On the structure of $C_0$-semigroups of holomorphic Carathéodory isometries in Hilbert space

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Abstract
We establish closed formulas for all strongly continuous one-parameter semigroups of holomorphic Carathéodory isometries of the unit ball of a Hilbert space in terms of spectral resolutions of skew self-adjoint dilations related to the Reich-Shoikhet non-linear infinitesimal generator.

Keywords: strongly continuous one-parameter group, holomorphic Carathéodory isometry, Hilbert space, dilation

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1. Introduction
Throughout this work $H$ denotes a fixed complex Hilbert space with scalar product $(x|y)$ being linear (i.e. C-linear) in $x$ and antilinear in $y$ and the canonical norm $\|x\| := (x|x)^{1/2}$. We also keep fixed the notations $B := \{x \in H : \|x\| < 1\}$, $a^* := [x \mapsto \langle x|a \rangle]$ for the open unit ball, and the adjoint representation of bounded linear functionals, respectively. We regard the elements $h, h^*$ ($h \in H$) as column resp. row matrices and, given a linear map $A : S \rightarrow H$ on some linear submanifold of $H$, we use the canonical $H \oplus \mathbb{C}$-split matrix identifications $x \oplus \xi \equiv \begin{bmatrix} x \\ \xi \end{bmatrix}$ resp. $\begin{bmatrix} A & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} x \oplus \xi \mapsto (Ax + b) \oplus (c^*x + d) \end{bmatrix}$ with $x \in S$, $b, c \in H$ and $\xi, d \in \mathbb{C}$. This gives rise to the familiar linear representation of fractional linear maps on $H$:

$$\mathfrak{F}_H \left( \begin{bmatrix} A & b \\ c & d \end{bmatrix} \right) := [x \mapsto (c^*x + d)^{-1}(Ax + b)].$$

Our object of chief interest will be the semigroup $\text{Iso}(d_B)$ of all holomorphic isometries of $B$ with respect to the Carathéodory metric $d_B$. Recall [3] that all its elements are fractional linear maps, namely they are compositions of Möbius transformations\(^1\) with linear isometries of $H$ (restricted to $B$). In 1987, in his pioneering work [10] Vesentini established that the correspondence

$$\mathfrak{F}^\#: [\mathcal{U}^t : t \in \mathbb{R}_+] \mapsto [\mathfrak{F}(\mathcal{U}^t)|B : t \in \mathbb{R}_+]$$

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\(^1\)Fractional linear transformations mapping $B$ injectively onto itself.

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maps the family \( C_0 S(\text{Iso}(H)) \) of all strongly continuous one-parameter semigroups of linear isometries of the indefinite norm \( ||x||^2 - ||\xi||^2 \) on \( H \oplus C \) into the family \( C_0 S(\text{Iso}(d_B)) \) of all strongly continuous one-parameter semigroups \( \{ \Psi^t : t \in \mathbb{R}_+ \} \subset \text{Iso}(d_B) \).

According to [10, Th.VII], for the corresponding non-linear objects \( \Psi^t := \mathfrak{g} (u^t) \) we have \( \{ p \in B : t \mapsto \Psi^t(p) \text{ is differentiable} \} = \{ x \in B : x \oplus 1 \in \text{dom}(A) \} \) which is dense in the ball \( B \). It is well-known [10, 4] that here we can identify the in \( H \oplus C \) densely defined linear operator \( A \) with an \( H \oplus C \)-split matrix if and only if the orbit \( t \mapsto \Psi^t(0) \) is differentiable. This happens if and only if the generator \( A \) has the form

\[
A = \begin{bmatrix}
iA + \nu & b \\
b^* & \nu
\end{bmatrix}, \quad \nu \in C, \ b \in H, \ A \in \text{Her}_s(H)
\]  

(1.1)

with \( \text{dom}(A) = \text{dom}(A) \oplus C \) where \( \text{Her}_s(H) \) stands for the family of all unbounded \( H \)-\textit{hermitian} operators (maximal symmetric, in \( H \) densely defined closed linear operators). Even the cases with non-differentiable 0-orbit can be treated by passing to a semigroup \((\Phi^t : t \in \mathbb{R}_+)\) of the form \( \Theta^{-1} \circ \Phi^t \circ \Theta \) with any Möbius transformation \( \Theta \) such that \( \Theta(0) \in \text{dom}(\Gamma) \). Since the Möbius group is transitive on \( B \), hence any strongly continuous one-parameter semigroup \( \{ \Psi^t : t \in \mathbb{R}_+ \} \subset \text{C}_0 (\text{Iso}(d_B)) \) is equivalent up to a Möbius transformation (\( \text{Möbius equivalent} \) for short in the sequel) to a semigroup \( \{ \Phi^t : t \in \mathbb{R}_+ \} \subset \text{C}_0 (\text{Iso}(d_B)) \) whose infinitesimal generator [7] has the form

\[
\Gamma(x) = \left. \frac{d}{dt} \right|_{t=0} \Phi^t = b - \langle x | b \rangle x + i A x, \quad x \in \text{dom}(R) \cap B
\]  

(1.2)

with some maximal symmetric operator \( A \) defined densely on \( H \) and some vector \( b \in H \). Also conversely, if \( i A \) is the infinitesimal generator for some strongly continuous one-parameter subsemigroup of \( \mathcal{L}(H) \) then, for any \( b \in H \), the vector field (1.2) is the infinitesimal generator of a strongly continuous one-parameter subsemigroup of \( \text{Iso}(d_B) \). It is worth to notice that Kaup [5, 6] achieved a far reaching Jordan theoretical analog of (1.2) describing the complete holomorphic vector field of the unit ball of JB*-triple and integrated them for the case \( A = 0 \) resulting in a fractional linear type formula for generalized Möbius transformations. However, strong continuity destroys such an elegant setting. In [11, 4] these considerations were extended to semigroups of fractional linear transformations arising from a strongly continuous one parameter semigroup of automorphisms of a Krein space, applied to the solutions of Ricatti type equations \( \dot{x} = \Gamma(x) \) with analogous vector fields to (1.2) in reflexive Hilbert \( C^* \)-modules, but without providing explicit algebraic formulas.

2. Results

Henceforth, for short, \( C_0 S \) [resp. \( C_0 G \)] will abbreviate the terms \textit{strongly continuous one-parameter semigroup} [-group]. We shall write \( \text{gen}[U^t : t \in \mathbb{R}_+] \) or \( \text{gen}[U^t : t \in \mathbb{R}] \)

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It seems as far no argument appeared in the literature concerning the plausible surjectivity of the map \( \mathfrak{g}^\# \). The question is rather harmless in our setting: in the case of the unit ball of a Hilbert space an argument with joint fixed points (Lemma 3.1) furnishes positive answer. However, e.g. in the case of the unit ball of \( \mathcal{L}(H) \) the surjectivity of the respective \( \mathfrak{g}^\# \) seems to be open and highly not trivial.
for the infinitesimal generator of the \( C_0S \{ U^t : t \in \mathbb{R}_+ \} \) or \( C_0G \{ \tilde{U}^t : t \in \mathbb{R} \} \), respectively. Given a closed subspace \( K \) in the Hilbert spaces \( H \) or \( \mathcal{C} \otimes H \) we shall write \( P_K \) the orthogonal projection onto \( K \) without danger of confusion.

