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ON SETS OF UNIQUENESS FOR COMPLETELY ADDITIVE ARITHMETIC FUNCTIONS

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Abstract.

Given a subgroup H of an abelian group G we deal with the problem to determine all the subsets $A \subset \mathbf{N}$ such that for any completely additive $f : \mathbf{N} \to G$ we have $f(A) \subset H$ whenever $f(A) \subset H$. Such sets are called sets of G/H-uniqueness. Here we give a characterization of sets of $\mathbf{Z}/(q\mathbf{Z})$-uniqueness and G-uniqueness (i.e. $G/\{0\}$-uniqueness), where G is a finite abelian group.

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1. INTRODUCTION

A function f mapping the natural numbers $\mathbf{N}$ into an abelian group G (with operation $+$) is said to be *completely additive* in case

$$f(mn) = f(m) + f(n)$$

holds for all $m, n \in \mathbf{N}$.

In an early paper Kátai [6] introduced the concept of *sets of uniqueness for completely additive functions.* This can be formulated in a more general setting: Given a subgroup H of G, determine all the subsets $A \subset \mathbf{N}$ such that for any completely additive $f : \mathbf{N} \to G$ we have $f(\mathbf{N}) \subset H$ whenever $f(A) \subset H$. By passing to the factor group G/H, the problem can be reformulated as to describe the sets $A \subset \mathbf{N}$ such that any completely additive function vanishing on A must vanish on the whole $\mathbf{N}$. Such sets are called *sets of G/H-uniqueness.*

In case $G = \mathbf{R}$ and $H = \{0\}$, Wolke [8] and, with a different proof, Indlekofer ([5], Theorem 1) showed that for a set A of $\mathbf{R}$-uniqueness every $n \in \mathbf{N}$ must be expressible as a finite product of rational powers of elements of A. Theorem 2 of the article [5] by Indlekofer proves that for $H = \mathbf{Z}$ the sets A of $\mathbf{R}/\mathbf{Z}$-uniqueness can be characterized by the property that every $n \in \mathbf{N}$ can be expressed as a finite product of *integer* powers of elements of A. A more specific form is given by the following

PROPOSITION. *Let $A \subset \mathbf{N}$. Then A is a set of $\mathbf{R}/\mathbf{Z}$ - uniqueness if and only if for each $n \in \mathbf{N}$ there exist $a_j \in A$, $\alpha_j \in \mathbf{Z}$ $(j = 1, \ldots, s)$ such that*

$$n = \prod_{j=1}^{s} a_j^{\alpha_j}.$$

The idea of the proof was to consider the multiplicative semigroup $\mathbf{N}$ as a generator family of the multiplicative group $Q_+ := \{m/n : \ m, n \in \mathbf{N}\}$ of positive rationals. The latter is isomorphic to the countably generated free (additive) abelian group $\bigoplus \mathbf{Z}$ by the function $\theta : Q_+ \to \bigoplus \mathbf{Z}$ mapping $\rho \in Q_+$ into the prime exponents of ρ, and the θ-image of a set of $\mathbf{R}/\mathbf{Z}$-uniqueness should generate the whole $\bigoplus \mathbf{Z}$.

REMARK. A form of this result would be implicit in Corollary of Dress and Volkmann [1]. However, the proof which they give is not complete. More detailed remarks and a counterexample may be found in Indlekofer [5] where a correct form of this result was first given. Hoffmann [3], who was apparently unaware of these papers presented a proof of this result, too. Kátai [7], Elliott [2] and Indlekofer [4] gave several examples for sets of $\mathbf{R}/\mathbf{Z}$- uniqueness.

In this paper we show that an analogous decomposition (see formula (1.2) below) characterizes the sets of $\mathbf{Z}/(q\mathbf{Z})$-uniqueness for natural numbers $q > 1$. Since A is a set of $\mathbf{Z}/(q\mathbf{Z})$-uniqueness if and only if every completely additive function $f : \mathbf{N} \to \mathbf{Z}$ taking values divisible by q on A takes values divisible by q on the whole $\mathbf{N}$, sets of $\mathbf{Z}/(q\mathbf{Z})$-uniqueness are usually called *mod_q-uniqueness* families.

Throughout this work let $q > 1$ be an arbitrarily fixed natural number. Our aim is the following characterization of *mod_q*- uniqueness families.

