ANNALES UNIV. SCI. BUDAPEST., **46** (2003), 149–156

ON THE JORDAN STRUCTURE OF TERNARY RINGS OF OPERATORS

By

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(Received May 14, 2004)

By a *ternary ring of operators (TRO)* we mean a norm-closed subspace in some $\mathscr{L}(H, K)$ (=$\{$bounded linear operators $H \to K\}$ with complex Hilbert spaces H, K) which is closed under the *ternary product* $[xyz] := xy^*z$. TRO's were introduced by Hestenes [9, 1962] who proved that, in the finite dimensional setting, TRO's can be faithfully represented as direct sums of spaces $\mathcal{M}_{m,n}(\mathbb{C})$ of $m \times n$ complex matrices. In infinite dimensions, Zettle [13, 1983] gave a characterization of TRO's among ternary Banach algebras, whence one could discover that Hilbert C*-modules are the same as TRO's. Henceforth many deep results have appeared studying TRO's and their applications, see [3, 2001], [11, 2002] and [6, 1999], among others showing that every TRO is isometrically isomorphic to a corner $pA(1-p)$ of a C*-algebra and that the ternary product is uniquely determined by the metric structure in a TRO. As a consequence, since the bidual of a C*-algebra is a W*-algebra, a TRO can be represented as a weak*-dense subTRO in $\bigoplus_{i \in I} p_i A_i (1-p_i)$, where $(A_i)_{i \in I}$ is the family of M-summands of A^{**}. The aim of this note is to show that this description can be refined somewhat to an infinite dimensional version of Hestenes' theorem. Namely we have the following

THEOREM 1.1. *Every TRO is isometrically isomorphic to a weak*-dense subTRO of the natural TRO of a direct sum* $\bigoplus_{i \in I} \mathscr{L}(H_i, K_i)$. *In particular,*

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[1] Supported by Ministerio de Educación y Cultura of Spain, Research Project BFM 2002-1529.

[2] Supported by Spanish–Hungarian Project TéT E-50/02 and Hungarian Research Grant OTKA T34267.

up to isometric isomorphisms, TRO's with predual are ℓ_∞-direct sums of $\mathcal{L}(H,K)$-spaces and a reflexive TRO is a finite ℓ_∞-direct sum of copies of $\mathcal{L}(H,K)$ spaces with $\dim K < \infty$.

Our proofs rely upon the Jordan theory of Banach spaces with symmetric unit ball, the so called *JB*-triples.* According to a result of Harris [7, 1973], TRO's when equipped with the Jordan triple product (*) $\{xyz\} := = (xy^*z + zy^*x)/2$ are JC*-triples and hence their unit ball is necessarily symmetric. Since the bidual of a C*-algebra is isometrically isomorphic to a weak*-closed subalgebra in some $\mathcal{L}(\widehat{H})$, the bidual of a TRO is a TRO again. Therefore, by Friedmann–Russo's Gelfand–Naimark type theorem for JB*-triples [4, 1985], it follows that any TRO E is isometrically isomorphic to a weak*-dense subTRO in the (ℓ^∞-direct) sum $\oplus_{j\in J}F_j$ of the minimal weak*-closed M-summands, the so called *Cartan factors,* of the bidual E^{**}, furthermore each Cartan factor F_j is a subTRO of E^{**}. From the theorem and its Jordan theoretical proof we obtain also the following characterization of TRO's among JB*-triples.

COROLLARY 1.2. *A JB*-triple E is the triple associated to a TRO if and only if in the canonical decomposition of the bidual $E^{**} = E_{\mathrm{at}} \oplus E_n$, the atomic ideal E_{at} consists only of Cartan factors of type 1. A TRO admits no Jordan*-representation (JB*-homomorphism) with weak*-dense range into a Cartan factor that is not of type 1.*

REMARK 1.3. From a holomorphic view point, JC*-triples (norm-closed subspaces of some $\mathcal{L}(H)$ closed under the Jordan-triple product $\{xyz\} := = xy^*z/2 + zy^*x/2$) are known as (isometric) copies of Banach spaces with symmetric unit balls which admit only vanishing Jordan representations in exceptional Cartan factors. It would be tempting to conjecture that TRO's are copies of those Banach spaces with symmetric unit ball whose Jordan representations in Cartan factors not isomorphic to some $\mathcal{L}(H,K)$ vanish. However this is not the case. Namely the assumption of the weak*-density of the range in Corollary 1.2 is indispensable: *There is an isometric JB*-homomorphism of the TRO $\mathcal{M}_n(\mathbb{C})$ of complex n-square matrices into the space $\mathcal{S}_{2n}(\mathbb{C})$ of symmetric 2n-square matrices.*

