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AN UPPER ESTIMATION FOR THE EIGENFREQUENCES OF VIBRATING LIAPUNOFF BODIES (FIRST BOUNDARY VALUE PROBLEM)

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1. For any bounded (open) domain $\Omega\subset R^n$ and for $j=1, 2, \ldots$ we define the j-th eigenfrequency $\Lambda_j(\Omega)$ of the homogeneous Ω shaped and at its boundary $\partial\Omega$ fixed vibrating body by

$$\Lambda_j(\Omega)=\inf_{L\in M_j}\sup_{f\in L}\Big(\int_\Omega \|\operatorname{grad} f(x)\|^2 dx\Big/\int_\Omega |f(x)|^2 dx\Big)^{1/2}, \tag{1}$$

where M_j denotes the collection of the j-dimensional subspaces of the Soboleff space $W_0^{1,2}$.*

As it is well-known (*cf.* [1]), if $\partial\Omega$ is an $(n-1)$-dimensional C^2-submanifold of R^n, then the eigenvalues of the boundary value problem

$$\Delta f+\Lambda^2\cdot f=0,\ f\in C_0^\infty(\overline{\Omega})$$

are given by (1). On the other hand, it is also shown (e.g. [1], [3]) that all the mappings $\Omega\mapsto\Lambda_j(\Omega)$ are continuous with respect to the topology on the set of the bounded R^n-domains defined by the usual Hausdorff dictance.

While for all dimensions it is clairified (*cf.* [2]) that

$$\Lambda_j(\Omega)\geq\Lambda_j\left(\left\{x\in R^n: \|x\|<\left(\frac{\operatorname{vol}_n\Omega}{\omega_n}\right)^{1/n}\right\}\right)\qquad (j=1,\ 2,\ \ldots)$$

where vol_n denotes the n-dimensional Hausdorff measure and $\omega_n\equiv\operatorname{vol}_n\{x\in R^n: \|x\|<1\}$, it is not at all known over two dimensions what kind of effective upper estimates can be given for the value of $\Lambda_j(\Omega)$ depending on some geometric parameters of Ω. However, for convex Ω-s, it was proved (cf. [3]) that the analogues of the best known two dimensional estimates (see [4]) hold in general (and can not be improved). The purpose of this paper is to extend a theorem of G. PÓLYA [5] concerning convex Ω-s to a larger class of geometrical figures (for generalized Liapunoff bodies, defined in the next sections).

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* i.e. $f\in W_0^{1,2}$ if $\operatorname{grad} f$ exists in the weak sense and belongs to $L^2(\Omega)$ and $\operatorname{supp} f$ is contained in some compact set which does not meet $\partial\Omega$.

2. A bounded domain $\Omega\,(\subset R^n)$ whose boundary is an $(n-1)$-dimensional C^2-submanifold in R^n is called a *Liapunoff body* if the Minkowskian curvature of $\partial\Omega$ with respect to the outward from Ω oriented normal vectors is non-negative at any point of $\partial\Omega$. (Remark here that the convexity of Ω is equivalent to the non-negativeness of the main curvatures of $\partial\Omega$ separately.)

According to some recent results in geometric measure theory, it is possible to give a generalization of the concept of Minkowskian curvature which applies to the boundary of any open subset $\Omega\subset R^n$. This can be carried out as follows:

It is shown in [6, Theorem 5] that by setting

$$K\equiv\{(x,k): x\in\partial\Omega,\ \|k\|=1,\ \exists\,\rho>0\ x+\rho k\in\Omega \text{ and } \operatorname{dist}(x+\rho k,\partial\Omega)=\rho\} \tag{2}$$

$$h(x,k)\equiv\sup\{\xi>0:\operatorname{dist}(x+\rho k,\partial\Omega)=\rho,\ \forall\,\rho\in[0,\xi]\}\qquad\text{for } (x,k)\in K, \tag{3}$$

one always can find a σ-finite Borel measure μ on K and Borel measurable functions $a_j: K\to R$ $(j=0,\dots,n-1)$ such that for all $f\in L^1(\Omega)$ we have

$$\int_\Omega f(y)\,dy=\int_K\int_0^{h(x,k)} f(x+\rho k)\sum_{j=0}^{n-1}a_j(x,k)\,\rho^j\,d\rho\,d\mu(x,k). \tag{4}$$