1.1 THEOREM. *The subset $A \subset \mathbf{N}$ is a mod_q-uniqueness family if and only if each natural number $n \in \mathbf{N}$ admits a decomposition of the form*

$$(1.2) \quad n = L^q \prod_{j=1}^{s} a_j^{r_j} \qquad L \in Q_+, \ a_j \in A, \ r_j \in \{0, \ldots, q-1\} \quad (j = 1 \ldots, s).$$

Obviously each set of $\mathbf{R}/\mathbf{Z}$-uniqueness is a mod_q-uniqueness family, but the converse is not true. Furthermore, a mod_q-uniqueness family is not necessarily a set of $\mathbf{R}$-uniqueness.

1.3 EXAMPLES. 1) Let p_1, p_2, p_3 be three different primes, and let

$$A = \{p_1^3, p_1^2 p_2, p_1^2 p_3\} \cup \mathbf{P} \backslash \{p_1, p_2, p_3\}$$

where $\mathbf{P}$ denotes the set of primes. Then the following holds:

(i) *A is a set of $\mathbf{Z}/(2\mathbf{Z})$-uniqueness,*

(ii) *A is a set of* **R**-*uniqueness,*

(iii) *A is not a set of* **R**/**Z**-*uniqueness.*

The proof of (ii) is obvious. Concerning (i) we observe that, for every completely additive $f : \mathbf{N} \to \mathbf{Z}$,

$$\begin{aligned} 3f(p_1) &\equiv 0 \bmod 2 \\ 2f(p_1) + f(p_2) &\equiv 0 \bmod 2 \\ 2f(p_1) + f(p_3) &\equiv 0 \bmod 2 \end{aligned}$$

implies

$$f(p_1) \equiv f(p_2) \equiv f(p_3) \equiv 0 \bmod 2.$$

For the proof of (iii) we define a completely additive function f by

$$f(p_1) = f(p_2) = f(p_3) = 2/3$$

and

$$f(p) = 0 \quad \text{for} \quad p \in \mathbf{P} \backslash \{p_1, p_2, p_3\}.$$

Then $f(A) = \{0, 2\} \subset \mathbf{Z}$ but $f(\mathbf{N}) \not\subset \mathbf{Z}$ which proves assertion (iii).

2) Let $\mathbf{P} = \{p_i\}$, $2 = p_1 < p_2 < \ldots$ and put $A = \{p_j p_{j+1}^q\}$. Then A is a mod_q-uniqueness family, but not a set of **R**-uniqueness. The first assertion is obvious. For the second assertion we define a completely additive function $f : \mathbf{N} \to \mathbf{R}$ by

$$f(p_j) = (-1)^{j-1} q^{-(j-1)} \quad \text{for} \quad j = 1, 2, \ldots$$

Then

$$f(p_j p_{j+1}^q) = (-1)^{j-1} q^{-(j-1)} - (-1)^{j-1} q^{-(j-1)} = 0$$

i.e. $f(A) = \{0\}$ but $f \neq 0$.

Actually the above theorem settles the case involving general finite Abelian groups.

1.4. COROLLARY. *Let G be a finite Abelian group. A subset $A \subset \mathbf{N}$ is a family of G-uniqueness (i.e. $G/\{0\}$-uniqueness) if and only if each natural number $n \in \mathbf{N}$ admits a decomposition of the form (1.2) with $q := \max_{g \in G} \text{order}(g)$.*

2. mod_q-UNIQUENESS IN TERMS OF EXTENDIBILITY OF GROUP HOMOMORPHISMS

Let $\mathbf{Z}_q := \mathbf{Z}/(q\mathbf{Z})$ be the cyclic group of order q. Thus the elements of $\mathbf{Z}_q$ are the cosets $n + q\mathbf{Z} = \{n + qm : m \in \mathbf{Z}\}$ $(n = 0, 1, \ldots, q-1)$. We shall write mod_q for the canonical map of $\mathbf{Z}$ onto $\mathbf{Z}_q$, i.e.

