REMARKS ON SUPERLINEAR OPERATORS

By

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In [1] E. M. NIKIŠIN introduced the notions of superlinear and positive superlinear operators concerning his investigation on Fourier series with respect to general orthonormal systems. According to [1], a mapping $T: E \to \mathcal{S}(0, 1)$ where E is any Banach space is by definition superlinear if for every $e \in E$ there exists a linear mapping $L_e: E \to S(0, 1)$ such that $L_e e = Te$ and $|L_e f| \le |Tf|$ for each $f \in E$. Furthermore, if $E = \mathcal{L}^p(X, \mu)$ for some $p \ge 1$ and for every $e \in E$, L_e can be chosen to be a positive linear mapping then T is called a positive superlinear operator ($\mathcal{L}^p \to \mathcal{S}$).

The aim of this paper is to examine these concepts in a vector lattice theoretical setting.

DEFINITION 1. Let E, F be a vector space and a vector lattice, respectively. A mapping $T: E \to F$ is *superlinear* if for every $e \in E$ there exists $L_e \in \mathcal{L}(E, F)$ such that $L_e e = Te$ and $|L_e| \le |T|$. (Throughout this work, we deal with real vector spaces. The symbol $|T|$ means the operator $f \to |Tf|$.)

PROPOSITION 1. Suppose the space F is oder complete (for def. see [3]). Then $T: E \to F$ is superlinear if and only if $|T|$ is a vector norm on E i.e. if, $|T(e_1+e_2)| \le |Te_1| + |Te_2|$ and $|T\lambda e_1| = |\lambda|\ |Te_1|$ $\forall e_1, e_2 \in E, \lambda \in \mathbf{R}$.

PROOF. Let $T: E \to F$ be superlinear. Then we can write $|T| = \sup_{e \in E} |L_e|$. But L is clearly a vector norm whenever $L: E \to F$ is linear.

Conversely, assume T is a vector norm on E and F is order complete. Given $e \in E$, define L_e^0 on the subspace $\mathbf{R}_e$ by $L_e^0 \lambda e \equiv \lambda Te$ ($\lambda \in \mathbf{R}$). We have $L_e^0 \le |T|$ on $\mathbf{R}_e$. Thus by the generalized Hahn-Banach theorem [2], L_e^0 admits a linear extension L_e such that $L_e \le |T|$. To complete the proof, we show $-L_e \le |T|$. Indeed, $-L_e f = L_e(-f) \le |T(-f)| = |Tf|$ $\forall_f \in E$.

DEFINITION 2. Let E, F be vector lattices. A mapping $T: E \to F$ is positive *superlinear* if for every $e \in E$ there exists $L_e \in \mathcal{L}_+(E, F)$ (i. e. $L_e p \ge 0$ whenever $p \in E_+$ (i. e. $p \ge 0$ in E) such that $L_e e = Te$ and $|L_e| \le |T|$.

THEOREM 1. *Let E, F be vector lattices, $T: E \to F$ a superlinear operator. Assume that the ordering of F is complete. Then equivalent are*

(a) *T is positive superlinear.*
(b) *$e_1 \leq e_2$ implies $Te_1 \leq |Te_2|$ and $-Te_2 \leq |Te_1|$ for all $e_1, e_2 \in E$.*
(c) *By setting $P \equiv |T|$ and $Qe \equiv \inf_{p \in E_+} P(e+p)$, $(e \in E)$, we have*

$$Qe \geq (Te) \vee ((-T-e)) \text{ for all } e \in E.$$

PROOF. (a)⇒(b): Suppose T is superlinear and $e_1 \leq e_2$ in E. Then choosing L_{e_1}, L_{e_2} in accordance with Definition 2, we obtain

$$Te_1 = L_{e_1} e_1 \leq L_{e_1} e_1 \leq |L_{21} e_2| \leq |Te_2|$$

nd

$$-Te_2 = -L_{e_2} e_2 = L_{e_2}(-e_2) \leq L_{e_2} e_1 \leq |L_{e_2} e_1| \leq |Te_1|.$$

(b)⇒(c): Let p be any element of E_+ and $e \in E$. An application of (b)) $e_1 \equiv e$ and $e_2 \equiv e+p$ yields $P(e+p) = |T(+e+p)| \geq Te$. Similarly, if $e_1 \equiv -e-p$, $e_1 \equiv -e$ we have $P(e+p) = |T(e+p)| = |T(-e-p)| \geq -T(-e)$.

