AN UPPER ESTIMATION FOR THE EIGENFREQUENCIES OF VIBRATING LIAPUNOFF BODIES (FIRST BOUNDARY VALUE PROBLEM)

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1. For any bounded (open) domain $\Omega \subset \mathbb{R}^n$ and for $j = 1, 2, \ldots$ we define the $j$-th eigenfrequency $\Lambda_j(\Omega)$ of the homogeneous $\Omega$ shaped and at its boundary $\partial \Omega$ fixed vibrating body by

$$
\Lambda_j(\Omega) = \inf_{M_j \in L_j} \sup_{f \in L_j} \left( \int_{\Omega} \| \nabla f(x) \|^2 dx \right)^{1/2},
$$

where $M_j$ denotes the collection of the $j$-dimensional subspaces of the Sobolev space $W_0^{1,2}$. *

As it is well-known (cf. [1]), if $\partial \Omega$ is an $(n-1)$-dimensional $C^2$-submanifold of $\mathbb{R}^n$, then the eigenvalues of the boundary value problem

$$
\Delta f + \lambda^2 f = 0, \quad f \in C_0^\infty(\overline{\Omega})
$$

are given by (1). On the other hand, it is also shown (e.g. [1], [3]) that all the mappings $\Omega \mapsto \Lambda_j(\Omega)$ are continuous with respect to the topology on the set of the bounded $\mathbb{R}^n$-domains defined by the usual Hausdorff distance.

While for all dimensions it is clarified (cf. [2]) that

$$
\Lambda_j(\Omega) \geq \Lambda_j \left( \left\{ x \in \mathbb{R}^n : \| x \| < \left( \frac{\text{vol}_{n-1}(\Omega)}{\omega_n} \right)^{1/n} \right\} \right) \quad (j = 1, 2, \ldots)
$$

where $\text{vol}_n$ denotes the $n$-dimensional Hausdorff measure and $\omega_n = \text{vol}_n(\{ x \in \mathbb{R}^n : \| x \| < 1 \})$, it is not at all known over two dimensions what kind of effective upper estimates can be given for the value of $\Lambda_j(\Omega)$ depending on some geometric parameters of $\Omega$. However, for convex $\Omega$-s, it was proved (cf. [3]) that the analogues of the best known two dimensional estimates (see [4]) hold in general (and can not be improved). The purpose of this paper is to extend a theorem of G. PÓLYA [5] concerning convex $\Omega$-s to a larger class of geometrical figures (for generalized Liapunoff bodies, defined in the next sections).

* i.e. $f \in W_0^{1,2}$ if $\nabla f$ exists in the weak sense and belongs to $L^2(\Omega)$ and supp $f$ is contained in some compact set which does not meet $\partial \Omega$. 

2. A bounded domain $\Omega(\subseteq \mathbb{R}^n)$ whose boundary is an $(n-1)$-dimensional $C^2$-submanifold in $\mathbb{R}^n$ is called a Liapunoff body if the Minkowskian curvature of $\partial \Omega$ with respect to the outward from $\Omega$ oriented normal vectors is non-negative at any point of $\partial \Omega$. (Remark here that the convexity of $\Omega$ is equivalent to the non-negativeness of the main curvatures of $\partial \Omega$ separately.)

According to some recent results in geometric measure theory, it is possible to give a generalization of the concept of Minkowskian curvature, which applies to the boundary of any open subset $\Omega(\subseteq \mathbb{R}^n)$. This can be carried out as follows:

It is shown in [6, Theorem 5] that by setting

(2) $K \equiv \{(x, k) : x \in \partial \Omega, \|k\| = 1, \exists \rho > 0 x + \rho k \in \Omega \text{ and } \text{dist} (x + \rho k, \partial \Omega) = \rho\}$

(3) $h(x, k) \equiv \sup \{\xi > 0 : \text{dist} (x + \rho k, \partial \Omega) = \rho, \forall \rho \in [0, \xi]\}$ for $(x, k) \in K$,

one always can find a $\sigma$-finite Borel measure $\mu$ on $K$ and Borel measurable functions $a_j : K \to \mathbb{R}$ $(j = 0, \ldots, n-1)$ such that for all $f \in L^1(\Omega)$ we have

(4) $\int_{\Omega} f(y) \, dy = \int_{K} \int_{\delta} f(x + \rho k) \sum_{j=0}^{n-1} a_j(x, k) \rho^j d\rho \, d\mu(x, k)$.

