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On the manifold of tripotents in JB*-triples

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Abstract

The manifold of tripotents in an arbitrary JB*-triple Z is considered, a natural affine connection is defined on it in terms of the Peirce projections of Z, and a precise description of its geodesics is given. Regarding this manifold as a fiber space by Neher's equivalence, the base space is a symmetric Kähler manifold when Z is a classical Cartan factor, and necessary and sufficient conditions are established for connected components of the manifold to admit a Riemann structure.
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1. Introduction

In [9] Hirzebruch proved that the manifold of minimal projections in a finite-dimensional simple formally real Jordan algebra is a compact Riemann symmetric space of rank 1, and that any such space arises in this way. Later on, in [14] Nomura estab-

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lished similar results for the manifold of minimal projections in a topologically simple real Jordan–Hilbert algebra. Recently, Jordan algebras and projections have been replaced by the more general notions of JB*-triples and tripotents, respectively. JB*-triples are precisely those complex Banach spaces whose open unit balls are homogeneous with respect to biholomorphic transformations.

In [1] an affine connection ∇ on $\mathcal{M}$, the manifold of tripotents in a JB*-triple Z, was defined in terms of the natural algebraic triple product structure of Z. Unfortunately, the description of the geodesics of ∇ given in [1, Theorem 2.7] by means of one-parameter groups of automorphisms of Z fails to be true in general since the corresponding second order differential equation is of sophisticated character. Our first goal is to develop a technique, based on exponential integrals, to find explicit formulas for the geodesics of ∇.

It is known that $\mathcal{M}$ is a fibre space with respect to Neher's relation of equivalence of tripotents. As proved by Kaup in [11], the base space $\mathbb{P}$ of that fibration is the manifold of all complemented principal inner ideals of Z, which is a closed complex submanifold of the Grassmannian $\mathbb{G} = \mathbb{G}(Z)$. The connected components of $\mathbb{P}$, which are orbits of Γ (the structure group of Z), are symmetric complex Banach manifolds on which Γ acts as a group of isometries, see [11]. We show that ∇ induces on these orbits a Γ-invariant torsion-free affine connection (also denoted by ∇) and compute its geodesics which turn out to be orbits of one-parameter subgroups of Γ.

All tripotents in the same equivalence class (in Neher's sense) have the same rank r ($0 \leqslant r \leqslant \infty$), that is constant over each connected component M of $\mathbb{P}$. It is reasonable to ask which of these connected components admit a Riemann structure. For Z a classical Cartan factor, we solve that problem with the aid of the concepts of *operator rank* and *operator corank*, and prove that M admits a Riemann structure if and only if either the operator rank or the operator corank are finite, in which case we prove that ∇ is the Levi-Civita and the Kähler connection of M. Some of these results were already known and due to E. Cartan in the $\mathbb{C}^n$ setting.

2. JB*-triples and tripotents

For a complex Banach space Z, denote by $\mathcal{L}(Z)$ the Banach algebra of all bounded linear operators on Z. A complex Banach space Z with a continuous mapping $(a,b,c) \mapsto \{abc\}$ from $Z \times Z \times Z$ to Z is called a *JB*-triple* if the following conditions are satisfied for all $a,b,c,d \in Z$, where the operator $a \Box b \in \mathcal{L}(Z)$ is defined by $z \mapsto \{abz\}$ and $[\,,]$ is the commutator product:

(1) $\{abc\}$ is symmetric complex linear in a, c and conjugate linear in b.
(2) $[a \Box b, c \Box d] = \{abc\} \Box d - c \Box \{dab\}$.
(3) $a \Box a$ is hermitian and has spectrum $\geqslant 0$.
(4) $\|\{aaa\}\| = \|a\|^3$.

If a complex vector space Z admits a JB*-triple structure, then the norm and the triple product determine each other. An *automorphism* is a bijection $\phi \in \mathcal{L}(Z)$ such that $\phi\{zzz\} = \{(\phi z)(\phi z)(\phi z)\}$ for $z \in Z$ which occurs if and only if ϕ is a surjective linear isometry of Z.

