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A NOTE ON INVARIANT SETS OF ITERATED FUNCTION SYSTEMS

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Abstract. We prove that the family of all invariant sets of iterated systems of contractions $\mathbf{R}^N \to \mathbf{R}^N$ is a nowhere dense F_σ type subset in the space of the nonempty compact subsets of $\mathbf{R}^N$ equipped with the Hausdorff metric.

An *iterated function system* (IFS for short) is a finite collection $(T_1, \dots, T_n)$ of weak contractions of a metric space X. By a *weak contraction* we mean a mapping $T\colon X \to X$ such that $d\big(T(x), T(y)\big) < d(x, y)$ for all $x, y \in X$, where d is the metric on X. A subset $A \subseteqq X$ is called an *invariant set* for the system if $A = T_1(A) \cup \dots \cup T_n(A)$. Given a real number $0 < r < 1$, a mapping T is called an *r-contraction* if $d\big(T(x), T(y)\big) < r \cdot d(x, y)$ for all x, y. The term *contraction* without adjectives refers to r-contraction for some $0 \leqq r < 1$ according to the most widespread terminology in the literature. It is known that if the space X is complete then, for any IFS of contractions, there exists a unique nonempty compact invariant set (see [1], [3], [4]). A general IFS may admit no invariant sets, however, it is not hard to see that if it has a compact invariant set then this must be unique. It is also known that any compact set in the euclidean spaces $\mathbf{R}^n$ can be arbitrarily closely approximated (in Hausdorff distance) by invariant sets of suitably chosen IFSs;

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see [1], [3]. Indeed, finite sets are dense among the compact subsets of X in Hausdorff distance and the finite set $A = \{a_1, \dots, a_n\}$ is the invariant set of the IFS $(T_1, \dots, T_n)$ with one point shrinkings $T_k \colon X \to \{a_k\}$.

It is a natural question to ask whether it is true that any compact set is actually the invariant set of some IFS. We show below that the answer to this question is no: we construct compact sets in $\mathbf{R}$ that are not invariant sets for any IFS. Starting from a one-dimensional example, one can obtain many more. We prove that the invariant sets form a nowhere dense set, which is of F_σ-type if only IFS of r-contractions are taken, among the compact subsets of $\mathbf{R}^n$, thus solving a problem raised by Edgar [2].

LEMMA 1. *There is a compact subset A of $\mathbf{R}$ such that A is not the invariant set of any iterated function system.*

PROOF. Define recursively the index sequence $n_1, n_2, \dots$ with the relations

$$n_1 := 1, \qquad n_k := (k+1)(n_1 + \dots + n_{k-1}) \quad \text{for} \quad k > 1.$$

Then there is a unique decreasing sequence $a_n \to 0$ such that, by setting $n_0 := 0$, for each index $k \geqq 1$ we have

$$a_{n_k} = 2^{1-k}, \quad a_i - a_{i-1} = \delta_k := 2^{1-k}(n_k - n_{k-1})^{-1} \quad \text{if} \quad n_{k-1} < i \leqq n_k.$$

In terms of this sequence, define

$$A := \{0\} \cup \{a_n \colon n = 1, 2, \dots\}, \quad A_k := \{a_i \colon n_{k-1} < i \leqq n_k\}, \quad a_\infty := 0.$$

Consider any weak contraction $T \colon A \to A$ and let i be any index with $n_{k-1} < i \leqq n_k$. Let also $T(a_i) = a_m$ and $T(a_{i-1}) = a_n$, where $m, n \in \mathbf{N} \cup \{\infty\}$. Then we have

$$|a_n - a_m| = \bigl|T(a_i) - T(a_{i-1})\bigr| < |a_i - a_{i-1}| = \delta_k.$$

It follows that either $n, m \geqq n_k$ or $n = m$. Therefore

$$\text{(1)} \quad \text{either} \quad T(A_k) \subset \{a_{n_k}\} \cup \bigcup_{j>k} A_j \cup \{0\} \quad \text{or} \quad T(A_k) = \{\text{one point}\} = T\{a_{n_k}\}.$$

On the other hand, since weak contractions are continuous, $T(a_n) \to T(0)$ as $n \to \infty$. Since 0 is the only accumulation point in A, either we have $T(0) = 0$ or $0 \neq T(0) = a_n$ except for finitely many indices n. If $T(0) = 0$ then, given any index n, the assumption $\bigl|T(a_n) - T(0)\bigr| < |a_n - 0|$ implies $T(a_n) = a_{n+d(n)}$ for some $d(n) > 0$. Thus we have also the alternatives

$$\text{(2)} \quad \text{either} \quad T(A) \text{ is finite} \quad \text{or} \quad T(0) = 0 \quad \text{and} \quad T(a_{n_k}) \in \bigcup_{j>k} A_j \cup \{0\} \quad \text{for all } k.$$

Assume that, in contrast to the statement of the lemma, A is the invariant set of some IFS $(T_1, \dots, T_N)$ consisting of weak contractions of A.