Here $d\mu$ and $a_0,\dots,a_{n-1}$ are necessarily determined only up to the signed measures

$$d\alpha_j\equiv a_j\,d\mu\qquad(j=0,\dots,n-1) \tag{5}$$

in the sense that if (4) is satisfied when $d\mu$ and $a_0,\dots a_{n-1}$ are replaced dy $d\tilde\mu$ and $\tilde a_0,\dots,\tilde a_{n-1}$, respectively, then we have

$$\int_E a_j\,d\mu=\int_E \tilde a_j\,d\tilde\mu\qquad(j=0,\dots,n-1)$$

for all such $E\subset K$ that $\int_E a_j\,d\mu$ or $\int_E \tilde a_j\,d\tilde\mu$ makes sense. Thus, for $d\mu$ (and hence also $d\tilde\mu$)-almost every $(x,k)\in K$, the polynomials $\rho\mapsto\sum_{j=0}^{n-1}a_j(x,k)\rho^j$ and $\rho\mapsto\sum_{j=0}^{n-1}\tilde a_j(x,k)\rho^j$ differ only in a positive constant factor.

We shall call the measure $d\alpha_j$ defined by (5), which depends only on the geometric parametars of Ω, the *j-th curvature measure of the boundary of* Ω. This terminology is motivated by the relation (6) below. The formula (4) can be considered as a generalization of the main theorem in [11].

In the classical case, when $\partial\Omega$ is C^2-smooth, we have $(x,k)\in K$ if and only if k is the toward Ω oriented normal vector (of unit length) of the surface $\partial\Omega$ at the point $x\,(\in\partial\Omega)$. Now there is a natural choice for $d\mu$ and $a_0,\dots,a_{n-1}$: We can define $d\mu$ by

$$\mu(E)=\operatorname{vol}_{n-1}\{x\in\partial\Omega:\exists\,k\ (x,k)\in E\}$$

(for the Borel measurable subsets E of K; vol_{n-1} denoting the $(n-1)$-dimensional Hausdorff measure). Then $a_0(x, k), \ldots, a_{n-1}(x, k)$ are the coefficients of the polynomial

$$(6) \qquad \rho \mapsto \sum_{j=0}^{n-1} a_j(x, k)\rho^j \equiv \prod_{i-1}^{n-1} (1-\rho k_i(x))$$

where $k_0(x), \ldots, k_{n-1}(x)$ denote the main curvatures with respect to the outer normal of $\partial\Omega$ at the point x. Thus, in this special case, the curvature measures $d\alpha_j$ (defined by (5) and (6)) are all absolutely continuous with respect to $d\alpha_0$ and the Minkowskian curvature $k_1+\cdots+k_{n-1}$ of $\partial\Omega$ coincides with $-\frac{d\alpha_1}{d\alpha_0}$. Therefore, to save the most properties of the classical case, we define generalized Liapunoff bodies in the following way:

Definition. A bounded domain Ω in R^n is said to be a *generalized Liapunoff body* if all its curvature measures $\alpha_j (j=0, \ldots, n-1)$ introduced above are absolutely continuous with respect to α_0 and the function $-\frac{d\alpha_1}{d\alpha_0}$ (which we shall call now the Minkowskian curvature of $\partial\Omega$) is non-negative.

Theorem 1. *If $\Omega \in R^n$ is a generalized Liapunoff body then the function $\rho \mapsto \mathrm{vol}_{n-1}\partial(\Omega_{-\rho})$ (where $\Omega_{-\rho}$ denotes the inner parallel domain of radius $\rho>0$ of Ω, i.e. $\Omega_{-\rho} \equiv \{x\in\Omega : \mathrm{dist}\,(x, \partial\Omega)>\rho\}$) is non-increasing for $0<\rho<\infty$.*

Proof. Let Ω denote a generalized Liapunoff body and define K and h as in (2) and (3). Choose $d\mu, a_0, \ldots, a_{n-1}$ so that (4) be satisfied. It is proved in [6, Theorem 5, Corollary] that here we necessarily have

$$(7) \qquad \sum_{j=0}^{n-1} a_j(x, k)\rho^j>0 \quad \text{whenever} \quad 0<\rho<h(x, k) \qquad ((x, k)\in K).$$

Remark that (7) is not a simple corollary of (4) and the positiveness of the operation $f \mapsto \int_\Omega f(y)dy$ because these facts ensure only $\sum_{j=0}^{n-1} a_j(x, k)\rho^j \geq 0$ for $0<\rho\leq h(x, k)$. It is easy to see from (2) and (3) that $\Omega_{-\rho}=\{x+\xi k\colon (x, k)\in K$ and $\rho<\xi\leq h(x, k)\}$and hence

$$(8) \qquad 1_{\Omega_{-\rho}}(x+\xi k = 1_{[\rho, h(x,k)]}(\xi) \text{ for } (x, k)\in K \text{ and } \xi\in[0, h(x, k)].$$

From (4) and (8) we obtain

$$(9) \qquad \mathrm{vol}_n \Omega_{-\rho} \int_K \int_\rho^\infty \varphi_{x,k}(\xi)\; d\xi\; d\mu(x, k)$$

where $\varphi_{x,k}(\xi)=\sum_{j=0}^{n-1} a_j(x, k)\xi^j \cdot 1_{[0, k(x,k)]}(\xi)$.

