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A NOTE ON KY FAN'S MINIMAX THEOREM

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Since the appearence of von Neumann's minimax theorem [6], several extensions of this principle were published. One of the main purposes of these articles was to eliminate the underlying vector space structure from the original hypothesis. Recently, the first author [5] gave a new elementary proof of von Neumann' minimax theorem that could be applied, as it was pointed out by the second author [7], after slight modifications also to quasiconvex-concave functions on intervals spaces. Recall that an interval space is a topological space X endowed with a mapping $[,]: X\times X\to\{\text{connected subsets of } X\}$ such that $x_1, x_2\in[x_1, x_2]=[x_2, x_1]$ for every $x_1, x_2\in X$. Here the set $[x_1, x_2]$ is called the interval between the points x_1 and x_2. The convexity of X-subsets and quasiconvexity of $X\to R$ functions can be defined in the usual manner: $K\subset X$ is convex (with respect to $[,]$) if $[x_1, x_2]\subset K$ for every $x_1, x_2\in K$ and $f: X\to R$ is quasiconvex if $f(z)\leqq\max\{f(x_1), f(x_2)\}$ whenever $x_1, x_2\in X$ and $z\in[x_1, x_2]$.

In 1952, KY FAN [3] proved the following abstract minimax theorem.

THEOREM A. *Given a compact topological space X, a discrete set Y and a function $f: X\times Y\to R$ with the properties*

(i) $\forall x_1, x_2\in X\ \forall\alpha_1, \alpha_2\geqq 0\ \alpha_1+\alpha_2=1\ \exists x\in X\ \forall y\in Y\ f(x, y)\geqq \sum_{j=1}^{2}\alpha_j f(x_j; y)$;

(ii) $\forall y_1, y\in Y\ \forall\beta_1, \beta_2\geqq 0\ \beta_1+\beta_2=1\ \exists y\in Y\ \forall x\in X\ f(x, y)\leqq \sum_{k=1}^{2}\beta_k f(x, y_k)$;

(iii) *the functions $x\mapsto f(x, y)$ are upper semicontinuous for any fixed $y\in Y$;*

we have

$$\sup_x \inf_y f(x, y) = \inf_y \max_x f(x, y). \tag{1}$$

Our first aim in this paper will be to clear up the relationship between Theorem A and the topological minimax theorems given in [7]. In Section 1 we point out that Theorem A is a consequence of a vector space minimax theorem due to BRÉZIS—NIRENBERG—STAMPACCHIA [1], which is a particular case of Theorem 2 in [7]. Moreover, Proposition 1 below reveals the linear character of Theorem A. In Section 2 we construct a bilinear counterexample concerning the extendibility of the Brézis—Nirenberg—Stampacchia minimax theorem on the basis of the "lifting" principle given in Section 1.[1] Finally, in Section 3 we show an illustrative example for a family of non-classical interval spaces and prove a Helly type theorem for them.

[1] This result answers negatively the last question in [7].

1. Throughout the whole section, let X be a fixed compact topological space, Y a discrete set and let f denote a $X\times Y\to\mathbf{R}$ function satisfying (i), (ii), (iii). Introduce the sets $\hat{X}\equiv\{$probability Radon measures on $X\}$ and $\hat{Y}\equiv\{$probability measures with finite support on $Y\}$. We shall consider $\hat{X}$ as a compact subset with respect to the weak* topology of $C(X)^*$ the dual space of $C(X)=\{$continuous $X\to\mathbf{R}$ functions with sup-norm$\}$. (By [2] p. 285 we have: $\mu_i\xrightarrow{w^*}\mu$ in $C(X)^*$ iff $\int h\,d\mu_i\to\int h\,d\mu$ for every $h\in C(X)$.) We define the function $\hat{f}$ on $\hat{X}\times\hat{Y}$ by $\hat{f}(\mu,\nu)=\int\int f(x,y)\,d\mu(x)\,d\nu(y)$. Observe that $\hat{f}$ may assume the value $-\infty$, however $\hat{f}(\mu,\nu)<+\infty$ for every $\mu\in\hat{X}$ and for every $\nu\in\hat{Y}$.

