in F$, the c-multiplication operator $R_c(a) := a \,\square\, c|_F$ satisfies $R_c(\{aca\})R_c(a) = \frac{2}{3}R_c(a)^3 + \frac{1}{3}R_c(\{ac\{aca\}\})$ (direct proof see e.g. [3, p. 263]). In particular, for $a := c := e$ we get $(\{eee\} \,\square\, e)(e \,\square\, e) = \frac{2}{3}(e \,\square\, e)^3 + \frac{1}{3}\{ee\{eee\}\} \,\square\, e$ on F, that is $A(A - \frac{\lambda}{2}\mathrm{id}_F)(A - \lambda \mathrm{id}_F) = 0$.

Definition 2. Two tripotents $a, b \in F$ are said to be *compatible* if $(a \,\square\, a)b = \lambda_{ab} b$ and $(b \,\square\, b)a = \lambda_{ba} a$ for some (uniquely determined) $\lambda_{ab}, \lambda_{ba} \in \mathbb{K}$. We call these scalars *structure coefficients* and we reserve the notation $\lambda_{ab}, \lambda_{ba}$ for them. Notice that weighted grids consists of pairwise compatible tripotents.

Let a, b be a compatible non-nil pair of tripotents. According to the above lemma, we can define their *Peirce coefficient* π_{ab} as $\pi_{ab} := [\mu \in \{0, 1, 2\} : 2\lambda_{ab} = \mu \mathrm{sgn}(a)]$. In terms of the Peirce coefficients we can extend McCrimmon's COG relations [4;5] as follows:

$$a\top b \text{ if } \pi_{ab}=\pi_{ba}=1, \qquad a\bot b \text{ if } \pi_{ab}=\pi_{ba}=0,$$
$$a\vdash b \text{ if } \pi_{ab}=2 \text{ and } \pi_{ba}=1, \quad a\approx b \text{ if } \pi_{ab}=\pi_{ba}=2.$$

If $a\top b$ we say that a and b are *collinear*, for $a\bot b$ we say a and b are *orthogonal*, if $a\vdash b$ we say that a *governs* b, for $a\approx b$ we say are *equivalent*. We write also $b\dashv a$ for $a\vdash b$.

Lemma 2. *If $a,b,c,\{abc\},g$ are pairwise compatible non-nil tripotents then*

$$\pi_{g\{abc\}}=\pi_{ga}-\pi_{gb}+\pi_{gc}.$$

Proof. By axiom we have $[[g\,\square\, g],a\,\square\, b]c=(\{gga\}\,\square\, b)c-(a\,\square\{bgg\})c$. That is, by setting $d:=\{abc\}$,

$$\{ggd\}=\{\{gga\}bc\}-\{a\{ggb\}c\}+\{ab\{ggc\}\}$$

$$\lambda_{gd}d=\lambda_{ga}d-\overline{\lambda_{gb}}d+\lambda_{gc}d=\lambda_{ga}d-\lambda_{gb}d+\lambda_{gc}d.$$

Lemma 3. (Third Tripotent Lemma). *Let a,b be two compatible tripotents. Then the element $c:=Q_b a(=\{bab\})$ is a tripotent with* $\mathrm{sgn}(c)=\mathrm{sgn}(a)\mathrm{sgn}(b^2)$ *if $\lambda_{ba}=\lambda_{bb}$ and $c\,\square\, c=0$ else. If a tripotent g is compatible with both a and b then g is compatible also with c. Furthermore*

$$c\,\square\, b=\gamma b\,\square\, a,\quad b\,\square\, c=\gamma a\,\square\, b\quad \textit{for}\quad \gamma:=2\lambda_{ba}-\lambda_{bb},$$

$$c\,\square\, c=2\lambda_{bb}\lambda_{ab}b\,\square\, b-\lambda_{bb}^2 a\,\square\, a\quad \textit{if}\quad \lambda_{ba}=\lambda_{bb}.$$

Proof. We have

$$\begin{aligned} c\,\square\, b &= \{bab\}\,\square\, b=[b\,\square\, a,b\,\square\, b]+b\,\square\{bba\}=-[b\,\square\, b,b\,\square\, a]+\lambda_{ba}b\,\square\, a=\\ &= -\{bba\}\,\square\, a+b\,\square\{abb\}+\lambda_{ba}b\,\square\, a=(-\lambda_{bb}+2\lambda_{ba})b\,\square\, a=\gamma b\,\square\, a,\end{aligned}$$

$$\begin{aligned} b\,\square\, c &= b\,\square\{bab\}=-[a\,\square\, b,b\,\square\, b]+\{abb\}\,\square\, b=[b\,\square\, b,a\,\square\, b]+\lambda_{ba}a\,\square\, b=\\ &= \{bba\}\,\square\, b-a\,\square\{bbb\}+\lambda_{ba}a\,\square\, b=(2\lambda_{ba}-\lambda_{bb})a\,\square\, b=\gamma a\,\square\, b,\end{aligned}$$

