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Symmetric continuous Reinhardt domains

By

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Abstract. We extend the concept of Reinhardt domains to complex function spaces and we give a complete parametric description of all bounded symmetric Reinhardt domains in a $\mathcal{C}_0$-space.

1. Introduction. A classical Reinhardt domain is an open connected subset D of $\mathbb{C}^n$ the space of all complex n-tuples being invariant under all coordinate multiplications $M_{\lambda_1,\dots,\lambda_n} : (z_1, \dots, z_n) \mapsto (\lambda_1 z_1, \dots, \lambda_n z_n)$ with $|\lambda_1|, \dots, |\lambda_n| = 1$. The Reinhardt domain $D \subset \mathbb{C}^n$ is called complete if $M_{\lambda_1,\dots,\lambda_n} D \subset D$ whenever $|\lambda_1|, \dots, |\lambda_n| \leqq 1$.

In 1974 Sunada [11] investigated the structure of bounded Reinhardt domains containing the origin from the viewpoint of biholomorphic equivalence. He was able to describe completely the symmetric Reihardt domains which, up to linear isomomorphism, turned to be direct products of Euclidean balls.

Our aim in this paper is to study infinite dimensional analogs of symmetric Reihardt domains. There are several ways of extending the definition of classical Reihardt domains. We are however motivated by a recent interesting work of Vigué [12] who considered continuous products of discs (of different radius). He obtained a rather surprising result that, despite the obvious symmetry of a disc, such a continuous product is not symmetric in general. We intend to give a definition of a continuous Reinhardt domain which includes the domains studied by Vigué in such a way that our results can also lead to some continuous mixing of Euclidean balls generalizing the work of Sunada.

In the classical definition of a Reinhardt domain n-tuples can be viewed as functions from the discrete space $\Omega := \{1, \dots, n\}$ to $\mathbb{C}$. In this terminology, complete Reinhardt domains are closely related with the ordering $f \geqq 0 \iff f(\omega) \geqq 0$ for all $\omega \in \Omega$. Namely $D \subset \mathcal{C}(\Omega)$ is a complete Reinhardt domain if $f \in D$ and $|g| \leqq |f|$ implies $g \in D$. This definition can simply be extended to any complex function lattice. In the setting of bounded continuous functions, the Gel'fand-Naimark theorem naturally leads to the lattice $\mathcal{C}_0(\Omega)$ of all continuous functions vanishing at infinity over a locally compact Hausdorff space. We

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note that in 1978 Braun-Kaup-Upmeier [1] introduced another interesting generalization of Reinhardt domains in infinite dimensional setting. This concept is different from ours and even in their later works their school changed the terminology so that today their concept is called bicircular domains.

In the present paper we concentrate on the symmetric case. We prove that continuous bounded symmetric Reinhardt domains are in certain sense made of Euclidean balls but in a more complicated manner than simple direct products. Moreover, for a given domain D, the dimensions of these balls are simultaneously bounded by a finite constant determined by some geometric parameters of D. It particular, in Theorem 1 we show that D can be explicitly described in the form $\{f : \sup_{j\in J} \sum_{\omega\in\Omega_j} m(\omega)|f(\omega)|^2 < 1\}$ for a suitable partition $\{\Omega_j : j \in J\}$ of the space Ω and a function $m : \Omega \to \mathbb{R}_+$.

The main tools we use in our arguments come from Jordan theory: by Kaup's celebrated Riemann Mapping Theorem [7], bounded symmetric circular domains can be regarded as unit balls of JB*-triples. The key point is our Corollary 4 which can be regarded as a generalization of the classical Banach-Stone Theorem (corresponding to the case $D = \{f : |f| < 1\}$). Here we identify the extreme points of the dual unit ball of the associated JB*-triple as certain finite linear combinations of point evaluations. Hence bidualization arguments lead to our conclusions. Concerning the JB*-triple theory applied, we refer to the survey [10].

2. Results.

Definition. Throughout this work let Ω be a locally compact Hausdorff topological space. By a *symmetric continuous Reinhardt domain* over Ω we mean a bounded symmetric domain $D \subset \mathcal{C}_0(\Omega)$ (the space of all continuous functions $\Omega \to \mathbb{C}$ with $\{\omega \in \Omega : |f(\omega)| \geqq \varepsilon\}$ compact $\subset \Omega$ for any $\varepsilon > 0$) equipped with the norm $\|f\|_\infty := \max|f|$) such that

$$f \in D,\ \ |g| \leqq |f| \Rightarrow g \in D \qquad (f, g \in \mathcal{C}_0(\Omega))\ .$$

Henceforth we fix a symmetric continuous Reinhardt domain D over Ω. It is well-known [7] that D is the unit ball for a so-called JB*-norm $\|.\|$ on $\mathcal{C}_0(\Omega)$ equivalent to $\|.\|_\infty$ and there exists (a unique) 3-variable operation $\{...\} : \mathcal{C}_0(\Omega)^3 \to \mathcal{C}_0(\Omega)$ such that

$(J1)$ $\{xyz\} = \{zyx\}$,

$(J2)$ $\{(\alpha_1 x_1 + \alpha_2 x_2)(\beta_1 y_1 + \beta_2 y_2)(\gamma_1 z_1 + \gamma_2 z_2)\} = \sum_{k,\ell,m=1}^{2} \alpha_k \overline{\beta_\ell} \gamma_m \{x_k y_\ell z_m\}$,

$(J3)$ $\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$,

$(J4)$ $\|\{xxx\}\| = \|x\|^3$,

$(J5)$ the spectrum of the operator $[x \mapsto \{aax\}]$is non-negative

for all $a, b, x, y, z \in \mathcal{C}(\Omega)$ and $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{C}$.

