LAA 7965
24/12/2003 DISK used
ARTICLE IN PRESS
No. of pages: 12, DTD=4.3.1

Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Linear Algebra and its Applications xx (2003) xxx–xxx
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Bicircular projections on some matrix and operator spaces

L.L. Stachó [a], B. Zalar [b,*]

[a] Bolyai Institute, University of Szeged, Aradi Vértanúk tere 1, 6270 Szeged, Hungary
[b] Department of Basic Sciences, Faculty of Civil Engineering, University of Maribor, Smetanova 17, 2000 Maribor, Slovenia

Received 7 January 2003; accepted 20 November 2003

Submitted by V. Mehrmann

Abstract

A projection on a complex Banach space is bicircular if its linear combinations with the complementary projection having coefficients of modulus one are isometries. Such projections are always bicontractive. In this paper we prove structure theorems for bicircular projections acting on the spaces of the full operator algebra, symmetric operators and antisymmetric operators.

1. Introduction

The study of bicircular projections is motivated by complex analysis, more specifically by the study of Reinhardt domains (see for example [12]). Their definition is however purely Banach theoretic. Let X be a complex Banach space in some norm $\|\dots\|$ and let $P : X \to X$ be a bounded linear projection. We always denote by $\overline{P}$ its complementary projection which is simply $1 - P$. Then we say that P is bicircular if all mappings of the form $e^{i\alpha} P + e^{i\beta}\overline{P}$ are isometric for all pairs of real numbers α, β. It is obvious that P is bicircular if and only if $\overline{P}$ is bicircular. It is also obvious that the definition need not be symmetric so we may require only that $P + e^{i\theta}\overline{P}$

* Corresponding author.
E-mail address: bojana.zalar@uni-mb.si (B. Zalar).

0024-3795/$ - see front matter © 2003 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2003.11.014

is isometric for every real θ. Every bicircular projection is contractive. This simple observation follows from the fact that

$$P = \frac{1}{2}(2P-1) + \frac{1}{2} = \frac{1}{2}\left(\mathrm{e}^{0\mathrm{i}}P + \mathrm{e}^{\mathrm{i}\pi}\overline{P}\right) + \frac{1}{2}.$$

The first natural problem is, being given a Banach space, to describe the structure of its bicircular projections. It is obvious that the answer depends on a given norm and can change if an equivalent norm is taken instead. We shall work with some matrix spaces so we note that we *always* consider any space of matrices to be equipped with the spectral norm so we view $M_n(\mathbb{C})$ as a special case of the algebra $B(H)$ of all bounded operators on a Hilbert space.

It is rather easy to see that every orthogonal projection on a Hilbert space is in fact bicircular. In the matrix case it can happen that both P and $\overline{P}$ are of norm one but are not bicircular. One example is

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \mapsto \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}.$$

In the present paper we investigate the structure of bicircular projections on the spaces of all square matrices, all symmetric matrices and all antisymmetric matrices. The study of bicircular projections on those spaces leads to a nice interplay between geometry and algebra so we feel this topic deserves further attention. The results we obtain are rather interesting—in particular for $M_n(\mathbb{C})$ there is a simple formula for bicircular projections which, except for $P \in \{0, 1\}$, does not preserve the subspaces of symmetric and antisymmetric matrices. For the space of symmetric matrices we found that 0 and 1 are the only bicircular projections. To some surprise the space of antisymmetric matrices admits nontrivial bicircular projections. They can be described by an algebraic formula using projections on the space $\mathbb{C}^n$ but only those whose range or kernel is one-dimensional.

It turned out that the result we obtained could be generalized to operators on Hilbert spaces. The essential ingredient is a proposition which shows that bicircularity implies the existence of an orthonormal base, consisting of eigenvalues of an auxiliary selfadjoint operator, thus permitting us to imitate original matrix calculations in the setting of $B(H)$.

2. Motivation and main results

Let H be a complex Hilbert space. It can be equipped with a conjugation, i.e. a conjugate linear isometric mapping $\alpha \mapsto \overline{\alpha}$ which satisfies $\overline{\overline{\alpha}} = \alpha$ for all $\alpha \in H$. If $H = \mathbb{C}^n$, the canonical conjugation is $(\alpha_1, \ldots, \alpha_n) \to (\overline{\alpha_1}, \ldots, \overline{\alpha_n})$. In general, by taking an orthonormal basis $\{e_i : i \in I\}$ in H, we can define $\overline{\alpha} := \sum_i \langle e_i, \alpha\rangle e_i$. The isometric property $\|\overline{\alpha}\| = \|\alpha\|$ can be linearized, using the conjugate linearity, into

$$\langle\overline{\alpha}, \overline{\beta}\rangle = \langle\beta, \alpha\rangle : \alpha, \beta \in H.$$

We say that the vector α from H is real if $\overline{\alpha} = \alpha$.

