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A NOTE ON KÖNIG'S MINIMAX THEOREM

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Recently G. Kassay [1] published an elementary proof of König's minimax theorem [2]. His method seems to be an interesting mixture of both of the so-called methods of level sets and cones, respectively. Formally, König's theorem is an extension of Ky Fan's classical minimax theorem [3] by restricting convexity to diadic rational convexity. It is well-known [4] that Ky Fan's theorem can be deduced from the Brézis–Nirenberg–Stampacchia level set minimax theorem by a function lifting. It is an old open question whether there is a short direct connection between König's and Ky Fan's minimax theorems.

The aim of this note is to show that the mentioned function lifting in [4] transforms a König-type saddle function into a Ky Fan-type saddle function with the same minimax values. A careful analysis of the proof of this fact leads also to new generalizations of König's theorem, which seem not be provable with a simple adaptation of Kassay's method.

Finally we remark that the question of König-type generalizations of M. Sion's minimax theorem [5] is still open.

1. On the continuity of convex functions

Throughout this section let V denote an arbitrary vector space and let τ be the finest locally convex topology on V. It is immediate that the absorbing convex subsets of V form a neighbourhood basis of 0 for τ. We say that a subset $S \subset V$ is a *simplex* in V if S is the convex hull of a set $B \subset V$ such that the system $\{b - b_0 : b \in B, b \neq b_0\}$ is linearly independent for all $b_0 \in B$.

1.1. LEMMA. *Assume S is a simplex in V, K is a convex subset of S and $x \in K$. Then the following statements are equivalent:*

(i) *K is a neighbourhood of x in the relative topology of τ on S,*

(ii) $\{v \in V : \exists \varepsilon > 0,\ x + \varepsilon v \in K\} = \{v \in V : \exists \varepsilon > 0,\ x + \varepsilon v \in S\}$.

PROOF. By shifting a suitable vertex of S into the origin and restricting ourselves to the subspace spanned by S, we may assume without loss of generality that $S = \operatorname{co}\big(B \cup \{0\}\big)$ the convex hull of some Hamel basis B of V with the origin and $x = \sum_{i=1}^{n} \beta_i b_i$ for some $b_1, \dots, b_n \in B$ and

$\beta_1, \ldots, \beta_n > 0$ with $\sum_{i=1}^n \beta_i = 1$. Let us write

$$C_0 := \{b_i - x : i = 1, \ldots, n-1\}, \quad C_1 := \{b - x : b \in B \setminus \{b_1, \ldots, b_n\}\}.$$

Then $C := \{-x\} \cup C_0 \cup C_1$ is again a Hamel basis of V and

$$(1.2) \qquad \{v \in V : \exists \varepsilon > 0,\ x + \varepsilon v \in S\} = \mathrm{co}\big((\mathbf{R}_+ C) \cup (-\mathbf{R}_+ C_0)\big).$$

Therefore for each $c \in C$ there exists $\varepsilon(c) > 0$ with $x + [0, \varepsilon(c)]\, c \subset K$ for $c \in C \setminus C_0$ and $x + [-\varepsilon(c), \varepsilon(c)]\, c \subset K$ for $c \in C_0$. Define

$$U := \mathrm{co}\Big(\bigcup_{c \in C} [-\varepsilon(c), \varepsilon(c)]\, c + x\Big).$$

Since C is a Hamel basis of V, U is a convex τ-neighbourhood of x and

$$U = \Big\{x + \sum_{c \in C} \lambda_c c : (c \mapsto \lambda_c) \in \Lambda,\ \sum_{c \in C} |\lambda_c| / \varepsilon(c) \leqq 1\Big\}$$

where $\Lambda := \{\text{functions } C \to \mathbf{R} \text{ with finite support}\}$. By (1.2) we obtain

$$U \cap S = \Big\{x + \sum_{c \in C} \lambda_c c : (c \mapsto \lambda_c) \in \Lambda,\ \lambda_c \geqq 0\ (c \in C),\ \sum_{c \in C} \lambda_c / \varepsilon(c) \leqq 1\Big\} =$$

$$= \mathrm{co}\Big(\bigcup_{c \in C} [0, \varepsilon(c)].c + x\Big).$$

Since $x, \varepsilon(c)c + x \in K$ $(c \in C)$, we have $U \cap S \subset K$ which completes the proof.