(c)⇒(a): By assumption, P is a vector norm on E. Hence for any α_1, $\alpha_2 \in \mathbf{R}_+$, $e_1, e_2 \in E$ and $p_1, p_2 \in E_+$,

$$\sum_{1,2} \alpha_j P(e_j + p_j) = \sum_{1,2} P(\alpha_j e_j + \alpha_j p_j) \geq P\Big(\sum_{1,2} \alpha_j e_j + \sum_{1,2} \alpha_j p_j\Big).$$

Since $\sum_{1,2} \alpha_j p_j \in E_+$, too, it follows that Q is also a vector norm on E. Let now $e \in E \setminus \{0\}$ be arbitrarily given and define $L_e^0: \mathbf{R}e \to F$ by $L_e^0(\lambda e) \equiv \lambda\, Te$ $(\lambda \in \mathbf{R})$. Observe that $Q(\lambda e) = \lambda Qe \geq$ by (c) $\geq \lambda((Te) \vee (-T(-e))) \geq \lambda Te = L_e^0(\lambda e)$ if $\lambda \geq 0$ and $Q(\lambda e) = |\lambda| Q(-e) \geq$ by(c) $\geq |\lambda|\ (T(-e) \vee (-Te)) = (|\lambda| T(-e)) \vee (\lambda Te) \geq Te = L_e^0(\lambda e)$ if $\lambda \leq 0$. Thus $L_e^0 \leq Q$ on $\mathbf{R}e$. By the generalized Hahn–Banach theorem [2], L_e^0 admits a linear extension L_e to such that $L_e \leq Q$.

Clearly $L_e \leq P$ since $Q \leq P$. On the other hand, $L_e \in \mathcal{L}_+(E, F)$ since $L_e(-p) \leq Q(-p) \leq P(p-p) = 0$ for all $p \in E_+$.

Next we turn our attention to the continuity properties of positive superlinear operators. It seems that those ranging in $S(0, 1)$ (as in Nikisin's original definition) are of particular importance among them because, as we see, positive superlinear operators between locally convex topological vector lattices are very rarely continuous unless being linear.

LEMMA 1. *If E, F are vector lattices and $T: E \to F$ is positive superlinear then T is convex, positive homogeneous and positive valued when restricted to E_+, furthermore $T(-p) = -T_{(p)}$ for all $p \in E_+$.*

PROOF. If $p \in E_+$ then $L_p\, p$, $L_{-p}\, p \geq 0$. But $L_p\, p = T_p$ and $L_{-p}(-p) = T_{(-p)}$ whence $T_p = |T_p|$ and $T(-p) = -|T(-p)| = -|T_p| = -T_p$. Thus T coincides with $|T|$ on E_+. This implies its convexity and homogeneity on E_+ since T is a vector norm.

LEMMA 2. Suppose $T: \mathbf{R}^2 \to \mathbf{R}$ is a continuous positive superlinear mapping. Then T is necessarily linear.

PROOF. We may assume also $T \neq 0$. Then we have $T(\lambda_1, \lambda_2) \neq 0$ for all $\lambda_1, \lambda_2 < 0$. Indeed, $T(\lambda_1, \lambda_2) = 0 < \lambda_1, \lambda_2$ would imply $|L_e(\lambda_1, \lambda_2)| \leq |T(\lambda_2, \lambda_2)| = 0$ i.e. $0 = L_e(\lambda_1, \lambda_2) = \lambda_1 L_e(1, 0) + \lambda_2 L_e(0, 1)$ and hence $L_e = 0$ (for $L_e \in \mathcal{L}_+(\mathbf{R}^2, \mathbf{R})$ by Definition 2) for all $e \in \mathbf{R}$. Thus, by Lemma 1, range T contains both positive and negative numbers. Since T is a vector norm, this means that the set $\mathcal{N} \equiv \{e : Te = 0\}$ is a 1 dimensional subspace of $\mathbf{R}^2$, disjoint from $(0, \infty) \times (0, \infty)$. Therefore we can find a linear functional $\varphi \in \mathcal{L}_+(\mathbf{R}^2, \mathbf{R})$ such that $\varphi(1, 1) = T(1, 1) > 0$ and $\mathcal{N} = \{e : \varphi e = 0\}$. Let $f \in \mathbf{R}^2$ be arbitrarily fixed and consider the linear functional L_f. Since $|L_f| \leq |T|$, we have $\{e : L_f e = 0\} \supset \mathcal{N}$. Hence for some $\lambda_f \in \mathbf{R}_+$, $L_f = \lambda_f \varphi$. To conclude, we prove $\lambda_f = \lambda_g$ for all $f\ g \notin \mathcal{N}$. We may assume $0 < \lambda_f \leq \lambda_g$. Then $\lambda_g |\varphi f| = |L_g f| \leq |Tf| = |L_f f| = \lambda_f |\varphi f|$ whence $\lambda_f = \lambda_g$ completing the proof.

THEOREM 2. *Let E, F be topological vector lattices and let $F_0^* \equiv \{\varphi \in F_+^* : \varphi|f| = |\varphi f|\ \ \forall f \in F\}$. If F_0^* sparates the points of F then each continuous positive superlinear map $T: E \to F$ is linear.*

PROOF. Let us fix any $\varphi \in F_0^*$. Observe that the functional $\varphi \circ T$ is also positive superlinear $(E \to \mathbf{R})$. In fact, given $e_1 \leq e_2$, from Theorem 1. (b) we obtain $Te_1 \leq |Te_2|$, $-Te_1 \leq |Te_2|$ whence $\varphi Te_1 \leq \varphi |Te_2| = |\varphi Te_2|$ and $-\varphi Te_2 \leq \varphi |Te_1| = |\varphi Te_1|$. Now from Lemma 2. we see that for any p_1, $p_2 \in E_+$, the functional $\mathbf{R}^2 \ni (\varrho_1, \varrho_2) \to \varphi T(\varrho_1 p_1 + \varrho_2 p_2)$ is linear. Thus for all $e_1, e_2 \in \mathbf{R}p_1 + \mathbf{R}p_2$ and $\lambda \in [0, 1]$,

$$\Phi\left(T\left(\frac{1}{2}e_1 + \frac{1}{2}e_2\right) - \frac{1}{2}Te_1 - \frac{1}{2}Te_2\right) = 0.$$