We shall write briefly E for the JB*-triple $(\mathcal{C}_0(\Omega), \|.\|, \{...\})$. Since the norms $\|.\|$, $\|.\|_\infty$ are equivalent on E, by the Riesz Representation Theorem [5], the dual space E' consists of all linear functionals of the form

$$f \mapsto \int f d\mu \qquad (\mu \in \mathcal{M}(\Omega))$$

where $\mathcal{M}(\Omega)$ denotes the family of all complex-valued regular measures with bounded variation on Ω. That is any element $\mu \in \mathcal{M}(\Omega)$ is a σ-additive function $\mathrm{Borel}(\Omega) \to \mathbb{C}$ such that $\mu(B) = \sup_{K \text{ compact} \subset B} \mu(K)$ for $B \in \mathrm{Borel}(\Omega) :=$ [the smallest σ-ring of subsets of Ω containing Ω and the compact subsets of Ω]. We denote the functional $[E \ni f \mapsto \int f\, d\mu]$ simply by $d\mu$, and conveniently we write $g d\mu := [f \mapsto \int fg\, d\mu]$ whenever $g \in \mathcal{B}(\Omega)$ where we write $\mathcal{B}(\Omega)$ for the family of all bounded complex-valued $Borel(\Omega)$-measurable functions. In terms of measures, the dual norm on E' is $\|d\mu\| := \sup_{f \in \mathrm{Ball}_1(E)} |\int f\, d\mu|$ where $\mathrm{Ball}_1(E) := \{x \in E : \|x\| \leqq 1\}$. It is well-known that the dual norm of $\|.\|_\infty$ is nothing else as the total variation (which is finite automatically due to the presence of the whole Ω in $\mathrm{Borel}(\Omega)$)

$$\|d\mu\|_1 = \sup \left\{ \sum_{K \in \mathcal{K}} |\mu(K)| : \mathcal{K} \text{ is a finite disjoint family of compact subsets of } \Omega \right\}.$$

In order to achieve a parametrized classification of all symmetric continuous Reinhardt domains over Ω, we shall describe the extreme points of $\mathrm{Ball}_1(E')$ in measure theoretical terms.

We can regard the space $\mathcal{B}(\Omega)$, which contains E, as a subspace of the bidual E'' by identifying the functional $[d\mu \mapsto \int f\, d\mu]$ with $f \in E$ as usual. In this sense the JB*-triple product $\{...\}$ extends from E to E'' in a separately weak*-continuous manner and E'' with the bidual norm and this extended triple product (denoted simply by $\|.\|$ and $\{...\}$, respectively) is a JB*-triple again [3, 2]. According to a celebrated lemma of Friedman-Russo [4, page 79], the atoms of the JB*-triple E'' and the extreme points of the unit ball of E' are in a remarkable one-to-one correspondence which can be stated as follows. Recall an atom of E'' is an element $a \in E''$ such that

$$\{aaa\} = a \neq 0 \quad \text{and} \quad \{axa\} = \overline{\phi_a(x)}a \qquad (x \in E'')$$

for some (necessarily unique) linear functional $\phi : E'' \to \mathbb{C}$. Then the extremal points of the dual unit ball are given by

$$\mathrm{ext}\, \mathrm{Ball}_1(E') = \left\{ \phi_a \big|_E : a \text{ atom} \in E'' \right\}.$$

Any function $u \in E$ gives rise to the multiplication operator $M_u : f \mapsto uf$ $(f \in E)$. Notice that for $|u| \leqq 1$ we have $|M_u f| \leqq |f|$ and hence $M_u D \subset D$. That is $M_u \mathrm{Ball}_1(E) \subset \mathrm{Ball}_1(E)$ for $\max |u| \leqq 1$ and, in general,

$$\|M_u\| \leqq \max |u| \qquad (u \in E).$$

Any function $g \in \mathcal{B}(\Omega)$ gives rise to the multipliers

$$M'_g(d\mu) := g\, d\mu \qquad (\mu \in \mathcal{M}(\Omega)),$$
$$M''_g(f) := [d\mu \mapsto f(g\, d\mu)] \qquad (f \in E'').$$

We note that $M'_g : E' \to E'$, $M''_g : E'' \to E''$ are linear operators and if $u \in E$ then M'_u, M''_u are the adjoint $E' \to E'$ and biadjoint $E'' \to E''$ of M_u, respectively. The norm of an adjoint is always the same as the original operator. In particular we have $\|M'_u\| = \|M''_u\| \leqq \max |u|$ for all $u \in E$.

Lemma 1. *In general,* $\|M'_g\| = \sup_{\|d\mu=1\|} \|g\, d\mu\| \leqq \sup |g|$ *and* $\|M''_g\| = \sup_{f \in \mathrm{Ball}_1(E'')} \|M_g f\| \leqq \sup |g|$.

Proof. Notice that M''_g is the adjoint of M'_g. Thus $\|M''_g\| = \|M'_g\|$.

