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Algebraically compact elements of JBW*-triples

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Introduction

Compact operators were originally introduced by F. RIESZ in the celebrated article [14] as linear operators on a Hilbert space $\mathscr{H}$ mapping the unit ball into a precompact set (in modern terminology). From the view point of the theory of C*-algebras such a definition has the virtual disadvantage of not being formulated in terms of the operator algebra rather in terms of a representation. As we shall see there are several ways of formulating the compactness of $a\in\mathscr{L}(\mathscr{H})$ in terms of $\mathscr{L}(\mathscr{H})$. E.g. $a\in\mathscr{L}(\mathscr{H})$ is compact iff the mapping $x\mapsto xax$ is weak* continous on $\overline{\mathrm{B}}_1\mathscr{L}(\mathscr{H})$ the closed unit ball of $\mathscr{L}(\mathscr{H})$ or equivalently if it is weak*$\to$weak continuous there. These statements, whose coincidence now is only ocassional and due mainly to the trivial factor structure of $\mathscr{L}(\mathscr{H})$, have a natural interpretation in frames of the complex dynamics of $\overline{\mathrm{B}}_1\mathscr{L}(\mathscr{H})$ and they furnish purely W*-algebraic definitions of compactness involving the predual. For a long time W*-algebras were considered as the most appropriate tools in fundating quantum mechanics. Nowadays, starting from the classical work [9], it seems that their role is taken by the algebraically more involved structures called JBW*-triples (Jordan—Banach triple-product star-algebras) which admit, in contrast, a very simple and natural function theoretic characterization of obviously high physical relevance as being dual Banach spaces whose unit ball is a symmetric domain or equivalently if the reversible complex dynamics on the unit ball is transitive [11], [17], [4]. The mentioned W*-algebraic definitions of compactness extend immediately to the setting of JBW*-triples requiring the weak* (or weak*$\to$weak) continuity of the mapping $x\mapsto\{xax\}$ on the unit ball where $\{abc\}$ denotes the three variable product (for def. see Section 1). Expectedly, as in the classical case, compact elements of JBW*-triples may play an especially important role in physical applications. Recently a

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complete Gelfand—Naimark theory is available for JBW*-triples [6], [7]. Our main purpose in this paper is to characterize various topological-algebraical notions of compactness in terms of factor splitting. It turns out that factor projections of compact elements belong to the atomic part of the space. It maybe somewhat surprising that infinite dimensional spin factors do not admit non-zero compact elements in the suggested sense. On the other hand, the minimal ideal spanned by spin free atomic tripotents can be described in terms of algebraic compactness.

In order to be self-contained and readable for non Jordan algebrists we separate a brief section providing Jordan theoretic background, and we present the core ideas of our considerations mainly by classical operator theoretical means through representations even in cases where a little more elegant unified Jordan theoretic treatment would be available.

1. Jordan—Banach spaces, JBW*-triples, Gelfand—Naimark representation

1.1. Definition. We call a Banach space E a *Jordan-Banach space* (JB-space) if for every $x\in B_1E(:=\{y\in E:\|y\|<1\})$ there exists $\Psi\in\mathrm{Aut}\ B_1E$ $(:=\{$biholomorphisms of $B_1E\})$ such that $\Psi(0)=x$.

By a remarkable theorem of W. Kaup [12], the category of JB-spaces admits an algebraic characterization: it coincides with the category of Jordan-Banach triple product star algebras or briefly JB*-triples axiomatized below:

1.1'. Definition. A Banach space E endowed with a continuous operation $\{\ \}: E^3\ni(x,y,z)\mapsto\{xyz\}$ of 3 variables is a *JB*-triple* if

(J1) *$\{xyz\}$ is symmetric and bilinear in x, z for fixed y and conjugate-linear in y for fixed x, z, with the *-norm property*

$$\|\{xxx\}\| = \|x\|^3 \quad (x\in E);$$

by setting $\delta_a\colon E\ni x\mapsto\{aax\}$ and $\varepsilon_a^\zeta:=\exp(\zeta\delta_a)$ $\left(=\sum_{n=0}^{\infty}(\zeta^n/n!)\delta_a^n\right)$,

(J2) *for $a\in E$, $\tau\in\mathbf{R}$ the operations $\varepsilon_a^{i\tau}$ are triple product isomorphisms i.e.*

$$\varepsilon_a^{i\tau}\{xyz\} = \{(\varepsilon_a^{i\tau}x)(\varepsilon_a^{i\tau}y)(\varepsilon_a^{i\tau}z)\} \qquad (x,y,z\in E),$$

(J3) $$\|\varepsilon_a^\zeta\| \leqq 1 \quad \textit{whenever} \quad \mathrm{Re}\,\zeta\leqq 0 \quad \textit{and} \quad a\in E.$$

1.2. Remark. The continuity of the triple product takes care of the well-definedness of ε_a^ζ. (J2) is equivalent to the algebraic relationship

(J2') $$i\delta_a\{xyz\} = \{(i\delta_a x)yz\}+\{x(i\delta_a y)z\}+\{xy(i\delta_a z)\}$$

i.e. each $i\delta_a$ is a derivation of the triple product. Hence by polarization we obtain

(J2″) $$\{a_1a_2\{xyz\}\} = \{\{a_1a_2x\}yz\} - \{x\{a_2a_1y\}z\} + \{xy\{a_1a_2z\}\}.$$

Axiom (J3) can be interpreted as follows (cf. [8]).

(J3′) *for each* $a\in E$, δ_a *is a positive E-hermitian operator i.e.* $\exp(i\tau\delta_a)$ *is a surjective linear isometry of E for all* $\tau\in\mathbf{R}$ *and* $Sp(\delta_a)\subset\mathbf{R}_+$.

