On the manifold of tripotents in $JB^*$-triples

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Abstract

The manifold of tripotents in an arbitrary $JB^*$-triple $Z$ is considered, a natural affine connection is defined on it in terms of the Peirce projections of $Z$, and a precise description of its geodesics is given. Regarding this manifold as a fiber space by Neher’s equivalence, the base space is a symmetric Kähler manifold when $Z$ is a classical Cartan factor, and necessary and sufficient conditions are established for connected components of the manifold to admit a Riemann structure.

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1. Introduction

In [9] Hirzebruch proved that the manifold of minimal projections in a finite-dimensional simple formally real Jordan algebra is a compact Riemann symmetric space of rank 1, and that any such space arises in this way. Later on, in [14] Nomura estab-
lished similar results for the manifold of minimal projections in a topologically simple real Jordan–Hilbert algebra. Recently, Jordan algebras and projections have been replaced by the more general notions of JB*-triples and tripotents, respectively. JB*-triples are precisely those complex Banach spaces whose open unit balls are homogeneous with respect to biholomorphic transformations.

In [1] an affine connection \( \nabla \) on \( \mathcal{M} \), the manifold of tripotents in a JB*-triple \( Z \), was defined in terms of the natural algebraic triple product structure of \( Z \). Unfortunately, the description of the geodesics of \( \nabla \) given in [1, Theorem 2.7] by means of one-parameter groups of automorphisms of \( Z \) fails to be true in general since the corresponding second order differential equation is of sophisticated character. Our first goal is to develop a technique, based on exponential integrals, to find explicit formulas for the geodesics of \( \nabla \).

It is known that \( \mathcal{M} \) is a fibre space with respect to Neher’s relation of equivalence of tripotents. As proved by Kaup in [11], the base space \( \mathbb{P} \) of that fibration is the manifold of all complemented principal inner ideals of \( Z \), which is a closed complex submanifold of the Grassmannian \( \mathcal{G} = \mathcal{G}(Z) \). The connected components of \( \mathbb{P} \), which are orbits of \( \Gamma \) (the structure group of \( Z \)), are symmetric complex Banach manifolds on which \( \Gamma \) acts as a group of isometries, see [11]. We show that \( \nabla \) induces on these orbits a \( \Gamma \)-invariant torsion-free affine connection (also denoted by \( \nabla \)) and compute its geodesics which turn out to be orbits of one-parameter subgroups of \( \Gamma \).

All tripotents in the same equivalence class (in Neher’s sense) have the same rank \( r \) (\( 0 \leq r \leq \infty \)), that is constant over each connected component \( M \) of \( \mathbb{P} \). It is reasonable to ask which of these connected components admit a Riemann structure. For \( Z \) a classical Cartan factor, we solve that problem with the aid of the concepts of operator rank and operator corank, and prove that \( M \) admits a Riemann structure if and only if either the operator rank or the operator corank are finite, in which case we prove that \( \nabla \) is the Levi-Civita and the Kähler connection of \( M \). Some of these results were already known and due to E. Cartan in the \( \mathbb{C}^n \) setting.

2. JB*-triples and tripotents

For a complex Banach space \( Z \), denote by \( \mathcal{L}(Z) \) the Banach algebra of all bounded linear operators on \( Z \). A complex Banach space \( Z \) with a continuous mapping \((a,b,c) \mapsto \{abc\}\) from \( Z \times Z \times Z \) to \( Z \) is called a JB*-triple if the following conditions are satisfied for all \( a,b,c,d \in Z \), where the operator \( a \Box b \in \mathcal{L}(Z) \) is defined by \( z \mapsto \{abz\} \) and \([,]\) is the commutator product:

1. \( \{abc\} \) is symmetric complex linear in \( a,c \) and conjugate linear in \( b \).
2. \( [a \Box b,c \Box d] = \{abc\} \Box d - c \Box [dab] \).
3. \( a \Box a \) is hermitian and has spectrum \( \geq 0 \).
4. \( \|\{aaa\}\| = \|a\|^3 \).

If a complex vector space \( Z \) admits a JB*-triple structure, then the norm and the triple product determine each other. An automorphism is a bijection \( \phi \in \mathcal{L}(Z) \) such that \( \phi\{zzz\} = \{(\phi z)(\phi z)(\phi z)\} \) for \( z \in Z \) which occurs if and only if \( \phi \) is a surjective linear isometry of \( Z \).
By $\text{Aut}^0(Z)$ we denote the connected component of the identity in the topological group $\text{Aut}(Z)$ of all automorphisms of $Z$ (see [7]). Two elements $x, y$ in $Z$ are orthogonal if $x \boxtimes y = 0$ and $e \in Z$ is called a tripotent if $\{eee\} = e$. The set of tripotents, denoted by $\text{Tri}(Z)$, is endowed with the induced topology of $Z$. Clearly $e = 0$ is an isolated point in $\text{Tri}(Z)$. For $e \in \text{Tri}(Z)$, a conjugate-linear operator $Q(e) \in \mathcal{L}(Z)$, that commutes with $e \boxtimes e$, is defined by $Q(e)z = \{eze\}$ for $z \in Z$. If $e \in \text{Tri}(Z)$, then the set of eigenvalues of $e \boxtimes e \in \mathcal{L}(Z)$ is contained in $[0, 1/2, 1]$ and we have the topological direct sum decomposition, called the Peirce decomposition of $Z$,