$$\mathrm{mod}_q(n) := n + q\mathbf{Z} \qquad (n \in \mathbf{Z}).$$

We shall view $\mathbf{Q}_+$ as multiplicative group generated by $\mathbf{N}$. The family

$$\mathbf{Q}_+^q := \{n^q/m^q : n, m \in \mathbf{N}\}$$

is a subgroup of $\mathbf{Q}_+$ and hence the family $\mathbf{Q}_+/\mathbf{Q}_+^q$ of all $\mathbf{Q}_+^q$-cosets is also an abelian group in a natural way. We shall write $e^{(q)}$ for the canonical homomorphism

$$e^{(q)} : \tfrac{n}{m} \to \tfrac{n}{m}\mathbf{Q}_+^q. \tag{2.1}$$

2.2. LEMMA. *Given a completely additive funtion $f : \mathbf{N} \to \mathbf{Z}$, there exists a unique homomorphism $\phi : \mathbf{Q}_+/\mathbf{Q}_+^q \to \mathbf{Z}_q$ such that*

$$\mathrm{mod}_q f = \phi \ o \ e^{(q)}. \tag{2.3}$$

Conversely, to every homomorphism $\phi : \mathbf{Q}_+^q \to \mathbf{Z}_q$ there exists some (not necessarily unique) completely additive $f : \mathbf{N} \to \mathbf{N}$ with (2.2).

PROOF. Let f be a completely additive function and suppose ϕ satisfies (2.2). Then necessarily

$$\phi\left(\tfrac{n}{m}\mathbf{Q}_+^q\right) = \mathrm{mod}_q(f(n) - f(m)) \qquad (m, n \in \mathbf{N}). \tag{2.4}$$

This shows the uniqueness of ϕ corresponding to f via (2.3). On the other hand, if $n, n', m, m' \in \mathbf{N}$ then

$$\begin{aligned} \frac{n}{m}\mathbf{Q}_+^q = \frac{n'}{m'}\mathbf{Q}_+^q &\iff \exists k, l \in \mathbf{N} \ \ \frac{nm'}{n'm} = \left(\frac{k}{l}\right)^q, \\ &\iff \exists k, l \in \mathbf{N} \ \ nm'l^q = n'mk^q. \end{aligned}$$

Thus

$$f(n) + f(m') + q \cdot f(l) = f(n') + f(m) + q \cdot f(k)$$

for some $k, l \in \mathbf{N}$ i.e.

$$\mathrm{mod}_q(f(n) - f(m)) = \mathrm{mod}_q(f(n') - f(m')) \quad \text{whenever} \quad e^{(q)}(n/m) = e^{(q)}(n'/m').$$

Therefore we may define a mapping ϕ on $\mathbf{Q}_+/\mathbf{Q}_+^q (= \{\frac{n}{m}\mathbf{Q}_+^q : n, m \in \mathbf{N}\})$ by the requirement (2.4). The obvious additive properties of (2.4) ensure that the mapping ϕ thus defined is a homomorphism $\mathbf{Q}_+/\mathbf{Q}_+^q \to \mathbf{Z}_q$.

For the proof of the converse statement let $(p_1, p_2, \ldots) := (2, 3, 5, 7, 11, \ldots)$ denote the sequence of primes. For each index choose a representant $z_i \in \phi(p_i\mathbf{Q}_+^q)$ $(i = 1, 2, \ldots)$. The uniqueness of the prime factorization in $\mathbf{N}$ implies immediately that the mapping

$$f : \prod_{i=1}^{s} p_i^{r_i} \mapsto \sum_{i=1}^{s} r_i \cdot z_i$$

is a completely additive function on $\mathbf{N}$ satisfying (2.3).

2.5. COROLLARY. *A subset $A \subset \mathbf{N}$ is a* mod_q*-uniqueness family if and only if the trivial homomorphism $\langle e^{(q)}(A)\rangle \to 0$ into $\mathbf{Z}_q$ of the subgroup $\langle e^{(q)}(A)\rangle$ (generated by the set $e^{(q)}(A)$ in $\mathbf{Q}_+/\mathbf{Q}_+^q$) admits only the trivial homomorphic extension to the whole $\mathbf{Q}_+/\mathbf{Q}_+^q$.*

PROOF. Suppose $A \subset \mathbf{N}$ is a family of mod_q-uniqueness and let $\phi : \mathbf{Q}_+/\mathbf{Q}_+^q \to \mathbf{Z}_q$ be a homomorphism vanishing on $\langle e^{(q)}(A)\rangle$. We can find a completely additive function $f : \mathbf{N} \to \mathbf{Z}$ satisfying (2.3). For every $a \in A$ we have $\mathrm{mod}_q f(a) = 0$. Since A is a family of mod_q-uniqueness, we must have $\mathrm{mod}_q f \equiv 0$. We know that ϕ is the only homomorphism $\mathbf{Q}_+/\mathbf{Q}_+^q \to \mathbf{Z}_q$ with (2.3). Since the trivial homomorphism satisfies (2.3), too, it follows $\phi \equiv 0$.