By $B(H)$ we denote the algebra of continuous linear operators acting on H equipped with the operator (spectral) norm. If a^* denotes the classical adjoint of the operator a (defined by $\langle a(\alpha), \beta\rangle = \langle \alpha, a^*(\beta)\rangle$), we can define the transposed operator by

$$a^{\mathrm{t}}(\alpha) = \overline{a^*(\overline{\alpha})}.$$

If $H = \mathbb{C}^n$ is equipped with the canonical conjugation, a^{t} represents the classical transposition $[\alpha_{ij}]^{\mathrm{t}} = [\alpha_{ji}]$. The mapping $a \mapsto a^{\mathrm{t}}$ is complex linear, isometric and commutes with the adjoint. The (complex) subspace of symmetric (antisymmetric) operators is defined by

$$S(H) = \{a \in B(H) : a^{\mathrm{t}} = a\},$$
$$A(H) = \{a \in B(H) : a^{\mathrm{t}} = -a\}.$$

Actually the space $A(H)$ is one of the classical Banach Lie algebras but we do not use this fact later. In the case of $B(H)$ we have natural candidates for bicircular projections. In fact the following observation is based on the identity $\|xx^*\| = \|x\|^2$ which holds for the spectral norm, so it is valid for all C*-algebras.

Observation 2.1. *If A is a C*-algebra and $p \in A$ is a selfadjoint projection, then the mappings defined by $Px = px$ and $Qx = xp$ are bicircular projections.*

Proof. We shall treat only the case of the operator P. Since $\overline{P}x = (1-p)x$, we have, for all real θ and $x \in A$,

$$\begin{aligned}\left\|Px + \mathrm{e}^{\mathrm{i}\theta}\overline{P}x\right\|^2 &= \left\|px + \mathrm{e}^{\mathrm{i}\theta}(1-p)x\right\|^2\\ &= \left\|(px + \mathrm{e}^{\mathrm{i}\theta}(1-p)x)^*(px + \mathrm{e}^{\mathrm{i}\theta}(1-p)x)\right\|\\ &= \left\|(x^*p + \mathrm{e}^{-\mathrm{i}\theta}x^*(1-p))(px + \mathrm{e}^{\mathrm{i}\theta}(1-p)x)\right\|\\ &= \left\|x^*px + x^*(1-p)x\right\| = \|x^*x\| = \|x\|^2\end{aligned}$$

so $P + \mathrm{e}^{\mathrm{i}\theta}\overline{P}$ is an isometry. □

For the case of $M_n(\mathbb{C})$ or, more general, for the C*-algebra $B(H)$ the converse is also true. We give a proof in the next section.

Theorem 2.2. *Let H be a complex Hilbert space and $P : B(H) \to B(H)$ a bicircular projection. Then there exists a selfadjoint projection $p \in B(H)$ such that either $Px = px$ $(x \in B(H))$ or $Px = xp$ $(x \in B(H))$.*

In the case of $S(H)$ and $A(H)$ we first note that mappings $x \mapsto px$ and $x \mapsto xp$ do not preserve those spaces even if $p^{\mathrm{t}} = p$. Their most natural completition is

LAA 7965
24/12/2003 DISK used
ARTICLE IN PRESS
No. of pages: 12, DTD=4.3.1

$Px = px + xp^{\mathrm{t}}$ where $p^* = p$. Such mapping is both $S(H)$ and $A(H)$ preserving but it is not always a projection. Since $P^2x = Px + 2pxp$ we are led to the equation $pxp = 0$ for all $x \in S(H)$ or all $x \in A(H)$. In the first case there is no nonzero solution so we get the following conclusion, which is proved in Section 4.

Theorem 2.3. *Let $P : S(H) \to S(H)$ be a bicircular projection. Then either $P = 0$ or $P = 1$.*

In the second case there are solutions which are rank one projections. They even give rise to bicircular projections as we observe below.

Observation 2.4. *Let $P : A(H) \to A(H)$ be a mapping defined by $Px = px + xp^{\mathrm{t}}$ where $p = \alpha \otimes \alpha$ for some unit vector $\alpha \in H$. Then P is a bicircular projection.*

Remark. We use a rather standard notation $\alpha \otimes \beta$ for the rank one operator defined by $(\alpha \otimes \beta)(\gamma) = \langle \gamma, \beta \rangle\, \alpha$.

Proof. Let x be an antisymmetric operator. Then

$$\begin{aligned}
\langle x\overline{\alpha}, \alpha \rangle &= -\langle x^{\mathrm{t}}\overline{\alpha}, \alpha \rangle = -\left\langle \overline{x^*(\overline{\overline{\alpha}})}, \alpha \right\rangle \\
&= -\left\langle \overline{x^*\alpha}, \alpha \right\rangle = -\langle \overline{\alpha}, x^*\alpha \rangle = -\langle x\overline{\alpha}, \alpha \rangle = 0
\end{aligned}$$

and therefore

$$\begin{aligned}
pxp^{\mathrm{t}} &= (\alpha \otimes \alpha)x(\overline{\alpha} \otimes \overline{\alpha}) \\
&= (\alpha \otimes \alpha)(x\overline{\alpha} \otimes \overline{\alpha}) \\
&= \langle x\overline{\alpha}, \alpha \rangle (\alpha \otimes \overline{\alpha}) = 0.
\end{aligned}$$