$$\begin{aligned} c\,\square\, c &= c\,\square\{bab\}=-[a\,\square\, b,c\,\square\, b]+\{abc\}\,\square\, b=-\gamma[a\,\square\, b,b\,\square\, a]+((c\,\square\, b)a)\,\square\, b=\\ &= -\gamma\{abb\}\,\square\, a+\gamma b\,\square\{aab\}+\gamma((b\,\square\, a)a)\,\square\, b=\\ &= -\gamma\lambda_{ba}a\,\square\, a+\gamma\lambda_{ab}b\,\square\, b+\gamma\lambda_{ab}b\,\square\, b.\end{aligned}$$

Consider any tripotent g compatible with a,b. Then $(c\,\square\, c)g=\gamma[2\lambda_{ab}\lambda_{bg}-\lambda_{ba}\lambda_{ag}]g$ and $(g\,\square\, g)c=\{gg\{bab\}\}=2\{\{ggb\}ab\}-\{b\{gga\}b\}=(2\lambda_{gb}-\lambda_{ga})c$. Assume $\lambda_{ba}=\lambda_{bb}$. Then $c\,\square\, c=2\lambda_{ab}\lambda_{bb}b\,\square\, b-\lambda_{bb}^2 a\,\square\, a$. In particular $(c\,\square\, c)a=\lambda_{bb}^2(2\lambda_{ab}-\lambda_{aa})a$ and $(c\,\square\, c)b=\lambda_{bb}^2\lambda_{ab}b$. Thus $\{ccc\}=(c\,\square\, c)\{bab\}=2\{\{ccb\}ab\}-\{b\{cca\}b\}=\lambda_{bb}^2[2\lambda_{ab}-2(\lambda_{ab}-\lambda_{aa})]\{bab\}=\lambda_{aa}\lambda_{bb}^2 c$. Hence c is a tripotent compatible with a,b and $\lambda_{cc}=\lambda_{aa}\lambda_{bb}^2$. Since also $(c\,\square\, c)g=\lambda_{bb}^2(2\lambda_{ab}\lambda_{bg}-\lambda_{ag})g$, the tripotent c is compatible with g. Assume that c is nonzero. Then $2\lambda_{gb}-\lambda_{ga}$ is an eigenvalue of $g\,\square\, g$. In particular for $g:=b$ we have $2\lambda_{bb}-\lambda_{ba},\lambda_{ba}\in\lambda_{bb}\{0,1/2,1\}$. Necessarily $\lambda_{ba}=\lambda_{bb}$.

Proposition 1. *Let a, b be compatible non-nil tripotents. Define recursively $a_1 := a$, $a_2 := b$,*

$$a_{n+1} := \{a_n a_{n-1} a_n\} \qquad (n > 2), \qquad a_{n-1} := \{a_n a_{n+1} a_n\} \qquad (n < 1)$$

and let $I := \{n \in \mathbb{Z} : a_n$ is nonzero$\}$. Then we have the following alternatives:

A1) $a \perp b$, $I = \{1,2\}$ and $a \,\square\, b = b \,\square\, a = 0$,

A2) $a \top b$, $I = \{1,2\}$ and the indexed set $G := \{a_w : w \in I\}$ is a weighted grid,

B1) $a \vdash b$, $I = \{0,1,2\}$ and the indexed set $G := \{a_w : w \in I\}$ is a weighted grid,

B2) $a \dashv b$, $I = \{1,2,3\}$ and the indexed set $G := \{a_w : w \in I\}$ is a weighted grid,

C) $a \approx b$ and $I = \mathbb{Z}$.

Proof. The index set is an interval in $\mathbb{Z}$ containing $\{1,2\}$. From the Third Tripotent Lemma it follows immediately by induction that the elements a_n $(n \in I)$ are pairwise compatible tripotents. For short, we write λ_{jk}, π_{jk} instead of the terms $\lambda_{a_j a_k}, \pi_{a_j a_k}$, respectively, in the sequel. If a, b are compatible with a tripotent g, then g is compatible with each a_n $(n \in I)$ and, by Lemma 2, the sequence $(\lambda_{g a_n} : n \in I)$ is an arithmetic sequence in $\mathbb{Z}$ with range in $\{\lambda_{gg}, \lambda_{gg}/2, 0\}$. Hence we only have the following possibilities:

A) $I = \{1,2\}$ i.e. $a_0 = a_3 = 0$ and $\pi_{21}, \pi_{12} < 2$;

B) $I = \{1,2,3\}$ i.e. $a_0 = 0$, a_3 is nonzero and

$$(\pi_{2k} : k \in I) = (2,2,2), \quad (\pi_{1k} : k \in I) = (2,1,0);$$

B') $I = \{0,1,2\}$ i.e. $a_3 = 0$, a_0 is nonzero and

$$(\pi_{1k} : k \in I) = (2,2,2), \quad (\pi_{2k} : k \in I) = (0,1,2);$$

C) $I \supset \{0,1,2,3\}$ i.e. a_0, a_3 are nonzero and $(\pi_{k0}, \dots, \pi_{k3}) = (2,2,2,2)$, $(k \in I)$.