Since F_0^* separates F, it follows

$$T\left(\frac{1}{2}e_1 + \frac{1}{2}e_2\right) = \frac{1}{2}Te_1 + \frac{1}{2}Te_2$$

i.e. the mapping T is linear when restricted to any 2 dimensional subspace of E spanned by positive elements. Thus if $e, f\ E$ then

$$T\left(\frac{1}{2}e + \frac{1}{2}f\right) = T\left(\frac{1}{2}(e_+ + f_+) + \frac{1}{2}(-e_- - f_-)\right) = \frac{1}{2}T(e_+ + f_+) +$$

$$+\frac{1}{2}T(-e_- - f_-) = \frac{1}{2}T(e_+ + f_+) - \frac{1}{2}T(e_- + f_-) = \left(\frac{1}{2}Te_+ + \frac{1}{2}Tf_+\right) -$$

$$-\left(\frac{1}{2}Te_- + \frac{1}{2}Tf_-\right) = \frac{1}{2}(Te_+ - Te_-) + \frac{1}{2}(Tf_+ - Tf_-) = \frac{1}{2}Te + \frac{1}{2}Tf$$

establishing the linearity of T.

COROLLARY 1. If E is a topological vector lattice and Ω is a compact to-ological space then each continuous positive superlinear map $E \to C(\Omega)$ is inear.

PROOF. The functionals $\delta_x \equiv [C(\Omega) \ni f \to f(x)]$ $(x \in \Omega)$ form a separating amily in $C(\Omega)$ and satisfy $\delta_x |f| = |f(x)| = |\delta_x f|$.

COROLLARY 2. If E is a topological vector lattice and μ is an arbitrary neasure then each continuous positive superlinear map $E \to L^\infty(\mu)$ is linear.

PROOF. By Kakutani's representation theorem on M-lattices [3], each $^\infty$ – space is isometrically order isomorphic to some $C(\Omega)$ space for suitable ompact topological space.

COROLLARY 3. If E is a topological vector lattice and $1 \leq p \leq \infty$ then very continuous positive superlinear map $E \to l^p$ is linear.

PROOF. Every continuous superlinear operator $T: E \to l^p$ can be viewed s a continuous positive superlinear $E \to l^\infty$. mapping.

The following question arises from the above corollaries: Is there any on-linear continuous positive superlinear operator $L^p(0, 1) \to L^q(0. 1)$ if $< \infty$? The answer is always affirmative in this case.

EXAMPLE. Let $1 \leq p \leq \infty$ and $1 \leq q < \infty$. The mapping $T: L^p(0, 1) \to$ $\to L^q(0, 1)$ defined by

$$Tf \equiv [(0,1) \ni t \to \begin{cases} t\int_0^{1/2} f \quad if \quad \left| t\int_0^{1/2} f \right| \geq \left| (1-t)\int_{1/2}^1 f \right| \\ (1-t)\int_{1/2}^1 f \quad \text{else} \end{cases}$$

s positive superlinear and continuous but non-linear.

PROOF. The non-linear character of T is obvious.

Continuity: Suppose $f_n \to f$ in $L^p(0, 1)$ $(n \to \infty)$.

Now

$$\int_0^{1/2} f_n \to \int_0^{1/2} f \quad \text{and} \quad \int_{1/2}^1 f_n \to \int_{1/2}^1 f, \ (n \to \infty).$$

Hence $Tf_n(t) \to Tf(t)$ whenever

$$\left| t\int_0^{1/2} f \right| \neq \left| (1-t)\int_{1/2}^1 f \right| \quad \text{or} \quad \left| t\int_0^{1/2} f \right| = \left| (1-t)\int_{1/2}^1 f \right| = 0.$$

i. e. almost everywhere. Since the sequence $\{|Tf_n|\}_1^\infty$ consists of functions majorized by the constant $\sup_n \int_0^1 |f_n|$, it follows

$$\|Tf_n - Tf\|_{L^q} = \left(\int_0^1 |Tf_n(t) - Tf(t)|^q \, dt \right)^{1/p} \to 0, \ (n \to \infty).$$

Positive superlinearity: Given $e \in L^P(0, 1)$, it is immediate that the linear mapping $L_e: L^P(0, 1) \to L^q(0, 1)$ defined by

$$L_e f \equiv [(0,1) \ni t \to \begin{cases} t\int_0^{1/2} f \quad \text{if} \quad \left| t\int_0^{1/2} e \right| \geq \left| (1-t)\int_{1/2}^1 e \right| \\ (1-t)\int_{1/2}^1 f \quad \text{else} \end{cases}$$

is positive and fulfills the requirements of Definition 2.

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