Recall that for μ-almost every $(x, k)\in K$, the polynomial $P_{x,k}:\xi\mapsto\sum_{j=0}^{n-1} a_j(x, k)\xi^j$ has only real roots (cf. [6, Theorem 5]) and that from the definition of Liapunoff bodies and (7) we have $P_{x,k}(0)>0$ and $P'_{x,k}(0)=a_1(x, k)=\left.\frac{d\alpha_1}{d\alpha_0}\right|_{(x,k)}\leq 0$ (for μ-almost every $(x, k)\in K$).

Since, in general, a polynomial $P:R\mapsto R$ having only real roots and such that $P(0)>0$ and $P'(0)\leq 0$ is constant or has a positive root and P decreases on $[0, \min\{\xi>0:P(\xi)=0\}]$ (cf. [10, Lemma]) it follows from (7) and the definition of $P_{x,k}$ that the functions $\varphi_{x,k}$ are monotone decreasing on the whole $[0, \infty)$ for μ-almost all $(x, k)\in K$. Therefore, from (9) we deduce that the function

$$\rho\mapsto -\frac{1}{2}\left(\left.\frac{d^+}{d\xi}\right|_\rho+\left.\frac{d^-}{d\xi}\right|_\rho\right)\operatorname{vol}_n\Omega_{-\xi}$$

is well-defined for all $\rho>0$ and it is decreasing.

However, it is shown in [7] that the $(n-1)$-dimensional Minkowski content of $\partial(\Omega_{-\rho})$ equals to $-\frac{1}{2}\left(\left.\frac{d^+}{d\xi}\right|_\rho+\left.\frac{d^-}{d\xi}\right|_\rho\right)\operatorname{vol}_n\Omega_{-\xi}$. Since the boundary of any bounded parallel set is easily an $(n-1)$-rectifiable subset of R^n (for definitions see [8]), a well-known theorem of M. KNESER (cf. [8]) implies that $\operatorname{vol}_{n-1}\partial\Omega_{-\rho}=$ $=(n-1)$-Minkovski content $(\partial(\Omega_{-\rho}))=-\frac{1}{2}\left(\left.\frac{d^+}{d\xi}\right|_\rho+\left.\frac{d^-}{d\xi}\right|_\rho\right)\operatorname{vol}_n\Omega_{-\xi}$. This completes the proof.

3. The following geometric estimation is given in [10] for the eigenfrequences $\Lambda_j(\Omega)$:

Theorem 2. *Let Ω be such a bounded domain in R^n that* $\sup_{\rho>0}\operatorname{vol}_{n-1}\partial(\Omega_{-\rho})<$ $<\infty$. *Then, by setting* $l(\Omega)\equiv\operatorname{vol}_n\Omega/\sup_{\rho>0}\operatorname{vol}_{n-1}\partial(\Omega_{-\rho})$, *we have*

$$\Lambda_1(\Omega)^2\leq\frac{\pi}{2}\cdot l(\Omega)^{-1}$$

The ideas of the proof of Theorem 2 are essentially based upon those of the article [5].

Thus Theorem 1 directely yields our chief observation

Theorem 3. *If Ω is a generalized Liapunoff body in R^n then*

$$\Lambda_1(\Omega)^2\leq\frac{\pi}{2}\cdot\frac{\lim_{\rho\downarrow 0}\operatorname{vol}_{n-1}\partial(\Omega_{-\rho})}{\operatorname{vol}_n\Omega} \tag{10}$$

In particular, if $\partial\Omega$ is a C^2-smooth hypersurface, then

$$\Lambda_1(\Omega)^2 \leq \frac{\pi}{2}\cdot\frac{\mathrm{vol}_{n-1}\partial\Omega}{\mathrm{vol}_n\Omega}$$

Remark. One can prove that for any generalized Liapunoff body $\Omega\subset R^n$ we have on the right hand side of (10)

$$\lim_{\rho\downarrow 0}\mathrm{vol}_{n-1}\partial(\Omega_{-\rho}) = \int_{\partial\Omega}\text{cardinality }\{k\colon (x, k)\in K\}d\,\mathrm{vol}_{n-1}(x).$$

