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On nonlinear projections of vector fields

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Abstract. *Given a locally Lipschitzian bounded vector field* X *in a neighborhood* U *of a closed domain* $\overline{D}$ *with Lipschitzian boundary and given a continuously differentiable projection* P *of* U *onto a* C^1*-submanifold of* U, *if* X *is complete in* $\overline{D}$ *then its projection* $P'X$ *is complete in* $P(U)\cap\overline{D}$. *We outline an application to the problem of contractive projections in Jordan theory.*

1. Introduction

In 1985 W. Kaup [6] solved the longstanding problem of what type of algebraic structure characterizes the image of a C^*-algebra by a contractive linear projection. One of the basic ingredients of his solution was the fact, established also by the author [8] in 1982, that the image of a complete holomorphic vector field in the unit ball of a (complex) Banach space is complete in the unit ball of the range by a contractive linear projection. Recently much interest is paid for the natural order-free generalizations of real C^*-algebras, the so-called real JB*-triples [4]. The problem if the range of a contractive linear projection of a real JB*-triple is a real JB*-triple is still open.

In this paper we prove a theorem concerning possibly nonlinear projections of locally Lipschitzian bounded vector fields on domains in Banach spaces. Our result may have independent interest in nonlinear real analysis even in finite dimensions besides its application to complete polynomial vector fields on the unit ball of a real JB*-triple as a first step toward the solution of the problem of contractive projections in real Jordan theory. We organize the paper to be self-contained.

2. Preliminaries, notations

Throughout the whole work D_0 denotes an (arbitrarily fixed) open subset in a Banach space E with norm $\|\ \|$ and D is an open subset of D_0 such that its boundary ∂D is Lipschitzian of codimension 1 contained in D_0. That is

$$\partial D\subset D_0\,,\quad U_a\cap\partial D=\{x\in U_a:\ \Psi_a(x)=0\},\ U_a\cap D=\{x\in U_a:\ \Psi_a(x)<0\}\quad(a\in\partial D)$$

for some family $\{\Psi_a:\ a\in\partial D\}$ of Lipschitzian functions of the form

$$\Psi_a:U_a\to\mathbb{R}\,,\quad a\in U_a\ \text{open}\subset D_0\,,\quad |\Psi_a(x)-\Psi_a(y)|\le\mu_a\|x-y\|\quad(x,y\in U_a)\,,$$
$$\Psi_a(x+\lambda e_a)=\Psi_a(x)+\lambda\ \text{ whenever }\ x,x+\lambda e_a\in U_a$$

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with suitable unit vectors $e_a \in E \quad (a \in \partial D)$. We regard any locally Lipschitzian bounded map

$$X : D_0 \to E\ , \qquad \begin{array}{l} \|X(a,t)\| \le M\ ,\ \|X(a,t) - X(a',t')\| \le M_{x,t}(\|a-a'\| + |b-b'|) \\ \text{for } a, a' \in D_0 \text{ and } t, t' \in \mathbb{R} \text{ with } \|a-a'\|, |t-t'| < \varepsilon_{x,t}. \end{array}$$

as a time-dependent vector field on D_0 with the value $X(a,t)$ at the point x and in the time t. By the classical Piccard-Lindelöf theorem on the existence and continuity of the maximal solution of ordinary differential equations [7, 1], under our hypothesis there exists a unique open subset $\Omega_X \subset D_0 \times \mathbb{R}$ along with a map $s_X : \Omega_X \to D_0$ such that for each $a \in D_0$ the section $\Omega_{X,a} := \{t \in \mathbb{R} :\ (a,t) \in \mathbb{R}\}$ is an interval containing 0 and

$$\text{(2.1)} \qquad \begin{array}{l} \frac{d}{dt} s_X(a,t) = X\big(s_X(a,t),t\big) \qquad (t \in \Omega_a)\ , \quad s_X(a,0) = a\ , \\ \left.\begin{array}{l} \lim_{t \uparrow \sup \Omega_a} s_X(a,t) \ \text{ exists and } \in \partial D_0 \ \text{ if } \sup \Omega_x < \infty\ , \\ \lim_{t \downarrow \inf \Omega_a} s_X(a,t) \ \text{ exists and } \in \partial D_0 \ \text{ if } \inf \Omega_x > -\infty \end{array}\right\} \end{array}$$