In this paper we develop a triangularization method leading to explicit algebraic formulas for a \( C_0S \) generated by a vector field (1.2) in terms of fixed points of \( \Gamma \) and quadratures of a \( C_0S \) formed by complex linear isometries of a 1-codimensional subspace of \( H \). As a consequence we conclude that any \( C_0S \) of holomorphic Carathéodory isometries of \( B \) admits a dilation to a \( C_0G \) of surjective holomorphic Carathéodory isometries of the unit ball of some covering Hilbert space. Our fixed point approach seems to be new even in finite dimensions (with uniformly continuous one-parameter groups).

Recall [3] that any Carathéodory isometry \( \Psi \in \text{Iso}(d_B) \) admits a continuous extension \( \overline{\Psi} \) to the closed unit ball \( \overline{B} \). Given a \( C_0S \) \( \Psi = \{ \Psi^t : t \in \mathbb{R}_+ \} \subset \text{Iso}(d_B) \), the extensions \( \overline{\Psi} := \{ \overline{\Psi} : t \in \mathbb{R}_+ \} \) form also a \( C_0S \) (see [8] in more general setting). According to [10, Section 7] \( \overline{\Psi} \) admits common fixed points whose family \( \text{Fix}(\overline{\Psi}) \) consists of one or two boundary points or it is the intersection of \( B \) with some closed complex-affine submanifold containing points from \( B \) in which case \( \overline{\Psi} \) is simply Möbius equivalent to a \( C_0S \) of linear isometries of \( H \) restricted to \( B \).

Our main goal is the classification of the remaining cases with explicit formulas up to Möbius equivalence.

**Theorem 2.1.** Suppose the vector field (1.2) is the infinitesimal generator of a \( C_0S \) \( \Phi := \{ \Phi^t : t \in \mathbb{R}_+ \} \subset \text{Iso}(d_B) \) having a common boundary fixed point \( \tau \in \text{Fix}(\overline{\Phi}) \cap \partial B \). Then we have

\[
P_{\Phi}(\xi x + x_0) = \left[ 1 - (1 - \xi)e^{-2\lambda t}/\varphi_{\lambda, \mu}(t, x_0, \xi) \right] \xi
\]

(2.2)

for all points \( x_0 + \xi x \in B \) with \( x_0 \perp \xi \) where \( H_0 := H \otimes \mathbb{C} \tau, \lambda := \text{Re}(\tau|b), \mu := \text{Im}(\tau|b), b_0 := P_{H_0}b, [V_0^t : t \in \mathbb{R}_+] \) is the \( C_0S \) of linear \( H_0 \)-isometries generated by the skew-\( H_0 \)-hermitian operator \( iP_{H_0}(A - \mu)H_0 \) and

\[
\varphi_{\lambda, \mu}(t, x_0, \xi) := 1 + (1 - \xi) \left( \left( \int_0^t e^{-2\lambda s} \int_0^\tau e^{\lambda r} V_0^r ds dr \right) b_0 \right) - (1 - e^{-2\lambda t})b_0 = \left( \int_0^t e^{-e^{-2\lambda t}} V_0^t \left. (x_0) \right| b_0 \right).
\]

(2.3)

**Remark 2.4.** The following converse can be discovered from the proofs later on (see Remark 3.13). Given any couple of vectors \( \tau, b_0 \in H \) such that \( ||\tau|| = 1 \) and \( b_0 \perp \tau \) along with any \( C_0S \) \( [V_0^t : t \in \mathbb{R}_+] \) of linear isometries of \( H_0 = H \otimes (\mathbb{C} \tau) \) and two real constants \( \lambda, \mu \), the maps (2.2) form a \( C_0S \) in \( \text{Iso}(d_B) \).

**Remark 2.5.** In case of \( \lambda \neq 0 \), one can express the integrated operators in (2.2) in terms of the resolvent \( R(\lambda, iS_0) \) of the \( H_0 \)-hermitian operator \( S_0 := \lambda^{-1}\text{gen}[V_0^t] : t \in \mathbb{R}_+ \). Namely we have \( \int_0^t e^{-\lambda r} V_0^r dr = (1 - e^{-\lambda V_0^t}) R(\lambda, iS_0), \int_0^t e^{-2\lambda r} \int_0^r e^{\lambda s} V_0^s ds dr = \frac{1}{2\lambda}(1 - e^{-2\lambda t}) R(\lambda, iS_0) - (1 - e^{-\lambda V_0^t}) R(\lambda, iS_0) R(\lambda, iS_0) R(\lambda, iS_0).

**Theorem 2.6.** Let \( \Psi := \{ \Psi^t : t \in \mathbb{R}_+ \} \subset \text{Iso}(d_B) \) be a \( C_0S \) with \( \tau \in \text{Fix}(\overline{\Psi}) \subset \partial B \). Then, with the notations of Theorem 2.1 we have the alternatives
(i) \( \text{Fix}(\Psi) \) consists of two points and \( \Psi \) is Möbius equivalent to some \( C_0 S \) \( \lfloor \Phi^t : t \in \mathbb{R}_+ \rfloor \subseteq \text{Iso}(d_B) \) of the form

\[
\Phi^t(\xi x + x_0) = \frac{\xi + \tanh(\lambda t)}{1 + \xi \tanh(\lambda t)} x + \frac{e^{-\lambda t}}{\cosh(\lambda t) + \xi \sinh(\lambda t)} V_0^t x_0; \tag{2.7}
\]

(ii) \( \{ \tau \} = \text{Fix}(\Psi) \), there is a \( \Psi \)-invariant disc of the form \( \emptyset \neq \{ \tau + C v \} \cap B \) and \( \Psi \) is Möbius equivalent to a \( C_0 S \) \( \lfloor \Phi^t : t \in \mathbb{R}_+ \rfloor \) of the form

\[
\Phi^t(\xi x + x_0) = \frac{1 + i \mu t}{1 - i \mu t} \frac{\xi - i \mu t/(1 + i \mu t)}{1 + i \mu t \xi/(1 - i \mu t)} x + \frac{1}{1 - i \mu t (1 - \xi)} V_0^t x_0; \tag{2.8}
\]

(iii) there is no \( \Psi \)-invariant disc of the form \( \emptyset \neq \{ \tau + C v \} \cap B \) and \( \Psi \) is Möbius equivalent to a \( C_0 S \) \( \lfloor \Phi^t : t \in \mathbb{R}_+ \rfloor \) with \( \lambda = 0 \).

**Remark 2.9.** In the setting of Theorem 2.1 a non-empty disc \( \{ \tau + C v \} \cap B \) is \( \lfloor \Phi^t : t \in \mathbb{R}_+ \rfloor \)-invariant if and only if \( \tau \not\in \text{dom}(A) \) and \( (i A + C v) v \in \mathbb{C} \tau \) as established in Lemma 4.1. Hence, in finite dimensions only cases (i),(ii) may appear. Example 4.2 with possible independent interest for physics or stochastic processes shows that case (iii) is not void.

