Jordan manifolds

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Abstract. The paper is a survey of the results of our forthcoming articles [10,11]. We introduce Jordan manifolds as Banach manifolds whose tangent spaces are endowed with Jordan triple products depending smoothly on the underlying points. As a chief natural application of this concept, we describe in detail the natural complex geometry bounded symmetric domains with their Harish-Chandra realizations as unit balls of JB*-triples and we extend the results to Jordan manifolds where the chart transition maps are generalized local Möbius transformations. We show examples of Jordan manifolds with various features giving rise to problems for further studies.

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1. Introduction

The theory of infinite dimensional Banach manifolds is faced with the problem that it is, in general, no longer locally Euclidean and so a good substitute for the Hilbertian structure has to be found. In a first step [14], the Hilbert space structure on the tangent bundle was replaced by Hilbert C*-modules, a concept of great importance in Noncommutative Geometry. The real source of geometrical concepts within this realm, however, is with the theory of JB*-triples, a class of spaces strictly larger then the class of Hilbert C*-modules. In this paper, we outline the results of our forthcoming [10,11], where we look at manifolds whose tangent spaces are equipped with algebraic Jordan structures providing a category which covers among others all Banach Finsler spaces with infinitesimal JB*-norms in a natural manner. More precisely, we introduce the concept of a Jordan manifold as a real or complex Banach manifold equipped with smoothly varying Jordan*-triple products on the tangent

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spaces. A Riemannian manifold $\mathcal{M}$ with scalar products $\langle .|. \rangle_p$ on the tangent spaces $T_p\mathcal{M}$ can be regarded as a Jordan manifold by means of the associated Jordan*-triple products $\{uvw\}_p := \frac{1}{2}\langle u|v\rangle_p w + \frac{1}{2}\langle w|v\rangle_p u$.

Jordan manifolds form a category with morphisms being smooth mappings whose derivatives are homomorphisms for the pointwise triple products. Consequently, we call a Jordan manifold homogeneous if its automorphism group is transitive, and symmetric if every point admits an automorphism whose derivative is minus-identity. Extending results from [13], the unique automorphism invariant connection on the unit ball of a Hilbert-C* module can be treated in the setting of JB*-triples in a more elegant way by means of local charts and triple products around the points achieved with adjoint actions of generalized Möbius transformations applied to the trivial identity coordinate around the origin. Actually, in Kaup's Hermitian symmetric spaces (a category including JB*-triples) [3,4] one can define a natural coordinate and triple product at every point in a purely geometric manner and the resulting coordinate transition maps prove to be generalized Moebius transformations giving rise to the description [10] of the automorphism invariant connection (in the sense of [6]) along with its associated curvature tensor. For the moment it seems to be an open question if the automorphisms of a symmetric real Jordan manifold form a Banach-Lie group with a construction analogous to Upmeier's topology [12] for Kaup's Hermitian symmetric spaces. We close the paper with examples from [11] of various Jordan manifolds exhibiting features for further research. For more details and proofs of the following results, the reader is referred to [10,11]

2. Preliminaries: bounded symmetric domains and JB*-triples

Throughout this work let $\mathbf{Z}$ denote an arbitrarily fixed complex Banach space with norm $\|\cdot\|$. We shall write $\mathrm{Ball}(\mathbf{Z})$ for its open unit ball $\mathrm{Ball}(\mathbf{Z}) := \{z \in \mathbf{Z} : \ \|z\| < 1\}$. By a *Jordan triple product* on $\mathbf{Z}$ we mean a continuous 3-variable operation $(x,y,z) \mapsto \{xyz\}$ being symmetric bilinear in the outer variables x,z and conjugate-linear in the inner variable y satisfying the *Jordan identity*

$$\text{(J)} \qquad \big\{ab\{xyz\}\big\} = \big\{\{abx\}yz\big\} - \big\{x\{bay\}z\big\} + \big\{xy\{abz\}\big\}$$

for all $a,b,x,y,z \in E$. We say that the Jordan triple $(\mathbf{Z}, \{\ldots\})$ is a *JB*-triple* if the *generalized C*-axiom*

$$\big\|\{zzz\}\big\| = \|z\|^3 , \qquad z \in \mathbf{Z}$$

holds and all the operations

$$D(a) : z \mapsto \{aaz\} , \qquad a \in \mathbf{Z}$$

are $\mathbf{Z}$-Hermitian with non-negative spectrum, that is

$$\big\| \exp\big(\zeta D(a)\big)\big\| \le 1 \quad \text{for every} \quad \zeta \in \mathbb{C} \quad \text{with} \quad \mathrm{Re}\,\zeta \le 0 \ .$$

2.1 Remark. (1) The Jordan identity is equivalent with the fact that all the operators $iD(a)$ $(a \in E)$ are derivations of the triple product, that is we have the identities

$$iD(a)\{xyz\} = \{[iD(a)x]yz\} - \{x[iD(a)y]z\} + \{xy[iD(a)z]\}.$$

(2) C*-algebras are Jordan triples with the triple product

$$\{xyz\} := \frac{1}{2}xy^*x + \frac{1}{2}zy^*x \ ,$$

moreover complex C*-algebras are JB*-triples with their natural norm.