where ∂D_0 stands for the boundary of D_0. In the sequel we reserve the notations $s_X, \Omega_X, \Omega_{X,a}$ for the maximal solution of the initial value problems $d/dt\ x(a,t) = X\big(x(a,t),t\big)$, $x(a,0) = a \in D_0$. Given a subset $S \subset D_0$, we say that the vector field X is *complete in* S if $\mathbb{R} = \Omega_{X,a}$ for all $a \in S$. By the boundary behaviour (2.1) we have

$$\text{(2.2)} \qquad \begin{array}{l} X \text{ complete in } D \iff X \text{ complete in } \overline{D} \iff X \text{ complete in } \partial D \\ \qquad \iff \text{for every } a \in \partial D \text{ there exists } t_0 > 0 \text{ with } \ s_X(a,t) \in \partial D \quad (|t| < t_0). \end{array}$$

Recall that a map $f : S \to F$ where $S \subset E$ and F is another Banach space (with norm $\|\ \|_F$) is *differentiable* (in Fréchet sense) with derivative $f'(a) \in \mathcal{L}(E,F) := \{\text{linear maps } E \to F\}$ at the point $a \in S$ if S is a neighborhood of the point a in E and

$$\lim_{h \to 0} \|f(a+h) - [f(a) + f'(a)h]\|_F \, \|h\|^{-1} = 0\ .$$

As usual, we regard $\mathcal{L}(E,F)$ as a Banach space with the operator norm $\|L\|_{E,F} := \sup\{\|Lx\|_F \,/\, \|x\| :\ 0 \ne x \in E\} \ \ (L \in \mathcal{L}(E,F))$.

A set $S \subset D_0$ is a C^k*-submanifold* of D_0 if there exists a closed linear subspace $F \subset E$ such that for every point $a \in S$ one can find neighborhood V_a open $\subset D_0$ of a along with a k-times continuously differentiable one-to-one map $\Phi_a : V_a \to E$ such that its range $\mathrm{ran}\Phi_a (:= \Phi_a(V_a) = \{\Phi_a(x) : x \in V_a\})$ is an open subset in E and its inverse Φ_a^{-1} is also k-times continuously differentiable on $\mathrm{ran}\Phi_a$ and $S \cap V_a = F \cap \mathrm{ran}\Phi_a$.

By a *projection* of D_0 we mean a mapping $P : D_0 \to D_0$ such that $P(a) = a$ for $a \in \mathrm{ran}P$.

With the standard techniques of differential geometry using local charts to describe the behaviour of curves tangent to vector fields we get the following.

2.3 Lemma. *Let P be a twice continuously differentiable projection of D_0 with bounded derivative onto a closed C^2-submanifold of D_0. Assume $X : D_0 \times \mathbb{R} \to E$ is a locally Lipschitzian bounded vector field. Then the vector field*

$$Y := P'X : (a,t) \mapsto P'(a)X(a,t)$$

is also locally Lipschitzian and bounded with

$$s_Y(a,t) \in \operatorname{ran}P \qquad \text{for } a \in \operatorname{ran}P \text{ and } t \in \Omega_{Y,a} . \tag{2.4}$$

Proof. We have $\mu := \sup_{a \in D_0} \|P'(a)\|_{E,E} < \infty$ and $\xi := \sup_{(a,t)\in D_0 \times \mathbb{R}} \|X(a,t)\| < \infty$. Then $\|Y(a,t)\| = \|P'(a)X(a,t)\| \le \|P'(a)\|_{E,E} \|X(a)\| \le \mu\xi$ $(a \in D_0,\ t \in \mathbb{R})$ showing the boundedness of the vector field Y.