Recall that, as an implicit simple special case\(^3\) of Deddens [1, Main Thm.], every \( C_0 S \) \( \lfloor U^t : t \in \mathbb{R}_+ \rfloor \) of isometries of \( H \) admits a *unitary group dilation* in the following sense: there exists a Hilbert space \( \tilde{H} \) containing \( H \) as a subspace along with a \( C_0 G \) \( \lfloor \tilde{U}^t : t \in \mathbb{R} \rfloor \) of unitary operators of \( \tilde{H} \) such that \( U^t = \tilde{U}^t | H (t \in \mathbb{R}_+) \). Applying a unitary dilation \( \lfloor \tilde{V}_0^t : r \in \mathbb{R} \rfloor \) of the isometry semigroup \( \lfloor V_0^t : t \in \mathbb{R}_+ \rfloor \) in (2.2), we readily obtain the following result with non-linear dilations.

**Corollary 2.10.** Given any \( C_0 S \) \( \lfloor \Psi^t : t \in \mathbb{R}_+ \rfloor \) of holomorphic Carathéodory isometries of \( B \), there is a strongly continuous one parameter group \( \lfloor \tilde{\Psi}^t : t \in \mathbb{R}_+ \rfloor \) of surjective holomorphic Carathéodory isometries of the unit ball \( \tilde{B} \) of some Hilbert space \( \tilde{H} \) containing \( H \) as a subspace such that \( \Psi^t = \tilde{\Psi}^t | B (t \in \mathbb{R}_+) \).

By means of the functional calculus of the skew self-adjoint generator \( i \tilde{S}_0 \) of the dilation group \( \lfloor \tilde{V}_0^t : t \in \mathbb{R} \rfloor \) of the \( C_0 S \) \( \lfloor V_0^t : t \in \mathbb{R}_+ \rfloor \) in the setting of Theorem 2.1, we get the following.

**Corollary 2.11.** In (2.2) we can write

\[
\varphi_{\lambda, \mu}(t, x_0, \xi) = \left( \left. f_1(t, \lambda, \tilde{S}_0) b_0 \right| b_0 \right) + (1 - \xi) \left( \left. f_2(t, \lambda, \tilde{S}_0) b_0 \right| b_0 \right) - (\lambda + i \mu) \int_0^t e^{-2 \lambda s} d s + 1,
\]

\[
f_{1, \mu}^{\tilde{H}}(x) = \varphi_{\lambda, \mu}(t, x_0, \xi)^{-1} \left( e^{-\lambda t} \exp(it \tilde{S}_0) x_0 + (1 - \xi) e^{-2 \lambda t} f_1(t, \lambda, \tilde{S}_0) b_0 \right)
\]

with the bounded analytic functions \( f_j(t, \lambda, \cdot) : \mathbb{R} \to \mathbb{C} (j = 1, 2; \lambda, t \in \mathbb{R}) \)

\[
f_1(t, \lambda, \sigma) := \frac{1 - e^{-(\lambda + i \sigma)}}{\lambda + i \sigma} = \sum_{n=1}^{\infty} (-1)^{n-1} (\lambda + i \sigma)^{n-1} \frac{n^2}{n!}, \quad f_2(t, \lambda, \sigma) := \frac{e^{-\lambda t}}{2 \lambda (\lambda + i \sigma)} \frac{1}{\lambda^2 + \sigma^2} \sum_{n=2}^{\infty} \left( \frac{(-2 \lambda)^n}{2 \lambda (\lambda + i \sigma)^n} - \frac{(-\lambda + i \sigma)^n}{\lambda^2 + \sigma^2} \right) \frac{n!}{n!};
\]

\(^3\) We begin Section 4 with an elementary proof in Banach space setting.
3. Triangularization with boundary fixed points

**Lemma 3.1.** Assume \( \Psi = [\Psi^t : t \in \mathbb{R}_+] \subset \text{Iso}(d_B) \) is a \( C_0 S \) where \( \Psi^t = \mathfrak{I}(U_t) | B \) with \( U_t \in \mathcal{L}(H \oplus C) \) (\( t \in \mathbb{R}_+ \)). Then there is a family \( [\mu_t : t \in \mathbb{R}_+] \subset C \setminus \{0\} \) such that \( [\mu_t U_t : t \in \mathbb{R}_+] \) is a \( C_0 S \) in \( \mathcal{L}(H \oplus C) \).

**Proof.** Let \( o \) be a common fixed point of the the transformations \( \overline{\Psi}^t = \mathfrak{I}(U_t) | \overline{B} \) (\( t \in \mathbb{R}_+ \)). We are going to show that the choice \( \mu_t := [U_t(o \oplus 1)]^{-1} \) entails \( \mu_t U_t(o \oplus 1) = o \oplus 1 \) suits our requirements. Consider the matrices \( \Psi^t := \mu_t U_t \). Clearly \( \mathfrak{I}(\Psi^t) = \mathfrak{I}(U_t) \) (\( t \in \mathbb{R}_+ \)). Since the map \( U \mapsto \mathfrak{I}(U)|B \) is a homomorphism with respect to compositions, and since its preimages are unique up to non-zero factors, we have \( \mathfrak{I}(\Psi^{t+s})|B = \Psi^t \circ \Psi^s = \mathfrak{I}(\Psi^t \Psi^s)|B \) and hence \( \Psi^{t+s} = d_{t,s} \Psi^t \Psi^s \) (\( t, s \in \mathbb{R} \)) with suitable constants \( d_{t,s} \neq 0 \). The fixed point property

\[
\Psi^t(\overline{\tau} \oplus 1) = \overline{\tau} \oplus 1 \quad (t \in \mathbb{R}_+) \tag{3.2}
\]

ensures that \( d_{t,s} \equiv 1 \) that is the family \( \{\Psi^t : t \in \mathbb{R}_+\} \) is a one-parameter matrix semigroup. To see its strong continuity, recall [3, Ch. VI] that the Möbius shifts

\[
\Theta_a := \mathfrak{M}_a, \quad M_a := \begin{bmatrix} Q_a & a \\ a^* & 1 \end{bmatrix}, \quad Q_a := P_{CA} + \sqrt{1 - |a|^2} (1 - P_{Ca}) \quad (a \in B) \tag{3.3}
\]

act transitively on \( B \). Thus, since every element of \( \text{Iso}(d_B) \) keeping the origin fixed is a restriction of a linear isometry of \( H \), we can write \( \Psi^t = \Theta_{a_t} \circ U_t \) where \( a_t := \Psi^t(0) \) and \( U_t \) is a suitable linear isometry of \( H \). Since \( U_t = \mathfrak{I}[U_0] \), with suitable constants \( \delta_t \neq 0 \) we can write

\[
\Psi^t := \delta_t M_{a_t} \begin{bmatrix} U_t & 0 \\ 0 & 1 \end{bmatrix} = \delta_t \begin{bmatrix} Q_{a_t} U_t & a_t \\ U_t^* a_t & 1 \end{bmatrix} \quad (t \in \mathbb{R}_+).
\]