$$Z = Z_1(e) \oplus Z_{1/2}(e) \oplus Z_0(e).$$

Here $Z_k(e)$ is the $k$-eigenspace of $e \boxtimes e$ and the Peirce projections $Z \rightarrow Z_k(e)$ with kernel $\bigoplus_{j \neq k} Z_j(e)$ are

$$P_1(e) = Q^2(e), \quad P_{1/2}(e) = 2(e \boxtimes e - Q^2(e)),$$

$$P_0(e) = 	ext{id} - 2e \boxtimes e + Q^2(e).$$

We shall use the Peirce rules $\{Z_l(e)Z_j(e)Z_k(e)\} \subset Z_{l-j+k}(e)$ where $Z_l(e) = \{0\}$ for $l \neq 0, 1/2, 1$. We note that $Z_1(e)$ is a complex unital JB$^*$-algebra in the product $a \circ b := \{aeb\}$ and involution $a^# := \{eae\}$. We have $Z_1(e) = A(e) \oplus iA(e)$ where

$$A(e) := \{z \in Z_1(e): z^# = z\}.$$

It is also customary to write $D(a, b)$ instead of $a \boxtimes b$ and $D(a)$ instead of $D(a, a)$ for $a, b \in Z$. A tripotent $e$ is said to be minimal or an atom in $Z$ if $e \neq 0$ and $P_1(e)Z = Ce$, and we let $\text{Min}(Z)$ be the set of them. A JB$^*$-triple $Z$ may have no nonzero tripotents. If $Z$ admits a (necessarily unique) predual space $Z_\ast$, then we say that $Z$ is a JBW$^*$-triple. The bidual $Z^{**}$ of a JB$^*$-triple is a JBW$^*$-triple and the canonical embedding $Z \hookrightarrow Z^{**}$ is a triple homomorphism. We let $Z^{**} = Z_\ast \oplus N$ denote the decomposition of the bidual $Z^{**}$ into its atomic and non-atomic ideals (see [4]). Here $Z_\ast = \bigoplus_{i \in I} F_i$ is the $\ell_\infty$-sum of the family of all minimal $\omega$-closed ideals $F_i$ in $Z^{**}$, each $F_i$ is a Cartan factor and $N$ contains no atoms. Every $e \in \text{Tri}(Z_\ast), e \neq 0$, has a decomposition of the form $e = \sum_{j \in J} e_j$ where the $e_j$ are pairwise orthogonal tripotent that are minimal in $Z^{**}$ and the series converges in the weak$^*$ topology of $Z^{**}$. The cardinality of $J$ is uniquely determined, and the rank of $e$ is defined to be that cardinality when finite and to be infinite otherwise. For $e \in N$ we set $\text{rank}(e) = 0$. Every JBW$^*$-triple $Z$ contains a (possibly empty) maximal family $(e_j)_{j \in J}$ of pairwise orthogonal minimal tripotents, and the (necessarily unique) cardinality of $J$ is the rank of $Z$. For details on JB$^*$-triples see [4,12].

3. The algebraic connection on the manifold of tripotents

Let $Z$ be a JB$^*$-triple and $\text{Tri}(Z)$ the set of all tripotents in $Z$ endowed with the relative topology of $Z$. Fix any nonzero tripotent $e_0 \in \text{Tri}(Z)$, and denote by $M$ the connected component of $e_0$ in $\text{Tri}(Z)$. Then all tripotents $e \in M$ have the same rank as $e_0$ and $\text{Aut}^0(Z)$ acts transitively on $M$ which is a real analytic manifold whose tangent space at a point $e$ is

$$T_eM = iA(e) \oplus Z_{1/2}(e).$$
For \( z = iv + u \in iA(e) \oplus Z_{1/2}(e) \) we set \( w := \frac{i}{2}v + 2u \) and \( K(e, z) := w \Box e - e \Box w \). Then [15, p. 25] a local chart of \( M \) at \( e \) in a suitable neighbourhood \( V \times U \) of \( (0, 0) \) in \( iA(e) \times Z_{1/2}(e) \) is given by

\[
  z \mapsto f(z) := \left[ \exp K(e, z) \right](e).
\]