This implies $P^2x = p^2x + x(p^{\mathrm{t}})^2 + 2pxp^{\mathrm{t}} = px + xp^{\mathrm{t}} = Px$ so P is in fact a projection on $A(H)$. Denote by q the complementary projection, i.e. $q = 1 - p$. Given $\theta \in \mathbb{R}$ we consider the element $u = \mathrm{e}^{-\mathrm{i}\theta/2}p + \mathrm{e}^{\mathrm{i}\theta/2}q$. Since $uu^* = (\mathrm{e}^{-\mathrm{i}\theta/2}p + \mathrm{e}^{\mathrm{i}\theta/2}q)(\mathrm{e}^{\mathrm{i}\theta/2}p + \mathrm{e}^{-\mathrm{i}\theta/2}q) = p + q = 1$, the element u is unitary which implies that the mapping $x \mapsto uxu^{\mathrm{t}}$ is isometric. However $pxp^{\mathrm{t}} = 0$ implies

$$\begin{aligned}
uxu^{\mathrm{t}} &= (\mathrm{e}^{-\mathrm{i}\theta/2}p + \mathrm{e}^{\mathrm{i}\theta/2}q)x\left(\mathrm{e}^{-\mathrm{i}\frac{\mathrm{i}\theta}{2}}p^{\mathrm{t}} + \mathrm{e}^{\mathrm{i}\theta/2}q^{\mathrm{t}}\right) \\
&= \mathrm{e}^{-\mathrm{i}\theta}pxp^{\mathrm{t}} + qxp^{\mathrm{t}} + pxq^{\mathrm{t}} + \mathrm{e}^{\mathrm{i}\theta}qxq^{\mathrm{t}} \\
&= (1-p)xp^{\mathrm{t}} + px(1-p^{\mathrm{t}}) + \mathrm{e}^{\mathrm{i}\theta}(1-p)x(1-p^{\mathrm{t}}) \\
&= xp^{\mathrm{t}} + px + \mathrm{e}^{\mathrm{i}\theta}(x - px - xp^{\mathrm{t}}) = Px + \mathrm{e}^{\mathrm{i}\theta}\overline{P}x. \qquad \square
\end{aligned}$$

In the last section we prove a converse to the above observation.

Theorem 2.5. *Let $P : A(H) \to A(H)$ be a bicircular projection. Then there exists a unit vector $\alpha \in H$ such that either $Px = px + xp^{\mathrm{t}}$ $(x \in A(H))$ or $(1 - P)x = px + xp^{\mathrm{t}}$ $(x \in A(H))$ where $p = \alpha \otimes \alpha$. In the second case we can also write $Px = qxq^{\mathrm{t}}$ where $q = 1 - p$.*

3. Proofs for the linear case

We divide the proof of Theorem 2.2 into two logical parts. We first prove its statement under the additional assumption that P is of the form $Px = ax + xb$. Next we show that bicircular projections satisfy a certain functional identity whose solutions on the algebra $B(H)$ are in fact mappings of the above type.

We first need an algebraic lemma which uses two well-known (and easy to prove) facts that the centre of the algebra $B(H)$ is $\mathbb{C} \cdot 1$ and that $axb = 0$ for all $x \in B(H)$ implies that either $a = 0$ or $b = 0$ (in ring theoretic terms this means that $B(H)$ is a prime ring).

Lemma 3.1. *Suppose that $a, b, c, d \in B(H)$ are such that $ax + xb + cxd = 0$ for all $x \in B(H)$. Then either $c \in \mathbb{C} \cdot 1$ or $d \in \mathbb{C} \cdot 1$.*

Proof. From the above equation we can deduce, once by putting xy instead of x and once by multiplying with y from the right,

$$axy + xyb + cxyd = 0 \quad (x \in B(H)),$$
$$axy + xby + cxdy = 0 \quad (x \in B(H))$$

which together implies $x[b, y] + cx[d, y] = 0$ $(x, y \in B(H))$. Now we once insert zx instead of x and once multiply by z from the left in order to obtain

$$zx[b, y] + czx[d, y] = 0 \quad (x, y, z \in B(H)),$$
$$zx[b, y] + zcx[d, y] = 0 \quad (x, y, z \in B(H))$$

which together implies $[c, z]x[d, y] = 0$ $(x, y, z \in B(H))$. If c is a multiple of identity, we are done. If this is not the case there exists such z_0 that $c_0 = [c, z_0] \neq 0$. Then $c_0x[d, y] = 0$ for all x implies $[d, y] = 0$ for all y, which implies that d is a multiple of identity. □

Proposition 3.2. *If $P : B(H) \to B(H)$ is a projection of the form $Px = ax + xb$ then there exists a projection $p \in B(H)$ such that either $Px = px$ or $Px = xp$.*

Proof. Since $P^2x = Px$ we have $(a^2 - a)x + x(b^2 - b) + 2axb = 0 (x \in B(H))$. Using the previous lemma it follows that either a or b must be the multiple of the identity. If a is central then $x(a^2 - a) + x(b^2 - b) + 2xab = 0$ $(x \in B(H))$ implies