Given any $(a,t) \in D_0 \times \mathbb{R}$, there exist $0 < \delta, \lambda < \infty$ with $\|X(a_1,t_1) - X(a_2,t_2)\| \le \lambda(\|a_1 - a_2\| + |t_1 - t_2|)$ and $\|P'(a_1) - P'(a_2)\| < (\|P''(a)\|_{E,\mathcal{L}(E,E)} + 1)\|a_1 - a_2\|$ for $\|a_1 - a\|, \|a_2 - a\|, |t_1 - t|, |t_2 - t| < \delta$. Then we have $\|Y(a_1,t_1) - Y(a_2,t_2)\| = \|P'(a_1)(X(a_1,t_1) - X(a_2,t_2)) - (P'(a_2) - P'(a_1))X(a_2,t_2)\| \le (\mu \cdot 2\lambda + \|P''(a)\|_{E,\mathcal{L}(E,E)} + 1)(\|a_1 - a_2\| + |t_1 - t_2|)$ whenever $\|a_1 - a\|, \|a_2 - a\|, |t_1 - t|, |t_2 - t| < \delta$. This establishes the locally Lipschitzian property of Y.

Fix $a \in \operatorname{ran}P$ arbitrarily. Since $\operatorname{ran}P$ is a closed C^2-submanifold of D_0,

$$\operatorname{ran}P \cap V = \Phi^{-1}(F \cap \widetilde{V}) , \quad \Phi(a) = 0$$

for some twice continuously differentiable one-to-one map $\Phi : V \to \widetilde{V}$ where $V, \widetilde{V}$ are open sets in E, $a \in V \subset D_0$, $0 \in \widetilde{V} = \operatorname{ran}\Phi$ and Φ^{-1} is twice continuously differentiable on $\widetilde{V}$. We can ~~choose V to be so small that~~ Φ, Φ', Φ'' and Y on V, respectively $\Phi^{-1}, [\Phi^{-1}]', [\Phi^{-1}]''$ on $\widetilde{V}$ are bounded and Lipschitzian. Choose t_0 such that we have $S_Y(a,t) \in V$ for $|t| < t_0$. To prove (2.4), it suffices to see that

$$\widetilde{y}(t) \in F \quad (|t| < t_0) \qquad \text{where} \quad \widetilde{y}(t) := \Phi(y(t)) \text{ and } y(t) := s_Y(a,t) \quad (|t| < t_0) .$$

Since $d/dt\ y(t) = [P'(y(t))]X(y(t),t)$ $(|t| < t_0)$ and $y(0) = a$,

$$\begin{aligned} &\frac{d}{dt}\widetilde{y}(t) = [\widetilde{P}'(\widetilde{y}(t))]\widetilde{X}(\widetilde{y}(t),t) \quad (|t| < t_0) , \quad \widetilde{y}(0) = 0 \\ &\text{where } \widetilde{P} := \Phi \circ P \circ \Phi^{-1} , \quad \widetilde{X}(\widetilde{p}) := [\Phi'(\Phi^{-1}(\widetilde{p}))]X(\Phi^{-1}(\widetilde{p})) \quad (\widetilde{p} \in \widetilde{V}) . \end{aligned} \tag{2.5}$$

We can use similar estimates leading to the boundedness and local Lipschitzianity of Y to prove that $\widetilde{Y}$ is bounded and Lipschitzian. Thus, by the Piccard-Lindelöf theorem, $\widetilde{y}$ is the *unique* solution of the initial value problem (2.5). On the other hand $\operatorname{ran}\widetilde{P} = \Phi(\operatorname{ran}P \cap V) \subset F$. Thus, since F is a closed linear subspace of E,

$$\operatorname{ran}\widetilde{Y} = \operatorname{ran}(\widetilde{P}'\widetilde{Y}) \subset \operatorname{ran}\widetilde{P}' \subset F .$$

However, then we can solve the initial value problem (2.5) within the subspace F. By the uniqueness of the solution of (2.5) it follows $\widetilde{y}(t) \in F$ $(|t| < t_0)$. Since also $\widetilde{y}(t) \in \widetilde{V}$ $(|t| < t_0)$, hence $y(t) = \Phi^{-1}(\widetilde{y}(t)) \in \Phi^{-1}(F \cap \widetilde{V}) = \mathrm{ran}P \cap V$. That is (2.4) holds. □