To show that $\|M'_g(d\mu)\| \leqq \sup |g| \|d\mu\|$, we proceed as follows. Fix $\mu \in \mathcal{M}(\Omega)$ arbitrarily. It is well-known that $d\mu = h|d\tilde{\mu}|$ for some function $h \in \mathcal{B}(\Omega)$ with $|h| = 1$ and

$$\tilde{\mu} \in \mathcal{M}(\Omega)_+ := \{\nu \in \mathcal{M}(\Omega) : \ \nu(B) \geqq 0 \ \ (B \in \mathrm{Borel}(\Omega))\}.$$

On the other hand, there exists a sequence $g_1, g_2, \ldots \in \mathcal{C}_0(\Omega)$ with $\max |g_1|, \max |g_2|, \ldots \leqq \sup |g|$ and $\tilde{\mu}\{\omega \in \Omega : \ g(\omega) \neq \lim_{n \to \infty} g_n(\omega)\} = 0$. Thus

$$\begin{aligned}
\|M'_g(d\mu)\| &= \|gh\, d\tilde{\mu}\| = \sup_{f \in D} \left| \int fgh\, d\tilde{\mu} \right| \\
&= \sup_{f \in D} \lim_{n \to \infty} \left| \int fg_n h\, d\tilde{\mu} \right| = \sup_{f \in D} \lim_{n \to \infty} |d\mu(M'_{g_n} f)| \\
&\leqq \sup_{f \in D} \lim_{n \to \infty} \|d\mu\|\, \|M_{g_n} f\| \leqq \sup_{f \in D} \lim_{n \to \infty} \|d\mu\| \max |g_n| \\
&= \max |g_n|\, \|d\mu\|. \quad \square
\end{aligned}$$

Given any set $S \in \mathrm{Borel}(\Omega)$, we write 1_S for the indicator function of S, that is $1_S(\omega) := [1 \text{ if } \omega \in S, \ 0 \text{ else}]$ $(\omega \in \Omega)$. By the previous lemma the operator

$$P_S := M''_{1_S}$$

is clearly a contractive projection $E'' \to E''$ since, in general

$$M'_{g_1} M'_{g_2} = M'_{g_1 g_2}, \quad M''_{g_1} M''_{g_2} = M''_{g_1 g_2}$$

and we always have $P_S P_T = P_{S \cap T}$ $(S, T \in \mathrm{Borel}(\Omega))$.

Recall that surjective linear isometries of JB*-triples are automorphisms of the underlying triple product [6]. In particular, because of the above lemma, M''_g is an automorphism of the JB*-triple E'' whenever $g \in \mathcal{B}(\omega)$ and $|g(\omega)| = 1$ for all $\omega \in \Omega$, since it is the inverse of the multiplier $M''_{\overline{g}}$.

Proposition 1. *Suppose $a \in E''$ is an atom and $S_1, \dots, S_N \in \mathrm{Borel}(\Omega)$ are disjoint sets such that $\bigcup_{k=1}^{N} S_k = \Omega$ and $P_k a \neq 0$ $(k = 1, \dots, N)$ with the projections $P_k := M''_{1_{S_k}}$. Then $\sum_k \|P_k a\|^2 = 1$ and $\|P_k a\|^{-1} P_k a$ is again an atom in E'' with $\phi_{\|P_k a\|^{-1} P_k a} = \|P_k a\|^{-1} \phi_a \circ P_k$ for any index k.*

Proof. For any $t_1, \dots, t_n \in \mathbb{R}$ the functions $g_{t_1,\dots,t_N} := \sum_{k=1}^{N} e^{it_k} 1_{S_k}$ have absolute value 1. Hence

$$A_{t_1,\dots,t_N} := \sum_{k=1}^{N} e^{it_k} P_k = M''_{g_{t_1,\dots,t_N}} \in \mathrm{Aut}(E'') \qquad (t_1, \dots, t_N \in \mathbb{R})$$

with the inverse

$$A^{-1}_{t_1,\dots,t_N} = M''_{\overline{g_{t_1,\dots,t_N}}} = \sum_{k=1}^{N} e^{-it_k} P_k.$$

It follows that the elements

$$a_{t_1,\dots,t_N} := A_{t_1,\dots,t_N} a \qquad (t_1, \dots, t_N \in \mathbb{R})$$

are all atoms of E'' since they are automorphic images of the atom a. Thus

$$\{a_{t_1,\dots,t_N} x a_{t_1,\dots,t_N}\} = \overline{\phi_{a_{t_1,\dots,t_N}}(x)} a_{t_1,\dots,t_N}$$

for any $x \in E''$ and $t_1, \dots, t_N \in \mathbb{R}$. On the other hand, here we have

$$\begin{aligned} \{a_{t_1,\dots,t_N} x a_{t_1,\dots,t_N}\} &= A_{t_1,\dots,t_N}\{a[A^{-1}_{t_1,\dots,t_N} x]a\} \\ &= \overline{\phi_a(A^{-1}_{t_1,\dots,t_N} x)} A_{t_1,\dots,t_N} a \\ &= \overline{\phi_a(A^{-1}_{t_1,\dots,t_N} x)} a_{t_1,\dots,t_N}. \end{aligned}$$