By $\mathrm{Aut}^\circ(Z)$ we denote the connected component of the identity in the topological group $\mathrm{Aut}(Z)$ of all automorphisms of Z (see [7]). Two elements x, y in Z are *orthogonal* if $x \square y = 0$ and $e \in Z$ is called a *tripotent* if $\{eee\} = e$. The set of tripotents, denoted by $\mathrm{Tri}(Z)$, is endowed with the induced topology of Z. Clearly $e = 0$ is an isolated point in $\mathrm{Tri}(Z)$. For $e \in \mathrm{Tri}(Z)$, a conjugate-linear operator $Q(e) \in \mathcal{L}(Z)$, that commutes with $e \square e$, is defined by $Q(e)z = \{eze\}$ for $z \in Z$. If $e \in \mathrm{Tri}(Z)$, then the set of eigenvalues of $e \square e \in \mathcal{L}(Z)$ is contained in $\{0, 1/2, 1\}$ and we have the topological direct sum decomposition, called the *Peirce decomposition* of Z,

$$Z = Z_1(e) \oplus Z_{1/2}(e) \oplus Z_0(e).$$

Here $Z_k(e)$ is the k-eigenspace of $e \square e$ and the *Peirce projections* $Z \to Z_k(e)$ with kernel $\bigoplus_{j \neq k} Z_j(e)$ are

$$P_1(e) = Q^2(e), \qquad P_{1/2}(e) = 2\big(e \square e - Q^2(e)\big),$$
$$P_0(e) = \mathrm{Id} - 2e \square e + Q^2(e).$$

We shall use the *Peirce rules* $\{Z_i(e)\, Z_j(e)\, Z_k(e)\} \subset Z_{i-j+k}(e)$ where $Z_l(e) = \{0\}$ for $l \neq 0, 1/2, 1$. We note that $Z_1(e)$ is a complex unital JB*-algebra in the product $a \circ b := \{aeb\}$ and involution $a^\# := \{eae\}$. We have $Z_1(e) = A(e) \oplus iA(e)$ where

$$A(e) := \{z \in Z_1(e):\ z^\# = z\}.$$

It is also customary to write $D(a, b)$ instead of $a \square b$ and $D(a)$ instead of $D(a, a)$ for $a, b \in Z$. A tripotent e is said to be *minimal* or an *atom* in Z if $e \neq 0$ and $P_1(e)Z = \mathbb{C}e$, and we let $\mathrm{Min}(Z)$ be the set of them. A JB*-triple Z may have no nonzero tripotents. If Z admits a (necessarily unique) predual space Z_*, then we say that Z is a JBW*-triple. The bidual Z^{**} of a JB*-triple is a JBW*-triple and the canonical embedding $Z \hookrightarrow Z^{**}$ is a triple homomorphism. We let $Z^{**} = Z_{\mathrm{a}} \oplus N$ denote the decomposition of the bidual Z^{**} into its atomic and non-atomic ideals (see [4]). Here $Z_{\mathrm{a}} = \bigoplus_{\iota \in I} F_\iota$ is the ℓ_∞-sum of the family of all minimal w^*-closed ideals F_ι in Z^{**}, each F_ι is a Cartan factor and N contains no atoms. Every $e \in \mathrm{Tri}(Z_a)$, $e \neq 0$, has a decomposition of the form $e = \sum_{j \in J} e_j$ where the e_j are pairwise orthogonal tripotents that are minimal in Z^{**} and the series converges in the weak* topology of Z^{**}. The cardinality of J is uniquely determined, and the *rank* of e is defined to be that cardinality when finite and to be infinite otherwise. For $e \in N$ we set $\mathrm{rank}(e) = 0$. Every JBW*-triple Z contains a (possibly empty) maximal family $(e_j)_{\iota \in J}$ of pairwise orthogonal minimal tripotents, and the (necessarily unique) cardinality of J is the *rank* of Z. For details on JB*-triples see [4,12].

3. The algebraic connection on the manifold of tripotents

Let Z be a JB*-triple and $\mathrm{Tri}(Z)$ the set of all tripotents in Z endowed with the relative topology of Z. Fix any nonzero tripotent $e_0 \in \mathrm{Tri}(Z)$, and denote by M the connected component of e_0 in $\mathrm{Tri}(Z)$. Then all tripotents $e \in M$ have the same rank as e_0 and $\mathrm{Aut}^\circ(Z)$ acts transitively on M which is a real analytic manifold whose tangent space at a point e is

$$T_eM = iA(e) \oplus Z_{1/2}(e).$$

For $z = iv + u \in iA(e) \oplus Z_{1/2}(e)$ we set $w := \frac{i}{2}v + 2u$ and $K(e,z) := w\Box e - e\Box w$. Then [15, p. 25] a local chart of M at e in a suitable neighbourhood $V \times U$ of $(0,0)$ in $iA(e) \times Z_{1/2}(e)$ is given by

$$z \mapsto f(z) := \big[\exp K(e,z)\big](e). \tag{1}$$

We denote by $P_e : Z \to iA(e) \oplus Z_{1/2}(e)$ the canonical projector from Z onto the tangent space $iA(e) \oplus Z_{1/2}(e)$ to M at e. By Peirce arithmetic, P_e is $\mathrm{Aut}^\circ(Z)$-invariant as it satisfies

$$P_{g(e)}g(z) = gP_e z, \quad g \in \mathrm{Aut}^\circ(Z),\ z \in Z.$$

Let $\mathfrak{D}(M)$ be the Lie algebra of all smooth vector fields on M. We define the *algebraic connection* ∇ on M by

$$(\nabla_X Y)_e := P_e\big(Y'_e X_e\big), \quad e \in M,\ X, Y \in \mathfrak{D}(M).$$