Without loss of generality, we may also assume that $0 = T_1(0) = \cdots = T_M(0)$ and $T_m(A)$ is finite for any $m > M$, that is $\bigcup_{M<m\leqq N} T_m(A) \subset A_1 \cup \dots \cup A_{K-1}$ for some index K with $K > N$. Then, using (1) and (2) we get

$$A_K = A_K \cap \bigcup_{m=1}^{N} T_m(A) = \bigcup_{m=1}^{M} \left[T_m(A) \cap A_K\right]$$

$$= \bigcup_{m=1}^{M} \bigcup_{j\in\mathbf{N}\cup\{\infty\}} \left[T_m(A_j) \cap A_K\right] \subset \bigcup_{m=1}^{M} \bigcup_{j<K} T_m(A_j).$$

However,

$$\#A_K = n_K - n_{K-1} > K \sum_{j<K} n_j \geqq K \sum_{j<K} \#A_j > M \sum_{j<K} \#A_j$$

$$\geqq \# \bigcup_{m=1}^{M} \bigcup_{j<K} \left[T_m(A_j)\right],$$

a contradiction. $\square$

Using the set just constructed in $\mathbf{R}$, we can obtain examples in $\mathbf{R}^n$.

LEMMA 2. *Let $A \subset [0,1]$ be the set constructed in Lemma 1. Suppose $\varepsilon > 0$, $\mathbf{u} \in \mathbf{R}^N$ is a unit vector and $\mathbf{Q} = \{\mathbf{q}_1, \dots, \mathbf{q}_m\}$ is a subset of $\mathbf{R}^N$ such that $\langle \mathbf{q}_\ell - \mathbf{q}_1, \mathbf{u}\rangle > \varepsilon$ ($\ell = 2, \dots, m$) and define $\mathbf{B} := \mathbf{Q} + \varepsilon A\mathbf{u}$. Then $\mathbf{B}$ is not the invariant set of any iterated function system of weak contractions.*

PROOF. Let $\mathbf{B}_\ell := \mathbf{q}_\ell + \varepsilon A\mathbf{u} = \{\mathbf{q}_\ell + \varepsilon\alpha\mathbf{u} \colon \alpha \in A\}$. Observe that

$$\mathbf{B} = \bigcup_{\ell=1}^{m} \mathbf{B}_\ell, \quad \mathbf{B}_\ell = \mathbf{S}_\ell(\varepsilon A) \quad \text{where} \quad \mathbf{S}_\ell \colon \mathbf{R} \to \mathbf{R}^N, \quad \mathbf{S}_\ell(\lambda) := \mathbf{q}_\ell + \lambda\mathbf{u}.$$

On the other hand, we have $\langle \mathbf{B}_\ell, \mathbf{u}\rangle \left(:= \left\{ \langle \mathbf{b}, \mathbf{u}\rangle \colon \mathbf{b} \in \mathbf{B}_\ell \right\}\right) = \langle \mathbf{q}_\ell, \mathbf{u}\rangle + \varepsilon A$ for every index ℓ and, by assumption, $\langle \mathbf{B}_\ell, \mathbf{u}\rangle - \langle \mathbf{q}_1, \mathbf{u}\rangle > \varepsilon$ (that is $\langle \mathbf{b} - \mathbf{q}_1, \mathbf{u}\rangle > \varepsilon$ for all $\mathbf{b} \in \mathbf{B}_\ell$) if $\ell > 1$. Hence

$$P(\mathbf{B}_1) = \varepsilon A, \quad P(\mathbf{B}_\ell) = \{\varepsilon\} \quad \text{for } \ell > 1$$

with the mapping $P \colon \mathbf{R}^N \to \mathbf{R}$, $P(\mathbf{v}) := \min\left\{\varepsilon, \langle \mathbf{v} - \mathbf{q}_1, \mathbf{u}\rangle\right\}$.