The value of \( \delta_t \) is determined unambiguously by (3.2): \( \delta_t = [1 + \langle a_t, \overline{a}_t \rangle]^{-1} \). Thus to complete the proof, it suffices to see the continuity of the functions \( t \mapsto a_t, \quad t \mapsto [U_t, Q_{a_t}] \) (\( x \in H \)). It is an immediate consequence of [2, App.A6] that the product \( t \mapsto A_t B_t \) is strongly continuous for any couple of uniformly bounded strongly continuous operator valued functions \( t \mapsto A_t \in \mathcal{L}(X_1, X_2), \quad t \mapsto B_t \in \mathcal{L}(X_2, X_3) \) in case of normed spaces \( X_t \). By assumption, the orbit \( t \mapsto a_t = \Psi^t(0) \) is a norm-continuous map \( \mathbb{R}_+ \to B \) entailing the norm continuity of the function \( t \mapsto Q_{a_t} \). We deduce the strong continuity of the \( H \)-isometry valued function \( t \mapsto U_t \) as follows. Consider any vector \( x \in H \). We may assume \( x \in B \) without loss of generality. Then, by the aid of the Möbius shifts (3.3) we can write

\[
U_t x = [\Theta_{a_t}^{-1} \circ \Psi_t](x) = \Theta_{-a_t} (\Psi_t(x)) \quad (t \in \mathbb{R}_+)
\]

whence the continuity of \( t \mapsto U_t x = (1 - \langle x, a_t \rangle)^{-1} [Q_{a_t} x - a_t] \) is immediate.

### 3.4. Standard notations, assumptions

Henceforth, for the proofs for Section 2, we assume without loss of generality the following facts.

(i) \( \Psi := [\Psi^t : t \in \mathbb{R}_+] \) is an arbitrarily given \( C_0 S \) of holomorphic Carathéodory isometries of \( B \) having no common fixed point within \( B \).
(ii) $\Phi^t := \Theta \circ \Psi^t \circ \Theta^{-1}$ $(t \in \mathbb{R}_+)$ with a suitable Möbius transformation $\Theta$;

(iii) the orbit $t \mapsto \Phi^t(0)$ is differentiable and $\Phi^t = \mathcal{Z}^t \mathcal{U}^t \mathcal{B}$ with some $C_0S$ $[U^t : t \in \mathbb{R}_+]$ of linear $H$-isometries,

$$
\mathcal{A} := \text{gen}[\mathcal{U}^t : t \in \mathbb{R}_+] = \begin{bmatrix} iA & b \\ b^* & 0 \end{bmatrix}, \quad b \in H, \quad iA = \text{gen}[U^t : t \in \mathbb{R}_+];
$$

(iv) $\tau \in \partial B$ is a joint boundary fixed point of the maps $\Phi^t$, we write

$$H_0 := H \oplus C\tau, \quad P := P_{C\tau}, \quad P_0 := P_{H_0} = 1 - P, \quad T : x \mapsto x + \tau, \quad T := \begin{bmatrix} \text{id}_H & \tau \\ 0 & 1 \end{bmatrix}.$$

**Proposition 3.5.** We have $\tau \in \text{dom}(A)$ with $A(\tau \oplus 1) = \nu(\tau \oplus 1)$ and $b = (\nu - iA)\tau$ for some $\nu \in \mathbb{C}$. The possibly unbounded operator $A_0 := P_0A|H_0 \cap \text{dom}(A)$ is $H_0$-hermitian and, in terms of $(C\tau \oplus H_0 \oplus \mathbb{C})$-matrices, we have

$$T^{-1}AT = \begin{bmatrix} -\nu & 0 & 0 \\ -b_0 & iA_0 & 0 \\ \nu & b_0 & \nu \end{bmatrix} \quad \text{where } b_0 := P_0b, \quad \nu = \langle \tau \vert b \rangle. \quad (3.6)$$

**Proof.** By assumption 3(ii), $\tau \oplus 1$ is a joint eigenvector of the linear operators $\mathcal{U}^t$. Hence $\mathcal{U}^t(\tau \oplus 1) = \zeta_t(\tau \oplus 1)$ $(t \in \mathbb{R}_+)$ with a continuous solution $[t \mapsto \zeta_t]$ of the Cauchy equation $\zeta_{s+t} = \zeta_s \zeta_t$. Thus for some $\nu \in \mathbb{C}$, $\zeta_t = e^{\nu t}$ and we have

$$\tau \oplus 1 \in \text{dom}(A) = \{ \mathfrak{j} : t \mapsto U^t \mathfrak{j} \text{ is differentiable} \}, \quad A(\tau \oplus 1) = \nu(\tau \oplus 1).$$

As a consequence, $\tau \in \text{dom}(A) = P_H\text{dom}(A)$ and the operator

$$\tilde{A}_0 := A - PA - AP + PAP = (1 - P)A(1 - P) = P_0AP_0$$

is a bounded perturbation ranging in $H_0$ of $A \in \text{Her}_0(H)$ with a self-adjoint operator of finite rank. Hence its restriction $A_0$ to $H_0$ is a well-defined unbounded $H_0$-hermitian operator. Since $A$ is a $(C\tau \oplus H_0)$-matrix operator, we can write

$$A = \begin{bmatrix} iA & b \\ b^* & 0 \end{bmatrix} = \begin{bmatrix} i\alpha & i\alpha_0 \\ i\beta & 0 \end{bmatrix}, \quad b_0 := P_0b, \quad \beta := \langle b \vert \tau \rangle, \quad \alpha_0 := P_0A\tau, \quad \alpha \nu := \langle A\tau \vert \tau \rangle$$

in terms of $(C\tau \oplus H_0 \oplus \mathbb{C})$-matrices. The eigenvector equation $A(\tau \oplus 1) = \nu(\tau \oplus 1)$ means $iAt + b = \nu \tau$ with $\langle \tau \vert b \rangle = \nu$ entailing $i\alpha + \beta = \nu$, $i\alpha_0 + b_0 = 0$, $\beta = \nu$. Since

$$T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

in $(C\tau \oplus H_0 \oplus \mathbb{C})$-matrix form, hence (3.6) is immediate.

**Notation 3.7.** Henceforth $[U^t_0 : t \in \mathbb{R}_+]$ denotes the $C_0S$ of $H_0$-isometries generated by the operator $iA_0 := P_0A|H_0 \cap \text{dom}(A)$. 

Lemma 3.8. Let $E_1, E_2$ be Banach spaces, $G := \begin{bmatrix} G_1 & 0 \\ H & G_2 \end{bmatrix}$ with $H \in \mathcal{L}(E_1, E_2)$ and $G_k := \text{gen}[W_k^t : t \in \mathbb{R}_+]$ for some $C_0S \ [W_k^t : t \in \mathbb{R}_+] \subset \mathcal{L}(E_k)$. Then the family

$$S^t := \begin{bmatrix} W_1^t \\ \int_0^t W_2^{t-s} H W_1^s ds \end{bmatrix} \quad (t \in \mathbb{R}_+)$$

is a $C_0S$ in $\mathcal{L}(E_1 \oplus E_2)$ such that $\text{gen}[S^t : t \in \mathbb{R}_+] = G$.