Then ∇ is a torsion-free $\mathrm{Aut}^\circ(Z)$-invariant affine connection on M. Recall a smooth curve $\gamma : I \to M$, where I is a neighbourhood of $0 \in \mathbb{R}$, is a ∇-geodesic if and only if

$$P_{\gamma(t)}\left(\frac{d^2}{dt^2}\gamma(t)\right) = 0.$$

Recall that $\mathrm{Aut}(Z)$ is a Banach–Lie group whose Lie algebra can naturally be identified with $\mathfrak{aut}(Z)$ the family of all skew Z-hermitian operators. In particular, any continuous mapping $F : \mathbb{R} \to \mathfrak{aut}(Z)$ gives rise to (uniquely determined) left and right primitive functions ${}^{\mathbf{L}}F : \mathbb{R} \to \mathrm{Aut}(Z)$, ${}^{\mathbf{R}}F : \mathbb{R} \to \mathrm{Aut}(Z)$ with the property ${}^{\mathbf{L}}F(0) = {}^{\mathbf{R}}F(0) = \mathrm{Id}$ and

$$\frac{d}{dt}{}^{\mathbf{L}}F(t) = \big[{}^{\mathbf{L}}F(t)\big]F(t), \qquad \frac{d}{dt}{}^{\mathbf{R}}F(t) = F(t)\big[{}^{\mathbf{R}}F(t)\big].$$

In the sequel we shall only use the left primitive functions.

Lemma 1. *Let Z be a JB*-triple and let the manifold M be as above. Given $e \in M$ and a smooth curve $a : \mathbb{R} \to iA(e) \oplus Z_{1/2}(e) = T_e M$, the curve*

$$\gamma(t) := g(t)e \quad \textit{where } g(t) := {}^{\mathbf{L}}K\big(e, a(t)\big)$$

in M is a ∇-geodesic if and only if $a(t) = iv_0 + \exp(-3tiv_0\Box e)u_0$ for some $v_0 \in A(e)$ and $u_0 \in Z_{1/2}(e)$.

Proof. We use the decomposition $a(t) = iv(t) + u(t)$ with $v(t) \in A(e)$ and $u(t) \in Z_{1/2}(e)$. An immediate calculation yields

$$K\big(e, a(t)\big)e = a(t),$$
$$K\big(e, a(t)\big)^2 e = K\big(e, a(t)\big)a(t) = -\alpha(t) + i\beta(t) + 2\{ueu\} - 2i\{euv\},$$

where

$$\alpha(t) := \frac{1}{2}\{vev\} + \frac{1}{2}\{evv\} + 2\{euu\}, \qquad \beta(t) := \frac{1}{2}\{veu\} + 2\{uev\} + \frac{1}{2}\{evu\}.$$

Here

$$\{ueu\} \in \big\{Z_{1/2}(e)Z_1(e)Z_{1/2}(e)\big\} \subset Z_0(e)$$

and therefore $P_e\{ueu\} = 0$. Also

$$\{euv\} \in \{Z_1 Z_{1/2} Z_1\} = Z_{1-1/2+1} = 0.$$

By Peirce rules, the summands in $\beta(t)$ belong to $Z_{1/2}$. Furthermore, since $\{veu\} \in Z_{1-1+1/2}$ and hence $\{e\{veu\}e\} \in Z_{1-1/2+1} = 0$, the Jordan identity yields

$$\{veu\} = \big\{\{eve\}eu\big\} = 2\big\{\{eeu\}ve\big\} - \big\{e\{veu\}e\big\} = 2\left\{\frac{1}{2}uve\right\}.$$

On the other hand, $\{vev\}, \{evv\} \in \{A(e)A(e)A(e)\} = A(e)$. By the Jordan identity we have

$$\{euu\} = \big\{uu\{eee\}\big\} = 2\big\{\{uue\}ee\big\} - \big\{e\{uue\}e\big\} = 2\{uue\} - Q(e)\{uue\}.$$

That is $\{euu\} = Q(e)\{euu\} \in A(e)$ and $\alpha(t) \in A(e)$. It follows

$$P_e K\big(e, a(t)\big)e = a(t), \qquad P_e K\big(e, a(t)\big)^2 e = i\beta(t) = 3i\big\{v(t)eu(t)\big\}.$$