(3) Complex Hilbert C*-modules are also Jordan triples with

$$\{xyz\} := \frac{1}{2}\langle x|y\rangle z + \frac{1}{2}\langle z|y\rangle x.$$

(4) If the unit ball $\mathbf{D} := \mathrm{Ball}(\mathbf{Z})$ is symmetric (that is for each point $a \in \mathbf{D}$, there is a biholomorphic automorphism $S_a : \mathbf{D} \leftrightarrow \mathbf{D}$ with Fréchet derivative $S_a'(a) = -\mathrm{Id}_{\mathbf{Z}}$ at a) then E can be equipped with a *unique* JB*-triple product.

(5) Any bounded symmetric domain $\mathbf{D}$ in a complex Banach space $\mathbf{E}$ can be mapped biholomorphically onto some bounded balanced convex symmetric domain (a so-called *Harish-Chandra realization*), that is onto the unit ball of some equivalent JB*-norm. Namely, by writing S_p and $\|.\|_a$ for the symmetry at $p \in \mathbf{D}$, given any point $o \in \mathbf{D}$, the is a biholomorphic transformation Φ_o mapping $\mathbf{D}$ onto the unit ball $\mathbf{B}_o$ of the infinitesimal Caratéodory norm of $\mathbf{D}$ at o along with a (unique) operation $\{xyz\}_o$ of three variables on $\mathbf{E}$ making $(\mathbf{E}, \|.\|_o)$ a JB*-triple such that

$$\phi_o'(x)(a - \{xax\}_o) = \frac{d}{d\tau}\Big|_{\tau=0+} \frac{1}{2} S_{o+\tau a} \circ S_o\big(\phi_o^{-1}(x)\big) \quad (x \in \mathbf{B}_o).$$

Henceforth let $(\mathbf{Z}, \{\ldots\})$ denote any given JB*-triple and let

$$\begin{array}{lcll} D(a,b) & : & z \mapsto \{a,b,z\}, & D(d) := D(a,a), \\ Q(a,b) & : & z \mapsto \{a,z,b\}, & Q(a) := Q(a,a), \\ B(a,b) & := & 1 - 2D(a,b) + Q(a)Q(b), & B(a) := B(a,a) \end{array}$$

be the usual *skew-derivations*, *quadratic representations* and *Bergman operators*, respectively. Recall that the transformation

$$T : c \mapsto \Big[\exp\Big(\big[c - Q(z)c\big]\frac{\partial}{\partial z}\Big)\Big]0$$

that is $T(x) = \big[$the value z_1 for the initial value problem $\frac{d}{dt}z_t = c - Q(z_t)c,\ z_0 = 0\big]$ is a well-defined real bianalytic mapping

$$T : \mathbf{Z} \longleftrightarrow \mathrm{Ball}(\mathbf{Z}) \ .$$

Given any point $a \in \mathrm{Ball}(\mathbf{Z})$, it is well-known [8, p. 27, 5] that the mapping

$$g_a := \exp\Big(\big[T^{-1}(a) - Q(z)T^{-1}(a)\big]\partial/\partial z\Big)$$

is a holomorphic automorphism of $\mathrm{Ball}(\mathbf{Z})$ and we have

$$g_a(z) = a + B(a)^{1/2}[1 + D(z,a)]^{-1}z, \qquad \|a\|, \|z\| < 1. \tag{2.2}$$

2.3 Definition. In the sequel we shall call the mappings $g_a \circ L$ composed with linear unitary operators of $\mathbf{Z}$ the *Möbius transformations* associated with the triple product $\{\ldots\}$. It is well-known [4] the group Aut $\mathrm{Ball}(\mathbf{Z})$ of all holomorphic automorphisms of the unit ball of a complex JB*-triple coincides with the set of all Möbius transformations of the underlying triple product.

2.4 Remark. In [4] one can find $g_a(z) = a + B(a)^{1/2}[1 - D(x,a)]^{-1}z$ which is obviously incorrect in the sign of the term $D(z,a)$ as on can see on the 1-dimensional example of the classical Möbius transformation $g_a(z) = (z+a)/(1+\overline{a}z)$ (with $a, z \in \mathrm{Ball}(\mathbb{C})$).

Next we are going to consider $\mathrm{Ball}(\mathbf{Z})$ as a complex manifold equipped with the charts $\big\{g_a^{-1} :\ a \in \mathrm{Ball}(\mathbf{Z})\big\}$. In this manner we get a natural generalization of the complex Poincaré model on the unit disc $\mathrm{Ball}(\mathbb{C})$ for the real 2-dimensional hyperbolic geometry.

Due to the possible lack of non-trivial smooth functions vanishing outside a ball, for a Banach manifold M, it is no longer convenient to apply the usual definition of a connection as a mapping $\nabla : TM \times TM \to TM$ as being a derivation for the first and linear in the second variable, both with respect to multiplication with smooth functions.