$a^2 - a + b^2 - b + 2ab = (a+b)^2 - (a+b) = 0$, so $p = a + b$ is a desired projection for which $Px = xp$. The case of b being central is similar. □

Remark 3.3. We will use the standard Jordan theoretical notation $\{xyz\} = \frac{1}{2}(xy^*z + zy^*x)$ (see for instance [10,11]) in all remaining sections. Recall that subspaces of $B(H)$ which are closed for $\{\cdots\}$ were called J*-algebras by Harris in [6]. He also showed [5] that unit balls of these J*-algebra subspaces have certain remarkable holomorphic properties, but we do not use this fact in our present paper. In the sequel we apply the following result which is included in [14, Theorem 2.1] where a self-contained proof is provided. We note that we believe the result below is well-known for a long time but were not able to trace the first reference. We note that it is also possible to describe J*-automorphisms of $B(H)$. The detailed proof in the context of JB*-triples can be found in [8]. We need a much weaker version than it appears in [8] or [14]. Its proof can easily be given with the classical tools of associative operator algebras so we include it.

Let H be a complex Hilbert space and let $D : B(H) \to B(H)$ be a bounded linear mapping satisfying

$$D(\{xyz\}) = \{D(x)yz\} + \{xD(y)z\} + \{xyD(z)\} \tag{1}$$

for all $x, y, z \in B(H)$. Then D is of the form $D(x) = ax + xb$ where $a^* = -a$ and $b^* = -b$. Note that a and b are not uniquely determined.

Proof. If we put $x = y = z = 1$ in (1) we obtain $D(1) + D(1)^* = 0$. Next we define $\delta(x) = D(x) + D(x^*)^*$ so that $\delta(1) = 0$. If we put first $x = z$, $y = 1$ in (1), then replace x by x^* and add the first expression with the adjungate of the second one, we obtain $\delta(x^2) = \delta(x)x + x\delta(x)$ which means that δ is a Jordan derivation of $B(H)$. It is well-known that Jordan derivation acting on C*-algebras and even general semiprime rings are usual derivations (see [3]). It is also well-known that derivations of $B(H)$ are inner so $\delta(x) = cx - xc$ for some $c \in B(H)$. If we put $x = z = 1$ and $y = x^*$ in (1) we obtain $D(x) = \frac{c+D(1)}{2}x + x\frac{D(1)-c}{2}$. □

The mappings of the above theorem are called J*-algebra derivations. Our next task is to show that bicircular projections can be connected with the above functional identity. As we use this observation for the symmetric and antisymmetric matrices as well we prefer to state it in the framework of J*-algebras.

Proposition 3.4. *Let $J \subset B(H)$ be a J*-algebra and $P : J \to J$ a bicircular projection. Then $D = \mathrm{i}P$ satisfies the functional identity of Remark* 3.3.

Proof. By [10, Proposition 5.5] every isometry A of a J*-algebra satisfies the identity $A(\{xyz\}) = \{A(x)A(y)A(z)\}$. If we compute this for $A = P + \mathrm{e}^{\mathrm{i}\theta}\overline{P}$ and compare the coefficients of $1, \mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-\mathrm{i}\theta}, \mathrm{e}^{2\mathrm{i}\theta}$ we obtain four identities

(I1) $\{P(x)\overline{P}(y)P(z)\} = 0$,

(I2) $\{\overline{P}(x)P(y)\overline{P}(z)\} = 0$,

(I3) $P\{xyz\} = \{P(x)P(y)P(z)\} + \{P(x)\overline{P}(y)\overline{P}(z)\} + \{\overline{P}(x)\overline{P}(y)P(z)\}$,

(I4) $\overline{P}\{xyz\} = \{\overline{P}(x)\overline{P}(y)\overline{P}(z)\} + \{\overline{P}(x)P(y)P(z)\} + \{P(x)P(y)\overline{P}(z)\}$.

Starting from (I3) we have

$$\begin{aligned}
P\{xyz\} =& \{P(x)P(y)P(z)\} + \{P(x)(y - P(y))(z - P(z))\} \\
&+ \{(x - P(x))(y - P(y))P(z)\} \\
=& 2\{P(x)P(y)P(z)\} - 2\{P(x)yP(z)\} + \{P(x)yz\} \\
&+ \{xyP(z)\} + \{P(x)P(y)P(z)\} - \{xP(y)P(z)\} - \{P(x)P(y)z\} \\
=& -2\{P(x)\overline{P}(y)P(z)\} + \{P(x)yz\} + \{xyP(z)\} + \big\{\overline{P}(x)P(y)\overline{P}(z)\big\} \\
&- \{xP(y)z\}.
\end{aligned}$$

Using (I1) and (I2) we have $P\{xyz\} = \{P(x)yz\} + \{xyP(z)\} - \{xP(y)z\}$. Because of the definition of $\{\cdots\}$ this bracket is conjugate linear in the middle term so

$$\begin{aligned}
\mathrm{i}P\{xyz\} =& \mathrm{i}\{P(x)yz\} + \mathrm{i}\{xyP(z)\} - \mathrm{i}\{xP(y)z\} \\
=& \{\mathrm{i}P(x)yz\} + \{xy\mathrm{i}P(z)\} + \{x\mathrm{i}P(y)z\}. \qquad \square
\end{aligned}$$

The proof of Theorem 2.2 is now obvious. Proposition 3.4 and Remark 3.3 together imply that a bicircular projection P is in fact of the form $P(x) = ax + xb$ and thus Proposition 3.2 concludes the proof.