As we know, the curve γ is ∇-geodesic if and only if

$$\begin{aligned}
0 &= P_{\gamma(t)} \frac{d^2}{dt^2}\gamma(t) = P_{g(t)e}\frac{d^2}{dt^2}g(t)e = g(t)P_e g(t)^{-1}\frac{d}{dt}\left[\frac{d}{dt}g(t)\right]e \\
&= g(t)P_e g(t)^{-1}\frac{d}{dt}\big[g(t)K\big(e, a(t)\big)\big]e \\
&= g(t)P_e g(t)^{-1}\left[\left(\frac{d}{dt}g(t)\right)K\big(e, a(t)\big) + g(t)\frac{d}{dt}K\big(e, a(t)\big)\right]e \\
&= g(t)P_e g(t)^{-1}\left[g(t)K\big(e, a(t)\big)^2 + g(t)K\left(e, \frac{d}{dt}a(t)\right)\right]e \\
&= g(t)\left[3i\big\{v(t)eu(t)\big\} + \frac{d}{dt}a(t)\right].
\end{aligned}$$

By passing to the components with respect to the decomposition $T_e M = iA(e) \oplus Z_{1/2}(e)$, we conclude that γ is a ∇-geodesic if and only if

$$\frac{d}{dt}v(t) = 0, \qquad \frac{d}{dt}u(t) = 3i\big(v(t)\,\square\,e\big)u(t),$$

that is if and only if $v(t) = v(0) =: v_0$ and $u(t) = \exp(3tiv_0\,\square\,e)u_0$ with $u_0 := u(0)$. $\square$

As a consequence we immediately get the following theorem.

Theorem 2. *Given any point $e \in M$ and a tangent vector $z \in iA(e) \oplus Z_{1/2}(e)$, there is a unique ∇-geodesic $\gamma^{\nabla}_{e,z} : I \to M$ with $\gamma^{\nabla}_{e,z}(0) = e$ and $\dot{\gamma}^{\nabla}_{e,z}(0) = z$, and we have the explicit formula*

$$\gamma^{\nabla}_{e,z}(t) = {}^{\mathrm{L}}K\big(e, iv + \exp(3tv\,\square\,e)u\big)$$

for the ∇-geodesics in terms of left primitive functions of $\mathbb{R} \to TM$ maps. In particular, the curve $\gamma_{e,z}(t) := \exp[tK(e,z)]e$ is a ∇-geodesic if and only if $\{veu\} = \{evu\} = 0$.

4. The base space of the manifold of tripotents in a JB*-triple

It is known [11] that M is a fiber space, the typical fiber being a manifold whose tangent space at e is $iA(e)$. We shall now study the *base* manifold of this fibre space. To each tripotent $e \in \mathrm{Tri}(Z)$ we associate $J_e = Q(e)Z$, the *principal inner ideal* generated by e, which is a complemented triple-ideal in Z. Let $\mathrm{Str}(Z)$ and Γ denote, respectively, the *structure group* of Z and its identity connected component. Γ is a complex Banach–Lie group whose Banach–Lie algebra $\mathfrak{str}(Z)$ is the complexification of $\mathfrak{aut}(Z)$. In contrast with $\mathrm{G} := \mathrm{Aut}^{\circ}(Z)$, Γ does not preserve the set of tripotents. However, it preserves $\mathrm{Reg}(Z) := \{a \in Z\colon a \in Q(a)Z\}$, the set of all *von Neumann regular* elements of Z, see [11]. In fact we have $\mathrm{Tri}(Z) \subset \mathrm{Reg}(Z)$ and $\mathrm{Reg}(Z)$ is the minimal Γ-invariant subset of Z that contains $\mathrm{Tri}(Z)$, i.e., $\mathrm{Reg}(Z) = \Gamma(\mathrm{Tri}(Z))$ is the orbit of $\mathrm{Tri}(Z)$ under Γ. The following result is known:

Lemma 3. *For $e, f \in \mathrm{Tri}(Z)$ the following conditions are equivalent:*

(a) *e and f generate the same principal inner ideal, i.e., $Q(e)Z = Q(f)Z$;*
(b) *$e \in Z_1(f)$ and $f \in Z_1(e)$;*
(c) *$D(e) = D(f)$;*
(d) *e and f have the same Peirce k-spaces ($k = 0, 1/2, 1$).*

Proof. The implications (b) ⇒ (c) ⇒ (d) are due to Neher [13, Theorem 2.3]. From (a) it immediately follows $e \in Z_1(f)Z$ and $f \in Z_1(e)$. By [11, Lemma 3.2(iv)] any von Neumann regular element (in particular, any tripotent) satisfies $Q(e)Z = Q^2(e)Z$, hence (d) ⇒ (a). □

