A counterexample concerning non-linear C0-semigroups of holomorphic Carathéodory isometries

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Abstract. We give an example for a C0-semigroup of non-linear 0-preserving holomorphic Carathéodory isometries of the unit ball.

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Introduction

According to Cartan's classical Linearization Theorem, given a bounded circular domain D in the N-dimensional complex space $\mathbb{C}^N$ any holomorphic mapping $F : D \to D$ with $F(0) = 0$ and preserving the Carathéodory (or Kobayashi) distance associated with D is necessarily linear and surjective [2]. In 1994, E. Vesentini [3, p. 508],[4, Section 3] found various examples showing that the infinite dimensional version of this fact is not valid in general Banach space setting. However, despite the main topics of his several works including [3,4], the provided examples (even with holomorphic families) seem not being suitable in constructing a C0-semigroup $[F^t : t \geq 0]$ of holomorphic 0-fixing Carathéodory isometries of a bounded circular domain in some Banach space. Our aim in this short note is a suitable construction with C0-semigroups.

Throughout this work let $\mathbf{E}$ denote a complex Banach space and define

$$\mathbf{X} := C_0(\mathbb{R}_+, \mathbf{E}) = \Big\{ x : \mathbb{R}_+ \to \mathbf{E} \Big| t \mapsto x(t) \text{ continuous, } \lim_{t\to\infty} x(t) = 0 \Big\}, \qquad \|x\| = \max_{t\geq 0} \|x(t)\|.$$

For the concepts of holomorphic maps between domains (open connected sets) contained in Banach spaces respectively the Carathéodory distance $d_{\mathbf{D}}$ on a bounded domain we refer to [2]. A domain $\mathbf{D} \subset \mathbf{E}$ is said to be circular if $\mathbf{D} = \zeta\mathbf{D}\big(= \{\zeta\mathbf{x} : \mathbf{x} \in \mathbf{D}\}\big)$ for any $\zeta \in \mathbb{C}$ with $|\zeta| = 1$. As usually, an indexed family $[F^t : t \geq 0]$ of maps $\mathbf{D} \to \mathbf{D}$ is a C0-semigroup if $F^0 = \mathrm{Id}_{\mathbf{D}} = [\mathbf{x} \mapsto \mathbf{x} : \mathbf{x} \in \mathbf{D}]$, $F^{s+t}(\mathbf{x}) = F^s\big(F^t(\mathbf{x})\big)$ for all $s, t \geq 0$ and $\mathbf{x} \in \mathbf{D}$ furthermore if the functions $[t \mapsto F^t(\mathbf{x}) : t \geq 0]$ are continuous for all fixed $\mathbf{x} \in \mathbf{D}$.

Lemma. *Let $\big[\varphi^t : t \in \mathbb{R}_+\big]$ be a C0-semigroup of $B(\mathbf{E})$-contractions. Then the maps $\Phi^t : B(\mathbf{X}) \to \mathbf{X}$ $(t \in \mathbb{R}_+)$ defined by*

$$\Phi^t(x) : \mathbb{R}_+ \ni \tau \mapsto \Big[\varphi^{t-\tau}\big(x(0)\big) \ \text{if } 0 \leq \tau \leq t, \quad x(\tau - t) \ \text{if } \tau \geq t\Big]$$

form a C0-semigroup of $B(\mathbf{X})$-isometries.

Proof. Consider any function $x \in B(\mathbf{X})$ and any parameter $t \in \mathbb{R}_+$. The function $\Phi^t(x)$ ranges in $B(\mathbf{X})$ with $\lim_{\tau\to\infty} \Phi^t(x)(\tau) = \lim_{\tau\to\infty} x(\tau - t) = 0$. The continuity of $\Phi^t(x)$ on the intervals $[0,t]$ resp. $[t,\infty]$ is immediate by its definition. Hence $\Phi^t(x) \in \mathbf{X}$ with well-defined $\max_{\tau\geq 0} \|x(\tau)\| < 1$. Given another function $y \in B(\mathbf{X})$, we have

$$\begin{aligned}
&\big\|\Phi^t(x) - \Phi^t(y)\big\| = \max\Big\{ \max_{0\leq\tau\leq t} \big\|\varphi^{t-\tau}\big(x(\tau)\big) - \varphi^{t-\tau}\big(y(\tau)\big)\big\|, \max_{\sigma\geq t} \big\|x(\sigma - t) - y(\sigma - t)\big\|\Big\} \leq \\
&\leq \max\Big\{ \max_{0\leq\tau\leq t} \big\|x(\tau) - y(\tau)\big)\big\|, \max_{\sigma\geq t} \big\|x(\sigma - t) - y(\sigma - t)\big\|\Big\} \leq \\
&= \max_{\tau\geq 0} \big\|x(\tau) - y(\tau)\big)\big\| = \|x - y\|.
\end{aligned}$$