If $A \subset \mathbf{N}$ is not a family of mod_q-uniqueness then we can choose a completely additive function $f : \mathbf{N} \to \mathbf{N}$ such that $\mathrm{mod}_q f(A) = 0$ but $\mathrm{mod}_q f \not\equiv 0$. The corresponding homomorphism $\phi : \mathbf{Q}_+/\mathbf{Q}_+^q \to \mathbf{Z}_q$ satisfying (2.3) is not trivial but it vanishes on $e^{(q}(A)$ and hence also on $\langle e^{(q)}(A)\rangle$.

3. EXTENDIBILITY OF HOMOMORPHISMS INTO FINITE ABELIAN GROUPS

3.1. DEFINITION. Let X and G be Abelian groups. We say that X is *G-injective* if all homomorphisms from subgroups of X into G admit homomorphic extensions to the whole X. The group X is *strongly G-injective* if all homomorphisms from proper subgroups of X into G admit non-trivial homomorphic extensions to the whole X.

In this section we shall be concerned with the description of strong $\mathbf{Z}_q$-injectivity. All the groups considered will be Abelian and we shall use additive notations. We write (k, l) and $[k, l]$ for the greatest common divisor and least common multiple of the numbers $k, l \in \mathbf{N}$, respectively. As usually, in a group G, the order of an element g is $\mathrm{order}(g) := \min\{k \in \mathbf{N} : k > 1, k \cdot g = 0\}$ with the convention $\min\emptyset := \infty$.

3.2. LEMMA. *Every homomorphism $X \to \mathbf{Z}_q$ vanishes on the subgroup*

$$N := \{x \in X : (q, \mathrm{order}(x)) = 1\}.$$

Proof. If $x, y \in \mathbf{N}$ and $(q, n) = (q, m) = 1$ with $n \cdot x = m \cdot y = 0$ then $(q, nm) = 1$ and

$nm \cdot (k \cdot x + l \cdot y) = 0 \quad (k, l \in \mathbf{Z})$. Thus N is a subgroup in X.

Let $\phi : X \to \mathbf{Z}_q$ be a homomorphism. Assume $x \in N, n \cdot x = 0$, $(q, n) = 1$ and let $\phi(x) = m + q\mathbf{Z}$. Then we get

$$q\mathbf{Z} = [0 \text{ in } \mathbf{Z}_q] = \phi(n \cdot x) = nm + q\mathbf{Z} \qquad \text{i.e. } q|nm \Rightarrow q|m.$$

Thus $\phi(x) = \frac{m}{q}q + q\mathbf{Z} = q\mathbf{Z} = 0$ in $\mathbf{Z}_q$.

3.3. COROLLARY. *If $\phi : X \to \mathbf{Z}_q$ is a homomorphism then for some homomorphism $\phi_0 : N \to \mathbf{Z}_q$ we have $\phi = \phi_0 \circ e$ where $e : x \mapsto x + N$ is the canonical map $X \to X/N$.*

3.4. REMARK. The group X/N in Corollary 3.3 consists of elements Y such that order$(Y) = \infty$ or such that order$(Y) < \infty$ and the prime divisiors of order(Y) divide q.

3.5. LEMMA. a) *The group $\mathbf{Z}$ is not $\mathbf{Z}_q$-injective.*

b) *Suppose we have $q = p^n u$, $r = p^m v$ where p is a prime, $m > n \geq 1$ and $(p, u) = (p, v) = 1$. Then $\mathbf{Z}_r$ is not $\mathbf{Z}_q$-injective.*

c) *If $r|q$ then $\mathbf{Z}_r$ is strongly $\mathbf{Z}_q$-injective.*

PROOF. a) Consider the homomorphism $\phi_0 : nq \mapsto n + q\mathbf{Z}$ of the subgroup $q\mathbf{Z}$ of $\mathbf{Z}$ into $\mathbf{Z}_q$. Any homomorphism $\phi : \mathbf{Z} \to \mathbf{Z}_q$ should satisfy $\phi(q) = q \cdot \phi(1) = q\mathbf{Z} = 0$ in $\mathbf{Z}_q$. Thus $q\mathbf{Z} = \phi(q) \neq \phi_0(q) = 1 + q\mathbf{Z}$ i.e. ϕ can not extend ϕ_0.