Of course, any element $a \in Z$ gives rise to a principal inner ideal in Z, namely the inner ideal $J_a = Q(a)Z$, but it may fail to be complemented in Z. In fact J_a is complemented if and only if $a \in \mathrm{Reg}(Z)$, and in that case there is a tripotent $e := \rho(a) \in \mathrm{Tri}(Z)$ whose inner ideal is the same as that of a [11, Lemma 3.2]. Yet, different tripotents e and f may give rise to the same inner ideal which occurs if and only if e and f are equivalent in the sense of Neher. Thus we can establish a bijection between the set $\mathbb{P}$ of all *complemented* principal inner ideals in Z,

$$\mathbb{P} := \{Q(e)Z\colon e \in \mathrm{Tri}(Z)\}, \tag{2}$$

and the set $\mathrm{Tri}(Z)/\sim$ of Neher's equivalence classes of tripotents, the bijection being $J_e \leftrightarrow \mathbf{e}$ where $\mathbf{e}$ stands for the equivalence class of e and $J_e := Q(e)Z$.

By [11], $\mathbb{P}$ is a subset of $\mathbb{G}$, the Grassmann manifold of (the Banach space) Z. In fact, $\mathbb{P}$ is a closed complex submanifold of $\mathbb{G}$, and for every point $J_e \in \mathbb{P}$ the tangent space to $\mathbb{P}$ at J_e can be identified with $Z_{1/2}(e)$ in the following manner: for $u \in Z_{1/2}(e)$, set $g_u := \exp D(u, e) \in \Gamma$. Then

$$g_u(J_e) = \operatorname{graph} g_u = \{g_u(x)\colon x \in J_e\} \in \mathbb{P}$$

and

$$N_{J_e} := \{g_u(J_e)\colon u \in Z_{1/2}(e)\} \subset \mathbb{P} \tag{3}$$

is a neighbourhood of J_e in $\mathbb{P}$. The canonical local chart of $\mathbb{P}$ at J_e is the mapping

$$u \mapsto g_u(J_e), \quad u \in Z_{1/2}(e). \tag{4}$$

The following corollary is contained in [11] though it is not explicitly written down.

Corollary 4. *The action of the complex Banach–Lie group Γ on $\mathbb{P}$ admits local holomorphic cross sections, more precisely: to every $J_e \in \mathbb{P}$ there is a neighbourhood N_{J_e} of J_e in $\mathbb{P}$ and a holomorphic function $\chi : N_{J_e} \to \Gamma$ such that $[\chi(J)](J_e) = J$ for all $J \in N_{J_e}$.*

Proof. Let N_{J_e} be the neighbourhood of J_e in $\mathbb{P}$ given by (3), in which the canonical chart is defined by (4). According to the previous discussion, for each point J in N_{J_e} there is a unique vector, say $u = u(J)$, in $Z_{1/2}(e)$ such that $g_{u(J)}(J_e) = J$. The mapping $\chi : N_{J_e} \to \Gamma$ given by

$$J \in N_{J_e} \mapsto u(J) \in Z_{1/2}(e) \mapsto g_{u(J)} \in \Gamma$$

is holomorphic on N_{J_e} and by construction satisfies $\chi(J_e) = g_{u(J)}(J_e) = J$. □

Since Γ is a Lie-subgroup of $\mathsf{GL}(Z)$, the general linear group of (the Banach space) Z, each element $g \in \Gamma$ induces a holomorphic automorphism of the manifold $\mathbb{P}$. In particular, if $g \in \Gamma$ takes a point J to J', then the tangent spaces to $\mathbb{P}$ at J and J' are isomorphic as Banach spaces. Via the holomorphic section $\chi : N_{J_e} \to \Gamma$ we can *unambiguously* identify the tangent spaces to $\mathbb{P}$ at all points J in N_{J_e} with the tangent space at J_e (that is, with $Z_{1/2}(e)$). Hence every *vector field* $X : \mathbb{P} \to T\mathbb{P}$ can be locally represented in N_{J_e} as a vector-valued function $X : N_{J_e} \to Z_{1/2}(e)$. Via the canonical inclusion $Z_{1/2}(e) \hookrightarrow Z$, every vector field $X : \mathbb{P} \to T\mathbb{P}$ will be locally represented in $N_{J_e} \subset \mathbb{P}$ by a Z-valued function $X : N_{J_e} \to Z$ such that the values that X takes at the points N_{J_e} belong to $Z_{1/2}(e)$. Again it will be convenient to simplify the notation and we shall write X_e instead of X_{J_e}, with which we implicitly identify the inner ideal J_e and the class $\mathbf{e}$ of tripotents e which generate it. This will lead to no confusion since all tripotents e in the class $\mathbf{e}$ have the same Peirce projectors and it makes sense to write $Q(e)$, $P_1(e)$, etc., no matter which representative e we have taken in $\mathbf{e}$.