It can also be shown [12], [3] but it is fairly not immediate that

$$\|\delta_a\| = \|a\|^2, \quad \|\{x, y, z\}\| \leqq \|x\|\cdot\|y\|\cdot\|z\| \qquad (a, x, y, z\in E).$$

Finally we note that the triple product is uniquely determined by the metric of E (cf. [12]).

1.3. Example. JB*-triples are natural generalizations of C*-algebras. Indeed, if E is a C*-algebra then the triple product

$$\{xyz\} := \frac{1}{2}xy^*z + \frac{1}{2}zy^*x$$

makes E a JB*-triple. In the sequel we consider C*-algebras always as JB*-triples with this triple product.

1.4. Definition. A JB*-triple E is called a *JBW*-triple* (Jordan—von Neumann triple) if $E\simeq F^*$ i.e. E is isometrically isomorphic to the dual of some Banach space F which is called a *predual* in this case. Any JBW*-triple has a unique *canonical predual* in the following sense [6]: There is a unique subspace denoted by E_* in the dual E^* of E such that for any predual F of E we have $F\simeq E_*$.

Theorem [2]. *The triple product* $\{\ \}$ *in a JBW*-triple E is* $\sigma(E, E_*)$*-continuous in each of its three variables, respectively.*

1.5. For our purposes we need only the following piece of structure theory of JBW*-triples [6], [1].

Let E be a JB*-triple. We shall use the concepts subtriple, ideal, l^∞-direct sum decomposition of E according to general category theory. I.e. a closed subspace F of E is a subtriple of E (denoted by $F\prec E$) if $\{FFF\}\subset F$, F is an ideal in E if $\{EEF\}$, $\{EFE\}$, $\{FEE\}\subset F$; furthermore E is the l^∞-directe sum of a family $\{F_i\colon i\in I\}$ of its subspaces (denoted by $E=\bigoplus_{i\in I} F_i$) if there exist linear projections $\pi_i\colon E\to F_i$ such that the mapping $x\mapsto(\pi_i x\colon i\in I)$ is a surjective isometry of E onto $\bigoplus^\infty_{i\in I} F_i := \{(x_i\colon i\in I)\in \times_{i\in I} F_i\colon \sup_{i\in I}\|x_i\|<\infty\}$ equipped with the norm $\|(x_i\colon i\in I)\| := \sup_{i\in I}\|x_i\|$.

Definition. An element $u \in E$ is called a *tripotent* if $\{uuu\}=u$. A non-zero tripotent $t \in E$ is called an *atom* in E if $\{x: \delta_t x = x\} = \mathbf{C}t$. Two elements $a, b \in E$ are said to be orthogonal (denoted by $a \perp b$) if $\delta_a b = b_b a = 0$. We shall write at $E := \{$*atoms of* $E\}$.

Example. For a Hilbert space $\mathscr{H}$, the tripotents of $\mathscr{L}(\mathscr{H})$ are exactly the partial isometries of $\mathscr{H}$. An operator u is an atom of $\mathscr{L}(\mathscr{H})$ iff for some unit vectors $\underline{e}, \underline{f} \in \mathscr{H}$ we have $u = \underline{e} \otimes \underline{f}^* (: \mathscr{H} \ni \underline{h} \mapsto \langle \underline{h}, \underline{f} \rangle \underline{e})$. For $a, b \in \mathscr{L}(\mathscr{H})$ we have $a \perp b$ iff $\operatorname{ran}(a) \perp \operatorname{ran}(b)$ and $\operatorname{ran}(a^*) \perp \operatorname{ran}(b^*)$ in $\mathscr{H}$. (Proof: Suppose $a \perp b$ in $\mathscr{L}(\mathscr{H})$. Then $0 = 2\{aab\} = aa^*b + ba^*a$ i.e. $aa^*b = -ba^*a$. It follows $(aa^*)^{2k} b = b(a^*a)^{2k}$ $(k = 0, 1, \ldots)$ whence $\varphi(aa^*)b = b\varphi(a^*a)$ for every even continuous function φ. In particular $aa^*b = ba^*a = 0$ with $\varphi(\cdot) := |\cdot|$. Hence the statement is immediate by polarization.)

Theorem [6], [1]. *Let E be a JBW*-triple, E_a the $\sigma(E, E_*)$-closed linear hull of its atoms and let $\mathscr{M}$ denote the family of minimal $\sigma(E, E_*)$-closed ideals of E. Then E_a is an ideal in E and*

$$E = E_a \oplus E_a^{\perp}, \quad E_a = \bigoplus_{F \in \mathscr{M}} F.$$

Each infinite dimensional $F \in \mathscr{M}$ is isometrically isomorphic to some Cartan factor of type 1, 2, 3 *or* 4 *(discussed in Section* 4*) and $E_a^{\perp}$ is isometrically isomorphic to a weak*-operator closed subtriple of some space $\mathscr{L}(\mathscr{H})$ with a Hilbert space $\mathscr{H}$.*

2. Dynamical characterization of algebraic compactness

Throughout the whole work let E denote a JBW*-triple. We shall always write w^*, w, n for the topologies $\sigma(E, E_*)$, $\sigma(E, E^*)$ and norm-topology, respectively, when there is no danger of confusion.

2.1. Definition. For a linear topology τ on E which is finer than w^* and coarser than n (i.e. $w^* \leqq \tau \leqq n$) we say that $a \in E$ is *τ-compact* if the mapping $a^*: x \mapsto \{xax\}$ is $(w^*) \to \tau$ continuous on $\overline{B_1}E$. We write $\operatorname{comp}_\tau E := \{$*$\tau$-compact elements of* $E\}$.