That is, for all real $t_1, \dots, t_N$ we have

$$\phi_{a_{t_1,\dots,t_N}}(x) = \phi_a(A^{-1}_{t_1,\dots,t_N} x) = \sum_{k=1}^{N} e^{-it_k} \phi_a(P_k x)$$

and

$$\left\{\left[\sum_k e^{it_k} P_k a\right] x \left[\sum_\ell e^{it_\ell} P_\ell a\right]\right\} = \overline{\sum_k e^{-it_k} \phi_a(P_k x)} \sum_\ell e^{it_\ell} P_\ell a.$$

Comparing the coefficients of the above trigonometric polynomials in the second identity, we get

$$\{(P_k a) x (P_\ell a)\} = \{(P_\ell a) x (P_k a)\} = \frac{1}{2}\overline{\phi_a(P_k a)} P_\ell a + \frac{1}{2}\overline{\phi_a(P_\ell a)} P_k a$$

for any $x \in E''$ and $k, \ell = 1, \dots, N$. In particular, for $k = \ell$, these relations mean $\{(P_k a)E''(P_k a)\} \subset \mathbb{C}P_k a$, that is each element $P_k a$ is some multiple of a suitable atom. Thus the element

$$a_k := \|P_k a\|^{-1} P_k a$$

is an atom in E''. The above equation further implies $\{(P_k a)(P_k a)(P_k a)\} = \overline{\phi_a(P_k a)} P_k a$. Since the number $\overline{\phi_a(P_k a)}$ belongs to the spectrum of the operator $[x \mapsto \{(P_k a)(P_k a)x\}]$, which is non-negative by (J5), we have

$$\|P_k a\|^3 = \|\{(P_k a)(P_k a)(P_k a)\}\| = \|\overline{\phi_a(P_k a)} P_k a\| = \phi_a(P_k a)\|P_k a\|.$$

Hence it follows $\|P_k a\| = \phi_a(P_k a)^{1/2}$. Thus, with the coefficients

$$\lambda_k := \phi_a(P_k a)^{1/2} > 0 \qquad (k = 1, \dots, N)$$

we have

$$a = a_{0,\dots,0} = \sum_k P_k a = \sum_k \lambda_k a_k, \quad \sum_k \lambda_k^2 = \sum_k \phi_a(P_k a) = \phi_a(a) = 1.$$

It only remains to relate the functionals ϕ_{a_k} to ϕ_a. Again the relation $\{(P_k a)x(P_k a)\} = \overline{\phi_a(P_k x)} P_k a$ entails

$$\begin{aligned} \overline{\phi_{a_k}(x)} a_k &= \{a_k x a_k\} = \{[\lambda_k^{-1} P_k a] x [\lambda_k^{-1} P_k a]\} \\ &= \lambda_k^{-2} \overline{\phi_a(P_k x)} P_k a = \lambda_k^{-1} \overline{\phi_a(P_k x)} a_k \end{aligned}$$

that is $\phi_{a_k} = \lambda_k^{-1} \phi_a \circ P_k$ $(k = 1, \dots, N)$. $\square$

Corollary 1. *The atoms $a_k := \|P_k a\|^{-1} P_k a$ $(k = 1, \dots, N)$ are pairwise collinear and*

$$\{a_k x a_\ell\} = \frac{1}{2}\overline{\phi_{a_k}(x)} a_\ell + \frac{1}{2}\overline{\phi_{a_\ell}(x)} a_k \qquad (x \in E'',\ k, \ell = 1, \dots, N).$$

If $\zeta_1, \dots, \zeta_N \in \mathbb{C}$ with $\sum_k |\zeta_k|^2 = 1$ the element $\sum_{k=1}^{N} \zeta_k a_k$ is an atom in E'' and

$$\phi_{\zeta_1 a_1 + \dots + \zeta_N a_N} = \sum_k \overline{\zeta_k} \phi_{a_k}.$$

Proof. We have $\phi_{a_k} = \lambda_k^{-1} \phi_a \circ P_k$ and $\phi_{e_k}(a_k) = 1$. Since $P_k a_k = a_k$ and $P_k P_\ell = 0$ for $k \neq \ell$,

$$\begin{aligned} \{a_k a_k a_\ell\} &= \frac{\lambda_k^{-1}\lambda_\ell^{-1}}{2} \big[\overline{\phi_a(P_\ell a_k)} P_k a + \overline{\phi_a(P_k a_k)} P_\ell a\big] \\ &= \frac{1}{2}\big[\overline{\phi_{a_k}(a_k)} a_\ell + \overline{\phi_{a_\ell}(0) a_k}\big] = \frac{1}{2} a_\ell \qquad (k \neq \ell) \end{aligned}$$

which means the collinearity of the family $\{a_k\}_{k=1}^N$. Then all the statements follow from the relationship

$$\left\{\left[\sum_k \zeta_k a_k\right] x \left[\sum_\ell \zeta_\ell a_\ell\right]\right\} = \sum_{k,\ell} \zeta_k \zeta_\ell \{a_k x a_\ell\}$$

$$= \sum_{k,\ell} \frac{1}{2} \zeta_k \zeta_\ell [\overline{\phi_{a_k}(x)} a_\ell + \overline{\phi_{a_\ell}(x)} a_k] = \overline{\sum_k \overline{\zeta_k} \phi_{a_k}(x)} \left[\sum_\ell \zeta_\ell a_\ell\right]. \quad \square$$