Next we examine cases A,B,B',C in more detail. A) We have the following subalternatives:

A1) $\lambda_{12} = 0$ or $\lambda_{21} = 0$. Suppose $\lambda_{12} = 0$. By the Third Tripotent Lemma, $0 = a_0 \,\square\, a_1 = (2\lambda_{12} - \lambda_{11}) a_1 \,\square\, a_2 = -\lambda_{11} a_1 \,\square\, a_2$. It follows $\lambda_{21} a_1 = \{a_2 a_2 a_1\} = (a_1 \,\square\, a_2) a_2 = 0$, i.e. $\lambda_{21} = 0$. Similarly, $\lambda_{21} = 0$ implies $a_2 \,\square\, a_1 = 0$ and $\lambda_{12} = 0$.

A2) $\pi_{12} = \pi_{21} = 1$. Then $a_1 \,\square\, a_2 : a_2 \mapsto \{a_2 a_2 a_1\} = [\mathrm{sgn}(a_2)/2] a_1 \mapsto 0$ and $a_2 \,\square\, a_1 : a_1 \mapsto [\mathrm{sgn}(a_1)/2] a_2 \mapsto 0$.

B) By the Third Tripotent Lemma, since $Q_{a_2} a_1 = a_3$ and $\lambda_{22} = \lambda_{21}$, we have $a_3 \,\square\, a_3 = 2\lambda_{12}\lambda_{22} a_2 \,\square\, a_2 - \lambda_{22}^2 a_1 \,\square\, a_1$ and $\lambda_{33} = \mathrm{sgn}(a_3) = \mathrm{sgn}(a_1)\mathrm{sgn}(a_2)^2 = \lambda_{11}\lambda_{22}^2$ which is nonzero. Also $(a_3 \,\square\, a_3) a_3 = \sigma\epsilon a_3$. We have $\pi_{13} = 0$. Thus $a_1 \perp a_3$ and from the argument in A1) we conclude that also $a_1 \,\square\, a_3 = a_3 \,\square\, a_1 = 0$ and hence $a_3 \perp a_1$ i.e. $\pi_{31} = 0$. By Lemma 2, the sequence $(\pi_{31}, \pi_{32}, \pi_{33})$ is arithmetic. By definition $\pi_{33} = 2$. Hence $(\pi_{31}, \pi_{32}, \pi_{33}) = (0,1,2)$. Finally we show that the family $\cup_{n=1}^{3}\mathbb{K}a_n$ is invariant by the operators $a_j \,\square\, a_k$. By the Third Tripotent Lemma,

$$a_3 \,\square\, a_2 = \lambda a_2 \,\square\, a_1 = \lambda_{22} a_2 \,\square\, a_1, \qquad a_2 \,\square\, a_3 = \lambda_{22} a_1 \,\square\, a_2,$$
$$(a_3 \,\square\, a_2)a_1 = \lambda_{22}(a_2 \,\square\, a_1)a_1 = \lambda_{22}\lambda_{11}a_2, \quad (a_3 \,\square\, a_2)a_2 = \lambda_{23}a_3, \quad (a_3 \,\square\, a_2)a_3 = 0,$$
$$(a_2 \,\square\, a_3)a_3 = \lambda_{32}a_2, \quad (a_2 \,\square\, a_3)a_2 = \lambda_{22}(a_1 \,\square\, a_2)a_2 = \lambda_{22}^2 a_1,$$
$$(a_2 \,\square\, a_3)a_1 = (a_1 \,\square\, a_3)a_2 = 0.$$

B') We can proceed as in B) with the index changes $1 \to 1, 2 \to 0,\ 3 \to -1$.

C) Observe that we can apply the previous arguments to the tripotents $a' := a_2$, $b' := a_3$ with the conclusion that the sequence $(a'_n : n \in \mathbb{Z})$ defined by an analogous recursion as that of $(a_n : n \in \mathbb{Z})$ from a, b can only satisfy alternative C) again. Notice that $a'_n = a_{n+1}$ for $n > 0$. It follows by induction that for any $n > 0$ the elements a_n are pairwise equivalent non-nil tripotents such that $a_n \,\square\, a_n$ is a linear combination of $a \,\square\, a$ and $b \,\square\, b$. Induction in the negative direction establishes the same conclusion for $n < 0$.