Assume that the contrary of the statement of Lemma 2 holds, that is $\mathbf{B} = \bigcup_{k=1}^{r} \mathbf{T}_k(\mathbf{B})$ where $\mathbf{T}_1, \dots, \mathbf{T}_r \colon \mathbf{B} \to \mathbf{B}$ are weak contractions of $\mathbf{B}$. Then $\mathbf{B} = \bigcup_{k=1}^{r} \bigcup_{\ell=1}^{m} \mathbf{T}_k(\mathbf{B}_\ell)$. Therefore, since $1 = a_1 = \max A$ by construction, we have

$$\varepsilon A = \bigcup_{k=1}^{r} \bigcup_{\ell=1}^{m} T_{k\ell}(\varepsilon A) \quad \text{where} \quad T_{k\ell} := P\mathbf{T}_k\mathbf{S}_\ell.$$

The mappings P, $\mathbf{S}_\ell$ are non-expansive and hence each $T_{k\ell} \colon \varepsilon A \to \varepsilon A$ is a weak contraction. This fact contradicts Lemma 1 (with εA instead of A and $T_{k\ell}$ instead of T_k). $\square$

Definition 3. Given any constant $r > 1$, by an *r-fractal* in $\mathbf{R}^N$ we mean a nonempty compact subset $\mathbf{B} \subset \mathbf{R}^N$ that is the invariant set of an IFS $(T_1, \dots, T_r)$ consisting of $(1 - r^{-1})$-contractions of $\mathbf{B}$. We write $\mathcal{F}_r^{(N)}$ for the set of all r-fractals and $\mathcal{F}^{(N)}$ for the family of all invariant sets of IFS by contractions of $\mathbf{R}^N$. Notice that $\mathcal{F}^{(N)} = \bigcup_{r>1} \mathcal{F}_r^{(N)}$.

Lemma 4. *The families $\mathcal{K}_n^{(N)} := \mathcal{F}_n^{(N)} \cap \mathcal{B}_n^{(N)}$ where $\mathcal{B}_n^{(N)} := \big\{\mathbf{B} \subset \mathbf{R}^N \colon \mathbf{B}$ compact, $\sup_{\mathbf{b}\in\mathbf{B}} \langle \mathbf{b}, \mathbf{b}\rangle^{1/2} \leqq n\big\}$ are compact in the Hausdorff distance d_N among the nonempty compact subsets of $\mathbf{R}^N$.*

Proof. Fix n, N arbitrarily. Suppose $\mathbf{B}^{(1)}, \mathbf{B}^{(2)}, \dots$ is a sequence in $\mathcal{K}_n^{(N)}$. We have to see that some of its subsequences converges to a set $\mathbf{B} \in \mathcal{K}_n^{(N)}$.

By assumption, there are $(1 - n^{-1})$-contractions $\mathbf{T}_k^{(i)} \colon \mathbf{B}^{(i)} \to \mathbf{B}^{(i)}$ $(1 \leqq k \leqq n,\ i = 1, 2, \dots)$ such that $\mathbf{B}^{(i)} = \bigcup_{k=1}^{n} \mathbf{T}_k^{(i)}\big(\mathbf{B}^{(i)}\big)$. Consider the sets

$$\operatorname{graph}\big(\mathbf{T}_k^{(i)}\big) := \Big\{\big(\mathbf{a}, \mathbf{T}_k^{(i)}(\mathbf{a})\big) \colon \mathbf{a} \in \mathbf{B}^{(i)}\Big\} \subset \mathbf{R}^N \times \mathbf{R}^N \equiv \mathbf{R}^{2N}.$$

It is well known that the family $\mathcal{B}_n^{(N)}$ is compact in d_N. Also $\operatorname{graph}\big(\mathbf{T}_k^{(i)}\big) \in \mathcal{B}_{2n}^{(2N)}$. Thus we may assume without loss of generality that

$$\lim_{i\to\infty} d_N\big(\mathbf{B}^{(i)}, \mathbf{B}\big) = 0, \quad \lim_{i\to\infty} d_{2N}\Big(\operatorname{graph}\big(\mathbf{T}_k^{(i)}\big), \mathbf{G}_k\Big) = 0, \quad 1 \leqq k \leqq n$$

for some nonempty compact sets $\mathbf{B} \in \mathcal{B}_n^{(N)}$ and $\mathbf{G}_1, \dots, \mathbf{G}_n \in \mathcal{B}_{2n}^{(2N)}$. It is also well known about limit sets in Hausdorff distance that here we have