Since $D = \mathrm{Ball}_1(E)$ is a bounded open subset of $\mathcal{C}_0(\Omega)$, for some $0 < \varepsilon \leqq M < \infty$ we have

$$\{x : \|x\|_\infty < \varepsilon\} \subset D \subset \{x : \|x\|_\infty < M\},$$

$$(*) \qquad \varepsilon \|d\mu\|_1 \leqq \|d\mu\| \leqq M \|d\mu\|_1 \qquad (\mu \in \mathcal{M}(\Omega)).$$

Lemma 2. *If* $d\mu \in \mathrm{ext}(\mathrm{Ball}_1(E))$ *and* $S_1, \dots, S_N \in \mathrm{Borel}(\Omega)$ *are pairwise disjoint sets such that* $1_{S_k} d\mu \neq 0$ $(k = 1, \dots, N)$ *then necessarily* $N \leqq (M/\varepsilon)^2$.

Proof. We may assume $\bigcup_{k=1}^N S_k = \Omega$ (by replacing S_N with $\Omega \setminus \bigcup_{k<N} S_k$ if necessary). For some atom $a \in E''$ we have $d\mu = \phi_a$. Then, with the projections $P_k := M''_{1_{S_k}}$, we have by Proposition 1

$$a_k := \|P_k a\|^{-1} P_k a \text{ is atom and } \phi_{a_k} = \|P_k a\|^{-1} \phi_a \circ P_k = \|P_k a\|^{-1} 1_{S_k} d\mu.$$

Thus, by Corollary 1,

$$b := \sum_k \zeta_k a_k \text{ is atom and}$$

$$\phi_b = N^{-1/2} \sum_k \phi_{a_k} = \left[N^{-1/2} \sum_k \|P_k a\|^{-1} 1_{S_k}\right] d\mu.$$

Since the sets $S_1, \dots, S_N$ are disjoint,

$$\|\phi_b\|_1 = \left\|\sum_k N^{-1/2}\right\| P_k a\|^{-1} 1_{S_k} d\mu\|_1 = \sum_k \|N^{-1/2} \|P_k a\|^{-1} 1_{S_k} d\mu\|_1$$

$$= N^{-1/2} \sum_k \|\phi_{a_k}\|_1.$$

Since $\|\phi_b\| = \|\phi_{a_1}\| = \dots = \|\phi a_N\| = 1$, and since, in general, $\varepsilon \|\phi\|_1 \leqq \|\phi\| \leqq M \|\phi\|_1$ $(\phi \in E')$, we have the following $\|.\|_1$-estimates

$$\frac{1}{\varepsilon} \geqq \|\phi_b\|_1 = N^{-1/2} \sum_k \|\phi_{a_k}\|_1 \geqq N^{-1/2} \sum_{k=1}^N \frac{1}{M} = \frac{N^{1/2}}{M}. \quad \square$$

Corollary 2. *Any measure $\mu \in \mathcal{M}(\Omega)$ with $d\mu \in$ ext(Ball$_1(E')$) has a finite support consisting of at most $(M/\varepsilon)^2$ points.*

Proof. Let $S_1, \dots, S_N$ be a partition of Ω with a maximal number of elements such that $1_{S_k} d\mu \neq 0$ $(k = 1, \dots, N)$. By Lemma 2 such partitions exist and necessarily $N \leqq (M/\varepsilon)^2$. For each index k, let

$$\mathcal{S}_k := \{S \in \text{Borel}(\Omega) : S \subset S_k, \ 1_S d\mu \neq 0\}.$$

Given any index k and a non-empty Borel(Ω)-measurable proper subset of S_k the partition $S_1, \dots, S_{k-1}, T, S_k \setminus T, S_{k+1}, \dots, S_N$ consists of more than N members, hence necessarily either $1_T d\mu = 0$ and $1_{S_k \setminus T} \neq 0$ or $1_T d\mu \neq 0$ and $1_{S_k \setminus T} = 0$. That is $\mathcal{S}_k$ is an ultrafilter in Borel(S_k). On the other hand, by the inner compact regularity of the measures in $\mathcal{M}(\Omega)$, for any $S \in \mathcal{S}_k$ there exists K compact $\subset S$ such that $1_K d\mu \neq 0$. Therefore $\{K \in \mathcal{S}_k : K \text{ compact} \subset S_k\}$ is a family with finite intersection property shrinking to a single point (as being a filter basis of a Borelian ultrafilter). Thus for some points $\omega_1 \in S_1, \dots, \omega_N \in S_N$ we have $\mathcal{S}_k = \{T \in \text{Borel}(S_k) : \omega_k \in T\}$ $(k = 1, \dots, N)$. It follows $d\mu = \sum_k 1_{\{\omega_k\}} d\mu$ and Support$(\mu) = \{\omega_1, \dots, \omega_N\}$. □

Henceforth, for any point $\omega \in \Omega$, we write

$$P_\omega := M''_{1_{\{\omega\}}}, \quad \overline{P_\omega} := \text{id} - P_\omega, \quad \delta_\omega := [\text{Borel}(\Omega) \ni S \mapsto 1 \text{ if } \omega \in S, \ 0 \text{ otherwise}].$$

Lemma 3. *For each $\omega \in \Omega$ the element $1_{\{\omega\}}$ is an atom of E''.*