4. Proofs for the symmetric and antisymmetric cases

Remark 4.1. Let J be one of the spaces $S(H)$ or $A(H)$, and let $P : J \to J$ be a bicircular projection. It is well-known and easy to see that both the symmetric and the antisymmetric matrices form a J*-algebra in the sense of Harris. By Proposition 3.4 the mapping $\mathrm{i}P$ is a J*-algebra derivation of X. Now we use a well-known (for the experts in Jordan theory at least) structure theorem whose origin we have traced to Kaup [9]. For detailed proof see also Upmeier [13, Lemma 2.6] in the case of $S(H)$ and de la Harpe [4] for the case of $A(H)$. The finite dimensional case was probably known much earlier. We note that this result can also be derived from [1] or [4] where it is proved that derivations of JB*-triples are strong limits of inner derivations. Note that inner derivations are finite real linear combinations of operators of the form $x \mapsto \mathrm{i}\{aax\} = \mathrm{i}\frac{1}{2}(aa^*x + xa^*a)$. Since $a^{\mathrm{t}} = \pm a$ for a which is an element of $S(H)$ or $A(H)$, we have $(aa^*)^{\mathrm{t}} = a^*a$ in both cases, so inner derivations of $S(H)$ and $A(H)$ can be written in the form $x \mapsto bx + xb^{\mathrm{t}}$.

Let $D : J \to J$ be a J-algebra derivation where J is one of the spaces $S(H)$ or $A(H)$. Then there exists an operator $a \in B(H)$ such that $a^* = -a$ and $D(x) = ax + xa^{\mathrm{t}}$ for all $x \in J$.*

Note that the proof of the above statement can also be given in a similar way to the proof in Remark 3.3 in order to make it more immediate to those not familiar with J*-algebras. As $S(H)$ is in fact a Jordan*-algebra, we may again define $\delta(x) = D(x) + D(x^*)^*$ and deduce from (1) that δ is a classical Jordan derivation which are well-known to be inner in the case of $S(H)$. In the case of $A(H)$ we can deduce from (1) that δ is a classical Lie derivation of the Lie algebra $A(H)$. In the case $\dim(H) < \infty$, the fact that Lie derivations of $A(H)$ are inner is classics and can be found in many textbooks. A generalization to infinite dimensions is given in [4, p. 82]. For a recent considerable generalization to the purely ring-theoretic setting see [2].

From the above remark it follows that the bicircular projection on J, which is of the form $P = -\mathrm{i}D$, can be written as $P(x) = cx + xc^{\mathrm{t}}$ where $c \in B(H)$ is a selfadjoint operator.

Proposition 4.2. *Suppose c is a selfadjoint operator such that the map $x \mapsto cx + xc^{\mathrm{t}}$ is a (bicircular) projection on J. Then every vector from H is a finite linear combination of eigenvectors of c.*

Proof. If the dimension of H is finite, there is an orthonormed base of eigenvectors of c. Otherwise we proceed as follows.

Since P is a projection on J, for any $x \in H$ we have

$$0 = P^2(x) - P(x) = (c^2 - c)x + x((c^{\mathrm{t}})^2 - c^{\mathrm{t}}) + 2cxc^{\mathrm{t}}.$$

Thus the operation $Dx = cxc^{\mathrm{t}} = -\frac{1}{2}[(c^2 - c)x + x((c^{\mathrm{t}})^2 - c^{\mathrm{t}})]$ is a J*-algebra derivation of J, due to the selfadjointness of $c^2 - c$. Therefore we have

$$\begin{aligned} x(Dy)^*x = &-D(xy^*x) + (Dx)y^*x \\ &+ xy^*(Dx) = 0 \quad \text{if } xy^* = y^*x = 0, \quad x, y \in J. \end{aligned}$$

In particular, if

$$\alpha_1, \alpha_2, \beta_1, \beta_2 \in H \quad \text{with } \{\alpha_1, \alpha_2\} \perp \{\beta_1, \beta_2\}$$

then for the choice

$$x = \alpha_1 \otimes \overline{\alpha_2} + \varepsilon\alpha_2 \otimes \overline{\alpha_1}, \quad y = \beta_1 \otimes \overline{\beta_2} + \varepsilon\beta_2 \otimes \overline{\beta_1},$$

where $\varepsilon = 1$ if $J = S(H)$ and $\varepsilon = -1$ if $J = A(H)$, and using the notations

$$\alpha_1' = \overline{\alpha_2}, \quad \alpha_2' = \varepsilon\overline{\alpha_1},$$

we get

$$\begin{aligned} 0 = x(Dy)^*x = x(cyc^{\mathrm{t}})^*x &= \sum_{k,l=1}^{2} (\alpha_k \otimes \alpha_k')(cyc^{\mathrm{t}})^*(\alpha_l \otimes \alpha_l') \\ &= \sum_{k,l=1}^{2} \langle \alpha_l, cyc^{\mathrm{t}}\alpha_k' \rangle \alpha_k \otimes \alpha_l'. \end{aligned}$$