2.5 Definition [6]. Let M be a manifold, modeled over the Banach space E, and denote the space of bounded bilinear mappings $E \times E \to E$ by $L^2(E,E)$. Then M is said to possess a connection if there is an atlas $\mathcal{U}$ for M so that for each $(U,\Phi) \in \mathcal{U}$ (where U is some open subset of M and Φ is a homeomorphism of U onto some open subset of E) there is a smooth mapping $\Gamma_\Phi : \Phi(U) \to L^2(E,E)$, called the *Christoffel symbol* of the connection on U, which under a change of coordinates $\psi : \Phi(U) \to E$ transforms according to

$$\Gamma_{\psi\circ\Phi}(\psi' u, \psi' v) = \Psi''(u,v) + \psi'\Gamma_\Phi(u,v)$$

for smooth vector fields u, v on $\Phi(U) \subset E$. The covariant derivative of a vector field Y in the direction of the vector field X is, locally, defined to be the principal part of

$$\nabla_X Y = dX(Y) - \Gamma(X, Y),$$

that is, if X, Y are smooth vector fields on M with $u := \Phi^{\#}X$ and $v := \Phi^{\#}Y$ (that is $u : \Phi(U) \ni \Phi(p) \mapsto \Phi'(p)X(p)$)) then

$$\begin{aligned} \Phi^{\#}\nabla_X Y &= \nabla_u v + \Gamma_\Phi(u, v) = \\ &= \Big[\Phi(U) \ni q \mapsto \frac{d}{dt}\Big|_{t=0} v\big(q + tu(q)\big) + \Gamma_\Phi\big(u(q), v(q)\big)\Big]. \end{aligned}$$

If a Banach Lie group G acts smoothly on M then, for each $g \in G$ a connection $g^*\nabla$ is defined by letting

$$g^*\nabla_X Y = \nabla_{g^*X} g^*Y, \qquad g^*X(gm) = d_m g X(m).$$

The Christoffel symbols then transform as in the definition above,

$$\begin{aligned} &\Gamma_{\Phi\circ g}(g(m))\big(g'(m)X(m), g'(m)Y(m)\big) = \\ &\qquad = g''(m)\big(X(m), Y(m)\big) + g'(m)\big[\Gamma_\Phi(m)\big(X(m), Y(m)\big)\big] \end{aligned}$$

at the points $m \in M$ and we say ∇ to be *invariant* under the action of G whenever $g^*\nabla = \nabla$ for all $g \in G$. We have

2.6 Theorem. *On* $U := \mathrm{Ball}(\mathbf{Z})$, *there exists exactly one Möbius invariant connection whose Christoffel symbol at* a *is, in terms of the invariant triple product structure,*

$$\begin{aligned} \{uvw\}_a &:= g_a'(0)\big\{[g_a'(0)^{-1}u][g_a'(0)^{-1}v][g_a'(0)^{-1}w]\big\} = \\ &= B(a)^{1/2}\big\{[B(a)^{-1/2}u][B(a)^{-1/2}v]B(a)^{-1/2}w\big\}, \end{aligned}$$

given by

$$\Gamma_{\mathrm{Id}}(a)(x, y) = 2\{x[g_a'(0)a]y\}_a.$$

Next we proceed to the geodesic equation of the connection ∇. Recall that a smooth curve $\gamma : I \to \mathrm{Ball}(Z)$ on an open real interval I around 0 is a ∇-geodesic if its derivative satisfies the equation $\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0$. According to Theorem 2.6, this means

$$\ddot\gamma(t) + 2B\big(\gamma(t)\big)^{1/2} D\Big(B\big(\gamma(t)\big)^{-1/2}\dot\gamma(t), \gamma(t)\Big) B\big(\gamma(t)\big)^{-1/2}\dot\gamma(t) = 0.$$

In the following, for any fixed vector $0 \neq v \in Z$, we denote by F_v the real JB*-subtriple generated by v, that is

$$F_v = \overline{\mathrm{Span}}_{\mathbb{R}}\big\{D(v)^n v : \ n = 0, 1, \dots\big\}.$$

Since $(\mathbf{Z}, \{\ldots\})$ is JB*-triple, the spectrum of the operator $D(v)|F_v$ is non-negative, and we write

$$\Omega_v := \big[\mathrm{Sp} D(v)^{1/2}|F_v\big] \setminus \{0\}.$$

It is well known that there is a real JB*-isomorphism $H_v : \mathrm{Re}\,\mathcal{C}_0(\Omega_v) \to F_v$ such that $H_v(\mathrm{Id}_{\Omega_v}) = v$ and $H_v(\varphi\psi\chi) = \{H_v(\varphi), H_v(\psi), H_v(\chi)\}$ for all functions $\varphi, \psi, \chi \in \mathrm{Re}\,\mathcal{C}_0(\Omega_v)$.

2.8 Theorem. *The (unique) maximal* ∇*-geodesic* $\gamma_{a,w}$ *with the properties* $\gamma_{a,w}(0) = a$ *and* $\dot\gamma(0) = w$ *has the form*

$$\mathbf{R} \ni t \mapsto g_a\Big(H_{g_a'(0)^{-1}w}\ \mathrm{artanh}\big(t\ \mathrm{Id}_{\Omega_{g_a'(0)^{-1}w}}\big)\Big).$$

This statement is obtained by first looking for particular solutions with the property $\gamma(0) = 0$.

2.7 Lemma. *The* ∇*-geodesic curves passing through the origin have the form*

$$\gamma_{0,v}(t) = H_v\ \mathrm{artanh}\big(t\ \mathrm{Id}_{\Omega_v}\big), \qquad t \in \mathbf{R},\ v \in Z$$

in terms of the Gelfand-Naimark representations H_v.