b) Consider the subgroup

$$X_0 := \{kp^{m-n}v + r\mathbf{Z} : k = 0, 1, \dots p^n - 1\}$$

of $\mathbf{Z}_r$ with the homomorphism

$$\phi_0 : kp^{m-n}v + r\mathbf{Z} \mapsto ku + q\mathbf{Z}$$

of X_0 into $\mathbf{Z}_q$. If ϕ is any homomorphism $\mathbf{Z}_r \to \mathbf{Z}_q$ extending ϕ_0 then $\phi(v + r\mathbf{Z}) = m + q\mathbf{Z}$ for some $m \in \mathbf{N}$ and

$$u + q\mathbf{Z} = \phi_0(p^{m-n}v + r\mathbf{Z}) = \phi(p^{m-n}v + r\mathbf{Z}) = p^{m-n}\phi(v + r\mathbf{Z}) = p^{m-n}m + q\mathbf{Z}$$

that is $u = p^{m-n}m + tq$ for some $t \in \mathbf{Z}$. However, $(p, u) = 1$ while $(p, p^{m-n}m + tq) = p$. This contradiction establishes b).

c) Assume $r|q$, let X_0 be a proper subgroup of $\mathbf{Z}_r$ and let $\phi_0 : X_0 \to \mathbf{Z}_q$ be a homomorphism. The group X_0 is cyclic, its order $r_0|r$ and hence also $r_0|q$. In particular

$$X_0 = \{k\frac{r}{r_0} + r\mathbf{Z} : k = 1, \dots, r_0\}$$

and we can write

$$\phi_0 : k\frac{r}{r_0} + r\mathbf{Z} \mapsto m + q\mathbf{Z} \quad \text{for some } m \in \{1, \ldots, q\}.$$

Then we have

$$q\mathbf{Z} = [0 \text{ in } \mathbf{Z}_q] = \phi_0(r\mathbf{Z}) = r_0 \cdot \phi_0\left(\frac{r}{r_0} + r\mathbf{Z}\right) = r_0 m + q\mathbf{Z}.$$

It follows $q|r_0 m$. Hence $\frac{q}{r_0}|m$ and $\frac{r}{r_0}|m$. Therefore the homomorphism

$$\phi : j + r\mathbf{Z} \mapsto j\frac{m}{r/r_0}$$

is a well-defined homomorphic extension of ϕ_0 from X_0 to $\mathbf{Z}_r = \{j + r\mathbf{Z} : j = 1, \ldots, r\}$.

If the homomorphism ϕ_0 is trivial then $m = q$ in the above construction. In this case $1 \le \frac{m}{r/r_0} = \frac{q}{r/r_0} < q$. Thus is $\phi(1 + r\mathbf{Z}) \ne q\mathbf{Z}$, i.e. the extension ϕ is not trivial.

3.6. LEMMA. *Let X and G be Abelian groups and let Y_1, Y_2 be subgroups of X. Suppose $\psi_1 : Y_1 \to G$ and $\psi_1 : Y_1 \to G$ are homomorphisms coinciding on $Y_1 \cap Y_2$. Then ψ_1 and ψ_2 admit a common homomorphic extension to $Y_1 + Y_2$.*

PROOF. If $y_1, y_1' \in Y_1$ and $y_2, y_2' \in Y_2$ satisfy $y_1 + y_2 = y_1' + y_2'$ then $y_1 - y_1' = y_2 - y_2' \in Y_1 \cap Y_2$ whence $\psi_1(y_1) - \psi_1(y_1') = \psi_2(y_2) - \psi_2(y_2')$ that is $\psi_1(y_1) + \psi_2(y_2) = \psi_1(y_1') + \psi_2(y_2')$. Therefore the mapping

$$\psi(y) := [\psi_1(y_1) + \psi_2(y_2) : y = y_1 + y_2,\ y_1 \in Y_1, y_2 \in Y_2] \qquad (y \in Y_1 + Y_2)$$

is a well defined homomorphic extension of ψ_1 and ψ_2.