Proof. The family $W := [W^t : t \in \mathbb{R}_+]$ where $W^t := W_1^t \oplus W_2^t$ is a $C_0S$ in $E_1 \oplus E_2$ and $G$ is a bounded perturbation of $\text{gen}(W) = G_1 \oplus G_2(= \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix})$ with $\text{dom}(G) = \text{dom}(G_1) \oplus \text{dom}(G_2)$ by the operator $\mathcal{H} := \begin{bmatrix} 0 & 0 \\ H & 0 \end{bmatrix}$. According to [2, Thm.III.1.10], for every fixed $\mathcal{J} = x \oplus y \in \text{dom}(\mathcal{G})$ we have

$$S_{\mathcal{J}}^t = \sum_{n=0}^{\infty} S_n(t) \quad \text{where} \quad S_0(t) := W^t \mathcal{J}, \ S_{n+1}(t) = \int_0^t W^{t-s} \mathcal{H} S_n(s)(s) ds.$$

Since $\mathcal{H}$ is an off-diagonal $2 \times 2$ triangular operator matrix, $S_n(t) = 0$ for $n > 1$.

3.9. Proof of Theorem 2.1

Since $T$ a bounded invertible $\mathbb{H} \oplus \mathbb{C}$-operator and $A = \text{gen}[U^t : t \in \mathbb{R}_+]$, we have

$$T^{-1}AT = \text{gen}[V^t : t \in \mathbb{R}_+] \quad \text{for} \quad V^t := T^{-1}U^t T.$$

Since $\Phi^t = \mathfrak{g}(U^t)|B \ (t \in \mathbb{R}_+)$, in terms of the translation $Tx := x + \nu$ we can interpret the $C_0S \ [V^t : t \in \mathbb{R}_+]$ as the linear representation by means of $\mathfrak{g}$ of the semigroup \([T^{-1} \circ \Phi^t \circ T : t \in \mathbb{R}_+]\) formed by holomorphic isometries of the shifted ball $B - \nu$ whose continuous extensions leave the origin fixed. Due to the projective identities

$$\mathfrak{g}(T^{-1}V T) = T^{-1} \circ \mathfrak{g}(V) \circ T \quad (V \in \mathcal{L}(B \oplus \mathbb{C})),$$

for the points $x \in T^{-1}B = B - \nu$ we have

$$\mathfrak{g}(V^t)(x) = [\mathfrak{g}(T^{-1}U^t T)](x) = [T^{-1} \circ \Phi \circ T](x) = \Phi(x + \nu) - \nu.$$

Therefore

$$\Phi^t(x) = \mathfrak{g}(V^t)(x - \nu) + \nu \quad (x \in B).$$

By the aid of Lemma 3.8 and (3.6) we calculate a quadrature form for $V^t$ as follows. Regarding the top left $2 \times 2$-corner of the matrix $T^{-1}AT$ we get

$$\begin{bmatrix} -\nu & 0 \\ -b_0 & i\lambda \end{bmatrix} = \text{gen}[V^t : t \in \mathbb{R}_+] \quad V^t = \begin{bmatrix} e^{-\nu t} & 0 \\ \int_0^t U_s^{t-s} e^{-\nu s}(-b_0) ds & U^t_0 \end{bmatrix}.$$  \hspace{1cm} (3.10)

Another application of Lemma 3.8 to $T^{-1}AT$ yields

$$V^t = \begin{bmatrix} V^t \\ \int_0^t e^{\nu(t-s)} b^* V^s ds \end{bmatrix} \quad (t \in \mathbb{R}_+).$$  \hspace{1cm} (3.11)
As a consequence of (3.11), since \( Y^t(x+1) = [V^t x] \otimes e^{it} \left[ \int_0^t \langle e^{-\tau s} V^\tau x | b \rangle \, d\tau + 1 \right] \), we get

\[
\Phi^t(x) = \frac{e^{-it} V^t (x - \tau)}{\langle \int_0^t e^{-\tau s} V^\tau (x - \tau) \, ds | b \rangle + 1} + \tau \quad (x \in B, t \in \mathbb{R}_+). \tag{3.12}
\]

We substitute (3.10) into (3.12) in terms of the new parametrization

\[
\lambda = \text{Re} \nu, \quad \mu := \text{Im} \nu, \quad V_0^t := e^{-i\mu t} U_0^t.
\]

Given any vector \( z = z_0 + \zeta \tau, z_0 \in H_0 \), and recalling the commutativity of convolutions,

\[
e^{-it} V^t z = \zeta e^{-2\lambda t} \left[ \tau - \int_0^t e^{\lambda s} V_0^s b_0 \, ds \right] + e^{-\lambda t} V_0^t z_0, \quad \int_0^t \langle e^{-\tau s} V^\tau z | b \rangle \, ds = \]

\[
= \zeta \left( \lambda + i\mu \right) e^{-2\lambda t} - \zeta \int_0^t e^{-2\lambda s} \int_0^s e^{\lambda r} \langle V_0^r b_0 | b_0 \rangle \, dr + \int_0^t e^{-\lambda s} \langle V_0^s z_0 | b_0 \rangle \, ds \, ds.
\]

The statement of Theorem 2.1 is immediate from (3.12) with \( z = x - \tau = x_0 + (\zeta - 1) \tau \).

**Remark 3.13.** It is discovered from the above proof that any tuple

\[
a := (H, \tau, [V_0^t : t \in \mathbb{R}_+], b_0, \lambda, \mu)
\]

with a Hilbert space \( H \), a unit vector \( \tau \in H \), a \( C_0^S \) \([V_0^t : t \in \mathbb{R}_+]\) of \( H_0 := H \ominus \mathbb{C} \tau\)-isometries, a vector \( b_0 \in H_0 \) and two real constants gives rise to a \( C_0^S \) \([\Phi^t_a : t \in \mathbb{R}_+]\) of holomorphic Carathéodory isometries of the open unit ball \( B \) of \( H \equiv \mathbb{C} \tau \oplus H_0 \) whose generator \( \Gamma(x) = \frac{d}{dt} \big|_{t=0+} \Phi^t_a(x) = \frac{d}{dt} \big|_{t=0+} \Theta(T^t T^{-1}) x \) has the form (1.2) with

\[
b = \left[ \begin{array}{c} \lambda - i\mu \\ b_0 \end{array} \right], \quad A = \left[ \begin{array}{c} 2\mu - i b_0 \\ b_0 \\ A_0 \end{array} \right], \quad iA_0 = \text{gen}[V_0^t : t \in \mathbb{R}_+] \tag{3.14}
\]

In particular we can extend \([\Phi^t_a : t \in \mathbb{R}_+]\) to a \( C_0^G \) \([\Phi^t_a : t \in \mathbb{R}] \subset \text{Iso}(d_B)\) if and only if \([V_0^t : t \in \mathbb{R}_+]\) consists of \( H_0\)-unitary operators (cf. [10, Thm.II]). Furthermore, given any tuple

\[
b := (H, A, \tau, \lambda)
\]

with a densely defined maximal symmetric linear \( H \)-operator \( A \), there is a unique \( C_0^S \)

\[
[\Phi^t_b : t \in \mathbb{R}_+] \subset \text{Iso}(d_B)\) whose infinitesimal generator is of the form (1.2) with \( b := (\nu - iA)e \) where \( \nu := \lambda + i\mu \) and \( \mu = \langle A \tau | \tau \rangle \).