The proof of the lemma follows eventually from the fact that the geodesic equation with initial conditions $\gamma(0) = 0$, $\gamma'(0) = v$ has a solution that remains within F_v.

The proof of Theorem 2.8 itself follows from Lemma 2.7 as a consequence of the well-known fact that due the Möbius invariance of the connection ∇ $h \circ \gamma$ is ∇-geodesic whenever γ is ∇-geodesic and $h \in$ Aut Ball(Z).

The expression for the geodesic in the above can be conveniently rewritten in terms of the power series for artanh. In fact, we have

$$\gamma_{0,v}(t) = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{2n+1} D(v,v)^{2n} v = \mathrm{artanh} tv.$$

In the same vein,

$$\gamma_{a,w}(t) = a + B(a)^{1/2}\big[1 + D(\mathrm{artanh} tB(a)^{1/2}v, a)\big]^{-1} \mathrm{artanh} tB(a)^{1/2}v.$$

3. Local Möbius transformations in real Jordan-Banach triples and Jordan-Möbius manifolds

Throughout this section $\mathbf{E}$ denotes a real Jordan-Banach triple with the norm $\|.\|$ and triple product $\{\ldots\}$, respectively. Thus we only assume that $\mathbf{E}$ is a real Banach space and $\{\ldots\}$ is a continuous real trilinear mapping $\mathbf{E}^3 \to \mathbf{E}$ satisfying the Jordan identity (J). We can take over the notations $D(a,b), Q(a,b), B(a,b)$ in the real setting without formal changes. In particular, all the operators $D(a,b) - D(b,a) : z \mapsto \{abz\} - \{baz\}$ belong to $\mathrm{Der}(\mathbf{E}, \{\ldots\})$ the set of the derivations of the triple product. Though not explicitly stated, a straightforward inspection of [4, Corollary 2.20] establishes the existence of a constant $\varepsilon > 0$ such that the transformations

$$H_v := \exp V_v \quad \text{with the vector fields} \quad V_v := \big[v - \{zvz\}\big]\partial/\partial z$$

are well-defined on the ball $\varepsilon\mathrm{Ball}(\mathbf{E})$ whenever $\|v\| < \varepsilon$, moreover they have the fractional linear Möbius form

$$\begin{aligned} &H_v(z) = g_{H_v(0)}(z) \quad \text{where} \\ (3.1) \qquad &g_a(z) := a + \lambda(a)[1 + D(z,a)]^{-1} z \;\; \text{where} \;\; \lambda(a) := H_v'(0) \in \mathcal{L}(\mathbf{E}). \end{aligned}$$

Notice that in the case of JB*-triples we can write $\lambda(a) = B(a)^{1/2}$ in termd of the Bergman operator. Besides the vector fields V_v of polynomial degree 2, let us introduce also the linear vector fields

$$L_\ell := \big[\ell z\big]\partial/\partial z, \quad \ell \in \mathrm{Der}(\mathbf{E}, \{\ldots\}).$$

For their Poisson commutators $\big[f(z)\partial/\partial z, g(z)\partial/\partial z\big] := \big(f'(z)g(z) - g'(z)f(z)\big)\partial/\partial z$, it is straightforward to check that we have

$$[L_\ell, V_v] = V_{\ell v}, \quad [L_\ell, L_m] = L_{[\ell,m]}, \quad [V_u, V_v] = L_{D(v,u) - D(u,v)}.$$

Therefore the real linear space

$$\big\{V_v + L_\ell : \; v \in Z, \; \ell \in \mathrm{Der}(\mathbf{E}, \{\ldots\})\big\}$$

equipped with the norm $\|V_v + L_\ell\| := \sup\{\|v - \{zvz\} + \ell z\| : \; \|z\| \le 1\}$ is a Banach-Lie algebra with the Poisson commutator. Thus, according to the Campbell-Hausdorff formula, for some sufficiently small constant $\delta > 0$ we have

$$\big[\exp(V_u + L_\ell)\exp(V_v + L_m)\big]z = \big[\exp(V_{w(u,v,\ell,m)} + L_{\Delta(u,v,\ell,m)}\big]z$$

with two suitable real-analytic mappings

$$\begin{aligned} w &: \big[\delta\,\mathrm{Ball}(\mathbf{E})\big]^2 \times \big[\delta\,\mathrm{Ball}\big(\mathrm{Der}(\mathbf{E}, \{\ldots\})\big)\big]^2 \longrightarrow \mathbf{E}, \\ \Delta &: \big[\delta\,\mathrm{Ball}(\mathbf{E})\big]^2 \times \big[\delta\,\mathrm{Ball}\big(\mathrm{Der}(\mathbf{E}, \{\ldots\})\big)\big]^2 \longrightarrow \mathrm{Der}(\mathbf{E}, \{\ldots\}) \end{aligned}$$

whenever $\|z\|, \|u\|, \|v\|, \|\ell\|, \|m\| < \delta$.