3.7. PROPOSITION. *The Abelian group X is strongly $\mathbf{Z}_q$-injective if and only if* $\text{order}(x)|q$ *for all $x \in X$.*

PROOF. *Necessity:* Suppose X is strongly $\mathbf{Z}_q$-injective. Then X is $\mathbf{Z}_q$-injective and hence every subgroup of X is $\mathbf{Z}_q$-injective. In particular for all $x \in X$, the cyclic subgroups $\langle x \rangle$ are $\mathbf{Z}_q$-injective. Now from Lemma 3.5 a) we see that $\text{order}(x) < \infty$ $(x \in X)$. Thus we can write

$$X = X_0 + N$$

where

$$\begin{aligned} X_0 &:= \{x \in X : \forall p \,\text{prime}\, p|\text{order}(x) \Rightarrow p|q\}, \\ N &:= \{x \in X : (q, \text{order}(x)) = 1\}. \end{aligned}$$

(Indeed, if $x \in X$ then we have a decomposition order $(x) = q'n$ where $n = \max\{s : s|\text{order}(x),\ (q, s) = 1\}$. Clearly, $\forall p$ prime $p|q' \Rightarrow p|q$ and $(q', n) = 1$. For some $k, l \in \mathbf{Z}, kn + lq' = 1$. Then $x = k \cdot x_1 + l \cdot x_0$ where $x_0 := n \cdot x$ and $x_1 := q' \cdot x$. Since order(x_0)

$= \text{order}(x)/n = q'$ and $\text{order}(x_1)/q' = n$, we have $x_0 \in X_0, x_1 \in N$ and $x \in \langle x_0 \rangle + \langle x_1 \rangle \subset X_0 + N$.) From Lemma 3.5 b) applied to the $\mathbf{Z}_q$-injective subgroups $\langle x \rangle$ with $x \in X_0$ it follows that $p^m | q$ whenever p is a prime with $p^m | \text{order}(x)$. That is $\text{order}(x)|q$ for $x \in X_0$. On the other hand, by Corollary 3.3, the trivial homomorphism of X_0 extends only trivially to $X = X_0 + N$. Hence $X + X_0$.

Sufficiency: Assume $\text{order}(x)|q$ $(x \in X)$ and let Y_0 be a proper subgroup of X. Consider any element $y_1 \in X$ lying outside Y_0. Let ψ_0 denote the trivial homomorphism of Y_0 into $\mathbf{Z}_q$ and set $Y_1 := \langle y_1 \rangle$. By Lemma 3.5 c), the trivial homomorphism of $Y_1 \cap Y_0$ admits a non-trivial extension $\psi_1 : Y_1 \to \mathbf{Z}_q$. By Lemma 3.6, there exists a homomorphism $\phi_0 : Y_0 + Y_1 \to \mathbf{Z}_q$ with $\phi_0|_{Y_0} = \psi_0$ and $\phi_0|_{Y_1} = \psi_1 \neq 0$. It remains to extend ϕ_0 homomorphically to X. The Zorn lemma establishes the existence of a maximal homomorphic extension $\phi : Y \to \mathbf{Z}_q$ of ϕ_0 where Y is a subgroup of X containing $Y_0 + Y_1$. Suppose $Y \neq X$. Then we can choose an element $y* \in X$ lying outside Y. However, now a similar construction to that of ϕ_0 gives a (non-trivial) homomorphic extension $\phi* : Y + \langle y* \rangle \to \mathbf{Z}_q$ of ϕ contradicting its maximality.

3.8. PROPOSITION. *The Abelian group X is $\mathbf{Z}_q$-injective if and only if* $(\text{order}(x), q^2)|q$ *for all $x \in X$.*

PROOF. *Necessity:* Let X be $\mathbf{Z}_q$-injective and consider any $x \in X$. The cyclic subgroup $\langle x \rangle$ is necessarily also $\mathbf{Z}_q$-injective. From Lemma 3.5 a) b) we deduce that $\text{order}(x) < \infty$ and $p^m|q$ whenever $p^m|\text{order}(x)$ for the prime divisors of q. This latter can be stated equivalently as $(\text{order}(x), q^2)|q$.

Sufficiency: Suppose $(\text{order}(x), q^2)|q$. With the subgroup N introduced in Lemma 3.2, we have $\text{order}(x + N)|q$ in X/N. Thus, by Proposition 3.7, the factor group X/N is (strongly) $\mathbf{Z}_q$-injective. Since any homomorphism $X \to \mathbf{Z}_q$ factorizes through X/N (Corollary 3.3), the $\mathbf{Z}_q$-injective of X follows.