Remark 1. We can summarize the results of this section in the context of the Peirce matrix of a finite weighted grid of non-nil tripotents as follows.

Let $G := \{g_w : w \in W\}$ be a finite weighted grid of non-nil tripotents where $W = \{w_1, \dots, w_n\}$. For short let us write $1, 2, \dots, n$ instead of $g_{w_1}, g_{w_2}, \dots, g_{w_n}$, respectively. Then the matrix $(\pi_{\mathfrak{j}\mathfrak{k}})_{j,k=1}^{n}$ of the Peirce coefficients of G satisfies the following rules.

M0) $\pi_{\mathfrak{j}\mathfrak{k}} \in \{0, 1, 2\}$ *and* $\pi_{\mathfrak{k}\mathfrak{k}} = 2$.

M1) $\pi_{\mathfrak{j}\mathfrak{k}} = 0$ *iff* $\pi_{\mathfrak{k}\mathfrak{j}} = 0$.

M2) if $\mathfrak{j} \dashv \mathfrak{k}$ *then there exists* m *such that* $\mathfrak{j} \perp \mathfrak{m} \dashv \mathfrak{k}$ *and for the rows* $\pi_{\mathfrak{i}\bullet} := (\pi_{\mathfrak{i}\mathfrak{t}})_{\mathfrak{t}=1}^{n}$ *In particular* $0 \le \pi_{\mathfrak{t}\mathfrak{k}} - \pi_{\mathfrak{t}\mathfrak{j}} \le 2$ $(\forall t)$ *if* $\pi_{\mathfrak{j}\mathfrak{k}} = 1$
$(\mathfrak{i} = \mathfrak{j}, \mathfrak{k}, \mathfrak{m})$ *we have* $\pi_{\mathfrak{m}\bullet} = \pi_{\mathfrak{k}\bullet} - \pi_{\mathfrak{j}\bullet}$.
and $\pi_{\mathfrak{k}\mathfrak{j}} = 2$.

M3) the columns $\pi_{\bullet\mathfrak{i}} := (\pi_{\mathfrak{t}\mathfrak{i}})_{\mathfrak{t}=1}^{n}$, $(\mathfrak{i} = 1, \dots, n)$ *satisfy* $\pi_{\bullet\mathfrak{m}} = \pi_{\bullet\mathfrak{i}} - \pi_{\bullet\mathfrak{j}} + \pi_{\bullet\mathfrak{k}}$ *whenever* $w_m = w_i - w_j + w_k$ *and* $\{\mathfrak{i}\mathfrak{j}\mathfrak{k}\}$ *is nonzero. In particular* $0 \le 2\pi_{\mathfrak{t}\mathfrak{k}} - \pi_{\mathfrak{t}\mathfrak{j}} \le 2$ $(\forall t)$ *if* $\pi_{\mathfrak{j}\mathfrak{k}} = 1$ *and* $\pi_{\mathfrak{k}\mathfrak{j}} = 2$.

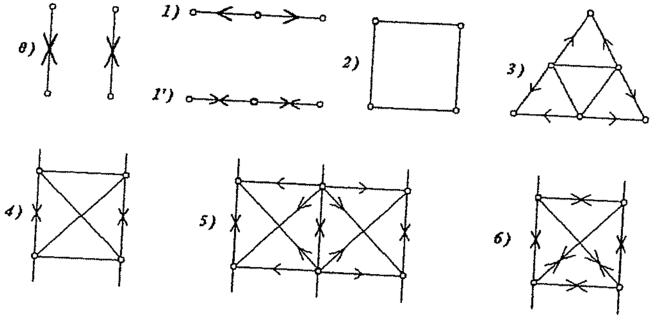
M4) $\pi_{\bullet\mathfrak{i}} = \pi_{\bullet\mathfrak{j}}$ *and* $w_i + n(w_i - w_j) \in W$ $(n \in \mathbb{Z})$ *whenever* $\pi_{p\mathfrak{i}\mathfrak{j}} = \pi_{\mathfrak{j}\mathfrak{i}} = 2$.

M4') In particular, if G *is finite,* $(\pi_{\mathfrak{i}\mathfrak{j}}, \pi_{\mathfrak{j}\mathfrak{i}}) \neq (2, 2)$ *for* $\mathfrak{i} \neq \mathfrak{j}$ *because equivalent couples cannot occur in finite weighted grids* .

If G is a family or pairwise compatible (generalized) tripotents, it is convenient to use an analogous graph representation as in [7] for the Peirce relation between its terms. That is, we can visualize the Peirce relation matrix of G as the graph G^* with signed edges whose vertices are the elements of G and, for $g, h \in G$, we write g — h if $g \top h$, g →— h if $g \vdash h$, g —×— h if $g \approx h$, and g is not connected to h for $g \perp h$.