2. Proofs

Before stating the proofs we recall some basic facts and notions involved. We know that given a surjective linear isometry $T: F_1 \to F_2$ between two TRO's, necessarily $T[xyz] = [(Tx)(Ty)(Tz)]$, $(x,y,z \in F_1)$. Furthermore if $F_i \subset \mathcal{L}(H_i, K_i)$, $(i \in I)$, are TRO's then their ℓ_∞-sum $\bigoplus_{i\in I} F_i$ is a TRO in the space $\mathcal{L}\big(\bigoplus_{i\in I}^2 H_i, \bigoplus_{i\in I}^2 K_i\big)$ with the ℓ_2-sums $\bigoplus_{i\in I}^2 H_i$ and $\bigoplus_{i\in I}^2 K_i$, and the natural pointwise operation $[(x_i)(y_i)(z_i)] := (x_i y_i z_i)$.

For later use, recall that JB*-triples can be equipped with a unique three variable operation $(x,y,z) \mapsto \{xyz\}$ which is symmetric linear in x,z and conjugate-linear in y satisfying among other axioms (for a complete list see [4]) the Jordan identity

$$\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$$

and the C*-axiom $\|\{xxx\}\| = \|x\|^3$. An element e in a JB*-triple is called a *tripotent* if $0 \neq e = \{eee\}$ in which case it has norm 1 and we write $Tri(E)$ for their family. Tripotents with respect to the Jordan triple product in a TRO are partial isometries. A tripotent e is said to be an *atom* in E if $\{eEe\} = \mathbb{C}e$ and we write $At(E)$ for the set of them. Recall that given $e, f \in Tri(E)$ we say that e *governs* f (written $e \vdash f$) if $e \in E_1(f) := \{x \in E: \{eex\} = x\}$ and $f \in E_{1/2}(e) := \{x \in E: \{eex\} = x/2\}$. We say that e,f are *collinear* (written $e \top f$) if $e \in E_{1/2}(f)$ and $f \in E_{1/2}(e)$.

In order to establish our main result we need some technical lemmas on JB*-triples.

LEMMA 2.1. *Let F be a TRO in $\mathcal{L}(H)$ and suppose $e,f \in Tri(F)$ are such that $\{eef\} = f/2$. Then the elements $x := ee^*f$ and $y := fe^*e$ are orthogonal tripotents in F that satisfy $f = x + y$.*

PROOF. By assumption $f = 2\{eef\} = ee^*f + fe^*e = x + y$. Hence $x = ee^*f = ee^*ee^*f + ee^*fe^*e = x + xe^*e$, that is $xe^*e = ee^*y = ee^*fe^*e = 0$. It follows $xy^* = ee^*fe^*ef^* = 0$, $yx^* = (xy^*)^* = 0$. Similarly $x^*y = f^*ee^*fe^*e = 0$, $y^*x = (x^*y)^* = 0$. Therefore

$$x + y = f = ff^*f = (x+y)(x+y)^*(x+y) =$$
$$= xx^*x + yy^*y,$$
$$ee^*(x+y) = ee^*xx^*x + e$$

since $x = ee^*x$ and $ee^*y = 0$. This means that $x = xx^*x$ and $y = yy^*y$, thus $x,y \in Tri(F)$. On the other hand $2\{xxy\} = x(x^*y) + (yx^*)x = x0 + 0x = 0$, that is $x \perp y$. ∎

LEMMA 2.2. *Let F be a TRO in $\mathcal{L}(H)$, and suppose $0 \neq e, f \in At(F)$ with $e \top f$. Then for the projections $p := ee^*$, $q := ff^*$, $P := e^*e$, $Q := f^*f$ we have either $p = q$ and $PQ = QP = 0$ or $P = Q$ and $pq = qp = 0$.*

PROOF. By Lemma 2.1 and since atoms are indecomposable into sums of non-zero orthogonal tripotents, the tripotents

$$x := ee^*f \quad y := fe^*e \quad X := ff^*e \quad Y := ef^*f$$

satisfy the alternatives

1) $x = f,\ y = 0,\ X = e,\ Y = 0,$ 2) $x = f,\ y = 0,\ X = 0,\ Y = e,$
3) $x = 0,\ y = f,\ X = e,\ Y = 0,$ 4) $x = 0,\ y = f,\ X = 0,\ Y = e.$

The alternative 2) implies $ee^*f = f, fe^*e = 0, ff^*e = 0, ef^*f = e$ and $ff^* = f * (ee^*f)^* = ff^*ee^* = (ff^*e)e^* = 0e^* = 0$ that is $f = 0$, contradicting the assumption $0 \neq f$.