1.3. COROLLARY. *If $\{g_i : i \in \mathcal{I}\}$ is a family of affine functions on V such that the function $f := \sup_{i \in \mathcal{I}} g_i$ is finite on the simplex S then f is continuous on S with respect to the relative topology of τ.*

PROOF. First of all remark that convex functions of one real variable are always upper semicontinuous. Hence, for any $x \in S$, $\eta > 0$ and $u \in \{v \in V : \exists \varepsilon > 0,\ x + \varepsilon v \in S\}$ there exists $\varepsilon > 0$ such that $f(x + \xi u) < \eta + f(x)$ for all $\xi \in [0, \varepsilon]$. Thus the convex level sets $K_\gamma := \{x \in S : f(x) < \gamma\}$ are all open in the relative topology of τ on S by 1.1. That is, the function f is upper semicontinuous. On the other hand, affine functions are all τ-continuous on V. Hence f as the supremum of a family of continuous functions is lower semicontinuous on S in the relative topology of τ.

2. König-convex and Ky Fan-convex mappings

2.1. DEFINITION. Let E be an ordered vector space and Z be any set. We say that a mapping $\Phi : Z \to E$ is *König-convex* if for every $z_1, z_2 \in Z$ there exists $z \in Z$ with $\Phi(z) \leqq (1/2)\Phi(z_1) + (1/2)\Phi(z_2)$. The mapping Φ is said to be *Ky Fan-convex* if for every $z_1, z_2 \in Z$ and $t \in [0,1]$ there exists $z \in Z$ with $\Phi(z) \leqq (1-t)\Phi(z_1) + t\Phi(z_2)$. If $-\Phi$ is König-convex (resp. Ky Fan-convex) then we say that Φ is *König-concave* (resp. *Ky Fan-concave*).

Throughout the whole work we write $\mathcal{D}$ for the field of diadic rationals.

2.2. LEMMA. *If $\Phi : Z \to E$ is König-convex then for every finite seqence $z_1, \dots, z_n \in Z$ and $0 \leqq \delta_1, \dots, \delta_n \in \mathcal{D}$ with $\sum_{i=1}^n t_i = 1$ there exists $z \in Z$ with $\Phi(z) \leqq \sum_{i=1}^n \delta_i \Phi(z_i)$.*

PROOF. Define $\overline{Z} :=$ {functions $Z \to \mathbf{R}$ with finite support} and

$$\overline{\Phi}(\overline{z}) := \sum_{z \in Z} \overline{z}(z)\Phi(z) \qquad (\overline{z} \in \overline{Z}).$$

Let $\overline{T} := \left\{ \overline{z} \in \overline{Z} : \exists z \in Z,\ \Phi(z) \leqq \overline{\Phi}(\overline{z}) \right\}$. We have to prove that

$$\overline{T} \supset \left\{ \overline{z} \in \overline{Z} : \operatorname{range}(\overline{Z}) \subset \mathcal{D},\ \overline{z} \geqq 0,\ \sum_{z \in Z} \overline{z}(z) = 1 \right\}. \tag{2.3}$$

By writing 1_z for the characteristic function of the set $\{z\}$, we have $1_z \in \overline{T}$ because $\overline{\Phi}(1_z) = \Phi(z)$ $(z \in Z)$. Furthermore, if $\overline{z}_1, \overline{z}_2 \in \overline{Z}$ then for some $z_1, z_2 \in Z$ we have $\Phi(z_i) \leqq \sum_{z \in Z} \overline{z}_i(z)\Phi(z)$ $(i = 1,2)$. Since Φ is König-convex, hence there exists $z_3 \in Z$ with

$$\Phi(z_3) \leqq \frac{1}{2}\Phi(z_1) + \frac{1}{2}\Phi(z_2) \leqq \sum_{z \in Z} \left(\frac{1}{2}\overline{z}_1(z) + \frac{1}{2}\overline{z}_2(z) \right) \Phi(z).$$

Thus $\overline{T} \supset (1/2)\overline{T} + (1/2)\overline{T}$ and $1_z \in \overline{T}$ $(z \in Z)$ whence 2.3 is immediate.