We denote by \( P_e : Z \to iA(e) \oplus Z_{1/2}(e) \) the canonical projector from \( Z \) onto the tangent space \( iA(e) \oplus Z_{1/2}(e) \) to \( M \) at \( e \). By Peirce arithmetic, \( P_e \) is \( \text{Aut}^0(Z) \)-invariant as it satisfies

\[
  P_g(e)g(z) = gP_ee, \quad g \in \text{Aut}^0(Z), \ z \in Z.
\]

Let \( \mathfrak{D}(M) \) be the Lie algebra of all smooth vector fields on \( M \). We define the \textit{algebraic connection} \( \nabla \) on \( M \) by

\[
  (\nabla_X Y)_e := P_e(Y_e^X e), \quad e \in M, \ X, Y \in \mathfrak{D}(M).
\]

Then \( \nabla \) is a torsion-free \( \text{Aut}^0(Z) \)-invariant affine connection on \( M \). Recall a smooth curve \( \gamma : I \to M \), where \( I \) is a neighbourhood of \( 0 \in \mathbb{R} \), is a \( \nabla \)-geodesic if and only if

\[
  P_{\gamma(t)}\left( \frac{d^2}{dt^2} \gamma(t) \right) = 0.
\]

Recall that \( \text{Aut}(Z) \) is a Banach–Lie group whose Lie algebra can naturally be identified with \( \text{aut}(Z) \) the family of all skew \( Z \)-hermitian operators. In particular, any continuous mapping \( F : \mathbb{R} \to \text{aut}(Z) \) gives rise to (uniquely determined) left and right primitive functions \( L^F : \mathbb{R} \to \text{Aut}(Z), \ R^F : \mathbb{R} \to \text{Aut}(Z) \) with the property \( L^F(0) = R^F(0) = \text{id} \) and

\[
  \frac{d}{dt} L^F(t) = \left[ L^F(t) \right] F(t), \quad \frac{d}{dt} R^F(t) = F(t) \left[ R^F(t) \right].
\]

In the sequel we shall only use the left primitive functions.

**Lemma 1.** Let \( Z \) be a \( JB^* \)-triple and let the manifold \( M \) be as above. Given \( e \in M \) and a smooth curve \( a : \mathbb{R} \to iA(e) \oplus Z_{1/2}(e) = T_eM \), the curve

\[
  \gamma(t) := g(t)e \quad \text{where} \quad g(t) := L^K(e, a(t))
\]

in \( M \) is a \( \nabla \)-geodesic if and only if \( a(t) = iv_0 + \exp(-3tv_0 \Box e)u_0 \) for some \( v_0 \in A(e) \) and \( u_0 \in Z_{1/2}(e) \).

**Proof.** We use the decomposition \( a(t) = iv(t) + u(t) \) with \( v(t) \in A(e) \) and \( u(t) \in Z_{1/2}(e) \). An immediate calculation yields

\[
  K(e, a(t))e = a(t),
\]

\[
  K(e, a(t))^2 e = K(e, a(t))a(t) = -\alpha(t) + i\beta(t) + 2\{uev\} - 2i\{euv\},
\]

where

\[
  \alpha(t) := \frac{1}{2}\{uev\} + \frac{1}{2}\{euv\} + 2\{euv\}, \quad \beta(t) := \frac{1}{2}\{uev\} + 2\{uev\} + \frac{1}{2}\{euv\}.
\]

Here

\[
  \{uev\} \in \left\{ Z_{1/2}(e)Z_1(e)Z_{1/2}(e) \right\} \subset Z_0(e)
\]
and therefore \( P_e\{ueu\} = 0 \). Also
\[
\{ueva\} \in \{Z_1Z_{1/2}Z_1\} = Z_{1-1/2+1} = 0.
\]
By Peirce rules, the summands in \( \beta(t) \) belong to \( Z_{1/2} \). Furthermore, since \( \{veu\} \in Z_{1-1+1/2} \) and hence \( \{e\{veu\}e\} \in Z_{1-1/2+1} = 0 \), the Jordan identity yields
\[
\{veu\} = \{e\{veu\}e\} = 2\{eue\} = 2\{uve\}.
\]
On the other hand, \( \{uev\}, \{euv\} \in \{A(e)A(e)A(e)\} = A(e) \). By the Jordan identity we have
\[
\{ueu\} = \{uu\{eee\}\} = 2\{uee\} = 2\{tie\} = 2\{uee\} = Q(e)\{uee\}.
\]
That is \( \{ueu\} = Q(e)\{ueu\} \in A(e) \) and \( \alpha(t) \in A(e) \). It follows
\[
P_eK(e, \alpha(t))e = a(t), \quad P_eK(e, \alpha(t))^2e = i\beta(t) = 3i\{v(t)ueu\}.
\]
As we know, the curve \( \gamma \) is \( \nabla \)-geodesic if and only if
\[
0 = P_\gamma\frac{d^2}{dt^2}\gamma(t) = P_\gamma\frac{d^2}{dt^2}g(t)e = g(t)P_eP_e^{-1}\frac{d}{dt}\left[\frac{d}{dt}g(t)\right]e
\]
\[
= g(t)P_e\frac{d}{dt}g(t)^{-1}\left[\left(\frac{d}{dt}g(t)\right)K(e, \alpha(t)) + g(t)\frac{d}{dt}K(e, \alpha(t))\right]e
\]
\[
= g(t)P_e\frac{d}{dt}g(t)^{-1}\left[\frac{d}{dt}g(t)K(e, \alpha(t))^2 + g(t)K(e, \alpha(t))^2\right]e
\]
\[
= g(t)\left[3i\{v(t)ueu\} + \frac{d}{dt}\alpha(t)\right].
\]
By passing to the components with respect to the decomposition \( T_eM = iA(e) \oplus Z_{1/2}(e) \), we conclude that \( \gamma \) is a \( \nabla \)-geodesic if and only if
\[
\frac{d}{dt}v(t) = 0, \quad \frac{d}{dt}u(t) = 3i(v(t)\square e)u(t),
\]
that is if and only if \( v(t) = v(0) =: v_0 \) and \( u(t) = \exp(3tv_0 \square e)u_0 \) with \( u_0 := u(0) \). \( \square \)