Here $d\mu$ and $a_0, \ldots, a_{n-1}$ are necessarily determined only up to the signed measures

(5) $d\alpha_j = a_j \, d\mu$ $(j = 0, \ldots, n-1)$

in the sense that if (4) is satisfied when $d\mu$ and $a_0, \ldots, a_{n-1}$ are replaced by $d\tilde{\mu}$ and $\tilde{a}_0, \ldots, \tilde{a}_{n-1}$, respectively, then we have

$$\int_{E} a_j d\mu = \int_{E} \tilde{a}_j d\tilde{\mu} \quad (j = 0, \ldots, n-1)$$

for all such $E \subseteq K$ that $\int_{E} a_j d\mu$ or $\int_{E} \tilde{a}_j d\tilde{\mu}$ makes sense. Thus, for $d\mu$ (and hence also $d\tilde{\mu}$)-almost every $(x, k) \in K$, the polynomials $\rho \mapsto \sum_{j=0}^{n-1} a_j(x, k) \rho^j$ and $\rho \mapsto \sum_{j=0}^{n-1} \tilde{a}_j(x, k) \rho^j$ differ only in a positive constant factor.

We shall call the measure $d\alpha_j$ defined by (5), which depends only on the geometric parameters of $\Omega$, the $j$-th curvature measure of the boundary of $\Omega$. This terminology is motivated by the relation (6) below. The formula (4) can be considered as a generalization of the main theorem in [11].

In the classical case, when $\partial \Omega$ is $C^2$-smooth, we have $(x, k) \in K$ if and only if $k$ is the outward $\Omega$ oriented normal vector (of unit length) of the surface $\partial \Omega$ at the point $x \in \partial \Omega$. Now there is a natural choice for $d\mu$ and $a_0, \ldots, a_{n-1}$: We can define $d\mu$ by

$$\mu(E) = \text{vol}_{n-1} \{x \in \partial \Omega : \exists k (x, k) \in E\}$$
(for the Borel measurable subsets $E$ of $K$; $\text{vol}_{n-1}$ denoting the $(n-1)$-dimensional Hausdorff measure). Then $a_0(x, k), \ldots, a_{n-1}(x, k)$ are the coefficients of the polynomial

$$\rho \mapsto \sum_{j=0}^{n-1} a_j(x, k) \rho^j \equiv \prod_{i=1}^{n-1} (1 - \rho k_i(x))$$

where $k_0(x), \ldots, k_{n-1}(x)$ denote the main curvatures with respect to the outer normal of $\partial \Omega$ at the point $x$. Thus, in this special case, the curvature measures $d\alpha_j$ (defined by (5) and (6)) are all absolutely continuous with respect to $d\alpha_0$ and the Minkowskian curvature $k_1 + \cdots + k_{n-1}$ of $\partial \Omega$ coincides with $-\frac{d\alpha_1}{d\alpha_0}$. Therefore, to save the most properties of the classical case, we define generalized Liapunoff bodies in the following way:

**Definition.** A bounded domain $\Omega$ in $\mathbb{R}^n$ is said to be a generalized Liapunoff body if all its curvature measures $\alpha_j (j = 0, \ldots, n-1)$ introduced above are absolutely continuous with respect to $\alpha_0$ and the function $-\frac{d\alpha_1}{d\alpha_0}$ (which we shall call now the Minkowskian curvature of $\partial \Omega$) is non-negative.

**Theorem.** If $\Omega \subseteq \mathbb{R}^n$ is a generalized Liapunoff body then the function $\rho \mapsto \text{vol}_{n-1, \partial} (\Omega_\rho)$ (where $\Omega_\rho$ denotes the inner parallel domain of radius $\rho > 0$ of $\Omega$, i.e. $\Omega_\rho = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \rho \}$) is non-increasing for $0 < \rho < \infty$.

**Proof.** Let $\Omega$ denote a generalized Liapunoff body and define $K$ and $h$ as in (2) and (3). Choose $d\mu, \alpha_0, \ldots, \alpha_{n-1}$ so that (4) be satisfied. It is proved in [6, Theorem 5, Corollary] that here we necessarily have

$$\sum_{j=0}^{n-1} a_j(x, k) \rho^j > 0 \quad \text{whenever} \quad 0 < \rho < h(x, k) \quad ((x, k) \in K).$$