Remark. It follows immediately from axiom (J1) by polarization that we have $a \in \operatorname{comp}_\tau E$ iff the *a-multiplication* $(x, y) \mapsto \{xay\}$ is $(w^*)^2 \to \tau$ continuous on $(\overline{B_1}E)^2$. In particular $a \in \operatorname{comp}_{w^*} E$ iff the a-multiplication is continuous at 0 when restricted to $\overline{B_1}E$, as a consequence of 1.4. Theorem.

Example. Let $\mathscr{H}$ be a Hilbert space, s the strong operator topology on $\mathscr{L}(\mathscr{H})$ and $c_0(\mathscr{H})$ the ideal of compact operators (in classical sense) in $\mathscr{L}(\mathscr{H})$. We have

$a\in c_0(\mathscr{H})$ iff the mapping $\mathscr{H}\ni h\mapsto ah$ is $\sigma(\mathscr{H},\mathscr{H}^*)\to\text{norm}(\mathscr{H})$ continuous. This means that $\text{comp}_s\,\mathscr{L}(\mathscr{H})=c_0(\mathscr{H})$. However, the definition of the topology s involves the underlying vector space $\mathscr{H}$, thus it is not a natural subject for our purposes (in contrast with w^*, w, n).

Lemma. *Let F be a w^*-closed subtriple of E. Then F is also a JBW*-triple and*

(i) *$\sigma(F, F_*)$ coincides with $\sigma(E, E_*)$ on $\overline{B_1}F$,*
(ii) $\text{comp}_\vartheta\, F\supset F\cap\text{comp}_\tau E$ *whenever the topology τ is finer than ϑ on F.*

Proof. In general, if H is a w*-closed subspace of E then the quotient space $E_*/\{\Phi\in E_*:\ \Phi H=0\}$ is a predual for H. Thus F is a JBW*-triple. Moreover, by [6],

$$F_* = \Big\{\Phi\in F^*:\ \sum_{u\in U}|\langle\Phi, u\rangle|<\infty \quad \text{for orthogonal tripotent families } U\Big\}.$$

Applying this also to E, we see that the topology $\sigma(F, F_*)$ is finer than $\sigma(E, E_*)$ on F. However, both of them are compact and Hausdorff on $\overline{B_1}F$ which establishes (i). Hence (ii) is immediate.

2.2. Lemma. *For any admissible τ (i.e. if $w^*\leqq\tau\leqq n$), $\text{comp}_\tau E$ is a norm-closed subtriple of E. Moreover $\{(\text{comp}_\tau E)E(\text{comp}_\tau E)\}\subset\text{comp}_\tau E$.*

Proof. By (J2″), for any fixed $a, c\in\text{comp}_\tau\, E$ and $b\in E$ we have

$$\{x\{abc\}y\} = \{\{bax\}cy\}-\{ba\{xcy\}\}+\{xc\{bay\}\} \qquad (x, y\in E).$$

The mapping $z\mapsto\{baz\}$ is $w^*\to\tau$ continuous and hence, by 2.1. Remark, the summands on the right hand side are all $(w^*)^2\to\tau$ continuous in (x, y). The norm-closedness of $\text{comp}_\tau E$ follows from the fact that if $\text{comp}_\tau E\ni a_n\to a$ then $\lim_{n\to\infty}\max_{x\in\overline{B_1}E}\|\{xa_nx\}-\{xax\}\|=\lim_{n\to\infty}\|a_n-a\|=0$. That is, $a^*|\overline{B_1}E$ (for def. see 2.1.) is the norm-uniform and hence (since $\tau\leqq n$) also the τ-uniform limit of the $w^*\to\tau$ continuous maps $a_n^*|\overline{B_1}E$. This implies the $w^*\to\tau$ continuity of $a^*|\overline{B_1}E$.

Remark. The classical compact operators form an ideal in $\mathscr{L}(\mathscr{H})$ also in the sense $\{\mathscr{L}(\mathscr{H})\mathscr{L}(\mathscr{H})c_0(\mathscr{H})\}$, $\{\mathscr{L}(\mathscr{H})c_0(\mathscr{H})\mathscr{L}(\mathscr{H})\}\subset c_0(\mathscr{H})$. Later we shall see that $\text{comp}_\tau E$ is indeed an ideal in E unless the topology τ has a rather asymmetric behaviour (cf. 5.4.).

2.3. The reversible complex dynamics associated with E is, in pure mathematical terms, the group $\text{Aut}\,B_1E$ of all one-one surjective holomorphic mappings $B_1E\to\to B_1E$ whose inverse is also holomorphic. It is well known [10] that for every $a\in E$, the initial value problem

$$\text{(1)} \qquad \frac{d}{dt}y_x(t) = a-\{y_x(t)ay_x(t)\};\quad y_x(0)=x$$

admits a solution y_x defined on the whole real line whenever $x \in \overline{B_1}E$ and the mappings $\Psi^t: \overline{B_1}E \ni x \mapsto y_x(t)$ form a 1-parameter subgroup of Aut B_1E when restricted to B_1E (the open unit ball). To express the dependence of Ψ^t on $a(\in E)$ we adopt the Lie theoretical notation $\exp[t(a-a^*)] := \Psi^t$.