Proof. By [6, Lemma 9] we can fix disjoint Borel subsets $B_1, B_2, \ldots$ of K and open sets $\Omega^{(1)}, \Omega^{(2)}, \ldots\subset R^n$ with positive reach (for def. see [6] or [11]) such that by setting $\rho_m \overset{\text{def}}{=} \inf\left\{\frac{1}{2}\text{ reach }\Omega^{(m)},\ h(x, k)\colon (x, k)\in B_m\right\}$ and $K_m \overset{\text{def}}{=} \{(x, k)\colon x\in\partial\Omega^{(m)},\ \|k\|=1,\ \exists\rho>0\ x+\rho k\in\Omega^{(m)},\ \mathrm{dist}(x, \partial\Omega^{(m)})=\rho\}$ we have

$$K=\bigcup_{m=1}^{\infty}B_m,\ \rho_m>0 \text{ and } B_m\subset K_m \qquad (m=1, 2, \ldots).$$

Using [6, Theorem A, B] we can see that for each point $y\in\partial(\Omega^{(m)}_{-\rho_m})$ there exists a unique pair $(x_m(y), k_m(y))$ in K_m with the property $y=x_m(y)+\rho_m k_m(y)$ and, by [8, 3.2.3], for any fixed $\xi\in R$, the mapping $T^m_\xi\colon y\mapsto y+\xi k(y)$ satisfies

$$\int_{T^m_\xi(S)}\mathrm{card}\,(T^m_\xi)^{-1}(z)\,\mathrm{vol}_{n-1}z=\int_S[1+(\xi-\rho_m)k^m_1(y)]\cdots[1+(\xi-\rho_m)k_{n-1}(y)]d\mathrm{vol}_{n-1}(y),$$

where $k^m_1, \ldots, k^m_{n-1}$ are the main curvatures of $\partial(\Omega^{(m)}_{-\rho_m})$ (cf. [6, Theorem B]) defined vol_{n-1} almost everywhere on $\partial(\Omega^{(m)}_{-\rho_m})$. Hence, in particular,

$$\int_{\{x:\exists k\,(x, k)\in B_m\}}\mathrm{card}\,\{k:(x, k)\in B_m\}\,d\mathrm{vol}_{k-1}(x)= \tag{11}$$

$$=\int_{\{x+\rho_m k:(x, k)\in B_m\}}[1-\rho_m k^m_1(y)]\cdots[1-\rho_m k^m_{n-1}(y)]d\mathrm{vol}_{n-1}(y).$$

The proof of the main Theorem in [6] shows (cf. [6, (5′), (5″)]) that the measures $a_j d\mu$ in formula (5) are given by

$$\sum_{j=0}^{n-1}\rho^j\int_B a_j d\mu=\int_{\{x+\rho_m k:(x, k)\in B\}}[1+(\rho-\rho_m)k^m_1(y)]\cdots[1+(\rho-\rho_m)k^m_{n-1}(y)]d\mathrm{vol}_{n-1}(y)$$

for $B\subset B_m$ and $\rho\in R$ $(m=1, 2, \ldots)$. Thus (11) yields

$$\int_{\partial\Omega}\mathrm{card}\,\{k:(x, k)\in K\}\,d\mathrm{vol}_{n-1}(x)=\int_K a_0(x, k)d\mu(x, k). \tag{12}$$

On the other hand, applying the functions $\varphi_{x,k}$ $((x, k)\in K)$ introduced in formula (9), we see

$$\int_K a_0 d\mu = \int_K \lim_{\rho\downarrow 0} \varphi_{x,k}(\rho)\, d\mu(x, k) = \lim_{\rho\downarrow 0} \int_K \varphi_{x,k}(\rho)\, d\mu(x, k)$$

since the functions $\varphi_{x,k}$ are monotone decreasing for all fixed $(x, k)\in K$. Now, to complete the proof, we need only to remark that, by (9) and by [8, 3.2.34], we have

$$\mathrm{vol}_{n-1}\,\partial\Omega_{-\rho} = -\frac{d}{d\rho}\mathrm{vol}_n\,\Omega_{-\rho} = -\frac{d}{d\rho}\int_K \int_\rho^\infty \varphi_{x,k}(\xi)\, d\xi\, d\mu(x, k) = \int_K \varphi_{x,k}(\rho)\, d\mu(x, k)$$

for almost every $\rho\in(0, \infty)$.

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