Direct Approach to Laurent Expansions

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An elementary proof without Cauchy's integral formula and contour integrals for the Laurent series expandibility of holomorphic functions on annuli is given.

The purpose of this note is to present an approach to the Laurent series of holomorphic functions, which may serve as a starting step of a course in complex analysis. Actually, our proof requires an additional hypothesis, the (local) boundedness of the derivative. However, one can eliminate this technical assumption immediately after our proof, by an application of Goursat's theorem [1] with squares. It may be worth to note that, with our approach, the introduction of contour integration may be postponed and somewhat simplified using a treatment with analytic functions and homotopy arguments concerning piecewise smooth curves similar to that in [1, Ch.IV]. Also the classification of isolated singularities along with the residue formula are available at an early stage. For the sake of a concise presentation we use Lebesgue's majorized convergence theorem. Assuming additionally the continuity of the derivatives, one can replace this tool with routine arguments involving only Riemann integration of continuous functions. (The number of analysis courses introducing immediately Lebesgue integration is rapidly growing, e.g. see [2].)

Before turning to our result, we provide some heuristics. The domain of convergence of a Laurent series on the complex plane $\mathbb{C}$ is an annulus. If

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \qquad (r < |z| < R)$$

then $a_n = (2\pi)^{-1} \int_0^{2\pi} f(\rho\, e^{i\varphi})\big(\rho\, e^{i\varphi}\big)^{-n}\, d\varphi \qquad (n = 0, \pm 1, \pm 2, \dots)$ with arbitrary radius $r < \rho < R$ because $\int_0^{2\pi} e^{in\varphi}\, d\varphi = \Big[2\pi \text{ if } n = 0,\ 0 \text{ if } n \neq 0\Big]$. It follows

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=-\infty}^{\infty} f(\rho_n e^{i\varphi})\big(\rho_n e^{i\varphi}\big)^{-n} z^n\, d\varphi$$

with uniform convergence in the integrand for $r < \inf_n \rho_n \le \sup_n \rho_n < R$. In particular, if $r < r' < |z| < R' < R$ then with $\rho_n := \Big[R' \text{ if } n \ge 0,\ r' \text{ if } n < 0\Big]$

we get

$$(1)\qquad f(z) = \frac{1}{2\pi}\int_0^{2\pi}\Big[\frac{f(R'e^{i\varphi})\,R'}{R'e^{i\varphi}-z} - \frac{f(r'e^{i\varphi})\,r'}{r'e^{i\varphi}-z}\Big]\,e^{i\varphi}d\varphi$$

since (for $0 < r' < |z| < R' < \infty$)

$$(2)\qquad \sum_{n=0}^{\infty}\big(R'e^{i\varphi}\big)^{-n}z^n = \frac{R'e^{i\varphi}}{R'e^{i\varphi}-z}\,,\qquad \sum_{n=-\infty}^{-1}\big(r'e^{i\varphi}\big)^{-n}z^n = \frac{r'e^{i\varphi}}{r'e^{i\varphi}-z}\,.$$

PROPOSITION. *Suppose $f : A \to \mathbb{C}$ is a holomorphic function with bounded derivative on the annulus $A := \{z \in \mathbb{C} :\ r < |z| < R\}$. Then for every $z \in A$*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \quad \text{where} \quad a_n = \frac{1}{2\pi}\int_0^{2\pi} f(\rho e^{i\varphi})\big(\rho e^{i\varphi}\big)^{-n}\,d\varphi$$

independently of the choice of the radius $\rho \in (r, R)$.

PROOF. Let us fix $z \in A$ arbitrarily. First we show that

$$(3)\qquad f(z) = \lim_{\substack{R'\searrow|z|\\ r'\nearrow|z|}} \frac{1}{2\pi}\int_0^{2\pi}\Big[\frac{f(R'e^{i\varphi})R'}{R'e^{i\varphi}-z} - \frac{f(r'e^{i\varphi})\,r'}{r'e^{i\varphi}-z}\Big]\,e^{i\varphi}\,d\varphi\,.$$

Indeed, from (2) it follows

$$\frac{1}{2\pi}\int_0^{2\pi}\Big[\frac{R'}{R'e^{i\varphi}-z} - \frac{r'}{r'e^{i\varphi}-z}\Big]\,e^{i\varphi}\,d\varphi = 1 \qquad (0 < r' < |z| < R' < \infty)\,.$$

Hence

$$(4)\qquad \begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi}\Big[\frac{f(R'e^{i\varphi})\,R'}{R'e^{i\varphi}-z} - \frac{f(r'e^{i\varphi})\,r'}{r'e^{i\varphi}-z}\Big]\,e^{i\varphi}\,d\varphi = \\ &= f(z) + \frac{1}{2\pi}\int_0^{2\pi}\Big[\frac{f(R'e^{i\varphi})-f(z)}{R'e^{i\varphi}-z}\,R' - \frac{f(r'e^{i\varphi})-f(z)}{r'e^{i\varphi}-z}\,r'\Big]\,e^{i\varphi}\,d\varphi \end{aligned}$$

for $r < r' < |z| < R' < R$. Observe that

$$\lim_{\substack{R'\searrow|z|\\ r'\nearrow|z|}}\Big[\frac{f(R'e^{i\varphi})-f(z)}{R'e^{i\varphi}-z}\,R' - \frac{f(r'e^{i\varphi})-f(z)}{r'e^{i\varphi}-z}\,r'\Big] = 0 \qquad \text{whenever} \quad |z|e^{i\varphi} \neq z\,.$$

The integrand on the right hand side of (4) is bounded by $(R'+r')\sup|f'| < \infty$. Thus we may apply Lebesgue's majorized convergence theorem to conclude that the right hand side of (4) converges to $f(z)$ for $r' \nearrow |z|$, $R' \searrow |z|$.