Observe that the linear maps $\alpha_k \otimes \alpha_l'$ are linearly independent if the vectors α_1, α_2 are linearly independent. In that case their coefficients vanish in the last expression. Hence, by the density of $\{(\alpha_1, \overline{\alpha_2}) : \alpha_1, \alpha_2 \text{ are linearly independent}\}$ in $H \times H$, by considering the coefficient of $\alpha_1 \otimes \alpha_l' (= \alpha_1 \otimes \overline{\alpha_2})$, we conclude

$$\begin{aligned} 0 &= \langle \alpha_1, cyc^{\mathrm{t}}\alpha_1' \rangle = \langle \alpha_1, c(\beta_1 \otimes \overline{\beta_2} + \varepsilon\beta_2 \otimes \overline{\beta_1})c^{\mathrm{t}}\overline{\alpha_2} \rangle \\ &= \langle \alpha_1, c(\beta_1 \otimes \overline{\beta_2} + \varepsilon\beta_2 \otimes \overline{\beta_1})\overline{c\alpha_2} \rangle \\ &= \langle \alpha_1, c[\langle c\beta_2, \alpha_2 \rangle \beta_1 + \varepsilon \langle c\beta_1, \alpha_2 \rangle \beta_2] \rangle \quad \text{whenever } \{\alpha_1, \alpha_2\} \perp \{\beta_1, \beta_2\} \text{ in } H. \end{aligned}$$

Thus for any fixed vectors $\beta_1, \beta_2 \in H$ we necessarily have

$$\begin{aligned} & c[\langle c\beta_2, \alpha_2 \rangle \beta_1 + \varepsilon \langle c\beta_1, \alpha_2 \rangle \beta_2] \in \{\beta_1, \beta_2\}^{\perp\perp} \\ & \quad = \operatorname{Span}\{\beta_1, \beta_2\} \quad \text{for every } \alpha_2 \in \{\beta_1, \beta_2\}^{\perp}. \end{aligned}$$

That is the $\mathbb{C}$-linear subspace

$$X_{\beta_1,\beta_2} := \{\langle c\beta_2, \alpha \rangle \beta_1 + \varepsilon \langle c\beta_1, \alpha \rangle \beta_2 : \alpha \in \{\beta_1, \beta_2\}^{\perp}\}$$

of $\operatorname{Span}\{\beta_1, \beta_2\}$ is always mapped by c into $\operatorname{Span}\{\beta_1, \beta_2\}$. Necessarily $\dim(X_{\beta_1,\beta_2}) < 2$, because the assumption $\dim(X_{\beta_1,\beta_2}) = 2$ would mean $X_{\beta_1,\beta_2} = \operatorname{Span}\{\beta_1, \beta_2\}$ entailing $c\beta_1, c\beta_2 \in \operatorname{Span}\{\beta_1, \beta_2\}$ and hence $X_{\beta_1,\beta_2} = 0$. For linearly independent vectors $\beta_1, \beta_2 \in H$ the conclusion $\dim(X_{\beta_1,\beta_2}) \leqslant 1$ means that for some constants $e_1, e_2 \in \mathbb{C}$ with $(e_1, e_2) \neq (0, 0)$ we have

$$e_2 \langle c\beta_2, \alpha \rangle + e_1 \langle c\beta_1, \alpha \rangle = 0 \quad \text{if } \alpha \in \{\beta_1, \beta_2\}^{\perp}.$$

This observation is equivalent to saying that

$$\begin{aligned} & c(e_1\beta_1 + e_2\beta_2) \in \operatorname{Span}\{\beta_1, \beta_2\} \quad \text{for some } e_1, e_2 \in \mathbb{C} \\ & \quad \text{with } |e_1|^2 + |e_2|^2 > 0 \quad \text{if } \beta_1, \beta_2 \in H \text{ are linearly independent.} \end{aligned}$$