3) implies $ee^*f = 0, fe^*e = f, ff^*e = e, ef^*f = 0$ and $ee^* = (ff^*e)e^* = e(ee^*f)^* = e0^* = 0$ that is $e = 0$, contradicting the assumption $0 \neq e$.

1) means $ee^*f = f, fe^*e = 0, ff^*e = e, ef^*f = 0$. Hence $q = ff^* = (ee^*f)f^* = (ee^*)(ff^*) = pq$ and also $q = ff^* = f(ee^*f)^* = (ff^*)(ee^*) = qp$. Therefore $p = ee^* = (ff^*e)e^* = (ff^*)(ee^*) = (ee^*)(ff^*) = ff^* = q$. On the other hand $PQ = (e^*e)(f^*f) = e^*(ef^*f) = e^*0 = 0$, $QP = (f^*f)(e^*e) = f^*(fe^*e) = f^*0 = 0$.

4) means $ee^*f = 0, fe^*e = f, ff^*e = 0, ef^*f = e$. Hence $P = e^*e = e^*(ef^*f) = (ee^*)(f^*f) = PQ$ and also $P = e^*e = (ef^*f)^*e = (f^*f)(e^*e) = QP$. Therefore $Q = f^*f = (fe^*e)^*f = e^*ef^*f = PQ = P$. On the other hand $qp = (ff^*)(ee^*) = (ff^*e)e^* = 0e^* = 0$ and $pq = (ee^*)(ff^*) = (ee^*f)f^* = 0f^* = 0$. ∎

COROLLARY 2.3. *If F is a TRO in $\mathcal{L}(H)$ and $0 \neq e_1, \ldots, e_N \in At(F)$ with $e_j \top e_k$ $(k \neq j)$ then either $p_1 = \cdots = p_N$ and $p'_k p'_j = 0$ $(k \neq j)$ or $p'_1 = \cdots = p'_N$ and $p_k p_j = 0$ $(k \neq j)$ for the projections $p_k := e_k e_k^*, p'_k := e_k^* e_k$ $(k = 1, \ldots, N)$.*

PROOF. By Lemma 2.2 we have the alternatives: 1) $p_1 = p_2$ and $p'_1 p'_2 = 0$ or 2) $p'_1 = p'_2$ and $p_1 p_2 = 0$.

1) Suppose $p_j \neq p_1$. Then $p'_j = p'_1$, $p_1 p_j = p_j p_1 = 0$ and also (since $p_j \neq p_2 = p_1$) $p'_j = p'_2$, $p_2 p_j = p_j p_2 = 0$. In particular $p'_j = p'_1 = p'_2$. By our assumption 1), $p'_1 p'_2 = 0$. But then $p'_1 = p'_2 = p'_1 p'_2 = 0$ that is $e_1^* e_1 = p'_1 = 0$ and $e_1 = 0$ which is impossible.

2) Similarly we can exclude $p'_j \neq p'_1$ in this case. ∎

LEMMA 2.4. *Let F be a TRO in $\mathcal{L}(H)$, and suppose $0 \neq e_1, e_2, e_3, e_4 \in$ $\in At(F)$. Then the situation $e_3 \perp e_4$, $e_k \top e_\ell$ $(k < \ell, (k,\ell) \neq (3,4))$ is impossible.*

PROOF. Let $p_k := e_k e_k^*$, $p_k' := e_k^* e_k$ $(k = 1,\ldots,4)$. We have the alternatives 1) $p_1 = p_2$ and $p_1' p_2' = p_2' p_1' = 0$ or 2) $p_1' = p_2'$ and $p_1 p_2 = p_2 p_1 = 0$.

Suppose 1). Since $e_1 \top e_2 \top e_3 \top e_1$, by Corollary 2.3 also $p_1 = p_3$. Since $e_1 \top e_2 \top e_4 \top e_1$, also $p_1 = p_4$. Thus 1) implies $p_1 = p_4$. However, the relationship $e_1 \perp e_4$ means (as it is well-known) that $0 = p_1 p_4 = p_4 p_1$ and $0 = p_1' p_4' = p_4' p_1'$. Therefore 1) is impossible. The case 2) can be treated analogously. ∎

We are now in the position to prove our maun result.