2.6 Remark. In finite dimensions the conditions of the Lemma can be weakened. Indeed *if* $\dim E < \infty$ *in the above Lemma it suffices to assume* P *to be continuously differentiable of and* $\mathrm{ran}P$ *to be a* C^1*-smooth submanifold of* D_0. In this case we can only assure that the vector field $\widetilde{P}'\widetilde{X}$ is well-defined and ranges continuously in F. Then instead of using The Piccard-Lindelöf theorem to establish the existence and uniqueness of the solution of (2.5), we can argue as follows. By Peano's theorem [2] on ordinary differential equations with finite dimensional continuous vector fields, (2.5) has a solution with values in F. For any solution $\widetilde{y}$ of (2.5) the function $z := \Phi^{-1} \circ \widetilde{y}$ is a solution of the initial problem $z(0) = a$, $d/dt\, z(t) = P'(z(t))X(z(t),t)$ $(|t| < t_0)$. However, this latter is unique and coincides necessarily with y whence (2.4) is immediate.

It is well-known [2] that Peano's theorem does not hold in general in infinite dimensional Banach space setting.

Question. What kind of weaker additional hypothesis are necessary for a continuously differentiable projection P and a locally Lipschitzian bounded vector field Z with the property $P'(a)Z(a,t) = Z(a,t)$ $(a \in \mathrm{ran}P,\ t \in \mathbb{R})$ to assure $s_Z(a,t) \in \mathrm{ran}P$ $(a \in \mathrm{ran}P,\ t \in \Omega_{Z,a})$?

3. Main result

3.1 Theorem. *Let* E *be a Banach space,* D, D_0 open $\subset E$ *such that* $\overline{D} \subset D_0$ *and* ∂D *is a Lipschitzian submanifold of codimension* 1 *in* D_0. *Assume* $X : D_0 \times \mathbb{R} \to E$ *is a locally Lipschitzian bounded vector field which is complete in* D *and let* $P : D_0 \to D_0$ *be a twice continuously differentiable projection such that* $\mathrm{ran}P$ *is a* C^2*-submanifold of* D_0. *Then the projected vector field* $Y(a,t) := P'(a)X(a,t)$ $(a \in D_0,\ t \in \mathbb{R})$ *is also complete in* $D \cap \mathrm{ran}P$.

Proof. According to Lemma 2.3 and (2.2), it suffices to establish that for any boundary point $a \in \partial D$ there exists $t_0 > 0$ such that the solution $y : (-t_0, t_0) \to D_0$ of the initial value problem

$$(3.2) \qquad \frac{d}{dt}y(t) = Y(y(t),t) \quad (|t| < t_0), \quad y(0) = a$$

ranges in ∂D.

Fix $a \in \partial D$ arbitrarily. Since ∂D is a Lipschitzian submanifold of codimension 1 in D_0, we can choose a bounded open convex subset U of D_0 along with a Lipschitzian function $\Psi : U \to \mathbb{R}$ and a unit vector $e \in E$ such that

$$\Psi(x + \lambda e) = \Psi(x) + \lambda \quad (x, x + \lambda e \in U), \quad a \in U \cap \partial D = \{x \in U : \Psi(x) = 0\}.$$

Observe that the mapping

$$R(x) := x - \Psi(x)e \qquad (x \in U)$$

is a Lipschitzian projection of U onto $U \cap \partial D$. Therefore the vector field $Z(x,t) := Y(R(x),t)$ $(x \in U,\ t \in \mathbb{R})$ is bounded and locally Lipschitzian on U. Thus, for a sufficiently small value $t_0 > 0$ which we fix henceforth, the initial value problem

$$\frac{d}{dt}u(t) = Y(R(u(t)),t)\ , \quad u(0) = a$$

has a unique solution $u : (-t_0, t_0) \to U$. We are going to show that

$$\Psi(u(t)) = 0 \qquad (|t| < t_0)\ . \tag{3.3}$$

Remark that from (3.3) it follows $R(u(t) = u(t)$ and hence $d/dt\ u(t) = Y(u(t),t)$ $(|t| < t_0)$. By the uniqueness of the solution of the initial value problem (3.2), we have necessarily $y(t) = u(t) = R(u(t)) \in \mathrm{ran}R \subset \partial D$ which completes the proof of the theorem.