$$\text{(3)} \quad \begin{cases} \mathbf{B} = \Big\{\mathbf{b} \in \mathbf{R}^N \colon \exists\, \mathbf{b}^{(1)}, \mathbf{b}^{(2)}, \dots \quad \mathbf{B}^{(i)} \ni \mathbf{b}^{(i)} \to \mathbf{b}\ (i \to \infty)\Big\}, \\ \mathbf{G}_k = \Big\{(\mathbf{a}, \mathbf{b}) \in \mathbf{R}^{2N} \colon \exists\, \mathbf{b}^{(1)}, \mathbf{b}^{(2)}, \dots \\ \qquad \mathbf{B}^{(i)} \ni \mathbf{b}^{(i)} \to \mathbf{a},\ \mathbf{T}_k^{(i)}\big(\mathbf{b}^{(i)}\big) \to \mathbf{b}\ (i \to \infty)\Big\}. \end{cases}$$

Using the coordinate projections $\Pi_1, \Pi_2 \colon \mathbf{R}^{2N} \to \mathbf{R}^N$, $\Pi_1(\mathbf{a},\mathbf{b}) := \mathbf{a}$, $\Pi_2(\mathbf{a},\mathbf{b}) := \mathbf{b}$, it is immediate that

$$\mathbf{B} = \lim_{i\to\infty} \mathbf{B}^{(i)} = \lim_{i\to\infty} \Pi_1 \operatorname{graph}\left(\mathbf{T}_k^{(i)}\right) = \Pi_1 \mathbf{G}_k, \quad 1 \leqq k \leqq n,$$

$$\mathbf{B} = \lim_{i\to\infty} \bigcup_{k=1}^{n} \mathbf{T}_k^{(i)}\left(\mathbf{B}^{(i)}\right) = \lim_{i\to\infty} \bigcup_{k=1}^{n} \Pi_2 \operatorname{graph}\left(\mathbf{T}_k^{(i)}\right)$$

$$= \Pi_2 \bigcup_{k=1}^{n} \lim_{i\to\infty} \operatorname{graph}\left(\mathbf{T}_k^{(i)}\right) = \Pi_2 \bigcup_{k=1}^{n} \mathbf{G}_k = \bigcup_{k=1}^{n} \Pi_2 \mathbf{G}_k$$

where the limits are taken in the respective Hausdorff metrics. On the other hand, from (3) and the fact that the mappings $\mathbf{T}_1^{(i)}, \dots, \mathbf{T}_n^{(i)}$ are $(1-n^{-1})$-contractions, we see that

$$(\mathbf{a},\mathbf{b}), (\mathbf{a}',\mathbf{b}') \in \mathbf{G}_k \Rightarrow \langle \mathbf{b}-\mathbf{b}', \mathbf{b}-\mathbf{b}'\rangle^{1/2} \leqq [1-n^{-1}]\langle \mathbf{a}-\mathbf{a}', \mathbf{a}-\mathbf{a}'\rangle^{1/2}.$$

Therefore each set $\mathbf{G}_k$ is the graph of some $(1-n^{-1})$-contraction $\mathbf{T}_k \colon \mathbf{B} \to \mathbf{B}$ and $\bigcup_{k=1}^n \mathbf{T}_k(\mathbf{B}) = \bigcup_{k=1}^n \Pi_2 \mathbf{G}_k = \mathbf{B}$. That is $\mathbf{B} \in \mathcal{F}_n^{(N)}$ which completes the proof. $\square$

THEOREM 5. *The set $\mathcal{F}^{(N)}$ of all invariant sets by IFS of contractions $\mathbf{R}^N \to \mathbf{R}^N$ is a nowhere dense F_σ-set among the compact subsets of $\mathbf{R}^N$ with respect to the Hausdorff metric d_N.*