#### 4. Invariant discs

**Lemma 4.1.** The \( C_0^S \) \([\Phi^t : t \in \mathbb{R}_+] \subset \text{Iso}(d_B)\) with generator (1.2) and joint boundary fixed point \( \tau \in \partial B \) admits no invariant disc of the form \( B \cap (\tau + C \tau) \neq \emptyset \) if and only if the operator \( iA + \langle \tau | b \rangle \) is not injective or \( \tau \in \text{range}(iA - \langle \tau | b \rangle) \).

**Proof.** Consider any vector \( v \in H \) such that \( \tau + v \in B \). The disc \( \Delta^\tau_{\nu} := B \cap (\tau + C \tau) \) is \([\Phi^t : t \in \mathbb{R}_+]\)-invariant if and only if the vector field (1.2) is tangent to it that is if \( b - (\tau + \tau v | b) (\tau + \tau v) + iA (\tau + \tau v) \in C v \) whenever \( \tau + \tau v \in B \). This happens if and only if \(-\langle v | b \rangle + iA v = \zeta v \) for some \( \zeta \in \mathbb{C} \) because we have \( \tau \in \text{dom}(\Gamma) = \text{dom}(A) \).
and $\Gamma(\tau) = b - \langle \tau | b \rangle + iA\tau = 0$ (due to the fact that the point $\tau$ is $[\overline{C\tau}] : t \in \mathbb{R}_+$-invariant). According to Proposition 3.5, here we have $b = (\nu - iA)\tau$ where $\nu = |\overline{\tau}|b$.

Therefore $\zeta v = -\langle \nu | (\nu - iA)\tau \rangle \tau + iA\tau = \langle \nu | \nu - iA \rangle v \tau + iA\tau$. Notice that, in general, $P_{\zeta v,x} = \langle \nu | (\nu - iA)\tau \rangle \tau + iA\tau$ is the disc $\Delta_2^\nu$ is $[\Phi^t : t \in \mathbb{R}_+]$-invariant and if and only if $-P_{\zeta v,x} + P_{\zeta v,Av} - \zeta v = 0$ i.e. $P_{\zeta v,Av} - \zeta v = 0$ and $P_{\zeta v,Av} - \zeta v = 0$ for some $\zeta \in \mathbb{C}$. By assumption $\Delta_2^\nu \neq 0$ which is possible if and only if $P_{\zeta v,v} \neq 0$ implying $\zeta = -\pi$. Hence we conclude that the $[\Phi^t : t \in \mathbb{R}_+]$-invariance of $\Delta_2^\nu$ is equivalent to the relation $P_{\zeta v,Av} - \zeta v = 0$ i.e. to $(iA + P)v = 0$ which completes the proof.

Example 4.2. The $C_0S$ of the type $\Psi_b = [\Psi^t_b : t \in \mathbb{R}_+]$ in Remark 3.13 with $H := L^2(\mathbb{R}), Af := [x \mapsto xf(x) \ (\text{dom}(A) := \{ f : \int_x^\infty |xf(x)|^2 dx < \infty \}), \lambda := 0$ and $\tau := (2\pi)^{-1} \int 2\pi x \exp(-(x - 1)^2)/2)$ admits no invariant 1-dimensional disc. Proof: We have $\langle Af | \tau \rangle = (2\pi)^{-1} \int 2\pi x \exp(-(x - 1)^2)/2 dx = (2\pi)^{1/2} - 1$. Thus, according to the construction of the $C_0S_{\Psi_b}, \nu = \langle \tau | b \rangle = i\mu = \langle \tau^{\nu} | \tau \rangle = i\nu \in i\mathbb{R}\{0\}$. The relation $(iA + \tau)v = \zeta v$ would imply $v = -i\zeta \exp(-(x - 1)^2)/2((x - \mu)$ would be in $L^2(\mathbb{R})$ which is possible only if $v = 0$.

4.3. Proof of Theorem 2.6

Recall [3] that the 1-dimensional complex affine discs the form $\Delta_\nu, := (p + C(q-p)) \cap B (q \neq p, q \in \partial B)$ are the ranges of complex geodesics for the Carathéodory distance $d_B$, and $d_B$-isometries preserve their family. In particular, in the case when $p \neq q \in \partial B$ are joint fixed points of the continuous extensions $\overline{\Psi}$ the disc $\Delta_\nu, is automatically $[\Psi^t : t \in \mathbb{R}_+]$-invariant. Suppose $\Psi^t(\Delta_\nu, = \Delta_\nu, (t \in \mathbb{R}_+)$. Then the restricted maps $\psi^t_{\nu, q} := \Psi^t | D_{\nu, q} form a $C_0S$ of holomorphic automorphisms of a 1-dimensional Hilbert ball, thus their continuous extensions $\psi^t_{\nu, q} to D_{\nu, q} admit at least one fixed point which is necessarily a joint fixed point for the maps $\overline{\Psi^t}$. A 1-dimensional application of Theorem 2.1 shows that all the orbits $t \mapsto \psi^t_{\nu, q}(x) = \Psi^t(x) (x \in \Delta_\nu, q$ are automatically real analytic. Hence, given any Möbius transformation $\Theta$, the $C_0S [\Phi^t : t \in \mathbb{R}_+] with $\Phi^t := \Theta \circ \Psi^t \circ \Theta^{-1}$ leaves the $d_B$-geodesic $D_{\Theta(q),\Theta(p)} invariant with differentiable 0-orbit $t \mapsto \Phi^t(0)$. Conversely, if $[\Phi^t : t \in \mathbb{R}_+] is a $C_0S$ leaving the disc $\Delta_\nu,-\tau(= |\tau| : |\tau| < 1)$ invariant and $\Phi^t(\tau) = \tau, \Psi^t = \Theta^{-1} \circ \phi^{-1} \circ \Theta (t \in \mathbb{R}_)$ then the image $\Theta(\Delta_\nu,-\tau(= |\tau|, |\tau| < 1)$ is a $[\overline{\Psi^t} : t \in \mathbb{R}_+]$-invariant 1-dimensional affine section of $\overline{B}$ containing a joint fixed point (namely $\Theta^{-1}(\tau(= |\tau|, |\tau| < 1))$ of $[\overline{\Psi^t} : t \in \mathbb{R}_+]$.