As a consequence we immediately get the following theorem.

**Theorem 2.** Given any point \( e \in M \) and a tangent vector \( z \in iA(e) \oplus Z_{1/2}(e) \), there is a unique \( \nabla \)-geodesic \( \gamma_{e,z}^\nabla : I \to M \) with \( \gamma_{e,z}^\nabla(0) = e \) and \( \gamma_{e,z}^\nabla'(0) = z \), and we have the explicit formula
\[
\gamma_{e,z}^\nabla(t) = L_K(e, i\gamma + \exp(3tv \square e)u)
\]
for the \( \nabla \)-geodesics in terms of left primitive functions of \( \mathbb{R} \to TM \) maps. In particular, the curve \( \gamma_{e,z}(t) := \exp(tK(e, z))e \) is a \( \nabla \)-geodesic if and only if \( \{veu\} = \{ueu\} = 0 \).
4. The base space of the manifold of tripotents in a JB*-triple

It is known \cite{11} that $M$ is a fiber space, the typical fiber being a manifold whose tangent space at $e$ is $iT(e)$. We shall now study the base manifold of this fibre space. To each tripo
totent $e \in \text{Tri}(Z)$ we associate $J_e = Q(e)Z$, the principal inner ideal generated by $e$, which is a complemented triple-ideal in $Z$. Let $\text{Str}(Z)$ and $\Gamma$ denote, respectively, the structure group of $Z$ and its identity connected component. $\Gamma$ is a complex Banach–Lie group whose Banach–Lie algebra $\text{str}(Z)$ is the complexification of $\text{aut}(Z)$. In contrast with $G := \text{Aut}^*(Z)$, $\Gamma$ does not preserve the set of tripotents. However, it preserves $\text{Reg}(Z) := \{a \in Z: a \in Q(a)Z\}$, the set of all von Neumann regular elements of $Z$, see \cite{11}. In fact we have $\text{Tri}(Z) \subset \text{Reg}(Z)$ and $\text{Reg}(Z)$ is the minimal $\Gamma$-invariant subset of $Z$ that contains $\text{Tri}(Z)$, i.e., $\text{Reg}(Z) = \Gamma(\text{Tri}(Z))$ is the orbit of $\text{Tri}(Z)$ under $\Gamma$. The following result is known:

**Lemma 3.** For $e, f \in \text{Tri}(Z)$ the following conditions are equivalent:

(a) $e$ and $f$ generate the same principal inner ideal, i.e., $Q(e)Z = Q(f)Z$;
(b) $e \in Z_1(f)$ and $f \in Z_1(e)$;
(c) $D(e) = D(f)$;
(d) $e$ and $f$ have the same Peirce $k$-spaces ($k = 0, 1/2, 1$).