All tripotents in the same equivalence class $\mathbf{e}$ have the same rank r $(0 \leqslant r \leqslant \infty)$, which is constant over each connected component of $\mathbb{P}$. If M is the component of $\mathbf{e} = J_e$ for some $e \in \mathsf{Tri}(Z)$, then M is a symmetric complex Banach manifold which is the manifold associated to the triple-dual of $Z_{1/2}(e)$. In particular, M is of compact type, hence every complex-valued holomorphic function on M is constant [3]. The following extends to our setting some classical results due to E. Cartan in $\mathbb{C}^n$ [8, Chapter IV].

We let $\mathfrak{D}(\mathbb{P})$ denote the Lie algebra of smooth vector fields on $\mathbb{P}$. Let Y'_e be the Fréchet derivative of $Y \in \mathfrak{D}(\mathbb{P})$ at e (more precisely, at $J_e \in \mathbb{P}$). Thus Y'_e is a bounded linear operator $Z_{1/2}(e) \to Z$ and it makes sense to take the projection $P_{1/2}(e)(Y'_e X_e) \in Z_{1/2}(e)$.

Definition 5. Let M be a connected component of $\mathbb{P}$. We define a connection ∇ on M by

$$(\nabla_X Y)_e := P_{1/2}(e)\big(Y'_e X_e\big), \quad X, Y \in \mathfrak{D}(M),\ e \in M. \tag{5}$$

It is a matter of routine to check that ∇ is an affine connection on M. For $g \in G$, and more generally for $g \in \Gamma$, we have (see [11, p. 573])

$$gQ(e)g^{-1} = Q\bigl(g(e)\bigr), \qquad gP_k g^{-1}(e) = P_k\bigl(g(e)\bigr) \quad (k = 0, 1/2, 1), \tag{6}$$

for all $e \in \mathrm{Tri}(Z)$. With this, one can check that ∇ is Γ-invariant and torsion-free, that is

$$g(\nabla_X Y) = \nabla_{g(X)} g(Y), \quad g \in \Gamma, \ X, Y \in \mathfrak{D}(M),$$

where $(gX)_e := g'_e(X_{g_e^{-1}})$ for $X \in \mathfrak{D}(M)$, and

$$T(X, Y) := \nabla_X Y - \nabla_Y X - [Y, Y] = 0, \quad X, Y \in \mathfrak{D}(M).$$

Fix a tripotent $e \in \mathrm{Tri}(Z)$ and a vector $u \in Z_{1/2}(e)$. For $t \in \mathbb{R}$ set

$$g_t(u) := \exp 2tD(u, e) \in \Gamma.$$

Thus $t \mapsto g_t(u)$ is a curve in the complex Lie group Γ. Since $\mathrm{Tri}(Z)$ is contained in $\mathrm{Reg}(Z)$ and the latter set is Γ-invariant, by evaluating at $e \in \mathrm{Tri}(Z)$, we get a curve $t \mapsto \gamma(t) := g_t(u)e$ in $\mathrm{Reg}(Z)$. Since every $a \in \mathrm{Reg}(Z)$ has been identified with the point $J_a \in \mathbb{P}$ (where $J_a = Q(a)Z$ is the inner ideal generated by a), we can lift $\gamma(t)$ to a curve in $\mathbb{P}$ by

$$t \mapsto \hat{\gamma}(t) := J_{\gamma(t)} = J_{g_t(u)e}, \quad t \in \mathbb{R}. \tag{7}$$

Theorem 6. *Let Z, $\mathbb{P}$, and M, respectively, be a JB*-triple, the base space of the manifold of tripotents in Z, and the connected component of $e \in \mathrm{Tri}(Z)$. The geodesics of the connection ∇ in M that have origin in J_e are the curves $t \mapsto \hat{\gamma}(t)$ in (7).*

Proof. The claim amounts to saying that $\hat{\gamma}(t)$ satisfies the second order ordinary differential equation

$$\bigl(\nabla_{\dot{\hat{\gamma}}(t)} \dot{\hat{\gamma}}(t)\bigr)_{\hat{\gamma}(t)} = 0, \quad t \in \mathbb{R}. \tag{8}$$

In the canonical local chart at $\gamma(t) = g_t(u)e$, (8) becomes $(\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t))_{\gamma(t)} = 0$ for $t \in \mathbb{R}$. Now

$$\dot{\gamma}(t) = \bigl(\exp 2tD(u, e)\bigr) D(u, e)(e) = g_t(u) D(u, e)(e) = g_t(u)e,$$
$$\ddot{\gamma}(t) = \bigl(\exp 2tD(u, e)\bigr) D(u, e)^2(e) = g_t(u) D(u, e)^2(e) = g_t(u) D(u, e)(u).$$