PROOF OF THEOREM 1.1. Let E be a TRO. We know that, without loss of generality, we may regard E^{**} as a weak* closed TRO in a space $\mathcal{L}(\widehat{H})$ with some Hilbert space $\widehat{H}$, moreover E is a weak* dense sub-TRO of E^{**} for the natural ternary product $[x,y,z] := xy^*z$, $(x,y,z \in \mathcal{L}(\widehat{H}))$. From a Jordan viewpoint, E^{**} is an ℓ^∞-direct sum of the form $E^{**} = E^{**}_{\text{at}} \oplus E^{**}_n$ where $E^{**}_{\text{at}} = \oplus_{j\in J} F_j$ and $\{F_j : j \in J\}$ is the family of all minimal atomic M-ideals of E^{**} with respect to the Jordan triple product $\{xyz\} :=$ $= \frac{1}{2}(xy^*z + zy^*x)$, $(x,y,z \in \mathcal{L}(\widehat{H}))$. Since the projection onto the atomic ideal $P_{\text{at}} : E^{**} \to \oplus_{j\in J} F_j$ is an isometric JB*-homomorphism which is a bijection on E, it suffices to see that each factor F_j is a Cartan factor of type 1. Concerning Cartan factors, by the familiar classification, each F_j is isometrically isomorphic to some of the following classical JB*-triples:

$\mathcal{L}(H_j, K_j)$ [type 1],

$\mathcal{L}_\pm(H_j) := \{x \in \mathcal{L}(H_j) : x = \pm\overline{x}^*\}$ [types 2,3] with a conjugation $x \mapsto \overline{x}$,

$\text{Spin}(H_j) := \left[H_j \text{ with } \{xyz\} := \langle x,y\rangle z + \langle z,y\rangle x - \langle x,\overline{z}\rangle\overline{y}\right]$ [type 4],

$\mathcal{M}_{1,2}(\mathbb{O})$ [type 5, of 16 dimensions], here $\mathbb{O}$ means the Cayley algebra of complex octonions.

$\mathcal{H}_3(\mathbb{O})$ [type 6, of 27 dimensions], the algebra of 3×3 hermitian matrices with entries in the octonions $\mathbb{O}$ equipped with the standard conjugation.

Our key observation is that, in all cases if F_j is not isomorphic to some $\mathcal{L}(H_j, K_j)$ then the standard covering atomic grid of F_j (see [12]) contains a couple of atoms e_1, e_2 with $e_1 \vdash e_2$ or it contains a family $\{e_1, e_2, e_3, e_4\}$ of atoms with $e_3 \perp e_4$, $e_k \top e_\ell$ $(k < \ell,\ (k,\ell) \neq (3,4))$. By the previous lemmas it immediately follows that this is impossible.

The statements concerning TRO's with predual are immediate. ∎

For the sake of completeness, we describe the mentioned systems $\{e_1, e_2\}$ respectively $\{e_1, \ldots, e_4\}$ of atoms for the types 2–6.

To this aim, let H be a Hilbert space, let $x \mapsto \overline{x}$ be a conjugation on H, let $\{h_m : m \in M\}$ be a complete orthonormal system in H such that $h_m = \overline{h_m}$, $(m \in M)$, and let $e \otimes f$ denote the operator $x \mapsto \langle x, e\rangle f$ on H.

Case type 2. With $e_1 := h_1 \otimes h_1$, $e_2 := h_1 \otimes h_2 + h_2 \otimes h_1$ we have $e_1, e_2 \in At(\mathscr{L}^-{}_+(H))$ and $e_1 \vdash e_2$.

Case type 3, $\dim E > 3$. With $e_1 := h_1 \otimes h_2 - h_2 \otimes h_1$, $e_2 := h_2 \otimes h_3 - h_3 \otimes h_2$, $e_3 := h_1 \otimes h_3 - h_3 \otimes h_1$, $e_4 := h_2 \otimes h_4 - h_4 \otimes h_2$ we have $e_1, e_2, e_3, e_4 \in At(\mathscr{L}(H))$ and $e_3 \perp e_4$, $e_k \top e_\ell$ $(k < \ell,\ (k,\ell) \neq (3,4))$.