Let $\beta \in H$ be arbitrarily given and consider the triple $(\beta, c\beta, c^2\beta)$. If $c^2\beta \in \operatorname{Span}\{\beta, c\beta\}$ then $\operatorname{Span}\{\beta, c\beta\}$ is an eigensubspace of c and hence the vector β is a linear combination of two eigenvectors of the selfadjoint c restricted to its two-dimensional eigensubspace $\operatorname{Span}\{\beta, c\beta\}$. If $c^2\beta \notin \operatorname{Span}\{\beta, c\beta\}$ then the triple $(\beta, c\beta, c^2\beta)$ is linearly independent. Thus in this case we may apply the previous observation with $\beta_1 := \beta$ and $\beta_2 := c^2\beta$ to conclude that $c(e_1\beta + e_2c^2\beta) \in \operatorname{Span}\{\beta, c^2\beta\}$ for some $(e_1, e_2) \neq (0, 0)$. Here the case $e_2 = 0$ is impossible by the linear independency of $(\beta, c\beta, c^2\beta)$. Therefore $c^3\beta \in \operatorname{Span}\{\beta, c\beta, c^2\beta\}$ which means that $\operatorname{Span}\{\beta, c\beta, c^2\beta\}$ is an eigensubspace of c and, in particular, the vector β is a linear combination of three eigenvectors of the selfadjoint c restricted to its three-dimensional eigensubspace $\operatorname{Span}\{\beta, c\beta, c^2\beta\}$. $\square$

Remark. Henceforth we assume without loss of generality that P is the operation $x \mapsto cx + xc^{\mathrm{t}}$ on $B(H)$ with a selfadjoint operator $c \in B(H)$ and $\{\alpha_k : k \in K\}$ is an orthonormed base of H formed by eigenvectors of c with $c\alpha_k = e_k\alpha_k$ where $e_k \in \mathbb{R}$.

Also we assume that each vector of H is a finite linear combination of eigenvectors of c.

Proof of Theorem 2.3. Let $J = S(H)$. We show that, in this case, either $P = 1$ or else $P = 0$. Since P is assumed to be a projection it follows $(c^2 - c)x + x((c^{\mathrm{t}})^2 - c^{\mathrm{t}}) + 2cxc^{\mathrm{t}} = 0$ $(x \in S(H))$. If we take a rank one operator $\beta \otimes \gamma$ on any Hilbert space then it is straightforward to compute that its transpose is $(\beta \otimes \gamma)^{\mathrm{t}} = \overline{\gamma} \otimes \overline{\beta}$. This implies that each $\alpha_k \otimes \overline{\alpha_k}$ belongs to $S(H)$ and so fulfills the above identity. Since $c(\alpha_k) = e_k\alpha_k$ we have, taking into account that $c^* = c$ and that e_k is a real number, $c^{\mathrm{t}}(\overline{\alpha_k}) = \overline{c^*(\overline{\overline{\alpha_k}})} = \overline{c(\alpha_k)} = \overline{e_k\alpha_k} = e_k\overline{\alpha_k}$ which finally yields $(4e_k^2 - 2e_k)\alpha_k \otimes \overline{\alpha_k} = 0$. This implies that all eigenvalues of the operator c belong to the set $\{0, 1/2\}$. It remains to be proved that the possibility of c having two different eigenvalues cannot happen. This clearly implies that either $c = 0$ or $c = 1/2$ which means that either $P = 0$ or $P = 1$.

Let us assume that $c(\alpha_k) = c^{\mathrm{t}}(\overline{\alpha_k}) = 0$, $c(\alpha_j) = \alpha_j/2$ and $c^{\mathrm{t}}(\overline{\alpha_j}) = \overline{\alpha_j}/2$. As c is selfadjoint α_k and α_j are automatically orthogonal. As $(\alpha_k \otimes \overline{\alpha_j} + \alpha_j \otimes \overline{\alpha_k})^{\mathrm{t}} = \overline{\overline{\alpha_j}} \otimes \overline{\alpha_k} + \overline{\overline{\alpha_k}} \otimes \overline{\alpha_j} = \alpha_k \otimes \overline{\alpha_j} + \alpha_j \otimes \overline{\alpha_k}$ it follows that $x = \alpha_k \otimes \overline{\alpha_j} + \alpha_j \otimes \overline{\alpha_k} \in S(H)$. If we compute the equality from the first paragraph of the proof for this particular x we obtain $\frac{1}{4}\alpha_k \otimes \overline{\alpha_j} + \frac{1}{4}\alpha_j \otimes \overline{\alpha_k} = 0$ which is clearly impossible. $\square$

Proof of Theorem 2.5. Assume $J = A(H)$. We use the same notations as in the previous proof. We show that one of the following statements holds:

$$P = 1,$$
$$P = 0,$$
$$P(x) = px + xp^{\mathrm{t}} \quad \text{where } p = \alpha \otimes \alpha \text{ is a rank one projection},$$
$$P(x) = qxq^{\mathrm{t}} \quad \text{where } q = 1 - \alpha \otimes \alpha \text{ is a co-rank one projection}.$$