Since trivially

$$\big\|\Phi^t(x) - \Phi^t(y)\big\| \geq \max_{\sigma\geq t} \big\|x(\sigma - t) - y(\sigma - t)\big\|\Big\} = \max_{\tau\geq 0} \big\|x(\tau) - y(\tau)\big\|\Big\} = \|x - y\|,$$

we conclude that each map Φ^t is a $B(\mathbf{X})$-isometry.
Next we check the semigroup property of $[\Phi^t : t \in \mathbb{R}_+]$. Let $s, t \geq$. Then we have

$$\begin{aligned}
\Phi^s \circ \Phi^t(x) : \tau \mapsto \Big[\varphi^{s-\tau}\big(\Phi^t(x)(0)\big) \text{ if } \tau \leq s, \quad \varphi^t(x)(\tau - s) \text{ if } \tau \geq s\Big], \\
\Phi^{s+t}(x) : \tau \mapsto \Big[\varphi^{(s+t)-\tau}\big(x(0)\big) \text{ if } \tau \leq s + t, \quad x\big(\tau - (s + t)\big) \text{ if } \tau \geq s + t\Big].
\end{aligned}$$

Thus if $0 \leq \tau \leq s$ then

$$\begin{aligned}
\Phi^s \circ \Phi^t(x)(\tau) = \varphi^{s-\tau}\Big(\Phi^t\big(x(0)\big)\Big) = \varphi^{s-\tau}\Big(\varphi^t\big(x(0)\big)\Big) = \\
= \varphi^{s-\tau} \circ \varphi^t\big(x(0)\big) = \varphi^{(s+t)-\tau}\big(x(0)\big) = \Phi^{s+t}(x)(\tau).
\end{aligned}$$

If $s \leq \tau \leq s + t$ then

$$\begin{aligned}
\Phi^s \circ \Phi^t(x)(\tau) = \Phi^t(x)(\tau - s) =^{\tau-s\leq t}= \varphi^{t-(\tau-s)}\big(x(0)\big) = \\
= \varphi^{(s+t)-\tau}\big(x(0)\big) = \Phi^{s+t}(x)(\tau).
\end{aligned}$$

If $s + t \leq \tau$ then

$$\Phi^s \circ \Phi^t(x)(\tau) = \Phi^t(x)(\tau - s) =^{\tau-s\geq t}= x\big((\tau - s) - t\big) = \Phi^{s+t}(x)(\tau).$$

We complete the proof by checking strong continuity, that is that $\|\Phi^t(x) - \Phi^s(x)\| \to 0$ whenever $s \to t$ in $\mathbb{R}_+$. Recall that the moduli of continuity

$$\Omega(z, \delta) := \max_{|t_1-t_2|\leq\delta} \|z(t_1) - z(t_2)\|, \qquad \omega(e, \delta) := \max_{|t_1-t_2|\leq\delta} \|\varphi^{t_1}(e) - \varphi^{t_2}(e)\|$$

of any function $z \in \mathbf{X}$ resp. any vector $e \in \mathbf{E}$ are well-defined and converge to 0 as $\delta \searrow 0$. Let $0 \le t_1 \le t_2$. Then we have

$$\Phi^{t_1}(x) - \Phi^{t_2}(x) = \begin{cases} \varphi^{t_2-\tau}(x(0)) - \varphi^{t_1-\tau}(x(0)) & \text{if } \tau \le t_1, \\ \varphi^{t_2-\tau}(x(0)) - x(\tau - t_1) & \text{if } t_1 \le \tau \le t_2, \\ x(\tau - t_2) - x(\tau - t_1) & \text{if } t_2 \le \tau. \end{cases}$$

Therefore

$$\|\Phi^{t_1}(x) - \Phi^{t_2}(x)\| \le \begin{cases} \omega\big(x(0), t_2 - t_1\big) & \text{if } \tau \le t_1, \\ \big\|\varphi^{t_2-\tau}\big(x(0)\big) - x(0)\big\| + \big\|x(\tau - t_1) - x(0)\big\| \le & \\ \qquad \le \omega\big(x(0), t_2 - t_1\big) + \Omega(x, t_2 - t_1) & \text{if } t_1 \le \tau \le t_2, \\ \Omega(x, t_2 - t_1) & \text{if } t_2 \le \tau. \end{cases}$$

Hence we see the uniform continuity of the function $t \mapsto \Phi^t(x)$ with modulus of continuity $\delta \mapsto \omega\big(x(0), \delta\big) + \Omega(x, \delta)$.