Lemma. *Given* $a \in E \setminus \{0\}$, *for* $|t| < \dfrac{\pi}{4\|a\|}$ *the mapping* $\exp[t(a-a^*)]$ *is the norm-uniform limit on* $\overline{B_1}E$ *of the series* $\sum_{n=0}^{\infty} t^n a_n$ *of polynomials of the a-multiplication (which is not necessarily associative) defined recursively by*

$$a_0(x) := x, \quad a_1(x) := a - \{xax\}, \quad a_{n+1}(x) := -\frac{1}{n+1} \sum_{k+l=n} \{a_k(x) a a_l(x)\} \quad (n \geqq 1).$$

Proof. For fixed $x \in E$, the function

$$z_x(t) := \sum_{n=0}^{\infty} t^n a_n(x) \qquad \big(|t| < \liminf_{n\to\infty} \|a_n(x)\|^{-1/n}\big)$$

satisfies (1). By setting

$$\alpha_0 := 1, \quad \alpha_1 := 2\|a\|, \quad \alpha_{n+1} := \frac{1}{n+1} \sum_{k+l=n} \alpha_k \|a\| \alpha_l \quad (n \geqq 1),$$

we see by induction that

$$\|a_n(x)\| \leqq \alpha_n \quad (n \geqq 0, \ \|x\| \leqq 1).$$

But the function

$$\alpha(t) := \sum_{n=0}^{\infty} \alpha_n t^n \qquad \big(|t| < \liminf_{n\to\infty} |\alpha_n|^{-1/n}\big)$$

is the solution of the initial value problem

$$\frac{d}{dt}\alpha(t) = \|a\| + \|a\|\alpha(t)^2; \quad \alpha(0) = 1$$

whence $\alpha(t) = \tan\left(\dfrac{\pi}{4} + t\|a\|\right)$ and $\liminf_{n\to\infty} |\alpha_n|^{-1/n} = \pi/4$.

2.4. Theorem. *We have* $a \in \mathrm{comp}_\tau E$ *iff the 1-parameter subgroup of* Aut B_1E *with infinitesimal generator* $B_1E \ni x \mapsto a - \{xax\}$ *consists of* $w^* \to \tau$ *continuous perturbations of* id$(: x \mapsto x)$.

Proof. First let $a \in \mathrm{comp}_\tau E$. By assumption $\tau \geqq w^*$. Obviously the constant mappings and id are $w^* \to w^*$ continuous, and polynomials of the a-multiplication (briefly *a-polynomials*) preserve boundedness. Therefore it follows by induction from

2.3. Lemma and 2.1. Remark that for $n \geqq 0$, $a_{n+1}(\cdot)$ is a $w^* \to \tau$ continuous map on bounded sets. Since the norm-topology on E is finer than τ, again 2.3. Lemma shows that $\exp[t(a-a^*)]-\mathrm{id}=\sum_{n=1}^{\infty} t^n a_n$ is the τ-uniform limit of $w^* \to \tau$ continuous mappings i.e. $\exp[t(a-a^*)]-\mathrm{id}$ is $w^* \to \tau$ continuous on $\overline{\mathrm{B}_1}E$ whenever $\|ta\| < \pi/4$. On the other hand, $\exp[kt(a-a^*)]=\{\exp[t(a-a^*)]\}^k$ $(t \in \mathbf{R};\ k=1, 2, \ldots)$. Thus if for fixed t and suitable $w^* \to \tau$ continuous maps Δ_j we have $\exp[jt(a-a^*)]=\mathrm{id}+\Delta_j$ $(j=1, \ldots, k)$ then also

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$$\exp[(k+1)t(a-a^*)] = (\mathrm{id}+\Delta_1)(\mathrm{id}+\Delta_k) = \mathrm{id}+\Delta_k+\Delta_1(\mathrm{id}+\Delta_k) =$$

$$= \mathrm{id}+\Delta_k+\Delta_1 \circ (w^* \to w^* \text{ cont. map}) = \mathrm{id}+(w^* \to \tau \text{ cont. map}).$$

I.e. $\exp[t(a-a^*)]=\mathrm{id}+(w^* \to \tau$ cont. map) for all $t \in \mathbf{R}$. Conversely, if $\Delta^t := \exp[t(a-a^*)]-\mathrm{id}$ is $w^* \to \tau$ continuous for all $t \in \mathbf{R}$, then by 2.3. Lemma the mapping $\Delta\colon \overline{\mathrm{B}_1}E \ni x \mapsto a-\{xax\}$ is the norm-uniform and hence also τ-uniform limit of $t^{-1}\Delta^t$ for $t \to 0$. I.e. now Δ is $w^* \to \tau$ continuous whence the τ-compactness of a is immediate.

2.5. Lemma. *Let $a \in E$. Then the following statements are equivalent*

(i) $a \in \mathrm{comp}_\tau E$,

(ii) $[\exp(a-a^*)](x) \in \mathrm{comp}_\tau E \quad (x \in \mathrm{comp}_\tau E,\ \|x\| \leqq 1)$,

(iii) $[\exp(a-a^*)](0) \in \mathrm{comp}_\tau E$.

Proof. (i)⇒(ii): the mapping $y \mapsto a-\{yay\}$ takes $\mathrm{comp}_\tau E$ into itself whenever $a \in \mathrm{comp}_\tau E$ (cf. 2.2. Lemma). Thus if $a, x \in \mathrm{comp}_\tau E$ then the maximal solution of the initial value problem 2.2. (1) ranges in $\mathrm{comp}_\tau E$. (ii)⇒(iii) is trivial. (iii)⇒(i): It is well-known (see e.g. [8]) that for all $a \in E$,

$$a = \sum_{n=1}^{\infty} \frac{1}{2n+1} \delta_b^n b \quad \text{where} \quad b := [\exp(a-a^*)](0).$$

Thus if $b \in \mathrm{comp}_\tau E$ then, by 2.2. Lemma, also $a \in \mathrm{comp}_\tau E$.

Remark. If $\mathrm{comp}_\tau E$ is an ideal in E then $a-\{EaE\} \subset \mathrm{comp}_\tau E$ if $a \in \mathrm{comp}_\tau E$. Thus $[\exp(a-a^*)](x)-x \in \mathrm{comp}_\tau E$ for all $x \in \overline{\mathrm{B}_1}E$ whenever $\mathrm{comp}_\tau E$ is an ideal n E and $a \in \mathrm{comp}_\tau E$.