From the Peirce decomposition of $D(u, e)(u)$ relative to e, calculated in [1, Lemma 2.6], and the assumption $u \in Z_{1/2}(e)$ we obtain $P_1(e)D(u, e)(u) = -2\{euu\}$. The main Jordan identity then yields $Q(e)\{euu\} = \{euu\}$, hence $P_1(e)D(u, e)(u) \in A(e)$ and so $P_1(e) \times D(u, e)(u) = 0$. Using the Γ-invariance of $P_1(e)$ and the property $P_1(e)D(u, e)(u) = 0$, we get

$$P_{1/2}\bigl(\gamma(t)\bigr)\dot{\gamma}(t) = P_{1/2}\bigl(g_t(u)e\bigr) g_t(u)e = g_t(u) P_{1/2}(e)u \in g_t(u) P_{1/2}(e) Z_{1/2}(e) = 0,$$
$$P_{1/2}\bigl(\gamma(t)\bigr)\ddot{\gamma}(t) = P_{1/2}\bigl(g_t(u)e\bigr) g_t(u) D(u, e)(u) = g_t(u) P_{1/2}(e) D(u, e)(u) = 0$$

and by (5) we finally have $\bigl(\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t)\bigr)_{\gamma(t)} = P_{1/2}(\dot{\gamma}(t))\ddot{\gamma}(t) = 0$. □

5. Manifolds of finite rank tripotents

Consider a JB*-triple Z, the base manifold $\mathbb{P}$ and the connected component M of J_e for a fixed tripotent $e \in \mathrm{Tri}(Z)$. When is it possible to introduce a Riemann (or a Kähler) manifold structure in M? For that the tangent space $T_eM \sim Z_{1/2}(e)$ has to be linearly homeomorphic to a Hilbert space, which occurs if and only if $Z_{1/2}(e)$ has finite rank [10]. For $e \in \mathrm{Min}(Z)$ we have $\operatorname{rank} Z_{1/2}(e) \leqslant 2$ by [11, Lemma 4.5], hence $Z_{1/2}(e)$ can either be a Hilbert space, an ℓ_∞ sum of two Hilbert spaces, or a complex spin factor and in all these cases M has a well-known Riemann structure. However, M may have a Riemann structure even if $e \notin \mathrm{Min}(Z)$.

In this section we answer this question when Z is a classical Cartan factor. Recall that classical Cartan factors come in four classes or types. Rectangular (or type I) Cartan factors are the spaces $Z := \mathcal{L}(H, K)$ where H and K are complex Hilbert spaces and $\dim H \leqslant \dim K$. Let H be equipped with a conjugation $\xi \to \bar{\xi}$ and let $z \to z'$ denote the associated transposition where $z'\xi := \overline{z^*\bar{\xi}}$ for $\xi \in H$ and $z \in \mathcal{L}(H)$. Then the classical symmetric and the anti-symmetric Cartan factors (or factors of types II and III) are defined as the spaces $Z := \{z \in \mathcal{L}(H)\colon z' = \varepsilon z\}$ where $\varepsilon = 1$ and $\varepsilon = -1$, respectively. Spin factors (or type IV Cartan factors) can be regarded as complex norm closed selfadjoint subspaces $Z \subset \mathcal{L}(H)$ such that $\{z^2\colon z \in Z\} \subset \mathbb{C}\mathrm{Id}$.

Definition 7 (cf. [5, p. 65]). For $a \in \mathfrak{A} := \mathcal{L}(H, K)$ we define the *operator rank* and *operator corank* by $\operatorname{rank_{op}}(a) := \dim a(H)$ and $\operatorname{corank_{op}}(a) := \max\{\dim \ker(a), \dim a(H)^\perp\}$.

A look to [12, Example 5.7] will illustrate this concept. Notice that $\operatorname{rank_{op}}(a^*) = \operatorname{rank_{op}}(a)$ and $\operatorname{corank_{op}}(a) = \operatorname{corank_{op}}(a^*)$, furthermore $\operatorname{rank_{op}}(a) + \operatorname{corank_{op}}(a) = \max\{\dim H, \dim K\}$. The operator rank and corank are lower semicontinuous functions on $\mathfrak{A}$ with values in $\mathbb{N} \cup \{\infty\}$.

Proposition 8. *Let Z be a JB*-triple and $e \in \mathrm{Tri}(Z)$. Then the following conditions are equivalent:*

(1) *The Peirce space $Z_{1/2}(e)$ is reflexive.*
(2) *$Z_{1/2}(e)$ is linearly homeomorphic to a Hilbert space.*
(3) $\operatorname{rank} Z_{1/2}(e) < \infty$.