3.2 Definition. By a *Möbius transformation* of $(\mathbf{E}, \{\ldots\})$ we mean a bianalytic mapping $\Phi : U \to \mathbf{E}$ defined in a neighborhood U of the origin in $\mathbf{E}$ such that for some $\ell \in \mathrm{Der}(\mathbf{E},\{\ldots\})$ we have $\Phi(z) = g_a(\exp \ell z)$, $z \in U$ with the customary notation in $g_a(z) := a + \lambda(a)[1 - D(z,a)]^{-1}$ established in (3.1).

The next two auxiliary results 3.3-4 with straightforward proofs establish in particular that composition preserve Möbius transformations.

3.3 Proposition. *There exists $\delta' > 0$ such that*

$$\|v\|, \|\ell\| < \delta' \text{ and } [\exp(V_v + L_\ell)]0 = 0 \quad \Longrightarrow \quad v = 0.$$

3.4 Corollary. *There exists $0 < \delta''(< \delta')$ such that given any $v \in Z$ and $\ell \in \mathrm{Der}(\mathbf{E},\{\ldots\})$ with $\|v\|, \|\ell\| < \delta''$, for some $m \in \mathrm{Der}(\mathbf{E},\{\ldots\})$ we have*

$$\bigl[\exp(V_v + L_\ell)\bigr] z = g_{a(v,\ell)}\bigl([\exp m] z\bigr), \quad \|z\| < \delta''$$

where $a(v,\ell) := \bigl[\exp(V_v + L_\ell)\bigr]0$. In particular, for sufficiently small vectors $p, q \in \mathbf{E}$,

$$g_p \circ g_q = g_{g_p(q)} \circ [\exp m_{p,q}]$$

with a real-analytic mapping $(p,q) \mapsto m_{p,q} \in \mathrm{Der} Z,\{\ldots\})$.

3.5 Definition. By a *Jordan-Möbius manifold* modeled with $(\mathbf{E},\{\ldots\})$ we mean a real-analytic manifold M equipped with an *inverse atlas* $\mathcal{X} = \{X_p :\ p \in M\}$ (that is a system of homeomorphisms between open subsets of M and $\mathbf{E}$ such that $\{X_p^{-1} :\ p \in M\}$ is an atlas on M in the usual sense) with the properties that for each couple of points $p, q \in M$ with $q \in \mathrm{dom}(X_p)$ we have

(3.6) $\qquad X_p(0) = p, \quad X_p^{-1} \circ X_q$ is a Möbius transformation.

In the sequel we shall write $U_p := \mathrm{dom}(X_p)$ and we assume that each of these sets is an open connected neighborhood of the origin in $\mathbf{E}$.

We say that M with inverse atlas $\mathcal{X} = \{X_p :\ p \in M\}$ satisfying (3.6) is a *uniform Jordan-Möbius manifold* if there exists a common constant $\varepsilon > 0$ such that

$$\mathrm{dom}(X_p \circ g) \supset \varepsilon\, \mathrm{Ball}(\mathbf{E}) \quad \text{whenever} \quad g \in \{\text{Möbius transf.}\} \text{ and } \|g(0)\| < \varepsilon.$$

In particular the region $\bigcup_{L \in \mathrm{Aut}(\mathbf{E},\{\ldots\})} \varepsilon L \mathrm{Ball}(\mathbf{E})$ is contained in $U_p = \mathrm{dom}(X_p)$ for any $p \in M$ in a uniform Jordan-Möbius manifold M.

3.7 Example. If $\mathbf{E} := \mathbf{Z}$ is a JB*-triple then its unit ball $M := \mathrm{Ball}(\mathbf{Z})$ is a uniform Jordan manifold with the charts $X_p := g_p | M$, $p \in M$. In this case we may choose $\varepsilon = 1$.

3.8 Example. Let $(\mathbf{E},\{\ldots\})$ be any Jordan-Banach triple. Then we can find a constant $\varrho > 0$ such that the sections of the mapping $[\varrho \mathrm{Ball}(\mathbf{E})]^2 \ni$

$(p,z)\mapsto g_p(z)$ are real-bianalytic for any fixed p and z, respectively. Then the ball $M := \varrho\mathrm{Ball}(\mathbf{E})$ with the topology from $\mathbf{E}$ and with the charts $X_p(z) := g_p(z)$ defined for $z \in U_p := \{u \in \mathbf{E} :\ \|u\|, \|g_p(u)\| < \varrho\}$ is a Jordan manifold which is not uniform in general.

3.8a Example. (Special case of 3.8 with Lorenz space). On the space $\mathbf{E} := \mathrm{Mat}(1,2,\mathbb{R})$ with the matrix $S := \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, define the tripe product $\{xyz\}_S := \frac{1}{2}\langle x|y\rangle_S z + \frac{1}{2}\langle z|y\rangle_S x = \frac{1}{2}xy^*Sz + \frac{1}{2}zy^*Sx$ by means of the indefinite scalar product $\langle x|y\rangle_S := xSy^* \in \mathbb{R}$ $(x,y\in\mathbf{E})$. Then the local Möbius transformations have the form

$$\begin{aligned} M_a(x) &= (1-aSa^*)^{-1/2}(x+a)(1+Sa^*x)^{-1}(1-Sa^*a)^{1/2} = \\ &= (1-aSa^*)^{-1/2}(1+xSa^*)^{-1}(x+a)(1-Sa^*a)^{1/2}\ . \end{aligned}$$