Proposition. *For any $\Psi \in \mathrm{Aut}\, \mathrm{B}_1 E$ the continuous extension of Ψ to $\overline{\mathrm{B}_1}E$ (which always exists) has fixed point whenever $\Psi(0) \in \mathrm{comp}_\tau E$.*

Proof. By Cartan's uniqueness theorem (see e.g. [8]), for some $a \in E$ and a surjective linear isometry Λ of E we have $\Psi=[\exp(a-a^*)]\Lambda|\mathrm{B}_1 E$ (in particular Ψ extends continuously to $\overline{\mathrm{B}_1}E$). If $\Psi(0) \in \mathrm{comp}_\tau E$ then, by 2.5. Lemma (iii),

$\exp(a-a^*)$ is a $w^* \to w^*$ continuous mapping (for $\tau \geqq w^*$). On the other hand, surjective linear isometries of E are $w^* \to w^*$ continuous [6]. Thus the continuous extension of Ψ to the w*-compact $\overline{B_1}E$ is $w^* \to w^*$ continuous if $\Psi(0) \in \operatorname{comp}_\tau E$. Now the statement follows from the classical Schauder—Tychonoff fixed point theorem.

3. The commutative case

Throughout this section let T denote a commutative von Neumann algebra. According to well-known tesults [16] we may fix a locally compact topological space Ω and a Radon measure μ on Ω such that $T \simeq L^\infty(\mu)$ (here $\simeq$ meaning isometric isomorphism) and

$$L^\infty(\mu)_* = \left\{\left[f \mapsto \int \varphi f \, d\mu\right]: \varphi \in L^1(\mu)\right\} \cong L^1(\mu).$$

We say that a subset $S \subset \Omega$ is a *μ-atom* if there is no $S_1 \subset S$ such that $0 < \mu(S_1) < \mu(S)$ and we call the countable (!) disjoint unions of μ-atoms *μ-atomic* sets.

3.1. Theorem. *Let a be a bounded μ-measurable function and define $S_a := \{\omega \in \Omega: a(\omega) \neq 0\}$. Then the following statements are equivalent*

(i) $a \in \operatorname{comp}_{w^*} L^\infty(\mu)$, [2)]

(ii) *$S_a \cap K$ is μ-atomic for $\mu(K) < \infty$ (K μ-measurable).*

Proof. Suppose first (ii) and consider an arbitrarily fixed $\varphi \in L^1(\mu)$. Then by setting $S_\varphi := \{\omega \in \Omega: \varphi(\omega) \neq 0\}$ we have $S_a \cap S_\varphi = \bigcup_{n=1}^{\infty} Z_n$ with suitable disjoint μ-atoms Z_n ($n = 1, 2, \ldots$). By 2.1. Remark we can establish (i) by showing

$$(1) \qquad \int \varphi \bar{a} x_i^2 \, d\mu \to 0 \quad \text{whenever} \quad |x_i| \leqq 1 \quad (i \in I)$$

and

$$\int \psi x_i \, d\mu \to 0 \quad (\psi \in L^1(\mu))$$

since now we have $\{xax\} = \bar{a} x^2$ $(x \in L^\infty(\mu))$.

Let $(x_i: i \in I)$ be a net in $L^\infty(\mu)$ satisfying the hypothesis part of (1). Since Z_n is a μ-atom, the functions φ, x_i, a are μ-almost everywhere constant on Z_n i.e. we may assume that

$$\varphi(\omega) = \beta_n, \quad x_i(\omega) = \xi_{i,n} \quad (i \in I),$$
$$a(\omega) = \alpha_n \quad \text{for} \quad \omega \in Z_n \quad (n = 1, 2, \ldots).$$

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2) More precisely we mean by this as usually that

$$\tilde{a} := \{f \text{ bounded}: f - a = 0 \;\; \mu\text{—a.e.}\} \in \operatorname{comp}_\tau L^\infty(\mu).$$

Let

$$v_n := \beta_n \bar{\alpha}_n \mu(Z_n) \quad (n = 1, 2, \ldots).$$

Observe that

$$\sum_{n=1}^{\infty} |v_n| = \sum_{n=1}^{\infty} |\beta_n \bar{\alpha}_n| \mu(Z_n) = \int_{S_a} |\varphi \bar{a}| \, d\mu < \infty$$

and

$$\int \varphi \bar{a} x_i^2 \, d\mu = \sum_{n=1}^{\infty} \beta_n \bar{\alpha}_n \xi_{i,n}^2 \mu(Z_n) = \sum_{n=1}^{\infty} \xi_{i,n}^2 v_n \quad (i \in I).$$

The assumption

$$\int \psi x_i \, d\mu \to 0 \quad (\psi \in L^1(\mu))$$

means in particular that

$$\lim_{i \in I} \xi_{i,n} = 0 \quad (n = 1, 2, \ldots).$$

It follows

$$\limsup_{i \in I} \Big| \sum_{n=1}^{\infty} \xi_{i,n}^2 v_n \Big| \leqq \limsup_{i \in I} \Big| \sum_{n>N} \xi_{i,n}^2 v_n \Big|$$

for any $N(<\infty)$. However

$$\Big| \sum_{n>N} \xi_{i,n}^2 v_n \Big| \leqq \sum_{n>N} |v_n| \to 0 \quad (N \to \infty).$$

(i)⇒(ii): Assume that $S_a \cap K$ is not μ-atomic for some $K(\subset \Omega)$ of finite μ-measure. Then removing a maximal disjoint system of μ-atoms with positive μ-measures from $S_a \cap K$ we see that there exists $S \subset S_a \cap K$ such that $\mu(S) > 0$ and for any μ-measurable $P \subset S$ we can fix a partition

$$P = P' \cup P'', \quad P' \cap P'' = \emptyset; \quad \mu(P') = \mu(P'') = \frac{1}{2} \mu(P).$$

