REMARKS ON SUPERLINEAR OPERATORS

By

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In [1] E. M. NIKIDSIN introduced the notions of superlinear and positive superlinear operators concerning his investigation on Fourier series with respect to general orthonormal systems. According to [1], a mapping $T : E \rightarrow S(0, 1)$ where $E$ is any Banach space is by definition superlinear if for every $e \in E$ there exists a linear mapping $L_e : E \rightarrow S(0, 1)$ such that $L_e e = Te$ and $|L_e f| \leq |Tf|$ for each $f \in E$. Furthermore, if $E = \mathcal{L}^p(X, \mu)$ for some $p \geq 1$ and for every $e \in E$, $L_e$ can be chosen to be a positive linear mapping then $T$ is called a positive superlinear operator ($\mathcal{L}^p - S$).

The aim of this paper is to examine these concepts in a vector lattice theoretical setting.

DEFINITION 1. Let $E, F$ be a vector space and a vector lattice, respectively. A mapping $T : E \rightarrow F$ is superlinear if for every $e \in E$ there exists $L_e \in \mathcal{L}(E, F)$ such that $L_e e = Te$ and $|L_e| = |T|$. (Throughout this work, we deal with real vector spaces. The symbol $|T|$ means the operator $f \mapsto |Tf|$.)

PROPOSITION 1. Suppose the space $F$ is order complete (for def. see [3]). Then $T : E \rightarrow F$ is superlinear if and only if $|T|$ is a vector norm on $E$ i.e. if, $|T(e_1 + e_2)| = |Te_1| + |Te_2|$ and $|T\lambda e_1| = |\lambda||Te_1|$ $\forall e_1, e_2 \in E, \lambda \in \mathbb{R}$.

PROOF. Let $T : E \rightarrow F$ be superlinear. Then we can write $|T| = \sup_{e \in E} |L_e|$. But $L$ is clearly a vector norm whenever $L : E \rightarrow F$ is linear.

Conversely, assume $T$ is a vector norm on $E$ and $F$ is order complete. Given $e \in E$, define $L_e^0$ on the subspace $\mathbb{R}_e$ by $L_e^0 e = \lambda Te$ ($\lambda \in \mathbb{R}$). We have $L_e^0 = |T|$ on $\mathbb{R}_e$. Thus by the generalized Hahn-Banach theorem [2], $L_e^0$ admits a linear extension $L_e$ such that $L_e \leq |T|$. To complete the proof, we show $-L_e \leq |T|$. Indeed, $-L_e f = L_e (-f) = |T (-f)| = |Tf|$ $\forall f \in E$.

DEFINITION 2. Let $E, F$ be vector lattices. A mapping $T : E \rightarrow F$ is positive superlinear if for every $e \in E$ there exists $L_e \in \mathcal{L}_+(E, F)$ (i.e. $L_e p \geq 0$ whenever $p \in E_+$ (i.e. $p \geq 0$ in $E$) such that $L_e e = Te$ and $|L_e| \leq |T|$.
Theorem 1. Let $E, F$ be vector lattices, $T:E\to F$ a superlinear operator. Assume that the ordering of $F$ is complete. Then equivalent are

(a) $T$ is positive superlinear.
(b) $e_\varepsilon\leq e_\delta$ implies $Te_\varepsilon\leq|Te_\delta|$ and $-Te_\delta\leq|Te_\varepsilon|$ for all $e_\varepsilon, e_\delta\in E$.
(c) By setting $P\equiv|T|$ and $Q\equiv\inf\{P(e+p), e\in E\}$, we have

$$Qe=\sqrt{(T(e)/|e|)}(T-e)$$

for all $e\in E$.