3.9 Example [2,9]. Let $(\mathbf{Z},\{\ldots\})$ be any complex JB*-triple such that the family $\mathrm{Tri}(\mathbf{Z},\{\ldots\}) := \big\{e\in\mathbf{Z} :\ \{eee\} = e \neq 0\big\}$ of its non-trivial tripotents is not void. Two tripotents $e,f\in\mathrm{Tri}(\mathbf{Z},\{\ldots\})$ are said to be *equivalent* ($e\sim f$ in notation) if $D(e,e) = D(f,f)$. Actually $\sim$ is an equivalence relation on $\mathrm{Tri}(\mathbf{Z},\{\ldots\})$. Let M be a connected component of $\mathrm{Tri}(\mathbf{Z},\{\ldots\})$ with respect to the topology inherited from $\mathbf{Z}$. Then the Peirce spaces $Z_{1/2}(e) := \{u\in Z :\ D(e,e)u = u/2\}$ $(e\in M)$ are all isomorphic, and the set $\mathbb{M} := \big\{\mathbf{e} :\ e\in M\big\}$ $(\mathbf{e} := \{f\in M :\ f\sim e\})$ of its equivalence classes can be regarded as a complex hermitian symmetric manifold modeled on $Z_{1/2}(e_0)$ with any $e_0\in M$ and being such that the automorphisms $\exp t[D(e,u)-D(u,e)]$ $\big(t\in\mathbb{R},\ u\in Z_{1/2}(e)\big)$ of $\mathbf{Z}$ acting on $\mathbb{M}$ form a continuous one-parameter subgroup of the Banach-Lie group $\mathrm{Aut}(\mathbb{M})$ of all biholomorphic automorphisms of $\mathbb{M}$ (with Upmeier's topology [12]). Given any $e\in M$, there is a neighborhood $\mathbf{U}$ of its equivalence class $\mathbf{e}$ in $\mathbb{M}$ along with a holomorphic chart map $\Phi : \mathbf{U}\to Z_{1/2}(e)$ such that each map $\Phi^{\#}\exp[D(e,u)-D(u,e)]$ $\big(u\in Z_{1/2}(e)\big)$ is a Möbius transformation with the triple product

$$\{u_1u_2u_3\} := \left.\frac{\partial^3}{\partial\zeta_1\partial\overline{\zeta_2}\partial\zeta_3}\right|_{\zeta_1=\zeta_2=\zeta_3=0} \Phi^{\#}\Big(\Big[P_{\zeta_1u_1}, \Big[P_{\zeta_2u_2}, P_{\zeta_3u_3}\Big]\Big]_e\Big)$$

in terms of the vector fields $P_u(\mathbf{f}) := \big[t\mapsto\exp t[D(e,u)-D(u,e)]\mathbf{f}\big]\in\mathrm{T}_{\mathbf{e}}\mathbb{M}$.

In the sequel let $\big(M,\{Y_p :\ p\in M\}\big)$ be a Jordan-Möbius manifold modeled on $(\mathbf{E},\{\ldots\})$. Henceforth, to simplify formulas, instead of the charts $Y_p : W_p\leftrightarrow U_p$ mapping open subsets of M onto open 0-neighborhoods of $\mathbf{E}$, conveniently we shall use the *inverse charts* that is homeomorphisms $X_p = Y_p^{-1} : \mathbf{E}\supset U_p\to M$ $(p\in M)$. By definition $X_p(0) = p$ and the transition maps $X_p^{-1}\circ X_q$ are Möbius transformations for couples of points lying sufficiently close together.

3.10 Theorem. *Let M be a Jordan-Möbius manifold modeled on $(\mathbf{E},\{\ldots\})$ with a system of inverse charts $\{X_p: \ p\in M\}$ having the properties* (3.6)*. Then there exists a (necessarily unique) connection ∇ on M such that its Christoffel symbol Γ with the charts $\Phi_p := X_p^{-1}$ satisfies*

$$\Gamma_{\Phi_p}(0)=0\ , \qquad p\in M.$$

Namely, if $p\in M$ is any point and the constant $\delta>0$ is so chosen that $\delta\,\mathrm{Ball}(\mathbf{E})\subset \mathrm{dom}(X_p)$ then for any couple of vectors $x,y\in\mathbf{E}$ and any point $a\in\delta\,\mathrm{Ball}(\mathbf{E})$ we have

$$\begin{aligned}\Gamma_{\Phi_p}(a)(x,y) &= g_a''(0)(g_{-a}'(0)x,g_{-a}'(0)y)=\\ &= 2\lambda(a)\big\{[\lambda(a)^{-1}x]a[\lambda(a)^{-1}y]\big\}\ .\end{aligned}$$

In particular $\Gamma_{\Phi_p}(a)(x,y)=2B(a)^{1/2}\{[B(a)^{-1/2}x]a[B(a)^{-1/2}y]\}$ in the case of $(\mathbf{E},\{\ldots\})$ being a complex JB-triple.*

The proof of the theorem is based on the technical results 3.3-4 and the below statements 3.11-12 which establish immediately that the calculations for the proof of Theorem 2.6 can be carried out locally even in the setting of general real Jordan-Möbius manifolds.