Now for any finite μ-measurable partition $\Pi = \{P_1, \ldots, P_N\}$ of S (i.e. $S = \bigcup_{k=1}^{N} P_k$; $P_k \cap P_l = \emptyset$ for $k \neq l$ and $P_1, \ldots, P_N$ are μ-measurable) define the function x_Π: $\Omega \to \{0, \pm 1\}$ by

$$x_\Pi := \sum_{k=1}^{N} (1_{(P_k)'} - 1_{(P_k)''})$$

where $1_P \colon \Omega \ni \omega \mapsto [1 \text{ if } \omega \in P, \ 0 \text{ else}]$ $(P \subset \Omega)$. If P is any μ-measurable set and Π is finer than $\{S \setminus P, S \cap P\}$ then then $\int x_\Pi 1_P \, d\mu = 0$. It follows

$$\lim_{\Pi} \int x_\Pi \varphi \, d\mu = 0 \qquad (\varphi \in L^1(\mu))$$

with the usual ordering of partitions (i.e. $\Pi_1 \geqq \Pi_2$ if for each $P_1 \in \Pi_1$ there is $P_2 \subset \Pi_2$ with $P_1 \subset P_2$). That is

$$x_\Pi \to 0 \quad w*$$

in $\overline{B_1} L^\infty(\mu)$. However, $\{x_\Pi a x_\Pi\} = x_\Pi^2 \bar{a} = 1_S \bar{a} \nrightarrow 0$ whence $a \notin \mathrm{comp}_{w*} L^\infty(\mu)$

3.2. Corollary. *We have*[3]

$$\mathrm{comp}_{w*} T = w^*\!-\mathrm{Span}\,(\mathrm{at}\, T),$$

moreover any $a \in \mathrm{comp}_{w*} T$ *can be written as*

$$a = w^*\!-\sum_{u \in \mathrm{at}_+ T} \alpha(u) u$$

where $\mathrm{at}_+ T := \{u \in \mathrm{at}\, T\colon u \geqq 0\}$, $\alpha(u)u := \delta_u a \quad (u \in \mathrm{at}_+ T)$.

Proof. This is nothing but an abstract reformulation of 3.1. Theorem. Namely, let Φ be an isometric C*-isomorphism between $L^\infty(\mu)$ and T. Then $\Phi\{1_S\colon S$ is a μ-atom$\} = \mathrm{at}_+ T$ and $\Phi\, \mathrm{comp}_{w*} L^\infty(\mu) = \mathrm{comp}_{w*} T$. Furthermore property (ii) can be reformulated as: $a \in L^\infty(\mu)$ is the w*-limit of the net ($1_K a$: K is a finite union of μ-atoms) and here the function $1_K a$ differs only on a set of μ-measure 0 from $\sum_{S\ \mu\text{-atom} \subset K} \alpha'(S) 1_S$ where $\alpha'(S) := \int_S a d\mu \cdot \mu(S)^{-1}$.

3.3. Corollary. *If T is a commutative von Neumann subalgebra of $\mathscr{L}(\mathscr{H})$ for a Hilbert space $\mathscr{H}$ and $a \in \mathrm{comp}_\tau T$ then $\mathscr{H}$ is spanned by the eigenvectors of the operator a.*

Proof. Let again Φ be an isometric C*-isomorphism between $L^\infty(\mu)$ and T and suppose $a \in \mathrm{comp}_\tau T$. Then $\Phi^{-1} a \in \mathrm{comp}_{w*} L^\infty(\mu)$ and so for $f := \Phi^{-1} a$ we have $f = w^*\!-\sum_{\gamma \in \Gamma} \gamma \cdot 1_\gamma$ where $S(\gamma) := f^{-1}\{\gamma\}$ $(\gamma \in \Gamma)$ and $\Gamma := \{\gamma \in \mathbf{C} \setminus \{0\}\colon \mu(S(\gamma)) > 0\}$. The operators $p_\gamma := \Phi(1_{S(\gamma)})$ $(\gamma \in \Gamma)$ are pairwise orthogonal projections and we have the spectral descomposition $a = w^*\!-\sum_{\gamma \in \Gamma} \gamma \cdot p_\gamma$ because the w*- and weak operator topologies coincide on bounded subsets of T.

4. w*-compactness in Cartan factors

On the basis of 3.3. we can easily describe the w^*-compact elements of Cartan factors of type 1, 2, 3, 4 which is a fundamental task for us in view of 1.5. Theorem. First we recall their definition (cf. [8]).

[3] The symbol $w^*\!-\Sigma$ stands for the w*-limit of finite partial sums.

4.1. Definition. The Cartan factors of *type 1* are the spaces

$$\mathscr{L}(\mathscr{H}, \mathscr{H}_1) \quad \text{with Hilbert spaces} \quad \mathscr{H}_1 \subset \mathscr{H}$$

considered as subtriples of $\mathscr{L}(\mathscr{H})$ (see 1.3.).

The Cartan factors of *type 2* resp. *type 3* are the spaces

$$\mathscr{L}_{\mathscr{B}}^{+}(\mathscr{H}) := \{x \in \mathscr{L}(\mathscr{H}) \colon \langle x\underline{e}, \underline{f}\rangle = \langle x\underline{f}, \underline{e}\rangle \ \ (\underline{e}, \underline{f} \in \mathscr{B})\},$$

$$\mathscr{L}_{\mathscr{B}}^{-}(\mathscr{H}) := \{x \in \mathscr{L}(\mathscr{H}) \colon \langle x\underline{e}, \underline{f}\rangle = -\langle x\underline{f}, \underline{e}\rangle \ \ (\underline{e}, \underline{f} \in \mathscr{B})\}$$

with a Hilbert space $\mathscr{H}$ and orthonormed basis $\mathscr{B}$ in $\mathscr{H}$ (considered as subtriples of $\mathscr{L}(\mathscr{H})$).