LEMMA 1. *For any fixed $\nu\in\hat{Y}$, the function $\mu\mapsto\hat{f}(\mu,\nu)$ is upper semicontinuous on $\hat{X}$.*

PROOF. It suffices to see only that for any $y\in Y$, the function $\mu\mapsto\int f(x,y)\,d\mu(x)$ is upper semicontinuous on X. To do this, let $\mu\in\hat{X}$ and $\gamma>\int f(x,y)\,d\mu(x)$ be arbitrarily given. Since, by assumption, the function $x\mapsto f(x,y)$ is upper semicontinuous, there exists $g\in C(X)$ such that $g\geqq f$ and $\int g\,d\mu<\gamma$. Indeed, the sets $H_{nk}\equiv\{x\in X: f(x,y)\geqq k/n\}$ are all compact and, by definition of $\int f(x,y)\,d\mu(x)$, the functions

$$g_n=-n+\sum_{k=-n^2}^{n^2-1}\frac{1}{n}1_{H_{nk}}$$

satisfy $g_n\geqq f(\cdot,y)$ for each $n>\max_x f(x,y)$ and $\int g_{2^n}\,d\mu\downarrow\int f(x,y)\,d\mu(x)$ $(n\to\infty)$. Since μ is Radon measure, for any pair n,k of indices, we can choose a compact set K_{nk} disjoint from H_{nk} such that $\mu((X\setminus H_{nk})\setminus K_{nk})<1/2n^2$. Then choose $h_{nk}\in C(X)$ so that $0\leqq h_{nk}\leqq 1$, $h_{nk}|K_{nk}=0$ and $h_{nk}|H_{nk}=1$. Set

$$h_n=-n+\sum_{k=-n^2}^{n^2-1}\frac{1}{n}h_{nk}.$$

Now $g_n\leqq h_n\in C(X)$ and we have $\int(h_n-g_n)\,d\mu<1/n$ $(n=1,2,\dots)$ whence $\gamma>\int h_{2^n}\,d\mu$ for sufficiently large indices n. Assume $\hat{X}\ni\mu_i\xrightarrow{w^*}\mu$. Then we have

$$\limsup_i\int f(x,y)\,d\mu_i\leqq\lim_i\int g\,d\mu_i=\int g\,d\mu<\gamma,$$

proving the lemma.

PROPOSITION 1. *We have*

$$\inf_\nu\max_\mu\hat{f}(\mu,\nu)=\inf_y\max_x f(x,y),\tag{2'}$$

$$\sup_\mu\inf_\nu\hat{f}(\mu,\nu)=\sup_x\inf_y f(x,y).\tag{2''}$$

PROOF. Statement (2′) is easy to verify. From (iii) we see that for any $\nu\in\hat{Y}$, there exists $y_\nu\in Y$ such that

$$\int f(x,y)\,d\nu(y)\geqq f(x,y_\nu)$$

for every $x \in X$ (cf. [3]). Hence

$$\inf_{\mu} \max f(\mu, v) = \inf_{y_v} \sup_{\mu} \int f(x, y_v)\, d\mu(x) = \inf_{y} \sup_{\mu} \int f(x, y)\, d\mu(x) = \inf_{y} \max_{x} f(x, y).$$

To prove (2″), we need the following observation.

LEMMA 2. *For any* $\mu \in \hat{X}$, *there exists a point* $x_\mu \in X$ *such that*

$$\int f(x, y)\, d\mu(x) \leqq f(x_\mu, y) \quad \textit{for every} \quad y \in Y.$$

PROOF. Fix $\mu \in \hat{X}$ arbitrarily and choose a net $\{\mu_i\}_{i \in I}$ in $\hat{X}$ consisting of finitely supported measures with the property

$$\liminf_i \int f(x, y)\, d\mu_i(x) \geqq \int f(x, y)\, d\mu(x) \quad \text{for every} \quad y \in Y. \tag{3}$$