A combinatorial study of $n \times n$ $(n < 6)$ matrices with the properties M0)-M4) (see [11]) establishes the following. *If* $G := \{g_w : w \in W\}$ *is a weighted grid and*

$u, v, w, z \in W$ form a parallelogram, then $P := \{g_u, g_v, g_w, g_z\}$ are vertices of a parallelogram in G^* of a subgraph of the form



According to classical grid terminology, P is a *quadrangle* in case 2), a *diamond* in case 3), and e.g. (g_u, g_v, g_w) is a triangle if g_v is the middle point and g_u, g_w are endpoints in case 1).

In [11, Appendix] also the geometric shapes of the figures $\{\pi_{\bullet g_w} : w \in W\}$ (as subsets in some $\mathbb{R}^N$) are described. In particular it turns out that quadrangles correspond to squares and the triangles in 2) carrying a diamond are regular.

Definition 3. Two weighted grids $G := g_w : w \in W\}$ and $H := \{h_z : z \in Z\}$ are *weight-equivalent*) (denoted by $G \simeq H$) if there exists a $1-1$ correspondence $\phi : W \to Z$ such that $g_w = h_{\phi(w)}$ $(w \in W)$.

Proposition 2. *Given a weighted grid $G := g_w : w \in W\}$, there is $G^* := \{g_z^* : z \in W^*\}$ such that $G^* \simeq G$ and the weight figure Z of any weight-equivalent realization $\{h_z : z \in Z\}$ of G is an affine image of W^*.*

Proof. Let U^* be the free real vector space whose generator symbols are the elements of G. Define

$$S^* := \mathrm{Span}_{\mathbb{R}}\{g_u - g_v + g_w - g_z \in U^* : 0 \neq \{g_u, g_v g_w\} \in \mathbb{K} g_z, g_u, g_v, g_w, g_z \in G\},$$
$$W^* := \{g + S^* \in U^*/S^* : g \in G\}.$$

Notice that these definitions are formulated purely in terms of the triple product $\{\ \}$ and the elements of G without relevant use of the indexing by W.

Given any $h^* \in S^*$, we can write $h^* = \sum_i \alpha_i(g_{u_i} - g_{v_i} + g_{w_i} - g_{z_i})$ with suitable real coefficients α_i and vectors $u_i, v_i, w_i, z_i \in W$ such that $u_i - v_i + w_i = z_i$ by the definition of weighted grids . Hence $\sum_j \gamma_j w_j = 0$ whenever $\sum_j \gamma_j g_{w_j} \in S^*$. It follows that the mapping

$$\phi : \sum_j \gamma_j g_{w_j} + S^* \mapsto \sum_j \gamma_j w_j$$

is well-defined and $\mathbb{R}$-linear $U^* \to \mathrm{Span}_{\mathbb{R}} W$ with $\phi(W^*) = W$. On the other hand, by setting

$$w^* := g_w + S^*(\in U^*), \qquad g^*_{w^*} := g_w \qquad (w \in W),$$

we have $\{g^*_{u^*} g^*_{v^*} g^*_{w^*}\} = \{g_u g_v g_w\} \in \mathbb{K} g_{u-v+w} = \mathbb{K} g^*_{u^*-v^*+w^*}$, $(u, v, w \in W)$, that is $\{g^*_{w^*} : w^* \in W^*\} \simeq \{g_w : w \in W\}$.

Definition 4. A figure W^* with the affine maximality property described in the above proposition is called a *grid figure* of G.

Corollary 1. *Let $G = \{g_w : w \in W\}$ be a finite weighted grid of non-nil tripotents with a grid figure W. Then the set $\widetilde{W} := \{\pi_{\circ g_w} : w \in W\}$ is linearly isomorphic to W.*

Proof. Given any $h \in G$, by the axiomatic identity, $(h \,\square\, h)\{g_u g_v g_w\} = (\lambda_{hg_u} - \lambda_{hg_v} + \lambda_{hg_w})\{g_u g_v g_w\} = \mathrm{sgn}(h)(\pi_{hg_u} - \pi_{hg_v} + \pi_{hg_w})\{g_u g_v g_w\}$, $(u, v, w \in W)$. Thus if the columns of the Peirce matrix are pairwise different, we may define

$$\tilde{g}_{\pi_{\circ g_w}} := g_w \qquad (w \in W)$$

with the effect that $\{\tilde{g}_{\tilde{w}} : \tilde{w} \in \widetilde{W}\} \simeq \{g_w : w \in W\}$. Since W is a grid figure, $\widetilde{W}$ is an $\mathbb{R}$-linear image of W. The only thing remained to prove is that the Peirce matrix has different columns for finite weighted grids . By Proposition 1C), finite weighted grids admit no equivalent couples of elements. Thus if $u, v \in W$ are different, then also g_u and g_v are different and hence at least one of the Peirce coefficients $\pi_{g_u g_v}$, $\pi_{g_v g_u}$ is < 2 while $\pi_{g_u g_u} = \pi_{g_v g_v} = 2$ and therefore $\pi_{\circ g_u}$ and $\pi_{\circ g_v}$ are different.