We prove (3.3) as follows. Since u has the continuous (moreover locally Lipschitzian) derivative $t \mapsto Y(R(u(t))$, the function $t \mapsto \Psi(u(t))$ is locally Lipschitzian. Recall that (locally) Lipschitzian functions of one real variable are absolutely continuous [7] and hence differentiable Lebesgue-almost everywhere satisfying Newton-Leibniz rule. Thus for (3.3) it suffices to see that

$$\limsup_{\delta \downarrow 0} \frac{1}{\delta}\left[\Psi\big(u(t+\varepsilon\delta)\big) - \Psi\big(u(t)\big)\right] \le 0 \qquad (|t| < t_0,\ \varepsilon = \pm 1)\ . \tag{3.4}$$

Fix $t \in (-t_0, t_0)$ and $\varepsilon \in \{\pm 1\}$ arbitrarily and write

$$a := R(u(t))\ , \quad v := Y(a,t)\ , \quad x(\delta) := s_X(a,\delta) \quad (\delta \in \Omega_{X,a})\ .$$

Observe that for $\delta \to 0$ we have

$$\begin{aligned}
&\frac{1}{|\delta|}\left|\Psi\big(u(t+\delta)\big) - \Psi\big(u(t)+\delta v\big)\right| \le \frac{M}{|\delta|}\left\|u(t+\delta) - [u(t)+\delta v]\right\| \to 0\ ,\\
&\Psi\big(u(t)+\delta v\big) - \Psi\big(u(t)\big) = \Psi\big([u(t)+\delta v] - \Psi(u(t))e\big) = \Psi\big(R(u(t))+\delta v\big) =\\
&\qquad\qquad = \Psi\big(a+\delta v\big) = \Psi\big(a+\delta v\big) - \Psi\big(a\big),\\
&\frac{1}{\delta}\left[P(x(\delta)) - a\right] = \frac{1}{\delta}\left[P(x(\delta)) - P(x(0))\right] \to\\
&\qquad \to P'\big(x(0)\big) = \frac{d}{d\tau}\Big|_{\tau=0} x(\tau) = P'(a)X(a,t) = Y(a,t) = v\ ,\\
&\frac{1}{|\delta|}\left|\Psi\big(P(x(\delta))\big) - \Psi\big(a+\delta v\big)\right| \le \frac{M}{|\delta|}\left\|P\big(x(\delta)\big) - [a+\delta v]\right\| \to 0
\end{aligned}$$

where M is the Lipschitz constant of Ψ. ¿From these estimates it readily follows

$$\limsup_{\delta \downarrow 0} \frac{1}{\delta}\left[\Psi\big(u(t+\varepsilon\delta)\big) - \Psi\big(u(t)\big)\right] = \limsup_{\delta \downarrow 0} \frac{1}{\delta}\Psi\big(P\big(x(\varepsilon\delta)\big)\big)\ . \tag{3.5}$$

By assumption, the vector field X is complete in D. Since $x(0) = a \in \partial D$, we have $x(\delta) \in \partial D \quad (\delta \in \Omega_{X,a})$. By assumption, the projection P maps the domain D into itself. Hence also $P(\partial D) \subset \overline{D}$. Therefore

$$P\big(x(\delta)\big) \in \overline{D} \quad \text{that is} \quad \Psi\big(P(x(\delta))\big) \leq 0 \qquad (\delta \in \Omega_{X,a}) .$$

Then (3.4) is immediate by the observation (3.5). □

4. Application in real Jordan theory

Let E be a complex Banach space and let $B(E) := \{x \in E : \ \|x\| < 1\}$ be its unit ball. By a remarkable result of W. Kaup [5, 3Ch. 10] the following statements are equivalent:

(i) $B(E)$ is (holomorphically) symmetric,

(ii) for each vector $e \in E$ there exists a vector field $X_e : E \to E$ of the form $X_e(x) = e + Q_e(x,x) \quad (x \in E)$ for some (necessarily unique) complex-bilinear continuous mapping $Q_e : E \times E \to E$,

(iii) there exists a (unique) operation of 3-variables on E denoted usually by $\{xy^*z\}$ which is complex-linear in the variables x, z, conjugate-linear in y and satisfies
$\|\{xx^*x\}\| = \|x\|^3 \quad (x \in E)$,
$\|\exp(\zeta L_e\|_{E,E} \leq 1 \quad (\mathrm{Re}\zeta \leq 0, \quad e \in E)$ for the linear operators $L_e(x) := \{ee^*x\}$,
$\{xy^*z\} = \{zy^*x\} \quad (x, y, z \in E)$,
$\{ab^*\{xx^*x\}\} = \{\{ab^*x\}y^*z\} - \{x\{ba^*y\}^*z\} + \{xy^*\{ab^*z\}\} \quad (a, b, x, y, z \in E)$.