This can be done since given $y_1, \dots, y_n \in Y$ and numbers η_k(possibly $-\infty$) such that $\eta_k < \int f(x, y)\, d\mu(x)$ $(k=1, \dots, n$; convention: $-\infty < -\infty)$, there exists $\varepsilon > 0$ such that

$$\sum_{j=-\infty}^{+\infty} j\varepsilon \mu(\{x\colon j\varepsilon \leqq f(x, y_k) < (j+1)\varepsilon\}) > \eta_k \quad (k = 1, \dots, n).$$

Therefore, denoting by $\{X_1, X_2, \dots\}$ the partition of X formed by the sets of the form $\{x\colon j_k\varepsilon \leqq f(x, y_k) \leqq (j_k+1)\varepsilon;\ k=1, \dots, n\}$ where $j_1, j_2, \dots, j_n$ range over $\{0, \pm 1, \pm 2, \dots\}$, we can find N such that

$$\sigma_k = \sum_{m=1}^{N} \mu(X_m) \cdot \inf_{x \in X_m} f(x, y_k) > \eta_k$$

and

$$\mu\Big(\bigcup_{m>N} X_m\Big) \cdot \inf_{x \in X_1} f(x, y_k) + \sigma_k > \eta_k \quad (k = 1, \dots, n).$$

Then, for each $m \in \{1, \dots, N\}$, take $x_m \in X_m$ arbitrarily and set

$$\mu_i \equiv \mu\Big(X_1 \cup \bigcup_{m>N} X_m\Big)\delta_{x_1} + \sum_{m=2}^{N} \mu(X_m)\delta_{x_m},$$

where i denotes the tuple $((y_1, \dots, y_n), (\eta_1, \dots, \eta_n))$ and δ_x stands for the probability measure supported by the point x. Now

$$\int f(x, y_k)\, d\mu_i(x) > \eta_k \quad (k = 1, \dots, n)$$

holds. That is the index net

$$I = \Big\{((y_1, \dots, y_n), (\eta_1, \dots, \eta_n))\colon n \in \mathbf{N},\ y_k \in Y,\ \eta_k < \int f(x, y_k)\, d\mu(x);\ k = 1, \dots, n\Big\}$$

has the ordering

$$((y_1, \dots, y_n), (\eta_1, \dots, \eta_n)) \leqq ((y'_1, \dots, y'_{n'}), (\eta'_1, \dots, \eta'_{n'}))$$

if and only if $n \leqq n'$, $y_k = y'_k$ and $\eta_k \leqq \eta'_k$ for $k=1, \dots, n$ with the above described measures μ_i satisfies (3).

By (i), for any index $i\in I$, there is an $x_i\in X$ such that $\int f(x, y)\,d\mu_i(x)\leqq f(x_i, y)$ for every $y\in Y$. By passing to a suitable subset, we may assume without loss of generality that $x_i\to x_0$ for some $x_0\in X$. Then for any $y\in Y$

$$\int f(x, y)\,d\mu(x) = \lim_i \int f(x, y)\,d\mu_i(x) \leqq \limsup_i f(x_i, y) \leqq f(x_0, y).$$

Thus the choice $x_\mu = x_0$ suits the requirements of the lemma.

From Lemma 2 we obtain (2″) as follows:

$$\sup_\mu \inf_\nu \hat{f}(\mu, \nu) = \sup_\mu \inf_\nu \sum_{y\in Y} \nu(\{y\})\cdot\int f(x, y)\,d\mu(x) =$$

$$= \sup_\mu \inf_\nu \int f(x, y)\,d\mu(x) = \sup_\mu \inf_y f(x_\mu, y) = \sup_x \inf_y f(x, y),$$

completing the proof of Proposition 1.

It is easy to see that Theorem 2 in [7] remains valid if the function f maps into the extended real line $\overline{\mathbf{R}} = \mathbf{R}\cup\{\pm\infty\}$. One needs only to apply Theorem 2 in [7] to the function arc tan f. In view of this remark, Theorem A follows from Lemma 1 and from Proposition 1.