2.4. LEMMA. *Let E be a function space (with its natural ordering) and let Z be a compact topological space. Assume $\Phi : Z \to E$ is a lower semicontinuous*[1] *König-convex mapping. Then Φ is necessarily Ky Fan-convex.*

PROOF. Fix any $z_0, z_1 \in Z$. By 2.2, for every $\delta \in \mathcal{D} \cap [0,1]$ there exists $z_\delta \in Z$ such that $\Phi(z_\delta) \leqq (1-\delta)\Phi(z_0) + \delta\Phi(z_1)$. Given any $t \in [0,1]$,

[1] I.e. if $E \subset$ {functions $\Omega \to \mathbf{R}$} then for each fixed $\omega \in \Omega$ the function $z \mapsto \Phi(z)(\omega)$ is lower semicontinuous on Z.

choose a sequence $\delta_1, \delta_2, \ldots \in \mathcal{D} \cap [0,1]$ such that $t = \lim_{n\to\infty} \delta_n$. By the compactness of the space Z, there exists an index net $(n_i : i \in \mathcal{I})$ with $\lim_{i\in\mathcal{I}} z_{\delta_{n_i}} = z^*$ for some $z^* \in Z$. Then

$$\Phi(z^*) \leqq \liminf_{i\in\mathcal{I}} \Phi(z_{\delta_{n_i}}) \leqq \liminf_{i\in\mathcal{I}} \left[(1-\delta_{n_i})\Phi(z_0) + \delta_{n_i}\Phi(z_1)\right] \leqq$$

$$\leqq (1-t)\Phi(z_0) + t\Phi(z_1).$$

3. König's theorem via Ky Fan's minimax theorem

3.1. DEFINITION. Henceforth let X denote a compact topological space, Y a non-empty set and let F be a function $X \times Y \to \mathbf{R}$. We write E_X (resp. E_Y) for the space of all real functions on X (resp. Y). We denote by τ_X the finest locally convex topology on the subspace $\overline{X} := \{\overline{x} : \operatorname{supp}(\overline{x})$ finite$\}$ and we embed the set X into $\overline{X}$ by identifying each point $x \in X$ with its characteristic function 1_x. In accordance with this embedding, we denote the simplex $\{\overline{x} \in \overline{X} : \overline{x} \geqq 0,\ \sum_{x\in X} \overline{x}(x) = 1\}$ by $\operatorname{co}(X)$. The objects $\overline{Y}, \tau_Y, \operatorname{co}(Y)$ are defined analogously.

We say that the function f is of *König-type* if the mapping $x \mapsto f(x,\cdot)$ is König-concave and upper semicontinuous from X into the function space E_Y (see footnote [1]) and $y \mapsto f(\cdot, y)$ is König-convex from Y into E_X.

Similarly we speak of fuctions of *Ky Fan-type* when replacing König-convexity (concavity) in the above definition by Ky Fan-convexity (concavity).

Finally we shall write shortly $\inf\sup f$ (resp. $\sup\inf f$) instead of $\inf_{y\in Y}\sup_{x\in X} f(x,y)$ (resp. $\sup_{x\in X}\inf_{y\in Y} f(x,y)$).

3.2. PROPOSITION. *Let $f : X \times Y \to \mathbf{R}$ be a function of König-type. Then the lifted function $\overline{f} : X \times \operatorname{co}(Y) \to \mathbf{R}$ defined by*

$$\overline{f}(x,\overline{y}) := := \sum_{y\in Y} \overline{y}(y) f(x,y) \ (x \in X,\ \overline{y} \in \operatorname{co}(Y))$$

is of Ky Fan-type and it satisfies

$$\inf\sup \overline{f} = \inf\sup f, \qquad \sup\inf \overline{f} = \sup\inf f.$$

PROOF. For any $x \in X$, clearly $\inf_{\overline{y}\in\operatorname{co}(Y)} \overline{f}(x,\overline{y}) = \inf_{y\in Y} f(x,y)$. Hence $\sup\inf \overline{f} = \sup\inf f$.

We have also $\inf\sup f = \inf_{y\in Y}\sup_{x\in X} \overline{f}(x, 1_y) \geqq \inf\sup \overline{f}$.