Proof of (i),(ii). It remains only to verify the possibility of the simplified representations (2.7),(2.8) by means of an appropriate choice for the coordinatizing Möbius transformation $\Theta$ in 3. By setting $x_0 := 0$ in (2.2), it is straightforward to check that a $C_0S [\Phi^t : t \in \mathbb{R}_+] of the form (2.2) leaves the disc $\Delta_\nu,-\tau$ invariant if and only if $b_0 = 0$ and $\Phi^t(\tau) = \omega_{\lambda, \mu}(t, \xi)\tau (|\xi| < 1)$ with the function

$$\omega_{\lambda, \mu}(t, \xi) := 1 - \frac{2\lambda(1-\xi)e^{-2\lambda t}}{2\lambda - (1-\xi)(\lambda + i\mu)(1-e^{-\lambda t})}.$$ 

It is also easy to see that the constant 1 is a joint fixed point of all functions $\omega_{\lambda, \mu}(t, \cdot)$ and, for fixed $\lambda, \mu \in \mathbb{R}$, the family $\omega_{\lambda, \mu}(t, \cdot) t \in \mathbb{R}_+$ admits another fixed point namely the constant $\xi_{\lambda, \mu} := \frac{i\mu}{\lambda + i\mu}$ with modulus 1 if and only if we have $\mu = 0$. Due to folklore 2-transitivity properties of the Möbius group (for direct proof see [8]), given any two couples $(e_1, e_2), (f_1, f_2) \in [\partial B]_2^2$ of distinct boundary points there exists a Möbius transformation $\Theta(e_1,f_1,e_2,f_2)$ with the effect $e_k \mapsto f_k (k = 1, 2)$. Thus in case if $[\overline{\Psi^t} : t \in \mathbb{R}_+] has only a
unique fixed point \( p \in \partial B \) but the disc \( \Delta_{p,q} \) is \([\Psi^t : t \in \mathbb{R}_+]\)-invariant, with any choice \( \Theta := \Theta^{(\mu,\sigma,\tau)} \) where \( |\sigma| = 1 \) we get a formula for \( \Phi^t \) by substituting \( b_0 = 0 \) and \( \mu = 0 \) in (2.2) which is (2.8).

If \( [\Psi^t : t \in \mathbb{R}_+] \) admits two distinct fixed points \( p, q \in \partial B \) then, as we have shown, the disc \( \Delta_{p,q} \) is automatically \([\Psi^t : t \in \mathbb{R}_+]\)-invariant, and with the choice \( \Theta := \Theta^{(\mu,\sigma,\tau)} \) we get a formula for \( \Phi^t \) by substituting \( b_0 = 0 \) and \( \mu = 0 \) in (2.2) establishing (2.7).

Proof of (iii). Suppose indirectly that \( 0 \neq \lambda = \text{Re}(\langle b \rangle) \). Then the skew symmetry of \( iA \) entails range\((iA - \langle b \rangle) = H \). By Lemma 4.1, we have a non-trivial \( \Gamma \)-invariant disc and we are in the settings of (i) or (iii). By assumption, (i) is not the case. However, in the case of (iii) we have \( \langle b \rangle \) is not automatically.

4.4. Proof for Remark 2.5 The operator \( S_0 \) is closed with dense domain in \( H_0 \). Since \( S_0 \) is also symmetric, both \( \pm iS_0 \) are dissipative (namely \( \text{Re}(\langle \pm iS_0x_0 \rangle = 0 \) for \( x_0 \in \text{dom}(S) \)) with the properties that both range\((\pm iS_0 + \delta) \) are dense in \( H \) for any \( \delta > 0 \) and that the operators \((\pm iS + \delta)^{-1} : \text{range}(S) \to H_0 \) are all bounded and densely defined.\(^4\)

Given \( \delta \in \mathbb{R} \setminus \{0\} \), by [2, II.Lemma 1.3], for any \( x_0 \in \text{range}(iS_0 + \delta) \) and \( t > 0 \) we have
\[
\int_0^t e^{-\delta \tau ^S_0 x_0} d\tau = \int_0^t e^{-\delta \tau ^S_0} (iS_0 - \delta)^{-1} x_0) d\tau = (e^{-\delta \tau ^S_0} - 1) (iS_0 - \delta)^{-1} x_0.
\]
The boundedness of both the operators \( V^t \) and the resolvent \( R(\delta, iS) \) \( \in \) \( \text{closure}\((\delta - iS_0)^{-1}\) \) establishes 2.5 for \( t \in \mathbb{R}_+ \) and \( 0 \neq \lambda \in \mathbb{R} \) with integrals of strongly continuous bounded operator valued functions.

5. Dilatation

**Lemma 5.1.** Let \([U^t : t \in \mathbb{R}_+]\) be a \( C_0S \) of linear isometries of a Banach space \( E \). Suppose \( \hat{E} \) is a subspace of another Banach space \( F \) and there is a surjective isometry \( V \in \mathcal{L}(F) \) such that \( U^t = V[E] \). Then there is a subspace \( E \subset \hat{E} \subset F \) along with a \( C_0S \) \([\hat{U}^t : t \in \mathbb{R}]\) of surjective linear isometries of \( \hat{E} \) such that \( U^t = \hat{U}^t[E] \) \( (t \in \mathbb{R}_+) \) with \( \text{dom}(\text{gen}[\hat{U}^t : t \in \mathbb{R}]) \supset \text{dom}(\text{gen}[U^t : t \in \mathbb{R}]) \)

**Proof.** Let \( \hat{E} := \text{closure}(E_{\infty}) \) \( \in F \) where \( E_{\infty} := \bigcup_{n=0}^{\infty} E_n := V^{-n}E \). By assumption \( V \in \mathcal{L}(E) \subset \hat{E} \) whence, by induction we conclude that the subspaces \( E_n \) \( (n \in \mathbb{Z}_+) \) form an increasing sequence. Therefore all the operators \( U^t_n := V^{-2n}U^{t+n}V^n[E_n] \) \( (t \geq -n, n \in \mathbb{Z}_+) \) are well-defined isometries \( E_n \to E_{[n-t]} \). We have \( U^t_n = U^t_{n+1} | E_n \) for all indices \( n \in \mathbb{Z}_+ \). Indeed if \( \hat{x} \in E_n \) and \( t \geq -n \) then
\[
U^t_{n+1} \hat{x} = V^{-2n-2}U^{t+n+2}V^n \hat{x} = V^{-2n-2}U^{t+n+1}U^1 V^n \hat{x} = V^{-2n-2}U^{t+n+2}V^n \hat{x} = V^{-2n-2}U^{t+n+1}U^1 V^n \hat{x} = U^t_n \hat{x}
\]
since \( V \) extends \( U^t \) and we have \( V^n \hat{x} \in E \) implying \( V^n \hat{x} = U^t \). Hence
\[
U^t_n \hat{x} := \lim_{n \to \infty} U^t_n \hat{x} = [U^t_n \hat{x} : n \in \mathbb{Z}_+, n \geq t] \quad (\hat{x} \in E_{\infty})
\]