3. Extension of box operators from the base triple

Proposition 3. *Let $u_1, u_2, u_3 \in W$. Assume $\pi_{31} > \pi_{32}$ and 1 is not orthogonal to 2 where $\mathfrak{k} := g_{u_k}$, $(k = 1, 2, 3)$ for short. Then $u_1 - u_2 + u_3$ is not in W and $u_4 := u_2 - u_1 + u_3 \in W$ with $4 := \{213\} \in \mathbb{K} g_{u_4} - \{0\}$ and*

$$4 \,\square\, 3 = \frac{1}{2}\lambda_{33} 2 \,\square\, 1, \qquad 3 \,\square\, 4 = \frac{1}{2}\lambda_{33} 1 \,\square\, 2, \qquad \lambda_{44} = \frac{1}{4}\lambda_{11}\lambda_{22}\lambda_{33},$$

$$\sum_{k=1}^{4} (-1)^k \gamma_k \lambda_{\mathfrak{k}\mathfrak{k}}^{-1} \mathfrak{k} \,\square\, \mathfrak{k} = 0 \text{ where } \gamma_{\mathfrak{k}} := 2 \text{ if } \exists \mathfrak{m}, \mathfrak{k} \dashv \mathfrak{m} \text{ and } \gamma_{\mathfrak{k}} := 1 \text{ else }.$$

Proof. Suppose $u_5 := u_1 - u_2 + u_3 \in W$. Then, with the term $5 := g_{u_5}$ we would have $\pi_{35} = \pi_{31} - \pi_{32} + \pi_{33} = \pi_{31} - \pi_{32} + 2 > 2$ which is impossible. Therefore $u_1 - u_2 + u_3$ is not in W and hence $\{123\} = 0$. By the Jordan identity

$$4 \,\square\, 3 = \{213\} \,\square\, 3 = [2 \,\square\, 1, 3 \,\square\, 3 + 3 \,\square\{321\}] =$$
$$= -[3 \,\square\, 3, 2 \,\square\, 1] = -\{332\} \,\square\, 1 + 2 \,\square\{133\} = (-\lambda_{32} + \lambda_{31}) 2 \,\square\, 1,$$

$$3 \,\square\, 4 = 3 \,\square\{213\} = 3 \,\square\{312\} = -[1 \,\square\, 2, 3 \,\square\, 3] + \{123\} \,\square\, 3 =$$
$$= [3 \,\square\, 3, 1 \,\square\, 2] = (\lambda_{31} - \lambda_{32}) 1 \,\square\, 2,$$

$$4\,\square\,4 = \{213\}\,\square\,4 = [2\,\square\,1, 3\,\square\,4] + 3\,\square\{421\} =$$
$$= (\lambda_{31} - \lambda_{32})[2\,\square\,1, 1\,\square\,2] + 3\,\square\{124\}.$$

Here we have $[2\,\square\,1, 1\,\square\,2] = \{211\}\,\square\,2 - 1\,\square\{221\} = \lambda_{12}2\,\square\,2 - \lambda_{21}1\,\square\,1$ and

$$\{124\} = \{12\{213\}\} = \{\{122\}13\} - \{2\{211\}3\} + \{21\{123\}\} =$$
$$= \lambda_{21}\{113\} - \lambda_{12}\{223\} = (\lambda_{21}\lambda_{13} - \lambda_{12}\lambda_{23})3.$$

Thus

$$4\,\square\,4 = (\lambda_{31} - \lambda_{32})\lambda_{12}2\,\square\,2 + (\lambda_{32} - \lambda_{31})\lambda_{21}1\,\square\,1 + (\lambda_{21}\lambda_{13} - \lambda_{12}\lambda_{23})3\,\square\,3.$$