The Cartan factors of *type 4* (or *spin factors* in other terminology) are the JB*-triples $\mathscr{H}_{\mathscr{B}}^{-}$ whose carrier space is a Hilbert space $\mathscr{H}$ endowed with a triple product $\{\ \}$ defined by the aid of an orthonormed basis $\mathscr{B}$ of $\mathscr{H}$ as follows.

$$2\{\underline{abx}\} := \langle \underline{a}, \underline{b}\rangle \underline{x} + \langle \underline{x}, \underline{b}\rangle \underline{a} - \langle \underline{a}, \bar{\underline{x}}^{\mathscr{B}}\rangle \bar{\underline{b}}^{\mathscr{B}} \quad (\underline{a}, \underline{b}, \underline{x} \in \mathscr{H})$$

where $\bar{\underline{x}}^{\mathscr{B}} := \sum_{\underline{e} \in \mathscr{B}} \langle \underline{e}, \underline{y}\rangle \underline{e} \quad (\underline{y} \in \mathscr{H})$.

Remark. (i) Type 1 factors are defined in most works as $\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ spaces with arbitrary Hilbert spaces $\mathscr{H}_1, \mathscr{H}_2$. However, by setting $\varkappa_j := \dim \mathscr{H}_j$ $(j=1, 2)$ we have $\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2) \simeq \mathscr{L}(\mathscr{H}_2, \mathscr{H}_1) \simeq \mathscr{L}(l^2(\max\{\varkappa_1, \varkappa_2\}), l^2(\min\{K_1, K_2\}))$,

(ii) The operation $-^{\mathscr{B}}$ is a conjugate linear involution. It is called the $\mathscr{B}$-conjugation. The norm in $\mathscr{H}_{\mathscr{B}}$ is defined as

$$\|\underline{x}\| := [\langle \underline{x}, \underline{x}\rangle + (\langle \underline{x}, \underline{x}\rangle^2 - |\langle \underline{x}, \bar{\underline{x}}^{\mathscr{B}}\rangle|^2)^{1/2}]^{1/2} \quad (\underline{x} \in \mathscr{H}).$$

(iii) The w^*-topology in a factor of type $\leqq 3$ coincides with the weak operator topology (abbreviated by *wop*) [6].

4.2. Throughout this subsection let $F_1 := \mathscr{L}(\mathscr{H}, \mathscr{H}_1)$, $F_2 := \mathscr{L}_{\mathscr{B}}^{+}(\mathscr{H})$, $F_3 := \mathscr{L}_{B}^{-}(\mathscr{H})$ be Cartan factors of type 1, 2, 3 in $\mathscr{L}(\mathscr{H})$, respectively.

Lemma. *Given* $a \in F_k$ *and two nets* $\overline{\mathrm{B}}_1\mathscr{H} \ni \underline{h}_i, \underline{f}_i \to 0$ $\sigma(\mathscr{H}, \mathscr{H}^*)$ *we can find a bounded net*

$$F_k \ni x_i \to 0 \quad w^*$$

and a unit vector $\underline{e} \in \mathscr{H}$ *such that*

$$\langle \{x_i a x_i\} \underline{e}, \underline{e}\rangle - \langle \underline{h}_i, a\underline{f}_i\rangle \to 0 \quad (k = 1, 2, 3).$$

Proof. For $x \in F_k$, $\underline{e} \in \mathscr{H}$ we have $\langle \{xax\}\underline{e}, \underline{e}\rangle = \langle x\underline{e}, a(x^*\underline{e})\rangle$.

(i) $k=1$. Let p denote the orthogonal projection of $\mathscr{H}$ onto $\mathscr{H}_1$. Then $\langle \underline{h}_i, a\underline{f}_i\rangle = \langle \mathrm{p}\underline{h}_i, a\underline{f}_i\rangle$ for all indices. Fix any $\underline{e} \in \mathscr{H}_1$ with $\|\underline{e}\| = 1$. Now the net

$$x_i := (\mathrm{p}\underline{h}_i) \otimes \underline{e}^* + \underline{e} \otimes \underline{f}^*$$

satisfies

$$\|x_i \underline{e} - p\underline{h}_i\| = \|\langle \underline{e}, \underline{f}_i \rangle \underline{e}\| \to 0,$$

$$\|x_i^* \underline{e} - \underline{f}_i\| = \|\langle \underline{e}, p\underline{h}_i \rangle \underline{e}\| \to 0,$$

$$2\overline{B}_1 F_1 \ni x_i \to 0 \quad w^*$$

(cf. Remark (iii)).