Proof. (a)$\Rightarrow$(b): Suppose $T$ is superlinear and $e_\varepsilon\leq e_\delta$ in $E$. Then choosing $L_{e_\delta}$, $L_{e_\varepsilon}$ in accordance with Definition 2, we obtain

$$Te_\varepsilon=L_{e_\delta}e_\varepsilon\leq L_{e_\varepsilon}e_\delta=|L_{e_\delta}e_\varepsilon|=|Te_\delta|$$

and

$$-Te_\delta=-L_{e_\delta}e_\delta=L_{e_\delta}(-e_\delta)\leq L_{e_\varepsilon}e_\delta=|L_{e_\delta}e_\varepsilon|=|Te_\varepsilon|.$$
Corollary 1. If $E$ is a topological vector lattice and $\mathcal{Q}$ is a compact topological space then each continuous positive superlinear map $E \to C(\mathcal{Q})$ is linear.

Proof. The functionals $\delta_x \equiv \{ \mathcal{Q}(x) \in f(x) \} \ (x \in \mathcal{Q})$ form a separating family in $C(\mathcal{Q})$ and satisfy $\delta_x[f] = \| f(x) \| = \| \delta_x \|_\mathcal{Q}$.

Corollary 2. If $E$ is a topological vector lattice and $\mu$ is an arbitrary measure then each continuous positive superlinear map $E \to L^\infty(\mu)$ is linear.

Proof. By Kakutani’s representation theorem on $M$-lattices [3], each $L^\infty$-space is isometrically order isomorphic to some $C(\mathcal{Q})$ space for suitable compact topological space.

Corollary 3. If $E$ is a topological vector lattice and $1 \leq p \leq \infty$ then every continuous positive superlinear map $E \to L^p$ is linear.

Proof. Every continuous superlinear operator $T : E \to L^p$ can be viewed as a continuous positive superlinear $E \to L^\ast$ mapping.

The following question arises from the above corollaries: Is there any non-linear continuous positive superlinear operator $L^\ast(0, 1) \to L^\ast(0, 1)$ if $1 < \infty$? The answer is always affirmative in this case.

Example. Let $1 \leq p \leq \infty$ and $1 < q < \infty$. The mapping $T : L^p(0, 1) \to L^q(0, 1)$ defined by

$$
Tf \equiv \begin{cases} 
\int_0^{1/2} f^2(t) \, dt & \text{if } t \int_0^{1/2} f \, dt \\
(1 - t) \int_{1/2}^1 f \, dt & \text{else}
\end{cases}
$$

is positive and continuous but non-linear.

Proof. The non-linear character of $T$ is obvious.

Continuity: Suppose $f_n \to f$ in $L^p(0, 1) (n \to \infty)$.

Now

$$
\int_0^{1/2} f_n \to \int_0^{1/2} f \quad \text{and} \quad \int_{1/2}^1 f_n \to \int_{1/2}^1 f, \ (n \to \infty).
$$

Hence $Tf_n(t) \to Tf(t)$ whenever

$$
\left| t \int_0^{1/2} f \right| \neq \left| (1 - t) \int_{1/2}^1 f \right| \quad \text{or} \quad \left| t \int_0^{1/2} f \right| = \left| (1 - t) \int_{1/2}^1 f \right| = 0.
$$

i. e. almost everywhere. Since the sequence $\{Tf_n\}_{n=1}^\infty$ consists of functions majorized by the constant $\sup_n \int_0^1 |f_n|$, it follows

$$
\| Tf_n - Tf \|_{L^q} \to 0, \ (n \to \infty).
$$

Positive superlinearity: Given $\epsilon \in L^\infty(0, 1)$, it is immediate that the linear mapping $L_\epsilon : L^\infty(0, 1) \to L^\infty(0, 1)$ defined by

$$
L_\epsilon f \equiv \begin{cases} 
\left. \int_0^{1/2} f \, dt \right| & \text{if } t \int_0^{1/2} e \, dt \\
(1 - t) \int_{1/2}^1 f \, dt & \text{else}
\end{cases}
$$

is positive and fulfills the requirements of Definition 2.

References