Proof. Given $f \in E''$, we have

$$P_\omega f(d\mu) = f(1_{\{\omega\}} d\mu) = f(\mu\{\omega\} d\delta_\omega) = f(d\delta_\omega) 1_{\{\omega\}}(d\mu).$$

Thus P_ω is a rank 1 projection $E'' \to E''$ with $\mathbb{C}1_{\{\omega\}} = \text{range}(P_\omega) = \text{kernel}(\overline{P_\omega})$. Since the projections $P_\omega, \overline{P_\omega}$ (being real multipliers of E'') are Jordan-derivations of the triple product $\{\dots\}$, for any $x \in E''$ we have

$$\begin{aligned} \overline{P_\omega}\{1_{\{\omega\}} x 1_{\{\omega\}}\} &= \{(\overline{P_\omega} 1_{\{\omega\}}) x 1_{\{\omega\}}\} - \{1_{\{\omega\}} (\overline{P_\omega} x) 1_{\{\omega\}}\} + \{1_{\{\omega\}} x (\overline{P_\omega} 1_{\{\omega\}})\} \\ &= -\{1_{\{\omega\}} (\overline{P_\omega} x) 1_{\{\omega\}}\}, \\ \overline{P_\omega}\{1_{\{\omega\}} x 1_{\{\omega\}}\} &= \overline{P_\omega}^2 \{1_{\{\omega\}} x 1_{\{\omega\}}\} = (-1)^2 \{1_{\{\omega\}} (\overline{P_\omega} x) 1_{\{\omega\}}\}. \end{aligned}$$

It follows $\overline{P_\omega}\{1_{\{\omega\}} E'' 1_{\{\omega\}}\} = 0$, $\{1_{\{\omega\}} E'' 1_{\{\omega\}}\} \subset \mathbb{C}1_{\{\omega\}}$ which means that $1_{\{\omega\}}$ is an atom of E''. □

Proposition 2. *The atomic part of E'' is an ℓ^∞-direct sum of Hilbert spaces of dimension $\leqq (M/\varepsilon)^2$, each of which is spanned by a collinear family of atoms of the form $e_\omega := \|1_{\{\omega\}}\|^{-1} 1_{\{\omega\}}$ $(\omega \in \Omega)$.*

Proof. We know [9] that

$$E''_{\mathrm{at}} := \text{weak*span}\{\text{atoms of } E''\} = \oplus_{j\in J} F_j$$

where the family $\{F_j : j \in J\}$ consists of the minimal ideals of E'' with respect to the triple product $\{...\}$ and $F_j = \text{weak*span}\{a \text{ atom} \in E'' : a \in F_j\}$ $(j \in J)$. Actually, by the two previous lemmas,

$$F_j = \text{weak*span}\{e_\omega : \omega \in \Omega_j\} \text{ where } \Omega_j = \{\omega \in \Omega : e_\omega \in F_j\} \quad (j \in J)$$

and $\Omega = \bigcup_{j\in J} \Omega_j$. Fix any index $j \in J$ arbitrarily. To complete the proof, we show that the family $\{e_\omega : \omega \in \Omega_j\}$ consists of at most $(M/\varepsilon)^2$ pairwise collinear atoms. In general, if $\omega, \eta \in \Omega$ then, since $i P_\eta$ is derivation

$$\begin{aligned} P_\eta\{e_\omega e_\omega e_\eta\} &= \{(P_\eta e_\omega) e_\omega e_\eta\} - \{e_\omega (P_\eta e_\omega) e_\eta\} + \{e_\omega e_\omega (P_\eta e_\eta)\} \\ &= \{e_\omega e_\omega e_\eta\} \in \mathbb{C} e_\eta. \end{aligned}$$

Thus the atoms e_ω $(\omega \in \Omega)$ are pairwise Peirce compatible and hence collinear or orthogonal (denoted by $\top$ resp. $\perp$) since the fact $\{x \in E'' : \{e_\omega e_\omega x\} = x\} = \mathbb{C} e_\omega$ $(\omega \in \Omega)$ excludes the possibilities of governing and association relations. Observe also that

$$e_\alpha \top e_\beta \top e_\gamma \Rightarrow e_\alpha \top e_\gamma \text{ or } \alpha = \gamma.$$

Indeed otherwise there would exist $\alpha, \beta, \gamma \in \Omega$ with $\alpha \neq \beta \neq \gamma \neq \alpha$ and $e_\alpha \top e_\beta \top e_\gamma \perp e_\alpha$. Then by the quadrangle lemma [9], the element $e := \{e_\alpha e_\beta e_\gamma\}$ would be a non-zero tripotent which is impossible, because

$$P_\beta e = \{(P_\beta e_\alpha) e_\beta e_\gamma\} - \{e_\alpha (P_\beta e_\beta e_\gamma\} + \{e_\alpha e_\beta (P_\beta e_\gamma)\} = -e$$

which would entail

$$e = -P_\beta e = -P_\beta(-P_\beta e) P_\beta^2 e = P_\beta e = 0.$$

Since F_j admits no orthogonal splitting, it follows $e_\alpha \top e_\beta$ for any couple $\alpha, \beta \in \Omega_j$ with $\alpha \neq \beta$. It is well-known [8] that, in this case, F_j can be equipped with an inner product $(.,.)_j$ with respect to which $\{e_\omega : \omega \in \Omega_j\}$ is a complete orthonormal family and $\|f\|^2 = (f, f)_j$ $(f \in F_j)$. Necessarily, every element $f \in F_j$ with $\|f\| = 1$ is an atom of E''. Therefore, by Lemma 2, $\text{cardinality}(\Omega_j) = \dim(F_j) \leqq (M/\varepsilon)^2$. $\square$

In terms of the Cartan factors F_j $(j \in J)$ of E'' we can summarize our considerations concerning the extreme points of the unit ball of E' as follows. Let

$$\Omega_j := \{\omega \in \Omega : 1_{\{\omega\}} \in F_j\} \quad (j \in J), \qquad m(\omega) := \|1_{\{\omega\}}\|^2 \quad (\omega \in \Omega).$$

Then we have the following extension of the Banach-Stone theorem.