Taking (2) into account, we have $f(z) = \lim_{\substack{R'\searrow|z|\\ r'\nearrow|z|}} \sum_{n=-\infty}^{\infty} I_n\big(\rho_n(r',R')\big)z^n$ where $\rho_n(r',R') := \big[R' \text{ if } n \geq 0,\ r' \text{ if } n < 0\big]$ and $I_n(\rho) := (2\pi)^{-1}\int_0^{2\pi} f(\rho e^{i\varphi})\big(\rho e^{i\varphi}\big)^{-n}d\varphi$

for $r < \rho < R$, $n = 0, \pm 1, \ldots$. Thus, to complete the proof, it suffices to establish that all the integrals $\rho \mapsto I_n(\rho)$ are constant.

Let us fix any index n and write $F(\zeta) := f(\zeta)\zeta^{-n}$ ($\zeta \in A$). Notice that the function F is holomorphic with locally bounded derivative and we have $I_n(\rho) = (2\pi)^{-1} \int_0^{2\pi} F(\rho e^{i\varphi})\, d\varphi$ ($r < \rho < R$). We can use again Lebesgue's majorized convergence theorem to establish that

$$\frac{I_n(\rho')-I_n(\rho)}{\rho'-\rho} = \frac{1}{2\pi}\int_0^{2\pi} \frac{F(\rho' e^{i\varphi})-F(\rho e^{i\varphi})}{\rho'-\rho}\, d\varphi \to \frac{1}{2\pi}\int_0^{2\pi} F'(\rho e^{i\varphi})\, e^{i\varphi} d\varphi \;(\rho' \to \rho).$$

On the other hand

$$\int_0^{2\pi} F'(\rho e^{i\varphi}) e^{2i\varphi}\, d\varphi = \frac{1}{i\rho}\int_0^{2\pi} \frac{\partial}{\partial\varphi} F(\rho e^{i\varphi})\, d\varphi = \frac{1}{i\rho} F(\rho\, e^{i\varphi})\Big|_{\varphi=0}^{2\pi} = 0 \,.$$

Thus the function $I_n : (r, R) \to \mathbb{C}$ is differentiable (in real sense) and its derivative vanishes. This completes the proof. □

REMARK. We sketch a way how to relax the assumption of the (local) boundedness of the derivative in the Proposition. For $\varepsilon > 0$, write $A_\varepsilon := \{z \in \mathbb{C} : \; r + \varepsilon < |z| < R - \varepsilon\}$ and for any $z \in \mathbb{C}$ let $Q(z, \varepsilon)$ denote the square $Q(z, \varepsilon) := \{\zeta \in \mathbb{C} : \; |\mathrm{Re}(\zeta - z)|\,, \; |\mathrm{Im}(\zeta - z)| < \varepsilon/2\}$. Let $0 < \delta < (R - r)/4$ be an arbitrarily fixed number. Given a holomorphic function $f : A \to \mathbb{C}$, consider the averages of the function f:

$$f_\varepsilon(z) := \frac{1}{\varepsilon^2}\int_{Q(z,\varepsilon)} f(x+iy)\, dx\, dy \qquad\qquad \big(z \in A_\delta\,,\; 0<\varepsilon<\delta\big)$$

where x, y denote the real and imaginary coordinates, respectively. All the functions f_ε have bounded continuous derivatives in real sense. The key observation is that

$$\frac{\partial}{\partial \bar z} f_\varepsilon(z) = \Big(\frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y}\Big) f_\varepsilon(z) = \frac{1}{2i\varepsilon^2}\oint_{\partial Q(z,\varepsilon)} f(\zeta)\, d\zeta = 0 \quad (z \in A_\delta,\; 0<\varepsilon<\delta)$$

since, by Goursat's theorem [1], the contour integral of a holomorphic function along the contour of a square vanishes. Thus all the functions f_ε are holomorphic and we may apply the Proposition to them. It follows that, by setting $r' := r + 2\delta$ and $R' := R - 2\delta$, for every fixed $z \in A_{2\delta}$ we have (1) with any f_ε instead of f. Since $\sup |f_\varepsilon(A_{2\delta})| \le \sup |f(A_\delta)| < \infty$ uniformly, by passing to the limit $\varepsilon \searrow 0$, Lebesgue's majorized convergence theorem yiels (1) for $z \in A_{2\delta}$. Taking (2) into account, we obtain a Laurent expansion for f on the annulus $A_{2\delta}$.

REFERENCES

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2. K.R. STROMBERG, 1981, *An Introduction to Classical Real Analysis*, Wadsworth, Endbrooks.