The Banach algebraic structures involving an operation of 3 variables with the properties described in (iii) are called *Jordan-Banach triple *-algebras* abbreviated usually as *JB^*-triples.* Any complex C^*-algebra A can be regarded as a JB*-triple with the triple product

$$\{xy^*z\} := \frac{1}{2}xy^*z + \frac{1}{2}zy^*x \qquad (x, y, z \in A) .$$

¿From the equivalence (ii) $\Leftrightarrow$ (iii) it follows that PE is a JB*-triple whenever P is a linear projection with the triple product $\{xy^*z\}$. Indeed, given any vector $e \in E$, the vector field $Y_e := PX_e$ is complete in $B(PE)(= P(B(E)))$ with $Y_e'(0) = Pe$. This is an immediate consequence of Theorem 3.1 by the following lemma which applies in *real* not only in complex Banach spaces.

4.1 Lemma. *Let E be a real Banach space with norm $\| \ \|$, let $P \in \mathcal{L}(E,E)$ be a projection with $\|P\|_{E,E} = 1$ and let $V : E \to E$ be a polynomial vector field* which is complete in $B(E)$. Then the vector field $Y(x) := PX(x) \quad (x \in E)$ is complete in $B(PE)$.*

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* *That is* $X(x) = C_0 + \sum_{k=1}^{n} C_k(\underbrace{x, \ldots, x}_{k}) \quad (x \in E)$ *for some* $n < \infty$, $C_0 \in E$ *and continuous* k*-linear mappings* $C_k : E^k \to E \quad (k = 1, \ldots, n)$.

Proof. Being linear, for the projection P we have $P'(x) = P \quad (x \in E)$. Thus P is trivially a continuously differentiable mapping whose range is a closed linear and hence in particular C^1-submanifold in E. Since $\|P\|_{E,E} = 1$, for the unit ball we have $B(PE) = P(B(E)) \subset B(E)$. Notice that the boundary $\partial B(E) = \{x \in E : \|x\| = 1\}$ is a Lipschitzian submanifold of E of codimension 1. We prove this latter fact as follows. Let any unit vector $a \in \partial B(E)$ be given. By the Hahn-Banach theorem we can chose a linear functional $\phi \in \mathcal{L}(E, \mathbb{R})$ such that $\phi(a) = \|\phi\|_{E,E} = \|a\| = 1$. Define

$$Q(z) := z - \phi(z)a \quad (z \in E) , \quad F := Q(E) = \{x \in E : \phi(x) = 0\} ,$$
$$K := \{x \in F : \|x\| < 1/3\} , \quad U := \{x + \lambda a : x \in K, \lambda > 0\} .$$

Remark that U is an open neighborhood of a in E. Observe that for some Lipschitzian (and concave) function $\psi : K \to \mathbb{R}$ we have

$$\begin{aligned} U \cap \partial B(E) &= \{x + \psi(x)a : x \in K\} = \\ &= \{z \in U : \Psi(z) = 0\} \quad \text{where } \Psi(z) := \phi(z) - \psi(Q(z)) \end{aligned} \tag{4.2}$$

and $\Psi(z+\lambda a) = \phi(z+\lambda a) - \psi(Q(z+\lambda a)) = \phi(z) + \lambda - \psi(Q(z)) = \Psi(z) + \lambda \quad (z, z+\lambda a \in U)$. Thus we can apply Theorem 3.1 with $D_0 := E$, $D := B(E)$, $X(x,t) := V(x)$ $(x \in E,\ t \in \mathbb{R})$ to conclude that the vector field Y is complete in $B(PE)$.