Corollary 4. *If D is a bounded continuous symmetric Reinhardt domain over Ω with* $(*)$ *then* $\text{cardinality}(\Omega_j) \leqq (M/\varepsilon)^2$ $(j \in J)$ *and*

$$\text{ext}(\text{Ball}_1(E')) = \bigcup_{j\in J} \left\{ \sum_{\omega\in\Omega_j} \zeta_\omega \delta_\omega : \sum_{\omega\in\Omega_j} m(\omega) |\zeta_\omega|^2 = 1 \right\}.$$

Theorem 3. *Let D be a bounded symmetric continuous Reinhardt domain over the locally compact topological space Ω and let $0 < \varepsilon \leqq M < \infty$ be constants such that $(*)$ holds. Then there exists a partition $\{\Omega_j : j \in J\}$ of Ω consisting of sets of $\leqq (M/\varepsilon)^2$ elements along with a function $m : \Omega \to [M^{-2}, \varepsilon^{-2}]$ such that*

$$(**)\qquad D = \left\{ f \in \mathcal{C}_0(\Omega) : \sup_{j \in J} \sum_{\omega \in \Omega_j} m(\omega)|f(\omega)|^2 < 1 \right\}.$$

For the triple product of the JB-triple whose unit ball is D we have*

$$(***)\qquad \{fgh\}(\omega) = \sum_{\eta \in \Omega_{j(\omega)}} \frac{m(\eta)}{2}[f(\eta)\overline{g(\eta)}h(\omega) + h(\eta)\overline{g(\eta)}f(\omega)]$$

for all $f, g, h \in \mathcal{C}_0(\Omega)$ and $\omega \in \Omega$ where $j(\omega)$ means the unique index with $\omega \in \Omega_{j(\omega)}$.

Proof. By Corollary 4, the characterization $(**)$ of D is immediate since

$$D = \mathrm{Ball}_1(E) = \left\{ f \in \mathcal{C}_0(\Omega) : \sup_{d\mu \in \mathrm{ext}\ \mathrm{Ball}_1(E')} \left| \int f\, d\mu \right| < 1 \right\}.$$

Moreover we have

$$\{x \in \mathcal{B}(\Omega) : \sup |x| \leqq \varepsilon\} \subset \mathrm{Ball}_1(\mathcal{B}(\Omega)) \subset \{x \in \mathcal{B}(\Omega) : \sup |x| \leqq M\}.$$

This implies

$$\varepsilon \leqq \sup |e_\omega| = \|1_{\{\omega\}}\|^{-1} = m(\omega)^{-1/2} \leqq M, \quad m(\omega) \in [M^{-2}, \varepsilon^{-2}] \quad (\omega \in \Omega).$$

Given $\omega \in \Omega_j$ and $f, g, h \in E$,

$$\{fgh\}(\omega) = d\delta_\omega(\{fgh\}) = d\delta_\omega P^{(j)}\{fgh\})$$

where $P^{(j)} := \sum_{\eta \in \Omega_j} P_\eta$ is the projection of E'' onto the factor $F_j = \sum_{\eta \in \Omega_j} \mathbb{C}1_{\{\omega\}}$ along $F_j^\perp = E''_{\mathrm{cont}} \oplus [\oplus_{k \in J \setminus \{j\}} F_k]$. Since the triple product acts componentwise on the Cartan factors of E'', we have $P^{(j)}\{fgh\} = \{(P^{(j)}f)(P^{(j)}g)(P^{(j)}h)\}$. Since the factor F_j is isomorphic to a Hilbert space where $\{m(\omega)^{-1/2}1_{\{\omega\}} : \omega \in \Omega_j\}$ is an orthonormal basis, $\{1_{\{\alpha\}}1_{\{\beta\}}1_{\{\gamma\}}\} = \frac{1}{2}\delta_\alpha\{\beta\}m(\alpha)1_{\{\gamma\}} + \frac{1}{2}\delta_\gamma\{\beta\}m(\gamma)1_{\{\alpha\}}$ $(\alpha, \beta, \gamma \in \Omega_j)$. Therefore

$$\begin{aligned}\{fgh\}(\omega) &= d\delta_\omega\{(P^{(j)}f)(P^{(j)}g)(P^{(j)}h)\} \\ &= d\delta_\omega \left\{ \sum_{\alpha \in \Omega_j} f(\alpha)1_{\{\alpha\}} \sum_{\beta \in \Omega_j} g(\beta)1_{\{\beta\}} \sum_{\gamma \in \Omega_j} h(\gamma)1_{\{\gamma\}} \right\}\end{aligned}$$

$$= d\delta_\omega \left[\frac{1}{2} \sum_{\beta,\gamma\in\Omega_j} m(\beta) f(\beta)\overline{g(\beta)} h(\gamma) 1_{\{\gamma\}} \right.$$

$$\left. + \frac{1}{2} \sum_{\alpha,\beta\in\Omega_j} m(\beta) h(\beta)\overline{g(\beta)} f(\alpha) 1_{\{\alpha\}} \right]$$

$$= \frac{1}{2} \sum_{\beta\in\Omega_j} m(\beta) f(\beta)\overline{g(\beta)} f(\omega) + \frac{1}{2} \sum_{\beta\in\Omega_j} m(\beta) h(\beta)\overline{g(\beta)} f(\omega). \quad \square$$