The cases where $\dim(H) \leqslant 2$ are trivial so we assume $\dim(H) \geqslant 3$ in the sequel. Let $e_1, e_2 \in \mathbb{R}$ be two eigenvalues of the operator c, not necessarily distinct and let $\alpha, \beta \in H$ be the corresponding orthogonal eigenvectors of norm one. Since P is assumed to be a projection it follows $(c^2 - c)x + x((c^{\mathrm{t}})^2 - c^{\mathrm{t}}) + 2cxc^{\mathrm{t}} = 0 (x \in A(H))$. It is obvious that the operator $x = \alpha \otimes \overline{\beta} - \beta \otimes \overline{\alpha} \neq 0$ belongs to $A(H)$ and so fulfills the above identity. If we compute the above expression we get $e_1^2 - e_1 + e_2^2 - e_2 + 2e_1e_2 = (e_1 + e_2)^2 - (e_1 + e_2) = 0$. From this it follows that the sum of any two eigenvalues is either 0 or 1. If we solve all systems of the form

$$e_j + e_k = \delta_1,$$
$$e_k + e_\ell = \delta_2,$$
$$e_j + e_\ell = \delta_3,$$

where $j, k, \ell \in K$ and $\delta_1, \delta_2, \delta_3 \in \{0, 1\}$ we obtain that all eigenvalues of c belong to the set $\{0, 1, 1/2, -1/2\}$.

If $e_{i_0} = 1$, the equations $e_{i_0} + e_j = 1 + e_j = 0$ have no solutions under the above constraints and therefore all remaining eigenvalues are zero. This means that, in this case, c is a rank one projection p and P is of the form $Px = px + xp^{\mathrm{t}}$. If $e_{i_0} = \max \mathrm{Sp}(c) = 1/2$ it follows that all remaining eigenvalues are $\pm 1/2$. Because of the above constraints it is impossible that two of them would have minus sign, so it follows that either all eigenvalues are 1/2 (in which case $c = 1/2$) or one is $-1/2$ and all others are 1/2 (in which case $c = (1/2) - p$ where p is a rank one projection). In the first case we have $Px = \frac{1}{2}x + \frac{1}{2}x = x$ while in the second case it follows $Px = \left(\frac{1}{2} - p\right)x + x\left(\frac{1}{2} - p^{\mathrm{t}}\right) = x - px - xp^{\mathrm{t}}$ so that the complementary projection is $\overline{P}x = px + xp^{\mathrm{t}}$. In the proof of Observation 2.4 we already saw that $pxp^{\mathrm{t}} = 0$ for all $x \in A(H)$. If we denote $q = 1 - p$ we obtain $qxq^{\mathrm{t}} = (1 - p)x(1 - p^{\mathrm{t}}) = x - px - xp^{\mathrm{t}} = Px$. The only remaining case is $e_{i_0} = \max \mathrm{Sp}(c) \leqslant 0$. It is clearly impossible that one of the remaining eigenvalues would be $-1/2$, so it follows $c = 0$. $\square$

Acknowledgements

The authors were supported by the Bilateral Slovenian–Hungarian Research Cooperation grant no. TET SLO-9/2000, OTKA T34267 of the Hungarian Scientific Research Foundation.

References

[1] T.J. Barton, Y. Friedman, Bounded derivations of JB*-triples, Quart. J. Math. Oxford 41 (1990) 255–268.
[2] K.I. Beidar, M. Brešar, M.A. Chebotar, W.S. Martindale 3rd, On Herstein's Lie map conjectures II, J. Algebra 238 (2001) 239–264.
[3] M. Brešar, Jordan derivations on semiprime rings, Proc. Amer. Math. Soc. 104 (1988) 1003–1006.
[4] P. de la Harpe, Classical Banach–Lie Algebras and Banach–Lie Groups of Operators in Hilbert Space, Lecture Notes in Mathematics 285, Springer, 1972.
[5] L.A. Harris, Bounded symmetric domains in infinite dimensional spaces, in: T.L. Hayden, T.J. Suffridge (Eds.), Infinite Dimensional Holomorphy, Lecture Notes in Mathematics 364, Springer, 1974, pp. 13–40.
[6] L.A. Harris, A generalization of C*-algebras, Proc. London Math. Soc. 42 (1981) 331–361.
[7] T. Ho, J.M. Moreno, A.M. Peralta, B. Russo, Derivations on real and complex JB*-triples, J. London Math. Soc. 65 (2002) 85–102.
[8] J. Isidro, L.L. Stachó, Holomorphic Automorphism Groups in Banach Spaces, North Holland Publishing Inc., 1985.
[9] W. Kaup, Über die Automorphismen Grassmannsche Mannigfaltigkeiten unendlicher Dimension, Math. Z. 144 (1975) 75–96.
[10] W. Kaup, A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces, Math. Z. 138 (1983) 503–529.

LAA 7965
24/12/2003 DISK used
ARTICLE IN PRESS
No. of pages: 12, DTD=4.3.1

[11] B. Russo, Structure of JB*-triples, in: W. Kaup, K. McCrimmon, H.P. Petersson (Eds.), Jordan Algebras, de Gruyter Publishing, 1994.
[12] L.L. Stachó, B. Zalar, Symmetric continuous Reinhardt domains, Arch. Math. Basel, in press.
[13] H. Upmeier, Derivations of Jordan C*-algebras, Math. Scand. 46 (1980) 251–264.
[14] B. Zalar, On the structure of automorphism and derivation pairs of B*-triple systems, in: Topics in Operator Theory, Operator Algebras and Applications (Timişoara 1994), Rom. Acad. Science, 1995, pp. 265–271.