**Proof.** The implications (b) $\Rightarrow$ (e) $\Rightarrow$ (d) are due to Neher \cite[Theorem 2.3]{13}. From (a) it immediately follows $e \in Z_1(f)Z$ and $f \in Z_1(e)$. By \cite[Lemma 3.2(iv)]{11} any von Neumann regular element (in particular, any tripo
totent) satisfies $Q(e)Z = Q^2(e)Z$, hence (d) $\Rightarrow$ (a). \hfill \Box

Of course, any element $a \in Z$ gives rise to a principal inner ideal in $Z$, namely the inner ideal $J_a = Q(a)Z$, but it may fail to be complemented in $Z$. In fact $J_a$ is complemented if and only if $a \in \text{Reg}(Z)$, and in that case there is a tripo
totent $e := \rho(a) \in \text{Tri}(Z)$ whose inner ideal is the same as that of $a$ \cite[Lemma 3.2]{11}. Yet, different tripotents $e$ and $f$ may give rise to the same inner ideal which occurs if and only if $e$ and $f$ are equivalent in the sense of Neher. Thus we can establish a bijection between the set $\mathbb{P}$ of all complemented principal inner ideals in $Z$,

\[ \mathbb{P} := \{ Q(e)Z: e \in \text{Tri}(Z) \}, \]

and the set $\text{Tri}(Z)/\sim$ of Neher's equivalence classes of tripotents, the bijection being $J_e \leftrightarrow e$ where $e$ stands for the equivalence class of $e$ and $J_e := Q(e)Z$.

By \cite{11}, $\mathbb{P}$ is a subset of $G$, the Grassmann manifold of (the Banach space) $Z$. In fact, $\mathbb{P}$ is a closed complex submanifold of $G$, and for every point $J_e \in \mathbb{P}$ the tangent space to $\mathbb{P}$ at $J_e$ can be identified with $Z_{1/2}(e)$ in the following manner: for $u \in Z_{1/2}(e)$, set $g_u := \exp D(u, e) \in \Gamma$. Then

\[ g_u(J_e) = \text{graph } g_u = \{ g_u(x): x \in J_e \} \in \mathbb{P} \]

and

\[ N_{J_e} := \{ g_u(J_e): u \in Z_{1/2}(e) \} \subset \mathbb{P} \]
is a neighbourhood of \( J_e \) in \( \mathbb{P} \). The canonical local chart of \( \mathbb{P} \) at \( J_e \) is the mapping

\[
u \mapsto g_u(J_e), \quad u \in Z_{1/2}(e).
\]  

The following corollary is contained in [11] though it is not explicitly written down.

**Corollary 4.** The action of the complex Banach–Lie group \( \Gamma \) on \( \mathbb{P} \) admits local holomorphic cross sections, more precisely: to every \( J_e \in \mathbb{P} \) there is a neighbourhood \( N_{J_e} \) of \( J_e \) in \( \mathbb{P} \) and a holomorphic function \( \chi : N_{J_e} \to \Gamma \) such that \( \chi(J_e)(J_e) = J \) for all \( J \in N_{J_e} \).

**Proof.** Let \( N_{J_e} \) be the neighbourhood of \( J_e \) in \( \mathbb{P} \) given by (3), in which the canonical chart is defined by (4). According to the previous discussion, for each point \( J \) in \( N_{J_e} \) there is a unique vector, say \( u = u(J) \), in \( Z_{1/2}(e) \) such that \( g_u(J)(J_e) = J \). The mapping \( \chi : N_{J_e} \to \Gamma \) given by

\[
J \in N_{J_e} \mapsto u(J) \in Z_{1/2}(e) \mapsto g_u(J) \in \Gamma
\]

is holomorphic on \( N_{J_e} \) and by construction satisfies \( \chi(J_e) = g_u(J)(J_e) = J \). \( \square \)

Since \( \Gamma \) is a Lie-subgroup of \( \text{GL}(Z) \), the general linear group of (the Banach space) \( Z \), each element \( g \in \Gamma \) induces a holomorphic automorphism of the manifold \( \mathbb{P} \). In particular, if \( g \in \Gamma \) takes a point \( J \) to \( J' \), then the tangent spaces to \( \mathbb{P} \) at \( J \) and \( J' \) are isomorphic as Banach spaces. Via the holomorphic section \( \chi : N_{J_e} \to \Gamma \) we can unambiguously identify the tangent spaces to \( \mathbb{P} \) at all points \( J \) in \( N_{J_e} \) with the tangent space at \( J_e \) (that is, with \( Z_{1/2}(e) \)). Hence every vector field \( X : \mathbb{P} \to T\mathbb{P} \) can be locally represented in \( N_{J_e} \) as a vector-valued function \( X : N_{J_e} \to Z_{1/2}(e) \). Via the canonical inclusion \( Z_{1/2}(e) \to Z \), every vector field \( X : \mathbb{P} \to T\mathbb{P} \) will be locally represented in \( N_{J_e} \subset \mathbb{P} \) by a \( Z \)-valued function \( X : N_{J_e} \to Z \) such that the values that \( X \) takes at the points \( N_{J_e} \) belong to \( Z_{1/2}(e) \). Again it will be convenient to simplify the notation and we shall write \( X_e \) instead of \( X_{J_e} \), with which we implicitly identify the inner ideal \( J_e \) and the class \( e \) of tripotents \( e \) which generate it. This will lead to no confusion since all tripotents \( e \) in the class \( e \) have the same Peirce projectors and it makes sense to write \( Q(e) \), \( P_1(e) \), etc., no matter which representative \( e \) we have taken in \( e \).