To prove the converse inequality, notice that $\operatorname{co}(Y)$ is a simplex in $\overline{Y}$ and for any $x \in X$, the function $g_x : \overline{y} \mapsto \sum_{y\in Y} \overline{y}(y) f(x,y)$ is affine on $\overline{Y}$. Moreover, if $\overline{y} \in \operatorname{co}(Y)$ and $\operatorname{supp}(\overline{y}) = \{y_1, \ldots, y_n\}$ then

$$g_x(\overline{y}) = \sum_{i=1}^{n} \overline{y}(y_i) f(x, y_i) \leqq \sum_{i=1}^{n} \max_{x' \in X} f(x', y_i) < \infty \qquad (x \in X)$$

since, for any fixed $y \in Y$, the function $x' \mapsto f(x', y)$ is upper semicontinuous on the compact space X. Thus we may apply 1.3 to conclude that the function $\overline{y} \mapsto \sup_{x \in X} \overline{f}(x, \overline{y})$ is convex and continuous when restricted to any finite dimensional affine section of $\operatorname{co}(Y)$. Fix again an arbitrary $\overline{y} \in \operatorname{co}(Y)$ and let $\operatorname{supp}(\overline{y}) = \{y_1, \ldots, y_n\}$. Given any $\varepsilon > 0$, we can choose diadic rationals $\delta_1, \ldots, \delta_n \geqq 0$ with $\sum_{i=1}^{n} \delta_i = 1$ such that

$$\sup_{x \in X} \overline{f}\left(x, \sum_{i=1}^{n} \delta_i 1_{y_i}\right) \leqq \sup_{x \in X} \overline{f}(x, \overline{y}) + \varepsilon.$$

Since the mapping $y \mapsto f(\cdot, y)$ is supposed to be König-convex, by 2.2 there exists $y^* \in Y$ with

$$f(\cdot, y^*) \leqq \sum_{i=1}^{n} \delta_i f(\cdot, y_i) = \overline{f}\left(\cdot, \sum_{i=1}^{n} \delta_i 1_{y_i}\right).$$

Therefore

$$\inf \sup f = \sup_{x \in X} f(x, y^*) \leqq \sup_{x \in X} \overline{f}(x, \overline{y}) + \varepsilon.$$

By the arbitrariness of $\overline{y} \in \operatorname{co}(Y)$ and $\varepsilon > 0$, hence $\inf \sup f \leqq \sup \inf \overline{f}$.

By 2.4, the mapping $x \mapsto f(x, \cdot)$ is also Ky Fan-concave. Hence, given any $t \in [0,1]$, $x_0, x_1 \in X$, there exists $x_t \in X$ with

$$f(x_t, y) \geqq (1-t) f(x_0, y) + t f(x_1, y) \qquad (y \in Y).$$

If $\overline{y} \in \operatorname{co}(Y)$ then

$$\overline{f}(x_t, \overline{y}) = \sum_{y \in Y} \overline{y}(y) f(x_t, y) \geqq$$

$$\geqq \sum_{y \in Y} \overline{y}(y) \big[(1-t) f(x_0, y) + t f(x_1, y) \big] = (1-t) \overline{f}(x_0, y) + t \overline{f}(x_1, y).$$

Thus the mapping $x \mapsto \overline{f}(x, \cdot)$ is Ky Fan-concave $X \to E_{\operatorname{co}(Y)}$.

For any fixed $\overline{y} \in \operatorname{co}(Y)$ the function $\overline{f}(\cdot, \overline{y}) = \sum_{y \in Y} \overline{y}(y) f(\cdot . y)$ is a finite convex combination of upper semicontinuous functions on X. Thus the mapping $x \mapsto \overline{f}(x, \cdot)$ is upper semicontinuous $X \to E_{\operatorname{co}(Y)}$.

Finally the mapping $\overline{y} \mapsto \overline{f}(\cdot, \overline{y})$ is affine $\operatorname{co}(Y) \to E_X$, whence it is in particular also Ky Fan-convex.

3.3. COROLLARY (König's theorem [2]). *If $f : X \times Y \to \mathbf{R}$ is a function of König-type then* $\inf \sup f = \sup \inf f$.

PROOF. We may apply Ky Fan's minimax theorem to the fuction $\overline{f}$ in 3.2. Hence $\inf\sup\overline{f} = \sup\inf\overline{f}$.

4. Generalizations

Throughout this section let X, Y denote two non-void sets, let f be a function $X \times Y \to \mathbf{R}$. We shall keep the notations $\overline{X}, \tau_X, \mathrm{co}(X)$ resp. $\overline{Y}, \tau_Y, \mathrm{co}(Y)$ established in 3.1. We denote by $\overline{f}$ the affine lifting

$$\overline{f}(x,\overline{y}) := \sum_{y\in Y} \overline{y}(y) f(x,y) \qquad (x \in X,\ \overline{y} \in \mathrm{co}(Y))$$

of the function f in the second variable to $X \times \mathrm{co}(Y)$.