3.11 Lemma. *Assume U,V are domains in a Banach space W and let $T:U\leftrightarrow V$ be a smooth diffeomorphism between them. Then given any couple $X,Y:U\to W$ of smooth vector fields on U, for their transforms $\widetilde{X}:=T^{\#}X$ respectively $\widetilde{Y}:=T^{\#}Y$ on V we have*

$$\begin{aligned}\widetilde{Y}'\widetilde{X}(v) &= T''\big(T^{-1}(v)\big)X\big(T^{-1}(v)\big)Y\big(T^{-1}(v)\big)+\\ &\quad+\ T'\big(T^{-1}(v)\big)Y'\big(T^{-1}(v)\big)X\big(T^{-1}(v)\big),\quad v\in V.\end{aligned}$$

3.12 Proposition, *Let $p\in M$ be an arbitrarily given point, let $\mathbf{X},\mathbf{Y}$ be two smooth vector fields on M and define $\mathbf{R}:=\nabla_{\mathbf{X}}\mathbf{Y}$. Then, for any vector $w\in U_p$, the image $R:=[X_p^{-1}]^{\#}\mathbf{R}$ of the vector field $\mathbf{R}$ by means of the local coordinate X_p can be expressed in terms of the image vector fields $X:=[X_p^{-1}]^{\#}\mathbf{X}$ and $Y:=[X_p^{-1}]^{\#}\mathbf{Y}$ and the chart transitions $H_w:=X_{X_p(w)}^{-1}\circ X_p$ as*

$$R(w)=H_w'(w)^{-1}H_w''(w)X(w)Y(w)+Y'(w)X(w)\ .$$

In Theorem 3.10 we could not include a statement about symmetry invariant connections because, unlike the unit ball of a complex JB*-triple, Jordan-Möbius manifolds need not be necessarily symmetric. We close this section by showing that the assumption of uniformness implies this property.

3.13 Theorem. *Connected uniform Jordan-Möbius manifolds are symmetric and they admit a unique symmetry invariant connection.*

Proof. Let M be a connected uniform Jordan-Möbius manifold modeled with $(\mathbf{E}, \{\ldots\})$ along with a constant $\varepsilon > 0$ and a system $\mathcal{X} = \{X_p : p \in M\}$ as in Definition 3.5. By Corollary 3.3, there exists $\delta \in (0, \varepsilon)$ such that $g_a\big(\delta \operatorname{Ball}(\mathbf{E})\big) \subset \varepsilon \operatorname{Ball}(\mathbf{E})$ for any vector $a \in \delta \operatorname{Ball}(\mathbf{E})$. Observe that, given any couple of points $p, q \in M$, we can find a finite sequence $v_1, \ldots, v_N \in \delta \operatorname{Ball}(\mathbf{E})$ such that the recursively defined sequence

$$p_0 := p, \quad p_{n+1} := X_{p_n}(v_n)$$

ends in $q = p_N$. Let us fix any point $p \in M$. We can see by induction that there exists a sequence $q_0 = p, q_1, \ldots, q_N \in M$ of points along with (linear) automorphisms $L_0 = \mathrm{Id}_{\mathbf{E}}, L_1, \ldots, L_N \in \operatorname{Aut}(\mathbf{E}, \{\ldots\})$ such that for the modified charts

$$Y_n := X_{q_n} \circ L_n, \qquad n = 1, \ldots n = 0, \ldots, N$$

we have

$$q_{n+1} = Y_n(-v_n), \quad Y_{n+1}^{-1} \circ Y_n = X_{q_n}^{-1} \circ X_{q_{n+1}}.$$

In view of 3.3-4, if we have another sequence $\widetilde{v}_1, \ldots, \widetilde{v}_{\widetilde{N}}$ and consider the corresponding points $\widetilde{p}_1, \ldots, \widetilde{p}_{\widetilde{N}}$ respectively $\widetilde{q}_1, \ldots, \widetilde{q}_{\widetilde{N}}$ with the above construction then the coincidence $p_N = \widetilde{p}_{\widetilde{N}}$ of the endpoints implies the coincidence $q_N = \widetilde{q}_{\widetilde{N}}$ as well. It is not hard to check that the thus well-defined transformation $S_p : p_N \mapsto q_N$ is a symmetry through the point p such that the maps $X_r^{-1} \circ S \circ X_p$ $(r \in M)$ are Möbius transformations. Finally we notice that $S_p \circ S_q$ is a Möbius transformation if its domain contains the origin of $\mathbf{E}$. □

4. Jordan manifolds

4.1 Definition. A connected manifold M is a *Jordan manifold* if each of its tangent spaces T_pM $(p \in M)$ is endowed with a triple product $\{\ldots\}_p$ and the mapping $(p, u, v, w) \mapsto \{uvw\}_p$ is continuously differentiable.

In the sequel we shall write $(M, \mathcal{A}, \mathcal{P})$ for the triple formed by the carrier space, atlas and system of triple products, respectively.