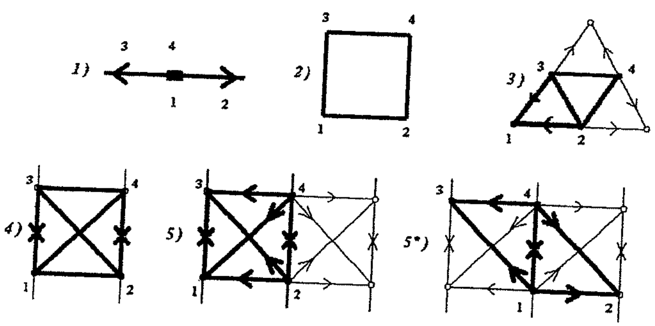
For $k = 1, 2, 3$ it follows

$$\lambda_{44}\mathfrak{k} = (4\,\square\,4)\mathfrak{k} = (\lambda_{31} - \lambda_{32})\lambda_{12}\lambda_{2\mathfrak{k}} + (\lambda_{32} - \lambda_{31})\lambda_{21}\lambda_{1\mathfrak{k}} + (\lambda_{21}\lambda_{13} - \lambda_{12}\lambda_{23})\lambda_{3\mathfrak{k}}.$$

Since $u_4 = u_2 - u_1 + u_3$, with $\epsilon_1 := -1$, $\epsilon_2 := \epsilon_3 := 1$ we have

$$\lambda_{44} = \lambda_{42} - \lambda_{41} + \lambda_{43} = \sum_{k=1}^{3} \epsilon_k[(\lambda_{31} - \lambda_{32})(\lambda_{12}\lambda_{2\mathfrak{k}} - \lambda_{21}\lambda_{1\mathfrak{k}}) + (\lambda_{21}\lambda_{13} - \lambda_{12}\lambda_{23})\lambda_{3\mathfrak{k}}] =$$
$$= \frac{1}{8}\lambda_{11}\lambda_{22}\lambda_{33}[(\pi_{31} - \pi_{32})[\pi_{12}(\pi_{22} - \pi_{21} + \pi_{23}) - \pi_{21}(\pi_{12} - \pi_{11} + \pi_{13})] +$$
$$+ (\pi_{21}\pi_{13} - \pi_{12}\pi_{23})(\pi_{32} - \pi_{31} + \pi_{33})].$$

Observe that, by Remark 1, the following cases are possible



It is straightforward to verify that in any of these cases we have $\pi_{31} - \pi_{32} = 1$ and $\lambda_{44} = \frac{1}{4}\lambda_{11}\lambda_{22}\lambda_{33}$. Substituting this into the expression of $4\,\square\,4$ in terms of $1\,\square\,1$, $2\,\square\,2$, $3\,\square\,3$ we get

$$\frac{1}{\lambda_{44}}4\,\square\,4 = \frac{\pi_{12}}{\lambda_{22}}2\,\square\,2 - \frac{\pi_{21}}{\lambda_{11}}1\,\square\,1 + \frac{\pi_{21}\pi_{13} - \pi_{12}\pi_{23}}{\lambda_{33}}3\,\square\,3.$$

Again, a case by case verification establishes that the coefficient of the term $\lambda_{\mathfrak{k}\mathfrak{k}}^{-1}\mathfrak{k}\,\square\,\mathfrak{k}$ has absolute value 2 if and only if $\mathfrak{k} \dashv \mathfrak{m}$ for some $m = 1, \ldots, 4$, otherwise its absolute value is 1.

Corollary 2. *Assume the base space F is spanned by a finite (or, in general, equivalence-free) weighted grid G of non-nil tripotents with grid figure W. Let $u, v, s, t \in W$ with $u - v = w - z$. Then $g_w \square g_z = \gamma g_u \square g_v$ and $g_z \square g_w = \overline{\gamma} g_v \square g_u$ for some nonzero $\gamma \in \mathbb{K}$.*

Proof. If $\#\{u, v, w, z\} \leq 2$ we have the trivial cases $u = w$, $v = z$ or $u = v$, $w = z$. If $\#\{u, v, w, z\} = 3$ then $v = w$ or $u = z$. Otherwise, according to Remark 1, the tripotents $\{g_u, g_v, g_w, g_z\}$ form a quadrangle or diamond or three of them form a triangle. In any case we can apply Proposition 3 with a suitable indexing $\{u_1, \ldots, u_4\}$ of the parallelogram $\{u, v, w, z\}$ (see cases o),1),2) on the figure in the proof of Proposition 3.

Remark 2. Concerning the rows of Peirce matrices, Proposition 3 has the following consequence. With the notations of Remark 1,

M5) *If $\pi_{ki} > \pi_{kj}$ then $w_j - w_i + w_k = w_l \in W$ for some index l and $\gamma_i \pi_{j\circ} - \gamma_j \pi_{j\circ} + \gamma_k \pi_{k\circ} - \gamma_j \pi_{l\circ} = 0$ where $\gamma_x := 2$ if $\pi_{xy} = 1$, $\pi_{yx} = 2$ for some $y \in \{i, j, k, l\}$ and $\gamma_x := 1$ else.*

It is straightforward to verify on the basis of Remark 1 that the generalized Peirce relations $\top$, $\perp$, *vdash*, $\dashv$ between the elements of an equivalence-free weighted grid of non-nil (generalized) tripotents satisfy all *abstract COG* axioms R_I, R_{II}, R_{III} in [7]. Hence the grid figure and the figure of the columns of the Peirce matrix of an equivalence-free weighted grid of non-nil generalized tripotents are $\mathbb{Z}$-linearly isomorphic to the 1-part of some 3-graded root system. Thus from the structure theory of roots systems we can conclude the following.