2. The following statement is a corollary of Theorem 2 in [7].

PROPOSITION 2. *Let X be a convex coompact subset of a topological vector space E and Y a convex subset in a vector space F, let E_0 be a linear submanifold of E such that the set $X_0 = E_0\cap X$ is dense in X. Suppose that $\hat{f}\colon X\times Y\to[-\infty, \infty)$ is an (extended valued) function with the properties* (iii) *and*

(iv) *$\hat{f}|X_0\times Y$ coincides with the restriction to $X_0\times Y$ of some bilinear mapping $E_0\times F\to\mathbf{R}$.*

Then (1) *holds (with $\hat{f}$ instead of f).*

PROOF. The function arctan $\hat{f}$ obviously fulfils the hypothesis of the Brézis—Nirenberg—Stampacchia minimax theorem (Corollary of Theorem 2 in [7]).

Next we shall investigate what happens when condition (iii) is relaxed from Proposition 2, but assuming the boundedness of f. Let $\mathbf{Z}^* = \{\infty, \pm 1, \pm 2, \dots\}$ denote the Alexandroff compactification of $\mathbf{Z}$ ($\mathbf{Z}$ standing for {integers}), $X = Y = \{$the probability measures on $\mathbf{Z}^*\}$. Define a topology on X and Y as follows: $\mu_i\to\mu$ iff $\int f\,d\mu_i\to\int f\,d\mu$ for every convergent f. Then let $f(\mu, \nu) = \iint K(m, n)\,d\mu(m)\,d\nu(n)$, where

$$K(m, n) = \begin{cases} 1 & \text{if} \quad n = \infty \\ 0 & \text{if} \quad n \neq \infty, \quad m = \infty \\ 1 & \text{if} \quad m+n \geqq 0; \quad m, n \neq \infty \\ 0 & \text{if} \quad m+n < 0; \quad m, n \neq \infty. \end{cases}$$

Obviously

$$0 \leqq \sup_\mu \inf_\nu f \leqq \sup_\mu \inf_{\delta_n} f = \sup_\mu \inf_n \int K(m, n)\,d\mu(m) =$$

$$= \sup_\mu \inf_n \begin{Bmatrix} 1 \quad (n = \infty) \\ \sum\limits_{m=-n}^{\infty} \mu(\{m\}) \end{Bmatrix} = \sup_\mu 0 = 0,$$

and

$$1 \geqq \inf_{\nu} \sup_{\mu} f \geqq \inf_{\nu} \sup_{\delta_m} f = \inf_{\nu} \sup_{m} \int K(m, n)\, d\nu(n) =$$

$$= \inf_{\nu} \sup_{m} \begin{Bmatrix} \nu(\{\infty\}) & (m = \infty) \\ \nu(\{\infty\}) + \sum_{n=-m}^{\infty} \nu(\{n\}) & \end{Bmatrix} = \inf_{\nu} 1 = 1.$$

Thus we have proved

PROPOSITION 3. *There exist a topological vector space E, convex compact subsets $X, Y \subset E$ and a bilinear function $\hat{f}: E \times E \to \mathbf{R}$ such that $\hat{f}$ is bounded on $X \times Y$ and*

$$\sup_{\mu \in X} \inf_{\nu \in Y} \hat{f}(\mu, \nu) < \inf_{\nu \in Y} \sup_{\mu \in X} \hat{f}(\mu, \nu).$$

3. Concerning Theorems 1 and 2 in [7], many mathematicians asked the authors to show examples of interval spaces, that could be interesting subjects for discrete geometrical investigations. We remark that an even more general concept of convexity than that of interval spaces is proposed in [4] for such studies. Here we show an interval structure on $\mathbf{R}^n$ that stands in a close relation with the natural lexicographic order of the space, admits convex sets of very different character from those in the classical sense but preserves Helly's theorem.