A *morphism* $F : M \to \widetilde{M}$ between two Jordan manifolds is a smooth mapping such that its derivatives give rise to triple product homomorphisms on the tangent spaces. Hence *isomorphisms* and *automorphisms* can be defined in a straightforward manner. For the group of automorphisms we shall use the notation

$$\operatorname{Aut}(M) := \big\{S : M \leftrightarrow M \text{ with } S, S^{-1} \text{ Jordan morphisms}\big\}.$$

Given two Jordan manifolds $(M, \overline{\mathcal{A}}, \mathcal{P})$ and $(N, \overline{\mathcal{B}}, \mathcal{Q})$ along with the domains $U \subset M$ and $V \subset N$, respectively, a continuously differentiable map $S : U \leftrightarrow V$ is said to be a *local Jordan automorphism* if S is a Jordan isomorphism between U and V as Jordan submanifolds. We write

$$\mathrm{Aut}_{\mathrm{loc}}(M) := \big\{\text{local Jordan automorphism in } M\big\}.$$

Notice that $S_1 \circ S_2 \in \mathrm{Aut}_{\mathrm{loc}}(M)$ whenever $S_1, S_2 \in \mathrm{Aut}_{\mathrm{loc}}(M)$ with $\mathrm{ran}(S_1) \cap \mathrm{dom}(S_2) \neq \emptyset$.

Henceforth $(M, \overline{\mathcal{A}}, \mathcal{P})$ will stand for an arbitrarily fixed Jordan manifold.

4.2 Example. Riemann spaces can be regarded as Jordan manifolds: If $\langle . | . \rangle_p$ is the inner product on T_pM then we take

$$\{uvw\}_p := \frac{1}{2}\langle u|v\rangle_p w + \frac{1}{2}\langle w|v\rangle_p u \ .$$

4.3 Example. The triple products in $\mathcal{P}$ need not be isomorphic to each other as we can see on the following 1-dimensional real Jordan manifold:

$$M := \mathbb{C} \quad \text{with} \quad \{uvw\}_p := \mathrm{Re}(p) u\overline{v}w.$$

On the other hand, with convergent subsequences of grids [7] in tangent spaces we can see the following.

4.3 Theorem. *If the members of $\mathcal{P}$ are finite dimensional JB*-triple products then they are isomorphic.*

4.4 Conjecture. *The assumption* $\dim(M) < \infty$ *in* 4.3 *can be dropped.*

4.5 Definition. A Jordan manifold M is *homogeneous* if $\mathrm{Aut}(M)$ is transitive on it that is for any pair $p, q \in M$ there exists $S \in \mathrm{Aut}(M)$ with $S(p) = q$. Analogously M is *locally homogeneous* if $\mathrm{Aut}_{\mathrm{loc}}(M)$ is transitive on it.

4.6 Example. The interval $(1, \infty)$ as real Jordan-Möbius manifold with the inverse charts $X_p(x) := (x + p)/(1 + xp)$ and the triple products $\{\ldots\}_p := [X_p^{-1}]^{\#}(xyz)$ i.e. $\{\ uvw\}_p = (1+p)^(1-p^2)^{-2} uvw$ $(1 < p \in \mathbb{R})$ is locally homogeneous but not homogeneous.

4.7 Example. The unit ball of a complex JB*-triple is a symmetric and hence necessarily homogeneous Jordan-Möbius manifold with the inverse charts g_a and triple products $\{\ldots\}_a$ in Theorem 2.6.

4.8 Question. *Are homogeneous complex Jordan manifolds all symmetric?*

4.9 Proposition. *There is a real-symmetric homogeneous Jordan manifold which is not of Jordan-Möbius type.*

Proof. The natural Jordan manifold structure of the following example studied by Corach-Porta-Recht (1993) [1] from Lie algebraic view points is suitable. Let A be a C*-algebra with unit 1 and consider the usual triple product $\{xyz\} := \frac{1}{2}xyz + \frac{1}{2}zyx$ on $\mathbf{E} := \{a \in A : \ a = a^*\}$. Let

$$\mathcal{M} := \{a \in A : \ a > 0\} = \{g^*g : \ g \text{ invertible}\}.$$

Observe that linear the transformations

$$L_g : M \leftrightarrow M, \ x \mapsto g^*xg$$

act transitively on M and the inversion

$$S : p \mapsto p^{-1}$$

is a real symmetry of M. Hence the maps

$$S_{g^*g} := L_g \circ S \circ L_{g^{-1}} : p \mapsto g^*gp^{-1}g^*g$$

form a family providing real symmetries at any point of M and the triple products $\{uvu\}_{g^*g} : = L_g\{(L_g^{-1}u)(L_g^{-1}u)(L_g^{-1}u)\} =$ are well-defined
$= u(g^*g)^{-1}v(g^*g)^{-1}u$ well-def.
(though the the points of M can be represented in several ways of the form g^*g). Moreover all these maps L_g, S_p $(p \in M)$ are $\mathcal{P}$-automorphisms. Therefore the vector fields

$$X_v := \frac{d}{d\tau}\Big|_{\tau=0+} S_{\exp(\tau v/2)} \circ S_1$$

are complete in M. Notice that they satisfy the identities $\exp(X_v)p = \exp(v/2)p\exp(v/2)$. On the other hand, we can deduce the following statement establishing the proposition:

In the chart $Y : \left[\exp(v - \{xvx\})\frac{\partial}{\partial x}\right]0 \mapsto [\exp X_v]1$ *the transformations* $\exp(X_w)$ $(w \in E)$ *are not in general of* $\{\ldots\}$*-Möbius type.* $\square$

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