Remark that (7) is not a simple corollary of (4) and the positiveness of the operation $f \mapsto \int f(y)dy$ because these facts ensure only $\sum_{j=0}^{n-1} a_j(x, k) \rho^j \geq 0$ for $0 < \rho \leq h(x, k)$. It is easy to see from (2) and (3) that $\Omega_\rho = \{ x + \xi : (x, k) \in K \}$ and $0 < \xi \leq h(x, k)$ and hence

$$1_{\Omega_\rho} (x + \xi) = 1_{B_\rho, \alpha_0 (x, k)} (\xi) \quad \text{for} \quad (x, k) \in K \quad \text{and} \quad \xi \in [0, h(x, k)].$$

From (4) and (8) we obtain

$$\text{vol}_n \Omega_\rho \int_{K_\rho} \int_{\mathbb{R}^n} \varphi_{x, k} (\xi) \ d\xi \ d\mu (x, k)$$

where $\varphi_{x, k} (\xi) = \sum_{j=0}^{n-1} a_j(x, k) \xi^j \cdot 1_{[0, \alpha_0 (x, k)]} (\xi)$. 

Recall that for \( \mu \)-almost every \((x, k) \in K\), the polynomial \( P_{x, k} : \xi \mapsto \sum_{j=0}^{n-1} a_j (x, k)\xi^j \) has only real roots (cf. \[6, \text{Theorem 5}\]) and that from the definition of Liapunoff bodies and (7) we have \( P_{x, k}(0) > 0 \) and \( P'_{x, k}(0) = a_1 (x, k) \frac{d a_1}{d a_0} (x, k) \leq 0 \)
(for \( \mu \)-almost every \((x, k) \in K\)).

Since, in general, a polynomial \( P : R \mapsto R \) having only real roots and such that \( P(0) > 0 \) and \( P'(0) \leq 0 \) is constant or has a positive root and \( P \) decreases on \([0, \min \{\xi > 0 : P(\xi) = 0\}]\) (cf. \[10, \text{Lemma}\]) it follows from (7) and the definition of \( P_{x, k} \) that the functions \( \varphi_{x, k} \) are monotone decreasing on the whole \([0, \infty)\) for \( \mu \)-almost all \((x, k) \in K\). Therefore, from (9) we deduce that the function

\[
\phi \mapsto -\frac{1}{2} \left( \frac{d^+}{d \xi^+} + \frac{d^-}{d \xi^-} \right) \text{vol}_n \Omega_{-\xi}
\]

is well-defined for all \( \phi > 0 \) and it is decreasing.

However, it is shown in \[7\] that the \((n-1)\)-dimensional Minkowski content of \( \partial (\Omega_{-\phi}) \) equals to

\[
-\frac{1}{2} \left( \frac{d^+}{d \xi^+} + \frac{d^-}{d \xi^-} \right) \text{vol}_n \Omega_{-\xi}.
\]

Since the boundary of any bounded parallel set is easily an \((n-1)\)-rectifiable subset of \( R^n \) (for definitions see \[8\]), a well-known theorem of M. KNESER (cf. \[8\]) implies that

\[
\text{vol}_{n-1} \partial (\Omega_{-\phi}) = (n-1)\text{-Minkowski content} \left( \partial (\Omega_{-\phi}) \right) = -\frac{1}{2} \left( \frac{d^+}{d \xi^+} + \frac{d^-}{d \xi^-} \right) \text{vol}_n \Omega_{-\xi}.
\]

This completes the proof.

3. The following geometric estimation is given in \[10\] for the eigenfrequencies \( \Lambda_1(\Omega)\):

**Theorem 2.** Let \( \Omega \) be such a bounded domain in \( R^n \) that \( \sup_{\phi > 0} \text{vol}_{n-1} \partial (\Omega_{-\phi}) < \infty \). Then, by setting \( I(\Omega) = \text{vol}_n \Omega / \sup_{\phi > 0} \text{vol}_{n-1} \partial (\Omega_{-\phi}) \), we have

\[
\Lambda_1(\Omega)^2 \leq \frac{\pi}{2} \cdot I(\Omega)^{-1}
\]

The ideas of the proof of Theorem 2 are essentially based upon those of the article \[5\].

Thus Theorem 1 directly yields our chief observation

**Theorem 3.** If \( \Omega \) is a generalized Liapunoff body in \( R^n \) then

\[
\Lambda_1(\Omega)^2 \leq \frac{\pi}{2} \cdot \lim_{\phi \to 0} \frac{\text{vol}_{n-1} \partial (\Omega_{-\phi})}{\text{vol}_n \Omega}
\]
In particular, if $\partial \Omega$ is a $C^2$-smooth hypersurface, then

$$\Lambda_4(\Omega) \leq \frac{\pi}{2} \frac{\text{vol}_{n-1}\partial \Omega}{\text{vol}_n \Omega}$$