All tripotents in the same equivalence class \( e \) have the same rank \( r \) (0 \( \leqslant r \leqslant \infty \)), which is constant over each connected component of \( \mathbb{P} \). If \( M \) is the component of \( e = J_e \) for some \( e \in \text{Tri}(Z) \), then \( M \) is a symmetric complex Banach manifold which is the manifold associated to the triple-dual of \( Z_{1/2}(e) \). In particular, \( M \) is of compact type, hence every complex-valued holomorphic function on \( M \) is constant [3]. The following extends to our setting some classical results due to E. Cartan in \( \mathbb{C}^n \) [8, Chapter IV].

We let \( \mathcal{D}(\mathbb{P}) \) denote the Lie algebra of smooth vector fields on \( \mathbb{P} \). Let \( Y'_e \) be the Fréchet derivative of \( Y \in \mathcal{D}(\mathbb{P}) \) at \( e \) (more precisely, at \( J_e \in \mathbb{P} \)). Thus \( Y'_e \) is a bounded linear operator \( Z_{1/2}(e) \to Z \) and it makes sense to take the projection \( P_{1/2}(e)(Y'_eX_e) \in Z_{1/2}(e) \).

**Definition 5.** Let \( M \) be a connected component of \( \mathbb{P} \). We define a connection \( \nabla \) on \( M \) by

\[
(\nabla_X Y)_e := P_{1/2}(e)(Y'_eX_e), \quad X, Y \in \mathcal{D}(M), \quad e \in M.
\]  

(5)
It is a matter of routine to check that $\nabla$ is an affine connection on $M$. For $g \in G$, and more generally for $g \in G$, we have (see [11, p. 573])
\begin{align*}
g Q (e) g^{-1} = Q (g (e)), \quad g P_k g^{-1} (e) = P_k (g (e)) \quad (k = 0, 1/2, 1), \quad (6)
\end{align*}
for all $e \in \text{Tri}(Z)$. With this, one can check that $\nabla$ is $\Gamma$-invariant and torsion-free, that is
\begin{align*}
g (\nabla_X Y) = \nabla_{g (X)} g (Y), \quad g \in \Gamma, \quad X, Y \in \mathcal{D}(M),
\end{align*}
where $(gX)_e := g'_e (X_{g' e})$ for $X \in \mathcal{D}(M)$, and
\begin{align*}
T (X, Y) := \nabla_X Y - \nabla_Y X - [Y, Y] = 0, \quad X, Y \in \mathcal{D}(M).
\end{align*}

Fix a tripotent $e \in \text{Tri}(Z)$ and a vector $u \in Z_{1/2}(e)$. For $t \in \mathbb{R}$ set
\begin{align*}
g_t (u) := \exp 2t D (u, e) \in \Gamma.
\end{align*}
Thus $t \mapsto g_t (u)$ is a curve in the complex Lie group $\Gamma$. Since $\text{Tri}(Z)$ is contained in $\text{Reg}(Z)$ and the latter set is $\Gamma$-invariant, by evaluating at $e \in \text{Tri}(Z)$, we get a curve $t \mapsto \gamma (t) := g_t (u) e \in \text{Reg}(Z)$. Since every $a \in \text{Reg}(Z)$ has been identified with the point $J_a \in \mathbb{P}$ (where $J_a = Q (a) Z$ is the inner ideal generated by $a$), we can lift $\gamma (t)$ to a curve in $\mathbb{P}$ by
\begin{align*}
t \mapsto \hat{\gamma} (t) := J_{\gamma (t)} = J_{g_t (u) e}, \quad t \in \mathbb{R}. \quad (7)
\end{align*}

**Theorem 6.** Let $Z$, $\mathbb{P}$, and $M$, respectively, be a JB*-triple, the base space of the manifold of tripotents in $Z$, and the connected component of $e \in \text{Tri}(Z)$. The geodesics of the connection $\nabla$ in $M$ that have origin in $J_e$ are the curves $t \mapsto \hat{\gamma} (t)$ in (7).