PROOF. We have $\mathcal{F}^{(N)} = \bigcup_{n=1}^\infty \mathcal{K}_n^{(N)}$. According to Lemma 4, each family $\mathcal{K}_n^{(N)}$ is closed with respect to d_N. Thus $\mathcal{F}^{(N)}$ is of F_σ type in the d_N-metric. It is well known that the family $\mathcal{Q}^{(N)}$ of all finite subsets of $\mathbf{R}^N$ is dense in $\mathcal{C}^{(N)} := \{\text{compact subsets of } \mathbf{R}^N\}$. Given any set $\mathbf{Q} \in \mathcal{Q}^{(N)}$ and a point $\mathbf{q} \in \mathbf{Q}$, we can choose a unit vector $\mathbf{u}_{\mathbf{Q},\mathbf{q}} \in \mathbf{R}^N$ such that $\varepsilon_{\mathbf{Q},\mathbf{q}} := \min_{\mathbf{p}\in\mathbf{Q}\setminus\{\mathbf{q}\}} \langle \mathbf{p}, \mathbf{u}_\mathbf{Q}\rangle - \langle \mathbf{q}, \mathbf{u}_\mathbf{Q}\rangle > 0$. By Lemma 2, for the sets $\mathbf{Q}_\varepsilon := \mathbf{Q} + \min\{\varepsilon, \varepsilon_{\mathbf{Q},\mathbf{q}}/2\} A \mathbf{u}_{\mathbf{Q},\mathbf{q}}$ with $\varepsilon > 0$ we have $\mathbf{Q}_\varepsilon \notin \mathcal{F}^{(N)}$. However, $d_N(\mathbf{Q}_\varepsilon, \mathbf{Q}) < \varepsilon$ for any $\varepsilon > 0$. Thus the set $\{\mathbf{Q}_\varepsilon \colon \mathbf{Q} \in \mathcal{Q}^{(N)}, \varepsilon > 0\}$ contained in the complement of $\mathcal{F}^{(N)}$ is dense in $\mathcal{C}^{(N)}$. Consequently, since $\mathcal{F}^{(N)}$ is of F_σ-type, it must be nowhere dense with respect to d_N. $\square$

REMARK. Actually, by Lemma 2, the sets $\mathbf{Q}_\varepsilon$ are not invariant sets of any IFS even by weak contractions. Therefore the family $\mathcal{F}^{(N,w)}$ of all invariant sets of IFS by weak contractions $\mathbf{R}^N \to \mathbf{R}^N$ is also not dense in Hausdorff distance in the space of all compact sets. However, $\mathcal{F}^{(N,w)}$ has a more sophisticated structure than being F_σ-type.

We give another example of a subset of $\mathbf{R}$ that is not the invariant set of any IFS consisting of r-contractions. We claim that the set $K = \{0, (\ln 3)^{-1}, (\ln 4)^{-1}, (\ln 5)^{-1}, \dots\}$ has this property. Notice that for the proof of Theorem 5 it suffices to use such a set only instead of that in Lemma 1 with the additional restrictiction of not being the invariant set of any IFS of weak contractions.

Let us assume, to the contrary, that $K = f_1(K) \cup \dots \cup f_n(K)$ for some $r(<1)$-contractions $f_1, \dots, f_n$. Let $a_k = \big(\ln(k+2)\big)^{-1}$, $k = 1, 2, \dots$. Define the *density* $d(B)$ of a subset $B \subseteqq K$ in the following way:

$$d(B) = \limsup_{N\to\infty} \frac{\sharp\{k\colon 1 \leqq k \leqq N \quad \text{and} \quad a_k \in B\}}{N}.$$

Clearly, we have $d\big(f_1(K)\big) + \dots + d\big(f_n(K)\big) \geqq 1$. Next we show that $d\big(f(K)\big) = 0$ for any weak contraction $f\colon K \to K$, thus obtaining a contradiction.

First suppose $f(0) \neq 0$. Then, by the continuity of f, the set $f(K)$ is easily seen to be finite. It follows that $d\big(f(K)\big) = 0$.

Now let us assume that $f(0) = 0$. Let $0 < r < 1$ denote the contraction factor of f, that is, we have $\big|f(x) - f(y)\big| \leqq r \cdot |x - y|$ for every x, y. If $1 \leqq k \leqq N$ and $a_k \in f(K)$ then $a_k = f(a_j)$ for some j. Then we have $a_k = |a_k - 0| \leqq r \cdot |a_j - 0| = r \cdot a_j$, that is, $\big(\ln(k+2)\big)^{-1} \leqq r \cdot \big(\ln(j+2)\big)^{-1}$, so $j + 2 \leqq (k+2)^r \leqq (N+2)^r$. It follows that

$$d\big(f(K)\big) \leqq \liminf_{N\to\infty} \frac{(N+2)^r - 2}{N},$$

and therefore $d\big(f(K)\big) = 0$.

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