4.1. PROPOSITION. *Assume the function* $f : X \times Y \to \mathbf{R}$ *has the following properties:*

(i) $\sup_{x\in X} f(x,y) < \infty \quad (y \in Y)$,

(ii) *the set* $\{\overline{y} \in \mathrm{co}(Y) : \exists\, y \in Y\ f(\cdot,y) \leqq \overline{f}(\cdot,\overline{y})\}$ *is dense in* $\mathrm{co}(Y)$ *with respect to* τ_Y.

Then we have $\inf\sup\overline{f} = \inf\sup f$ *and* $\sup\inf\overline{f} = \sup\inf f$.

PROOF. The simple arguments at the beginning of the proof of 3.2 show that $\sup\inf\overline{f} = \sup\inf f$ and $\inf\sup f \geqq \inf\sup\overline{f}$.

Since $\mathrm{co}(Y)$ is a simplex in $\overline{Y}$ and since the family $\{\overline{f}(x,\cdot) : x \in X\}$ of affine functions on X is bounded from above (by assumption (i)) for each $\overline{y} \in \mathrm{co}(Y)$, it follows from 1.3 that the function $\mathrm{co}(Y) \ni \overline{y} \mapsto \sup_{x\in X}\overline{f}(x,\overline{y})$ is continuous with respect to the topology τ_Y. Then, given any $\varepsilon > 0$ and $\overline{y} \in \mathrm{co}(Y)$, by assumption (ii) there exist $y^* \in Y$ and $\overline{y}^* \in \mathrm{co}(Y)$ with

$$\sup_{x\in X}\overline{f}(x,\overline{y}^*) \leqq \sup_{x\in X}\overline{f}(x,\overline{y}) + \varepsilon \quad \text{and} \quad f(\cdot\,,y^*) \leqq \overline{f}(\cdot\,,\overline{y}^*).$$

Thus $\inf\sup f \leqq \sup_{x\in X} f(\cdot\,,y^*) \leqq \sup_{x\in X}\overline{f}(x,\overline{y}^*) \leqq \sup_{x\in X}\overline{f}(x,\overline{y}) + \varepsilon$ for every $\overline{y} \in \mathrm{co}(Y)$ and $\varepsilon > 0$. This implies $\inf\sup f \leqq \inf\sup\overline{f}$.

4.2. THEOREM. *Let X be a compact topological space, Y an abstract set and* $f : X \times Y \to \mathbf{R}$ *be a function satisfying* 4.1(ii) *and such that the mapping* $x \mapsto f(x,\cdot)$ *is Ky Fan-concave and upper semicontinuous (cf. footnote* [1]*). Then* $\inf\sup f = \sup\inf f$.

PROOF. The lifted function $\overline{f} : X \times \mathrm{co}(Y) \to \mathbf{R}$ is of Ky Fan-type (for definition see 3.1). Hence, by Ky Fan's minimax theorem $\inf\sup\overline{f} = \sup\inf\overline{f}$. Since for every fixed $y \in Y$, the function $x \mapsto f(x,y)$ is upper semicontinuous on the compact space X, also 4.1(i) holds. Thus, by 4.1, also $\inf\sup f = \inf\sup\overline{f} = \sup\inf\overline{f} = \sup\inf f$.

4.3. REMARK. Several equivalent but seemingly weaker formulations can be given for the conditions of 4.2.

(i) The Ky Fan-concavity of $x \mapsto f(x,\cdot)$ can be replaced by König-concavity in view of 2.4.

(ii) Observe that, by writing $\mathcal{M}_Y := \{\text{functions } Y \to (0,\infty)\}$, the family of all figures

$$U_\mu := \Big\{\overline{y} \in \overline{Y} : \sum_{y\in Y} |\overline{y}(y)|\,\mu(x) < 1\Big\} \qquad (\mu \in \mathcal{M}_Y)$$

forms a neighbourhood basis of 0 for the topology τ_Y on the space $\overline{Y}$. Therefore condition 4.1(ii) can be formulated elementarily as follows:

For every finite family $\{y_1,\dots,y_n\} \subset Y$ and $t_1,\dots,t_n \geqq 0$ with $\sum_{i=1}^n t_i = 1$ and for every $\mu \in \mathcal{M}_Y$ there exist $y^* \in Y$ and $\{(y_i', t_i') : i = 1,\dots,n'\} \subset Y \times \mathbf{R}_+$ such that $n' \geqq n$, $y_i' = y_i$ $(i = 1,\dots,n)$, $\sum_{i=1}^{n'} t_i' = 1$,

$$f(\cdot,y^*) \leqq \sum_{i=1}^{n'} t_i' f(\cdot,y_i') \quad \text{and} \quad \sum_{i\leqq n} |t_i - t_i'|\mu(y_i) + \sum_{i>n} t_i'\mu(y_i') < 1.$$

4.4. COROLLARY. *If X is a compact space, Y is a set and $f : X \times Y \to \mathbf{R}$ is a function such that*

$\{\overline{y} \in \operatorname{co}(Y) : \exists y^* \in Y,\ f(\cdot,y^*) \leqq \sum_{y\in Y} \overline{y}(y) f(\cdot,y)\}$ is dense in $\operatorname{co}(Y)$ with respect to the topology τ_Y,

$\{\overline{x} \in \operatorname{co}(X) : \exists x^* \in X,\ f(\cdot,x^*) \geqq \sum_{x\in X} \overline{x}(x) f(x,\cdot)\}$ is dense in $\operatorname{co}(X)$ with respect to the topology τ_X and the mapping $x \mapsto f(x,\cdot)$ is continuous (cf. footnote [1]) then $\inf\sup f = \sup\inf f$.

PROOF. In view of 4.3(i) we need only to verify the König concavity of $x \mapsto f(x,\cdot)$.

Let $x_1, x_2 \in X$ be arbitrarily fixed. We have to find $x^* \in X$ such that $f(x^*,y) \geqq \big(f(x_1,y) + f(x_2,y)\big)/2$ for all $y \in Y$.

Given any $\varepsilon > 0$ and finite subset $F \subset Y$, define

$$\mu_{\varepsilon,F}(x) := \sum_{y\in F} \max|f(x,y)|/\varepsilon \qquad (x \in X).$$

By the continuity of the mapping $x \mapsto f(x,\cdot)$, the function $\mu_{\varepsilon,F}$ belongs to $\mathcal{M}_X$ (for definition see 4.3(ii)). By assumption, we can choose $x_{\varepsilon,F} \in X$ and $\overline{x}_{\varepsilon,F} \in \operatorname{co}(X)$ such that

$$f(\overline{x}_{\varepsilon,F},\cdot) \geqq \sum_{x\in X} \overline{x}_{\varepsilon,F}(x) f(x,\cdot) \quad \text{and} \quad \overline{x}_{\varepsilon,F} - \overline{x}^* \in U_{\mu_{\varepsilon,f}}$$

where $\overline{x}^* := (1/2)1_{x_1} + (1/2)1_{x_1}$ and $U_{\mu_\varepsilon,f}$ denotes the τ_X-neighbourhood of 0 defined in 4.3(ii). It follows from the definition of $U_{\mu_\varepsilon,f}$ that

$$\left|\sum_{x\in X}\overline{x}_{\varepsilon,F}(x)f(x,y)-\sum_{x\in X}\overline{x}^*(x)f(x,y)\right|\leqq$$

$$\leqq\sum_{x\in X}\left|\overline{x}_{\varepsilon,F}(x)-\overline{x}^*(x)\right|\max\left|f(\cdot,y)\right|\leqq\varepsilon\quad(y\in F).$$

In particular

$$f(x_{\varepsilon,F},y)\geqq\sum_{x\in X}\overline{x}_{\varepsilon,F}(x)f(x,y)\geqq\sum_{x\in X}\overline{x}^*(x)f(x,y)-\varepsilon=$$

$$=\frac{1}{2}f(x_1,y)+\frac{1}{2}f(x_2,y)-\varepsilon\quad(y\in F).$$

If x^* is an accumulation point (with respect to the topology of X) of the net $(x_{\varepsilon,F}:\varepsilon>0, F$ finite $\subset Y)$ then, by the continuity of the functions $x\mapsto$ $\mapsto f(x,y)$ $(y\in Y)$ on the space X, we have $f(x^*,y)\geqq\big(f(x_1,y)+f(x_2,y)\big)/2$ for all $y\in Y$.

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