Remark. One can prove that for any generalized Liapunoff body $\Omega \subset R^n$ we have on the right hand side of (10)

$$\lim_{\rho \to 0} \text{vol}_{n-1}\partial (\Omega - \rho) = \int \text{cardinality } \{k : (x, k) \in K\} d\text{vol}_{n-1}(x).$$

Proof. By [6, Lemma 9] we can fix disjoint Borel subsets $B_1, B_2, \ldots$ of $K$ and open sets $\Omega^{(0)}, \Omega^{(2)}, \ldots \subset R^n$ with positive reach (for def. see [6] or [11]) such that by setting $\rho_m \overset{\text{def}}{=} \inf \left\{ \frac{1}{2} \text{ reach } \Omega^{(m)}, (x, k) : (x, k) \in B_m \right\}$ and $K_m \overset{\text{def}}{=} \{(x, k) : x \in \partial \Omega^{(m)}, \|k\| = 1, \exists \rho > 0 x + \rho k \in \Omega^{(m)}, \text{dist } (x, \partial \Omega^{(m)}) = \rho \}$ we have

$$K = \bigcup_{m=1}^\infty B_m, \rho_m > 0 \text{ and } B_m \subset K_m \quad (m = 1, 2, \ldots).$$

Using [6, Theorem A, B] we can see that for each point $y \in \partial (\Omega^{(m)} \setminus \rho_m)$ there exists a unique pair $(x_m(y), k_m(y))$ in $K_m$ with the property $y = x_m(y) + \rho_m k_m(y)$ and, by [8, 3.2.3], for any fixed $\xi \in R$, the mapping $T_{\xi}^m : y \mapsto y + \xi k(y)$ satisfies

$$\int_{T_{\xi}^m(y)} \text{card } \{k : (x, k) \in B_m\} d\text{vol}_{n-1}(x) = \int_S [1 + (\xi - \rho_m) k_m(y)] \cdots [1 + (\xi - \rho_m) k_{n-1}(y)] d\text{vol}_{n-1}(y),$$

where $k_1, \ldots, k_{n-1}$ are the main curvatures of $\partial (\Omega^{(m)} \setminus \rho_m)$ (cf. [6, Theorem B]). defined vol$_{n-1}$ almost everywhere on $\partial (\Omega^{(m)} \setminus \rho_m)$. Hence, in particular,

$$\int_{\{k : (x, k) \in B_m\}} \text{card } \{k : (x, k) \in B_m\} d\text{vol}_{n-1}(x) = \int_{\{x + \rho_m k : (x, k) \in B_m\}} [1 - \rho_m k_1(y)] \cdots [1 - \rho_m k_{n-1}(y)] d\text{vol}_{n-1}(y).$$

The proof of the main Theorem in [6] shows (cf. [6, (5'), (5'')]) that the measures $a_{0}d\mu$ in formula (5) are given by

$$\sum_{l=0}^{n-1} \rho_l \int_{B_m} a_{0}d\mu = \int_{\{x + \rho_m k : (x, k) \in B_m\}} [1 + (\rho - \rho_m) k_m] \cdots [1 + (\rho - \rho_m) k_{n-1}(y)] d\text{vol}_{n-1}(y),$$

for $B \subset B_m$ and $\rho \in R$ ($m = 1, 2, \ldots$). Thus (11) yields

$$\int_{\partial \Omega} \text{card } \{k : (x, k) \in K\} d\text{vol}_{n-1}(x) = \int_{K} a_{0}(x, k) d\mu(x, k).$$
On the other hand, applying the functions \( \varphi_{x, k} \) \( ((x, k) \in K) \) introduced in formula (9), we see
\[
\int_{K} a_{\alpha} d \mu = \lim_{\varphi \to 0} \int_{K} \varphi_{x, k}(\xi) d \mu (x, k) = \lim_{\varphi \to 0} \int_{K} \varphi_{x, k}(\xi) d \mu (x, k)
\]
since the functions \( \varphi_{x, k} \) are monotone decreasing for all fixed \((x, k) \in K\). Now, to complete the proof, we need only to remark that, by (9) and by [8, 3.2.34], we have
\[
\text{vol}_{n-1} \partial \Omega_{\varphi} = - \frac{d}{d \varphi} \text{vol}_{n} \Omega_{\varphi} = \frac{d}{d \varphi} \int_{K} \varphi_{x, k}(\xi) d \xi d \mu (x, k) = \int_{K} \varphi_{x, k}(\xi) d \mu (x, k)
\]
for almost every \( \varphi \in (0, \infty) \).

REFERENCES


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