We define the interval spaces $(\mathbf{R}^n, [,]_n)$ recursively as follows: for $x, y \in \mathbf{R}$ define

$$[x, y]_1 = \{\text{the usual closed interval between } x \text{ and } y\}.$$

Suppose, $(\mathbf{R}^n, [,]_n)$ is defined and set

$$[(x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1})]_{n+1} \equiv$$

$$\equiv (\{(x_1, \ldots, x_n)\} \times [x_{n+1}, y_{n+1}]_1) \cup ([(x_1, \ldots, x_n), (y_1, \ldots, y_n)]_n \times \{y_{n+1}\}),$$

whenever $(x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1}) \in \mathbf{R}^{n+1}$ and $x_{n+1} \leqq y_{n+1}$.

Now we prove

THEOREM. *Assume $K_1, \ldots, K_N \subset \mathbf{R}^n$ are convex sets with respect to $[,]_n$ and*

$$\bigcap_{m=1}^{n+1} K_{i_m} \neq \emptyset \quad \textit{whenever} \quad 1 \leqq i_1 \leqq \ldots \leqq i_{n+1} \leqq N.$$

Then

$$\bigcap_{i=1}^{N} K_i \neq \emptyset.$$

PROOF. For $n=1$ the statement is the same as that of Helly's theorem. Now suppose that the theorem holds for all $n<r$. It is well known (cf. [4]) that to carry

out the induction step it suffices to prove only

$$\bigcap_{i=1}^{r+2} K_i \neq \varnothing.$$

whenever $K_1, \ldots, K_{r+2}$ are $[\,,]_r$-convex sets and

$$\bigcap_{\substack{i=1\\ i\neq m}}^{r+2} K_i \neq \varnothing \quad (m = 1, 2, \ldots, r+2). \tag{4}$$

Thus let $K_1, \ldots, K_{r+2}$ be fixed $[\,,]_r$-convex sets such that (4) is fulfilled. For any $m\in\{1, \ldots, r+2\}$ choose

$$x^m \equiv (x_1^m, \ldots, x_r^m)\in \bigcap_{\substack{i=1\\ i\neq m}}^{r+2} K_i.$$

We may assume $x_r^1 \leqq \ldots \leqq x_r^{r+2}$. Now observe that the points $y^m \equiv (x_1^m, \ldots, x_{r-1}^m, x_r^{r+1})$ $(m=1, \ldots, r+1)$ satisfy

$$y^m \in \bigcap_{\substack{i=1\\ i\neq m}}^{r+2} K_i \quad (m = 1, \ldots, r+1).$$

In fact, we have $x^{r+2}, x^m \in K_i$ whenever $i \neq m, r+2$, whence

$$y^m \in \{(x_1^m, \ldots, x_{r-1}^m)\}\times[x_r^m, x_r^{r+2}]_1 \subset [x^m, x^{r+2}]_r \subset K_i \quad \text{if} \quad i \neq m,\ r+2.$$

Furthermore

$$y^m \in \{(x_1^m, \ldots, x_{r-1}^m)\times[x_r^m, x_r^{r+1}]_1 \subset [x^m, x^{r+1}]_r \subset K_{r+2} \quad (m = 1, \ldots, r+1).$$

Thus the sets

$$L_i \equiv \{(x_1, \ldots, x_{r-1})\colon (x_1, \ldots, x_{r-1}, x_r^{r+1})\in K_i \cap K_{r+2}\} \quad (m = 1, \ldots, r+1)$$

fulfil

$$\bigcap_{\substack{i=1\\ i\neq m}}^{r+1} L_i \neq \varnothing \quad \text{for} \quad m = 1, \ldots, r+1.$$

Since

$$L_i \times \{x_r^{r+1}\} \equiv (\mathbf{R}^{r-1}\times \{x_r^{r+1}\}) \cap K_i \cap K_{r+1}$$

the set L_i is $[\,,]_{r-1}$-convex $(i=1, \ldots, r+1)$. Therefore by hypothesis, $\bigcap_{i=1}^{r+1} L_i \neq \varnothing$ whence $\bigcap_{i=1}^{r+2} K_i = \varnothing$. Theorem 1 is proved.

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