Case type 4, $\dim E > 3$. With $e_k := 2^{-1/2}(h_k + i h_4)$, $(k = 1,2,3)$ and $e_4 := 2^{-1/2}(h_3 - i h_3)$ we have $e_1, e_2, e_3, e_4 \in At(\mathrm{Spin}(H))$ and $e_3 \perp e_4$, $e_k \top e_\ell$, $(k < \ell,\ (k,\ell) \neq (3,4))$.

In the cases of types 5–6 the standard grid of the unit matrices contains 8 atoms spanning a spin factor (type 4) of 8 dimensions. So as in the previous case, again there are atoms $e_1, \ldots, e_4$ with $e_3 \perp e_4$, $e_k \top e_\ell$ $(k < \ell,\ (k,\ell) \neq (3,4))$.

LEMMA 2.5 *If G is a Cartan factor then the atomic part of G^{**} is a copy of G.*

PROOF. We have $G^{**} = G_n^{**} \oplus \bigoplus_{j \in J} G_j$ where G_n^{**} is a non-atomic JBW*-triple and each G_j is a Cartan factor. Also there is an isometric JB*-homomorphism $U : G \to G^{**}$ onto some weak*-closed JB*-subtriple of G^{**}. Let π_j denote the canonical projection $G^{**} \to G_j$ and consider the representation $U_j := \pi_j U$ of G. The kernel K_j of U_j is a weak*-closed ideal in G. Since G is a factor, we have either $K_j = \{0\}$ or $K_j = G$. Since UG is weak*-dense in G^{**}, necessarily $U_j G \neq \{0\}$ and this excludes the possibility of $K_j = G$. Thus $K_j = \{0\}$, that is, the JB*-homomorphism U_j is injective. By a theorem of Horn–Dang–Neher on normal representations [10], injective JB*-homomorphisms are isometries. Thus $U_j G$ is a copy of G lying weak*-dense in the Cartan factor G_j. This is possible only if $U_j G = G_j$ and $U : G \leftrightarrow G_j$ is a JB*-isomorphism. By writing π for the canonical projection

$G^{**} \to \bigoplus_{j\in J} G_j$, it follows that πU is not weak*-dense in $\bigoplus_{j\in J} G_j$ unless the index set J is a singleton. ∎

PROOF OF COROLLARY 1.2. Let E be a TRO, G a Cartan factor and consider a JB*-homomorphism $T : E \to G$. It is well-known that the bidual operator $T^{**} : E^{**} \to G^{**}$ is also a JB*-homomorphism. We have $E^{**} = E_n^{**} \oplus \bigoplus_{i\in I} E_i$ where each term E_i is a Cartan factor and E_n^{**} is a non-atomic JBW*-triple. By the previous lemma, we may assume that $G^{**} = G_{\mathrm{at}}^{**} \oplus G$ and, with the canonical projection $\pi : G^{**} \to G$, the operator πT^{**} is a JB*-homomorphism $E^{**} \to G$ which maps E onto a weak*-closed subtriple of G. Since πT^{**} is weak*-continuous, it follows that $\pi T^{**} E^{**} = G$. The kernel K of the operator πT^{**} is a weak*-closed ideal of E^{**}. It is well known [1, 1985] that $E^{**} = K \oplus K^{\perp}$ where $K^{\perp} := \{x \in E^{**} : \{efx\} = 0,\ \ e,f \in K\}$ is a weak*-closed ideal in E^{**}. Moreover, πT^{**} is an isometry on $K^{\perp}$ because injective JB*-homomorphisms are isometric [10]. Since $G = \pi T^{**} E = \pi T^{**} K^{\perp}$, the weak*-closed ideal $K^{\perp}$ must be a copy of the Cartan factor G. Hence $K^{\perp}$ is a minimal weak*-closed ideal in E^{**} and so $G \simeq K^{\perp} = E_i$ for some $i \in I$. By the theorem, each factor E_i is of type 1, hence so must be G. ∎

PROOF FOR REMARK 1.3. Let $e^{k\ell}$ denote the $n \times n$-matrix with 1 at the position (k,ℓ) and with 0 at other entries and let $s^{k\ell}$ be the symmetric $(2n) \times (2n)$-matrix with 1 at the positions $(2k-1, 2\ell)$ and $(2\ell, 2k-1)$ and 0 elsewhere. It is straightforward to verify that the linear extension T of the map $[e^{k\ell} \mapsto s^{k\ell} : 1 \le k,\ell \le n]$ satisfies the identity $T(xy^*z + zy^*x) = (Tx)(Ty)^*(Tz) + (Tz)(Ty)^*(Tx)$ (by checking it for $n := 3$ and the unit matrices without loss of generality). ∎

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