Remark. The conclusion of the above lemma holds even if $\mathbf{E}$ is assumed to be a *normed* space and not necessarily a Banach space.

Corollary. *If the maps φ^t are holomorphic then each Φ^t is a holomorphic $d_{B(\mathbf{X})}$-isometry because $d_{B(\mathbf{X})}\big(x, y\big) = \max_{\tau \ge 0} d_\Delta\big(x(\tau), y(\tau)\big)$ and the maps φ^t are $d_{B(\mathbf{E})}$-contractions.*

Remark. It is well-known [1] that, given a continuously differentiable function $f : \mathbb{R}_+ \to \mathbf{E}$ where $\mathbf{E}$ is a Banach space, we have

$$\frac{d^+}{dt}\big\|f(t)\big\| := \limsup_{h \searrow 0} \big[\|f(t+h)\| - \|f(t)\|\big]/h = \sup_{L \in \mathcal{S}(f(t))} \mathrm{Re}\big\langle L, f'(t)\big\rangle$$

in terms of the family of supporting bounded linear functionals

$$\mathcal{S}(y) := \big\{L \in \mathbf{E}^* : \|L\| = 1,\ \langle L, y\rangle = \|y\|\big\} \qquad (y \in \mathbf{E}).$$

In particular f is non-increasing whenever $\mathrm{Re}\big\langle L, f'(t)\big\rangle \le 0$ for any $t \in \mathbb{R}_+$ and for any functional $L \in \mathcal{S}(f(t))$.

Lemma. *Let $V : U \to \mathbf{E}$ be a bounded continuously differentiable map (regarded as a vector field) on some open neighborhood U of the closed unit ball $\overline{B(\mathbf{E})}$ with $V(0) = 0$ and let $\mu \ge \sup_{e_1, e_2 \in B(\mathbf{E})} \big\|V(e_1) - V(e_2)\big\|$. Then the maximal flow of the vector field $W : B(\mathbf{E}) \ni e \mapsto V(e) - \mu e$ is a well-defined uniformly continuous one-parameter semigroup $[\varphi^t : t \in \mathbb{R}_+]$ consisting of contractive (non-expansive) self maps of $B(\mathbf{E})$.*

Proof. By definition, any flow of W is a family $[\varphi^t : t \in I]$ of self maps $\varphi^t : B(\mathbf{E}) \to B(\mathbf{E})$ where I is some (relatively) open subinterval of $\mathbb{R}_+$ and, for any point $e \in B(\mathbf{E})$, the function $I \ni t \mapsto \varphi^t(e)$ is the solution of the initial value problem

$$(*) \qquad \frac{d}{dt} z(t) = W\big(z(t)\big), \quad z(0) = e.$$

By writing I_e for the maximal solution of $(*)$, it is well-known that $\sup I_e > 0$ in any case, furthermore we have $\lim_{t\to\sup I_e} \|z(t)\| = 1$ whenever $\sup I_e < \infty$.
Let $e_1, e_2 \in B(\mathbf{E})$ and consider the function $f(t) := \varphi^t(e_1) - \varphi^t(e_2)$ defined on the interval $I_{e_1} \cap I_{e_2}$. Observe that, given any functional $L \in \mathcal{S}\big(\varphi^t(e_1) - \varphi^t(e_2)\big)$, we have

$$\begin{aligned}
&\mathrm{Re}\big\langle L, f'(t)\big\rangle = \mathrm{Re}\big\langle L, W\big(\varphi^t(e_1)\big) - W\big(\varphi^t(e_2)\big)\big\rangle = \\
&= \mathrm{Re}\big\langle L, V\big(\varphi^t(e_1)\big) - V\big(\varphi^t(e_2)\big)\big\rangle - \mu \mathrm{Re}\big\langle L, \varphi^t(e_1) - \varphi^t(e_2)\big\rangle = \\
&= \mathrm{Re}\big\langle L, V\big(\varphi^t(e_1)\big) - V\big(\varphi^t(e_2)\big)\big\rangle - \mu\big\|\varphi^t(e_1) - \varphi^t(e_2)\big\| \le \\
&\le \mu\big\|\varphi^t(e_1) - \varphi^t(e_2)\big\| - \mu\big\|\varphi^t(e_1) - \varphi^t(e_2)\big\| = 0.
\end{aligned}$$