Cartan factors of type IV *satisfy the above conditions. For Cartan factors of types* I–III, *these conditions are equivalent to*

(4) $\operatorname{rank_{op}}(e) < \infty$ *or* $\operatorname{corank_{op}}(e) < \infty$.

Proof. The equivalences (1) ⇔ (2) ⇔ (3) are well-known (e.g., [10] or [6, Theorem 6.2]) as is the assertion concerning spin factors. Let $p_1 := ee^*$ and $p_2 := e^*e$ denote the initial and final projections of the tripotent (partial isometry) e. Then

$$Z_{1/2}(e) = Z \cap \left[p_1\mathfrak{A}(\mathbb{1} - p_2) \oplus (\mathbb{1} - p_1)\mathfrak{A}p_2\right].$$

If (4) holds, then $p_1\mathfrak{A}(\mathbb{1}-p_2)$, which is linearly homeomorphic to $\mathcal{L}(p_1H,(\mathbb{1}-p_2)K)$, is linearly isomorphic to a Hilbert space because then $\dim p_1(H)<\infty$ or $\dim(\mathbb{1}-p_2)(K)<\infty$. Similarly $(\mathbb{1}-p_1)\mathfrak{A}p_2$ is linearly isomorphic to a Hilbert space. Hence $Z_{1/2}(e)$ is the direct sum of two Hilbert spaces, and so it is reflexive.

For the converse we make a type by type discussion.

Type I. In this case we have $Z_{1/2}(e)=p_1\mathfrak{A}(\mathbb{1}-p_2)\oplus(\mathbb{1}-p_1)\mathfrak{A}p_2$ where both direct summands are reflexive. Hence $\operatorname{rank}_{\mathrm{op}}(e)<\infty$ or $\operatorname{corank}_{\mathrm{op}}(e)<\infty$.

Types II and III. In these cases we have $p_1=\varepsilon p_2':=p$ and

$$Z_{1/2}(e)=\{x+\varepsilon x'\colon x=px(\mathbb{1}-p),\ x\in\mathcal{L}(H)\}\approx\{x\in\mathcal{L}(H)\colon x=px(\mathbb{1}-p)\}$$

is reflexive. Hence $\dim p(H)<\infty$ or $\dim(\mathbb{1}-p)(H)<\infty$. This completes the proof. □

From now on we assume that Z is a classical Cartan factor and that $e\in\operatorname{Tri}(Z)$ has finite rank r, and return to study the connected component M of the point $J_e\in\mathbb{P}$. Now also $s:=\operatorname{rank}Z_{1/2}(e)$ is finite. If $u\in Z_{1/2}(e)$ and $u=\sum_k\alpha_ke_k$ is a *spectral resolution* of u, then the sum

$$\langle u,u\rangle:=\sum_1^s\alpha_k\bar{\alpha}_k, \tag{9}$$

does not depend on the frame $(e_1,\dots,e_s)$ we have chosen, and the *algebraic inner product* in $Z_{1/2}(e)$ is defined by polarization in (9). Moreover, we have

$$\|u\|^2\leqslant\langle u,u\rangle\leqslant s\|u\|^2,\quad u\in Z_{1/2}(e),$$

so that $Z_{1/2}(e)$, the tangent space to M at J_e, is linearly homeomorphic to a Hilbert space under the *algebraic norm* (see [2, p. 161]). The map $\nu:TM\to\mathbb{R}$ which in the canonical chart $N_{J_e}\times Z_{1/2}(e)$ of TM at the point (J_e,T_eM) is given by

$$\nu(x,u):=\langle u,u\rangle,\quad x\in N_{J_e},\ u\in Z_{1/2}(e),$$

is a norm on M and (M,ν) is a Hilbert manifold. We can define a Riemann metric on M by

$$g_e(X,Y):=\langle X_e,Y_e\rangle,\quad X,Y\in\mathfrak{D}(M),\ e\in M.$$

Remark that g is *hermitian*, i.e., we have $g_e(iX,iY)=g_e(X,Y)$, and that it has been defined in algebraic terms. Moreover, ∇ is compatible with the Riemann structure, i.e.,

$$Xg(Y,W)=g(\nabla_XY,W)+g(Y,\nabla_XW),\quad X,Y,W\in\mathfrak{D}(M).$$

Therefore ∇ is the only Levi-Civita connection on M. On the other hand, ∇ satisfies

$$\nabla_X(iY)=i\nabla_XY,\quad X,Y\in\mathfrak{D}(M),$$

hence it is the only hermitian connection on M. Thus the Levi-Civita and the hermitian connection are the same in this case, and so ∇ is the Kähler connection on M.

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