**Proof.** The claim amounts to saying that $\hat{\gamma} (t)$ satisfies the second order ordinary differential equation
\begin{align*}
(\nabla_{\hat{\gamma} (t)} \dot{\hat{\gamma}} (t))_{\hat{\gamma} (t)} = 0, \quad t \in \mathbb{R}. \quad (8)
\end{align*}
In the canonical local chart at $\gamma (t) = g_t (u) e$, (8) becomes $(\nabla_{\hat{\gamma} (t)} \dot{\hat{\gamma}} (t))_{\gamma (t)} = 0$ for $t \in \mathbb{R}$. Now
\begin{align*}
\dot{\gamma} (t) &= (\exp 2t D (u, e)) D (u, e) (e) = g_t (u) D (u, e) (e) = g_t (u) e, \\
\ddot{\gamma} (t) &= (\exp 2t D (u, e)) D (u, e) (e) = g_t (u) D (u, e)^2 (e) = g_t (u) D (u, e) (u). \\
\end{align*}
From the Peirce decomposition of $D (u, e) (u)$ relative to $e$, calculated in [1, Lemma 2.6], and the assumption $u \in Z_{1/2}(e)$ we obtain $P_{1/2} (e) D (u, e) (u) = -2 [e u u]$. The main Jordan identity then yields $Q (e) [e u u] = [e u u]$, hence $P_{1/2} (e) D (u, e) (u) \in A (e)$ and so $P_{1/2} (e) \times D (u, e) (u) = 0$. Using the $\Gamma$-invariance of $P_{1/2} (e)$ and the property $P_{1/2} (e) D (u, e) (u) = 0$, we get
\begin{align*}
P_{1/2} (\gamma (t)) \dot{\gamma} (t) &= P_{1/2} (g_t (u) e) g_t (u) e = g_t (u) P_{1/2} (e) u \in g_t (u) P_{1/2} (e) Z_{1/2}(e) = 0, \\
P_{1/2} (\gamma (t)) \ddot{\gamma} (t) &= P_{1/2} (g_t (u) e) g_t (u) D (u, e) (u) = g_t (u) P_{1/2} (e) D (u, e) (u) = 0
\end{align*}
and by (5) we finally have $(\nabla_{\hat{\gamma} (t)} \dot{\hat{\gamma}} (t))_{\gamma (t)} = P_{1/2} (\dot{\gamma} (t)) \ddot{\gamma} (t) = 0$. \qed
5. Manifolds of finite rank tripotents

Consider a JB*-triple $Z$, the base manifold $\mathbb{P}$ and the connected component $M$ of $J_e$ for a fixed tripotent $e \in \text{Tri}(Z)$. When is it possible to introduce a Riemann (or a Kähler) manifold structure in $M$? For that the tangent space $T_eM \sim Z_{1/2}(e)$ has to be linearly homeomorphic to a Hilbert space, which occurs if and only if $Z_{1/2}(e)$ has finite rank [10]. For $e \in \text{Min}(Z)$ we have rank $Z_{1/2}(e) \leq 2$ by [11, Lemma 4.5], hence $Z_{1/2}(e)$ can either be a Hilbert space, an $\ell_\infty$ sum of two Hilbert spaces, or a complex spin factor and in all these cases $M$ has a well-known Riemann structure. However, $M$ may have a Riemann structure even if $e \notin \text{Min}(Z)$.

In this section we answer this question when $Z$ is a classical Cartan factor. Recall that classical Cartan factors come in four classes or types. Rectangular (or type I) Cartan factors are the spaces $Z := \mathcal{L}(H, K)$ where $H$ and $K$ are complex Hilbert spaces and $\dim H \leq \dim K$. Let $H$ be equipped with a conjugation $\xi \mapsto \overline{\xi}$ and let $z \mapsto z'$ denote the associated transposition where $z'\xi := z^*\overline{\xi}$ for $\xi \in H$ and $z \in \mathcal{L}(H)$. Then the classical symmetric and the anti-symmetric Cartan factors (or factors of types II and III) are defined as the spaces $Z := \{z \in \mathcal{L}(H) : z' = \varepsilon z\}$ where $\varepsilon = 1$ and $\varepsilon = -1$, respectively. Spin factors (or type IV Cartan factors) can be regarded as complex norm closed selfadjoint subspaces $Z \subset \mathcal{L}(H)$ such that $\{z^2 : z \in Z\} \subset \mathbb{C}d$.

**Definition 7** (cf. [5, p. 65]). For $a \in \mathcal{A} := \mathcal{L}(H, K)$ we define the operator rank and operator corank by $\text{rank}_{\text{op}}(a) := \dim a(H)$ and $\text{corank}_{\text{op}}(a) := \max\{\dim \ker(a), \dim a(H)^\perp\}$.

A look to [12, Example 5.7] will illustrate this concept. Notice that $\text{rank}_{\text{op}}(a^*) = \text{rank}_{\text{op}}(a)$ and $\text{corank}_{\text{op}}(a) = \text{corank}_{\text{op}}(a^*)$, furthermore $\text{rank}_{\text{op}}(a) + \text{corank}_{\text{op}}(a) = \max\{\dim H, \dim K\}$. The operator rank and corank are lower semicontinuous functions on $\mathcal{A}$ with values in $\mathbb{N} \cup \{\infty\}$.