For the sake of self-containedness we include a detailed proof of (4.2). Given any point $z \in U$ we have $\|Q(z)\| < 1/3$ and hence the for the ball of $2K + Q(z)$ codimension 1 and radius $2/3$ we have $K \subset 2K + Q(z) \subset B(E)$. Therefore, since $B(E)$ is convex and open, for the cones

$$C_z^{(-)} := \mathrm{co}([2K + Q(z)] \cup \{z\}) = \{x + \lambda a : 0 < \lambda < \psi_z^{(-)}(x),\ x \in 2K + Q(z)\}$$

where $\psi_z^{(-)}(x) := \phi(z) - (3\phi(z)/2)\,\|x - Q(z)\|$ and

$$C_z^{(+)} := z + \mathbb{R}(C_z^{(-)} - z) = \{x + \lambda a : \lambda > \psi_z^{(+)}(x),\ x \in F\}$$

where $\psi_z^{(-)}(x) := \phi(z) - (3\phi(z)/2)\,\|x - Q(z)\|$ we have

$$C_z^{(-)} \subset B(E) , \quad C_z^{(+)} \subset E \setminus \overline{B(E)} \qquad \text{for any } z \in U \cap \partial B(E) . \tag{4.3}$$

Since $a \in U \cap \partial B(E)$, it follows in particular

$$\begin{aligned} &\emptyset \neq \{x + \lambda a : 0 < \lambda < \psi_a^{(-)}(x)\} \subset B(E) , \\ &\emptyset \neq \{x + \lambda a : \lambda > \psi_a^{(+)}(x)\} \subset E \setminus \overline{B(E)} \end{aligned} \qquad (x \in K) .$$

Hence for every $x \in K$ the segment $I(x) := \{x + \lambda a : \psi_a^{(-)}(x) \le \lambda \le \psi_a^{(+)}(x)\}$ contains some point from $\partial B(E)$. On the other hand, by (4.3), for any $x \in K$ the intersection $I(x) \cap \partial B(E)$ consists of a unique point and

$$\begin{aligned} I(x) \cap \partial B(E) = I(x) \setminus \bigcup_{z \in U \cap \partial B(E)} (C_z^{(-)} \cup C_z^{(+)}) = \\ = \{x + [\sup_{z \in U \cap \partial B(E)} \psi_z^{(-)}(x)]a\} = \{x + [\inf_{z \in U \cap \partial B(E)} \psi_z^{(+)}(x)]a\} . \end{aligned}$$

Thus (4.2) holds with the function

$$\psi(x) := \inf_{z\in U\cap\partial B(E)} \psi_z^{(+)}(x) = \sup_{z\in U\cap\partial B(E)} \psi_z^{(-)}(x)]a\} .$$

Given any $x, y \in K$ with $\psi(y) > \psi(x)$, by setting $z := x + \psi(x)a$ we have

$$0 \le \psi(y) - \psi(x) = \psi(y) - \psi_z^{(+)}(x) \le \psi_z^{(+)}(y) - \psi_z^{(+)}(x) = \phi(z)\,\|y-x\| \le \|y-x\|$$

which shows the Lipschitzian property of ψ. □

4.4 Remark. Though Jordan theory had its origins, inspired by early quantum mechanics, in the study of the algebraic structure of symmetric operators on a real Hilbert space, as far most investigations involving topology were carried out in the setting of complex Banach spaces and manifolds. The systematic study of real JB * -triples began perhaps just in 1995 with the paper [4].

By a *real JB* -triple* we mean a real Banach space E equipped with a continuous operation $\{xy^*x\}$ of three variables such that the norm $\|\ \|$ of E and the triple product $\{xy^*x\}$ admits an extension to the complexification $E\oplus(iE)$ with the properties described in (iii). Lemma 4.1 yields immediately the following.

4.5 Corollary. *If E is a real JB* -triple with norm $\|\ \|$ and triple product $\{xy^*z\}$, respectively, then for any contractive linear projection $P: E \to E$ the (time-independent) vector fields*

$$PX_e : x \mapsto Pe - P\{xe^*x\} \qquad (e \in PE)$$

are complete in $B(PE)$ the unit ball of the range of P. □

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