4. PROOF OF THEOREM 1.1. AND COROLLARY 1.4

PROOF OF THEOREM 1.1.

Let $(p_1, p_2, \ldots)$ denote the sequence of primes. Since every positive rational number $R \in \mathbf{Q}_+$ can be written in a unique way in the form

$$R = \prod_{i=1}^{\infty} p_i^{n_i} \qquad n_1, n_2, \ldots \in \mathbf{Z}, \ \lim_{i \to \infty} n_i = 0$$

the multiplicative group of $\mathbf{Q}_+$ is isomorphic to the additive group of $\mathcal{Z} := \{(n_1, n_2, \ldots) : n_i \in \mathbf{Z}, \lim_i n_i = 0\}$ of all integer valued sequences with finite support. Therefore the multiplicative group $\mathbf{Q}_+/\mathbf{Q}_+^q$ is isomorphic to the additive group $\mathcal{Z}_q := \{(n_1 + q\mathbf{Z}, n_2 + q\mathbf{Z}, \ldots) : n_i \in \mathbf{Z}, \lim_i n_i = 0\}$ of all $\mathbf{Z}_q$-valued sequences with finite support. Since the order of every element in $\mathcal{Z}_q$ is obviously a divisor of the number q, the same holds in $\mathbf{Q}_+/\mathbf{Q}_+^q$. Thus, by Proposition 3.7, the multiplicative group $\mathbf{Q}_+/\mathbf{Q}_+^q$ is strongly $\mathbf{Z}_q$-injective. Hence Corollary 2.5 shows that a subset $A \subset \mathbf{N}$ is mod_q-uniqueness family if and only if

$$(4.1) \qquad \langle e^{(q)}(A) \rangle = \mathbf{Q}_+/\mathbf{Q}_+^q$$

with the canonical homomorphism (2.1). The statement (1.2) is an elementary transcription of (4.1).

PROOF OF COROLLARY 1.4.

The finite Abelian group G is the direct sum of (finitely many) cyclic subgroups say $G_1, \ldots, G_t$. The canonical projections $\pi_i : G \to G_i$ are homomorphisms and a homomorphism $\phi : X \to G$ vanishes if and only if $\pi_i \circ \phi = 0$ for $i = 1, \ldots, t$. Therefore a subset $A \subset \mathbf{N}$ is a family of G-uniqueness if and only if A is a family of G_i-uniqueness simultaneously for every index $i = 1, \ldots, t$. By writing q_i for the cardinality of G_i, this means that A is a family of mod_{q_i}-uniqueness for all $i = 1, \ldots, t$.

Let us denote by q the least common multiple of $q_1, \ldots, q_t$. It is well-known that $q = \max_{g \in G} \text{order}(g)$.

Assume first that $A \subset \mathbf{N}$ is a family of mod_{q_i}-uniqueness simultaneously for $i = 1, \ldots, t$. Consider any completely additive function $f : \mathbf{N} \to \mathbf{Z}$ such that $q|f(a)$ for all $a \in A$. Then $q_i|f(a)$ for all $a \in A$ and $i = 1, \ldots, t$. Consequently $q_i|f(n)$ for all $n \in \mathbf{N}$ and $i = 1, \ldots, t$ and hence $q|f(n)$ for all $n \in \mathbf{N}$. That is the set A is a family of mod_q-uniqueness whenever it is a family of G-uniqueness.

Conversely, let A be a family of mod_q-uniqueness. Consider any number $n \in \mathbf{N}$. By (1.2), we can write

$$n = L^q \prod_{j=1}^{s} a_j^{r_j}$$

with some $L \in \mathbf{Q}_+$ and finite sequences $a_1, \ldots, a_s \in A$ and $r_1, \ldots, r_s \in \mathbf{N}$. However, then we have automatically

$$n = (L^{q/q_i})^{q_i} \prod_{j=1}^{a} a_j^{r_j} \qquad (i = 1, \ldots, t)$$

which shows by Theorem 1.1 that A is a mod_{q_i}-uniqueness family for $i = 1, \ldots, t$. Thus A is a family of G-uniqueness whenever it is a family of mod_q-uniqueness.

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