Hence we conclude that the function $t \mapsto f(t)$ is decreasing, in particular we have the contraction property $\big\|\varphi^t(e_1) - \varphi^t(e_2)\big\| \le \big\|\varphi^0(e_1) - \varphi^0(e_2)\big\| = \big\|e_1 - e_2\big\|$ for $t \in I_{e_1} \cap I_{e_2}$. By assumption $W(0) = V(0) = 0$ implying $\varphi^t(0) \equiv 0$ with $I_0 = [0, \infty) = \mathbb{R}_+$. Hence we see also that $\big\|\varphi^t(e)\big\| = \big\|\varphi^t(e) - \varphi^t(0)\big\| \le \big\|e - 0\big\| = \big\|e\big\| < 1$ for all $e \in B(\mathbf{E})$ and $t \in I_e$. This is possible only if $\sup I_e = \infty$. Therefore the maximal flow of W is defined for all (time) parameters $t \in \mathbb{R}_+$ and consists of $B(\mathbf{E})$-contractions φ^t.
It is well-known that flows parametrized on $\mathbb{R}_+$ are strongly continuous semigroups automatically. The uniform continuity of in our case is a consequence of the fact that $\big\|\varphi^{t_2}(e) - \varphi^{t_1}(e)\big\| \le \int_{t_1}^{t_2} \big\|\frac{d}{dt}\varphi^t(e)\big\|dt = \int_{t_1}^{t_2} \big\|W\big(\varphi^t(e)\big)\big\|dt \le \int_{t_1}^{t_2} 4\mu\ dt \quad (0 \le t_1 \le t_2)$, which shows that $\omega(e,\delta) \le 4\mu\delta \quad (e \in B(\mathbf{E}),\ \delta \in \mathbb{R}_+)$.

Example. Let $\mathbf{E} := \mathbb{C}$ with $B(\mathbf{E}) = \Delta = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and let $V(z) \equiv z^2$. Since $|z_1^2 - z_2^2| = |z_1 - z_2| \cdot |z_1 + z_2| \le 2|z_1 - z_2|$, we can apply the above Lemma with $W(z) := z^2 - 2z$. For the flow $[\varphi^t : t \in \mathbb{R}_+]$ of W we obtain the holomorphic maps

$$\varphi^t(z) = \frac{2z}{\big(1 - e^{2t}\big)z + 2e^{2t}} \qquad (z \in \Delta,\ t \ge 0).$$

Indeed, the solution of the initial value problem

$$(**) \qquad \frac{d}{dt}x(t) = x(t)^2 - 2x(t), \quad x(0) = z$$

is $x(t) = 2z/\big[\big(1 - e^{2t}\big)z + 2e^{2t}\big]$ as one can check by direct computation. As for heuristics, we get a real valued solution with real calculus for $(**)$ with initial values $-1 < z < 1$, and the obtained formula extends holomorphically to Δ.

Theorem. *Given a complex Banach space* $\mathbf{E}$*, there is a C0-semigroup of non-linear holomorphic* 0*-preserving norm and Carathéodory isometries of the open unit ball of the function space* $\mathbf{X} := C_0(\mathbb{R}_+, \mathbf{E})$.

Proof. We can apply the construction of the first Lemma with a semigroup $[\varphi^t : t \in \mathbb{R}_+]$ obtained with the construction of the 2nd Lemma with any $\mathbf{E}$-polynomial vector field V.

Example. Let $\mathbf{E} := \mathbb{C}$ and $\mathbf{X} := C_0(\mathbb{R}_+, \mathbb{C})$. Then the maps

$$\Phi^t(x) : \mathbb{R}_+ \ni \tau \mapsto \left[\frac{2x(0)}{\big(1 - e^{2(t-\tau)}\big)x(0) + 2e^{2(t-\tau)}} \text{ if } \tau \le t, \quad x(\tau - t) \text{ if } \tau \ge t\right]$$

form a C0-semigroup of non-linear holomorphic 0-preserving norm and Carathéodory isometries of the unit ball $B(\mathbf{X})$.

Question. Is any holomorphic norm-isometry of the unit ball of a complex Banach space automatically a Carathéodory isometry as well?

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