\(^4\) Indeed \( y_0 \perp \text{range}(\pm iS_0 + \delta) \) means \( 0 = (\pm iS_0x_0 - \delta x_0|y_0) \) that is \( 0 = (x_0| iS_0y_0 - \delta y_0) \) for \( (x_0 \in \text{dom}(S)) \) entailing \( iS_0y_0 + \delta y_0 = 0 \) with \( \delta \|y_0\|^2 = \pm (S_0y_0|y_0) \in \mathbb{R} \) which is possible only if \( y = 0 \). Thus by the Lumer-Phillips theorem [2, II.Thm.3.15], also the operator \(-iS_0 \) generates a strongly continuous contraction (actually isometry) semigroup and all the values \( 0 \neq \delta \in \mathbb{R} \) belong to the resolvent set of \( iS \).
is a well-defined linear isometry of the linear manifold $E_\infty$ for any $t \in \mathbb{R}$. Since 
$\text{range}(U_t^n) \supset V^{-2nU[t]^{+n}}E = V^{[t]^{-n}}E$ for $t \geq -n$, we have 
$\text{range}(U_t^n) = E_\infty$ ($t \in \mathbb{R}$). Thus the operators $\tilde{U}^t := \text{closure}(U_t^n)$ ($t \in \mathbb{R}$) are well-defined surjective linear $E_\infty$-isometries, each of which extending the respective $U^t$. We check they form a $C_0G$ as follows. Since $[U^t : t \in \mathbb{R}]$ is an equilipschitzian family, it suffices to see that its 
restriction $[\tilde{U}^t : t \in \mathbb{R}]$ to the dense submanifold $E_\infty$ of $E$ is a $C_0G$. Given $s,t \in \mathbb{R}$ and $x \in E_t$, we have $\tilde{U}^t x = V^{-2nU[t]^{+n}}V^n x \in E_{2n}$ whenever $n \geq \ell t$ and 
$U^s(\tilde{U}^t x) = V^{-2nU[t]^{+n}}V^mU^s \tilde{U}^t x$ whenever $m \geq \max\{-s,2n\}$. It follows $\tilde{U}^t \tilde{U}^t x = \tilde{U}^{s+t} x$ because hence, with $k \geq 2(|s| + |t| + \ell)$ we have 
\[
\tilde{U}^s(\tilde{U}^t x) = V^{-4kU[s]^{+2k}}V^{2kU[t]^{+k}}V^{k}x = \quad \\
= V^{-4kU[s]^{+t+k}}V^{2k}V^{k}x = V^{-4kU[s]^{+t+k}}V^{2k}xV^{k} = \tilde{U}^{s+t} x.
\]

To see strong continuity, consider any vector $x \in E_\ell$. Then for any integer $n \geq \ell$ the orbit $(-n,\infty) \ni t \mapsto \tilde{U}^t x = V^{-2nU[t]^{+n}}V^n x$ is continuous since $V^{-2n}$ is an isometry and $(V^n x) \in E$. Hence we can also the required generator domain inclusion property: with $\tilde{x} := x \in \text{dom}(\text{gen}[U^t : t \in \mathbb{R}])$ we have $V^n x = \tilde{U}^t x \in \text{dom}(\text{gen}[U^t : t \in \mathbb{R}])$ entailing even the differentiability of the orbits $(-n,\infty) \ni t \mapsto \tilde{U}^t x$.

In particular, since every linear isometry of a Hilbert space admits a unitary dilation [9], in our setting of interest we conclude the following.

**Corollary 5.2.** If $[U^t : t \in \mathbb{R}_+]$ is a $C_0S$ of linear $H$-isometries, there exists a Hilbert space $\tilde{H}$ containing $H$ as a subspace along with a $C_0G$ $[\tilde{U}^t : t \in \mathbb{R}]$ of $\tilde{H}$-unitary operators such that $U^t = \tilde{U}^t|H$ ($t \in \mathbb{R}_+$) whose generator is an extension of $\text{gen}[U^t : t \in \mathbb{R}_+]$.

### 5.3. Proof of Corollaries 2.10-11

Given any Hilbert space $H$ containing $H$ as a subspace, every M"obius transformation of $H$ extends to a M"obius transformation of $\tilde{H}$. Hence it suffices to see only that any $C_0S$ of the form of Theorem 2.1 admits a group a dilation of the same algebraic form in a larger Hilbert space. Let $[\Phi^t : t \in \mathbb{R}_+]$ be given as in Theorem 2.1. According to Corollary 3.13, for some tuple $a := (H,\pi,[V^t_0 : t \in \mathbb{R}_+],b_0,\lambda,\mu)$ we have $\Phi^t = \Phi^t_a = \widehat{\text{gen}}([U^t]) (t \in \mathbb{R}_+)$ with 
\[
\text{gen}[U^t : t \in \mathbb{R}_+] = \begin{bmatrix} iR & b \\ \ast & 0 \end{bmatrix} = \begin{bmatrix} i(S_0 + \mu) & -b_0 \\ b_0^* & 2i\mu \\ b_0 & \lambda - i\mu \end{bmatrix}.
\]  

In terms of $[H_0 \oplus (\mathbb{C}r) \oplus \mathbb{C}]$-matrices. Let $[\tilde{V}^t_0 : t \in \mathbb{R}]$ be the dilation $C_0G$ of $[V^t_0 : t \in \mathbb{R}_+]$ consisting of unitary operators of a covering Hilbert space $\tilde{H}_0$ of $H_0$ with the skew self-adjoint extension $i\tilde{S}_0 = \text{gen}[V^t_0 : t \in \mathbb{R}]$ of $iS_0$ guaranteed by Corollary 5.2. Also by Remark 3.13 the tuple $a := (\tilde{H},\pi,[\tilde{V}^t_0 : t \in \mathbb{R}_+],b_0,\lambda,\mu)$ where $\tilde{H} := \tilde{H}_0 \oplus (\mathbb{C}r)$ gives rise to a $C_0G$ $\Phi_a^t : t \in \mathbb{R}$ such $\Phi_a^t = \tilde{\text{gen}}([U^t]) (t \in \mathbb{R})$ whose infinitesimal generator can be written in the form of the right hand side of (5.4) when the entry $S_0$ is replaced with $\tilde{S}_0$.

Hence, by Theorem 2.1, the transformations $\Phi_a^t$ can be written in the form (2.2) with $\tilde{S}_0$ in place of $S_0$ and $\tilde{V}^t_0$ in place of $V^t_0$. Since $\tilde{V}^t_0|H_0 = V^t_0 (t \in \mathbb{R}_+)$, it readily follows $\Phi_a^t|H = \Phi_a^t (t \in \mathbb{R}_+)$ which completes the proof of Corollary 2.10.
To prove Corollary 2.11, consider any $C_b^0 S_\Phi t a : t \in \mathbb{R}_+$ with its dilation group $[\Phi t a : t \in \mathbb{R}_+]$ as above. By construction, the dilation $C_b G [\tilde{V}_0^t : t \in \mathbb{R}]$ consists of $C_b^0$-unitary operators. Thus, in view of Stone’s classical theorem, we can apply the functional calculus [9] with its skew self-adjoint generator $i S_0$ when evaluating the transformations $\Phi t a$ by means of (3.12). Actually, for any $t \in \mathbb{R}$ we have

$$\int_0^t e^{-\lambda \tau} \tilde{V}_0^\tau d\tau = g_1(t, S_0), \quad \int_0^t e^{-2\lambda \tau} \int_0^\tau e^{\lambda \sigma} d\sigma d\tau = g_2(t, S_0)$$

with the functions $s \mapsto \int_0^s e^{-\lambda \tau} e^{i\tau s} d\tau$ resp. $s \mapsto \int_0^s e^{-2\lambda \tau} \int_0^\tau e^{\lambda \sigma} e^{i\tau s} d\sigma d\tau$ which are real analytic $\mathbb{R} \to \mathbb{C}$. Straightforward calculation establishes their algebraic form and the Taylor series appearing in Corollary 2.11.

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