Corollary 3. *Given a finite or equivalence-free weighted grid $G := \{g_w : w \in W\}$ of non-nil generalized tripotents, its Peirce matrix is isomorphic to the direct sum of a family of Peirce matrices of (possibly infinite dimensional) real Cartan factors.*

Proof of Theorem 1.

We may assume that $F = \mathrm{Span}G$ where $G = \{g_w : w \in W\}$ is a weighted grid with grid figure W. It is well-known that $a \square b = 0$ whenever a, b are two orthogonal tripotents (the proof in [8] stated for weakly associative complex partial J*-triples applies in general). Therefore, by setting

$$\Delta := \{u - v : u, v \in W, u \not\perp u\},$$

we have

$$\mathrm{Span}F \square F = \sum_{u,v \in W} \mathbb{K} g_u \square g_v = \sum_{w \in W} \mathbb{K} g_w \square g_w \oplus \Big(\bigoplus_{d \in \Delta} \sum_{u - v = d} \mathbb{K} g_u \square g_v \Big).$$

By Corollary 2, each space $\mathcal{L}_d := \sum_{u-v=d} \mathbb{K} g_u \square g_v$ ($d \in \Delta(W)$) is 1-dimensional and it consists of the multiples of an operator L_d such that $L_d : \mathbb{K} g_w \to K g_{w+d}$

($w \in W$). Let $A := \sum_{u,v \in W} \alpha_{uv} g_u \,\square\, g_v$ and suppose $A|_F = 0$. Since the restricted operators $L_d|_F$ ($d \in \Delta$) form trivially a linearly independent system (as weighted shifts into different directions), necessarily $A \in \sum_{u \in W} \mathbb{K} g_u \,\square\, g_u$. By Corollary 3 we may suppose that the grid figure W is the set of columns of the Peirce matrix of a (complex or real) Cartan factor. Concerning Peirce matrices of Cartan factors, it is shown in [11, Appendix] that there is a directed connected graph P in W (a path with at most one junction) such that $\Delta_0 := \{u - v : (u,v) \text{ edge of } P\}$ is a root base for Δ in the sense that each $d \in \Delta$ is a positive or negative integer linear combination from Δ_0. Hence the set of all vertices of P (denoted also by P) is an $\mathbb{R}$-linear basis in $\mathrm{Span}_{\mathbb{R}} W$ and given any $w \in W$ and for any $v, w \in W$ there exists a finite sequence $u_1, v_1, \dots, u_n, v_n \in P$ such that $u - v = \sum_k (u_k - v_k)$ where each term (u_k, v_k) or (v_k, u_k) is an edge of P. According to Remark 1, $\{g_u, g_v, g_w, g_z\}$ is always a triangle, quadrangle or diamond whenever $u - v + w - z = 0$ and $u \neq v, z$ for $u, v, w, z \in W$. Therefore *given any point* $w \in W - P$, *there exists a finite sequence* $w_0, u_1, v_1, \dots, w_{n-1}, u_{n-1}, v_{n-1}, w_n$ *such that* $w_0 \in P$, $w_n = w$, $u_i, v_i \in P$ *and* $\{w_{i-1}, u_i, v_i, w_i\}$ *is a triangle, quadrangle or diamond with* $w_i = w_{i-1} + (u_i - v_i)$ ($i = 1, \dots, n$). By Proposition 3 it follows that each operator $g_w \,\square\, g_w$ ($w \in W$) is a linear combination from $\mathcal{D}_0 := \{g_p \,\square\, g_p : p \in P\}$. In particular $A = \sum_{p \in P} \widetilde{\alpha}_p g_p \,\square\, g_p$ with suitable constants $\widetilde{\alpha}_p \in \mathbb{K}$. Since Cartan factors are Hermitian $(\pi_{uv})_{u,v \in W} = S(\pi_{vu})_{u,v \in W} S^{-1}$ for some diagonal matrix S (cf [11, Lemma 3.7]). Since the columns $\{\pi_{\bullet p} : p \in P\}$ are linearly independent, the rows $\{\pi_{p \bullet} : p \in P\}$ and consequently the operators $\{g_p \,\square\, g_p|_F : p \in P\}$ are also linearly independent. Therefore $A|_F = 0$ implies $\widetilde{\alpha}_p = 0$ ($p \in P$).

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Bolyai Institute Attila Jozsef. University of Szeged (JATE), 6720 Szeged, Hungary