It is now natural to investigate necessary and sufficient conditions for a function $m : \Omega \to \mathbb{R}_+$ and a partition $\{\Omega_j : j \in J\}$ such that the set D given by $(**)$ is a bounded symmetric continuous Reinhardt domain over Ω.

Theorem 2. *Given a function $m : \Omega \to \mathbb{R}$ such that $0 < \inf m$, $\sup m < \infty$ along with a partition $\{\Omega_j : j \in J\}$ of Ω such that* $\sup\{\text{cardinality}(\Omega_j) : j \in J\} < \infty$, *the set D defined by $(**)$ is a bounded symmetric continuous Reinhardt domain over Ω and $(***)$ is the triple product of the JB*-triple whose unit ball is D if and only if all the functions $\omega \mapsto \sum_{\eta\in\Omega_{j(\omega)}} m(\eta)|f(\eta)|^2$ $(f \in C_0(\Omega))$ are continuous.*

Proof. Given a real-valued function $m : \omega \to \mathbb{R}_+$ with finite positive bounds from above and below along with a partition $\{\Omega_j : j \in J\}$ the cardinality of whose members does not exceed a finite bound, the operation $(***)$ makes the space $\widehat{E}$ of all bounded functions $\Omega \to \mathbb{C}$ into a JB*-triple with the norm

$$|||f||| := \left[\sup_{j\in J} \sum_{\omega\in\Omega_j} m(\omega)|f(\omega)|^2 \right]^{1/2} \qquad (f \in \widehat{E}).$$

By the boundedness conditions on m and $\{\Omega_j : j \in J\}$, this norm is equivalent to the usual sup-norm of $\widehat{E}$. Since the set D given by $(**)$ is the intersection of the $|||.|||$-unit ball with the closed subspace $C_0(\Omega)$, it is a bounded symmetric continuous Reinhardt domain over Ω if and only if the triple product $(***)$ preserves C_0-functions. Thus it suffices to verify that $f \in C_0(\Omega)$ implies $\{f\}^3 = \{fff\} \in C_0(\Omega)$. Observe that, given any function $f \in \widehat{E}$, $\{f\}^3(\omega) = g_f(\omega) f(\omega)$ where $g_f(\omega) := \sum_{\eta\in\Omega_{j(\omega)}} m(\eta)|f(\eta)|^2$ $(\omega \in \Omega)$. The function g_f is bounded in any case. Thus if g_f is continuous, we obviously have $\{f\}^3 = g_f f \in C_0(\Omega)$ for $f \in C_0(\Omega)$. Conversely, suppose $\{f\}^3 \in C_0(\Omega)$ for every $f \in C_0(\Omega)$, and fix $f \in C_0(\Omega)$ arbitrarily. We complete the proof by establishing the continuity of g_f. Since $\{f\}^3 = g_f f$, the function g_f is trivially continuous at the points $\omega \in \Omega$ where $f(\omega) \neq 0$. Consider a point ω where $f(\omega) = 0$. Since Ω is a locally compact Hausdorff space, we can find a function $h \in C_0(\Omega)$ with $h(\omega) = 1$. By assumption, $\{f + h/n\}^3 \in C_0(\Omega)$ for all natural numbers n and $[f + h/n](\omega) = 1/n \neq 0$. It follows that each of the functions $g_{f+h/n}$ is continuous at ω. By our boundedness conditions, the sequence $g_{f+h/n}$ converges uniformly (with respect to the sup-norm) to g_f which implies the continuity of g_f at ω. $\square$

Notice that, in general, the function m need not be continuous on Ω. If we consider the interval $\Omega := [-1, 1]$ we can define the triple product and the norm by

$$\{f\}^3(x) := \left[\frac{1}{2}|f(x)|^2 + \frac{1}{2}|f(-x)|^2\right] f(x),$$
$$\|f\|^2 := \sup_{-1 \leqq x \leqq 1} |f(x)|^2 + |f(-x)|^2.$$

It is easy to see that the unit ball of this norm is a bounded symmetric continuous Reinhardt domain over Ω corresponding to the partition $\{\{x, -x\} : 0 \leqq x \leqq 1\}$ with the function $m(x) := [1 \text{ if } x \neq 0,\ 2 \text{ if } x = 0]$.

The investigation of all possible partitions and functions m which give rise to the same domain D seems to be an interesting problem for the future research. In our terminology, the special case where Ω is compact and all the members of the underlying partition are singletons was recently investigated by J.-P. Vigué [12]. According to his result, the functions m_1 and m_2 determine the same symmetric domain if and only if $\liminf m_1 = \liminf m_2$ is a strictly positive continuous function.

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