**Proposition 8.** Let $Z$ be a JB*-triple and $e \in \text{Tri}(Z)$. Then the following conditions are equivalent:

1. The Peirce space $Z_{1/2}(e)$ is reflexive.
2. $Z_{1/2}(e)$ is linearly homeomorphic to a Hilbert space.
3. $\text{rank} Z_{1/2}(e) < \infty$.

Cartan factors of type IV satisfy the above conditions. For Cartan factors of types I–III, these conditions are equivalent to

4. $\text{rank}_{\text{op}}(e) < \infty$ or $\text{corank}_{\text{op}}(e) < \infty$.

**Proof.** The equivalences (1) $\iff$ (2) follow from [10], [6, Theorem 6.2]) as is the assertion concerning spin factors. Let $p_1 := ee^*$ and $p_2 := e^*e$ denote the initial and final projections of the tripotent (partial isometry) $e$. Then

$$Z_{1/2}(e) = Z \cap [p_1 \mathcal{A}(1 - p_2) \oplus (1 - p_1) \mathcal{A} p_2].$$
If (4) holds, then \( p_1 \mathcal{A}(1 - p_2) \), which is linearly homeomorphic to \( \mathcal{L}(p_1 H, (1 - p_2)K) \), is linearly isomorphic to a Hilbert space because then \( \dim p_1(H) < \infty \) or \( \dim(1 - p_2)(K) < \infty \). Similarly \( (1 - p_1)\mathcal{A}_2 \) is linearly isomorphic to a Hilbert space. Hence \( Z_{1/2}(e) \) is the direct sum of two Hilbert spaces, and so it is reflexive.

For the converse we make a type by type discussion.

Type I. In this case we have \( Z_{1/2}(e) = p_1 \mathcal{A}(1 - p_2) \oplus (1 - p_1)\mathcal{A}_2 \) where both direct summands are reflexive. Hence \( \text{rank}_{\text{op}}(e) < \infty \) or \( \text{corank}_{\text{op}}(e) < \infty \).

Types II and III. In these cases we have \( p_1 = \epsilon p_1' := p \) and

\[
Z_{1/2}(e) = \{ x + \epsilon x' : x = px(1 - p), \ x \in \mathcal{L}(H) \} \approx \{ x \in \mathcal{L}(H) : x = px(1 - p) \}
\]

is reflexive. Hence \( \dim p(H) < \infty \) or \( \dim(1 - p)(H) < \infty \). This completes the proof. \( \Box \)

From now on we assume that \( Z \) is a classical Cartan factor and that \( e \in \text{Tri}(Z) \) has finite rank \( r \), and return to study the connected component \( M \) of the point \( J_e \in \mathcal{P} \). Now also \( s := \text{rank} Z_{1/2}(e) \) is finite. If \( u \in Z_{1/2}(e) \) and \( u = \sum_k \alpha_k e_k \) is a spectral resolution of \( u \), then the sum

\[
\langle u, u \rangle := \sum_k \alpha_k \tilde{\alpha}_k,
\]

\[ (9) \]
does not depend on the frame \((e_1, \ldots, e_s)\) we have chosen, and the algebraic inner product in \( Z_{1/2}(e) \) is defined by polarization in (9). Moreover, we have

\[
\|u\|^2 \leq \langle u, u \rangle \leq s\|u\|^2, \quad u \in Z_{1/2}(e),
\]

so that \( Z_{1/2}(e) \), the tangent space to \( M \) at \( J_e \), is linearly homeomorphic to a Hilbert space under the algebraic norm (see [2, p. 161]). The map \( \nu : TM \to \mathbb{R} \) which in the canonical chart \( N_{J_e} \times Z_{1/2}(e) \) of \( TM \) at the point \( (J_e, T_e M) \) is given by

\[
\nu(x, u) := \langle u, u \rangle, \quad x \in N_{J_e}, u \in Z_{1/2}(e),
\]

is a norm on \( M \) and \((M, \nu)\) is a Hilbert manifold. We can define a Riemann metric on \( M \) by

\[
g_e(X, Y) := \langle X_e, Y_e \rangle, \quad X, Y \in \mathcal{D}(M), \quad e \in M.
\]

Remark that \( g \) is hermitian, i.e., we have \( g_e(iX, iY) = g_e(X, Y) \), and that it has been defined in algebraic terms. Moreover, \( \nabla \) is compatible with the Riemann structure, i.e.,

\[
X g(Y, W) = g(\nabla_X Y, W) + g(Y, \nabla_X W), \quad X, Y, W \in \mathcal{D}(M).
\]

Therefore \( \nabla \) is the only Levi-Civita connection on \( M \). On the other hand, \( \nabla \) satisfies

\[
\nabla_X(iY) = i\nabla_X Y, \quad X, Y \in \mathcal{D}(M),
\]

hence it is the only hermitian connection on \( M \). Thus the Levi-Civita and the hermitian connection are the same in this case, and so \( \nabla \) is the Kähler connection on \( M \).

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