(ii) $k=2$. We write simply $^-$ instead of $^{-\mathscr{B}}$ and choose $\underline{e} \in \mathscr{H}$ such that

$$\|\underline{e}\| = 1, \quad \langle \bar{\underline{e}}, \underline{e} \rangle = 0$$

(this is possible if $\dim \mathscr{H} > 1$). Now the net

$$x_i := \underline{h}_i \otimes \underline{e}^* + \bar{\underline{e}} \otimes \bar{\underline{h}}_i^* + \underline{e} \otimes \underline{f}_i^* + \bar{\underline{f}}_i \otimes \bar{\underline{e}}^*$$

satisfies

$$4\overline{B}_1 F_1 \ni x_i \to 0 \quad w^*$$

and

$$\|x_i \underline{e} - \underline{h}_i\| = \|\langle \underline{e}, \underline{f}_i \rangle \underline{e} + \langle \underline{e}, \underline{h}_i \rangle \underline{e}\| \to 0,$$

$$\|x_i^* \underline{e} - \underline{f}_i\| = \|\langle \underline{e}, \underline{h}_i \rangle \underline{e} + \langle \underline{e}, \bar{\underline{f}}_i \rangle \bar{\underline{e}}\| \to 0.$$

(iii) $k=3$. The same construction applies as in (ii) with

$$x_i := \underline{h}_i \otimes \underline{e}^* - \bar{\underline{e}} \otimes \bar{\underline{h}}_i^* + \underline{e} \otimes \underline{f}_i^* - \bar{\underline{f}}_i \otimes \bar{\underline{e}}^*.$$

Proposition. *We have* $\mathrm{comp}_{w^*} F_k = F_k \cap c_0(\mathscr{H})$ $(k=1, 2, 3)$[4].

Proof. By 2.1. Remark and 4.1. Remark (iii), we have

$$\mathrm{comp}_{w^*} F_k = \{a \in F_k \colon \langle \{x_i a x_i\} \underline{e}, \underline{e} \rangle \to 0 \ (\underline{e} \in \mathscr{H}) \ \text{ whenever } \ \overline{B}_1 F_k \ni x_i \to 0 \ w^*\}.$$

Now applying 4.2. Lemma twice we see

$$\mathrm{comp}_{w^*} F_k = \{a \in F_k \colon \langle \underline{h}_i, a \underline{f}_i \rangle \to 0 \ \text{ if } \ \overline{B}_1 \mathscr{H} \ni \underline{h}_i, \underline{f}_i \to 0 \ \sigma(\mathscr{H}, \mathscr{H}^*)\} =$$
$$= F_k \cap \mathrm{comp}_{w^*} \mathscr{L}(\mathscr{H}) \quad (k = 1, 2, 3).$$

We establish that $\mathrm{comp}_{w^*} \mathscr{L}(\mathscr{H}) = c_0(\mathscr{H})$ as follows:

For any $\underline{e}, \underline{f}, \underline{h} \in \mathscr{H}$ we have

$$\langle \{x_i (\underline{e} \otimes \underline{e}^*) x_i\} \underline{f}, \underline{h} \rangle = \langle (x_i \underline{e}) \otimes (x_i^* \underline{e})^* \underline{f}, \underline{h} \rangle = \langle x_i \underline{e}, \underline{h} \rangle \langle \underline{f}, x_i^* \underline{e} \rangle \to 0$$

if $\overline{B}_1 \mathscr{L}(\mathscr{H}) \ni x_i \to 0 \ w^*$. Thus by 2.2. Lemma,

$$c_0(\mathscr{H}) = \mathrm{Span}\, \{\underline{e} \otimes \underline{e}^* \colon \underline{e} \in \mathscr{H}\} \subset \mathrm{comp}_{w^*} \mathscr{L}(\mathscr{H}).$$

Conversely, let $a \in \mathrm{comp}_{w^*} \mathscr{L}(\mathscr{H})$. Since $\{x a^* x\} = \{x^* a x^*\}^*$ $(x \in \mathscr{L}(\mathscr{H}))$ and $\overline{B}_1 \mathscr{L}(\mathscr{H}) \ni x_i \to 0 w^*$ iff $\overline{B}^1 \mathscr{L}(\mathscr{H}) \ni x_i^* \to 0 w^*$, we have also $a^* \in \mathrm{comp}_{w^*} \mathscr{L}(\mathscr{H})$.

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[4]) Here we write $c_0(\mathscr{H}) := \{\textit{compact operators}\ \mathscr{H} \to \mathscr{H}\}$.

Again by 2.2. Lemma, for $b:=1/2aa^*+1/2a^*a$ we have $b=\{a1a^*\}\in\mathrm{comp}_{w^*}\mathcal{L}(\mathcal{H})$. Consider the w*-closed linear hull T of $\{b^n\colon n=0,1,\ldots\}$. T is a commutative von Neumann subalgebra and hence a JBW*-subtriple of $\mathcal{L}(\mathcal{H})$. By 2.1. Lemma $b\in\mathrm{comp}_{w^*}T$ and so by 3.3. Corollary we can find an orthonormed basis $\mathcal{B}$ in $\mathcal{H}$ and $\lambda\colon\mathcal{B}\to\mathbf{R}_+$ such that

$$b\underline{e}=\lambda(\underline{e})\underline{e}\quad(\underline{e}\in\mathcal{B}).$$

Assume now that $b\notin c_0(\mathcal{H})$. Then there is $\lambda>0$ and an infinite sequence $\underline{e}_1,\underline{e}_2,\ldots$ $\ldots\in\mathcal{B}$ with $\lambda(\underline{e}_n)\to\lambda$. But then

$$t_n:=\underline{e}_1\otimes\underline{e}_n^*+\underline{e}_n\otimes\underline{e}_1^*\to 0\quad w^*$$

in $\overline{\mathrm{B}}_1\mathcal{L}(\mathcal{H})$ while

$$\{t_nbt_n\}=\lambda(\underline{e}_1)\underline{e}_n\otimes\underline{e}_n^*+\lambda(\underline{e}_n)\underline{e}_1\otimes\underline{e}_1^*\to\lambda\underline{e}_1\otimes\underline{e}_1^*\neq 0\quad w^*,$$

contradicting $b\in\mathrm{comp}_{w^*}\mathcal{L}(\mathcal{H})$. Hence $b\in c_0(\mathcal{H})$ and therefore also $a\in c_0(\mathcal{H})$.

4.3. Proposition. *In infinite dimensional spin factors F, we have* $\mathrm{comp}_{w^*}F=0$.

Proof. According to 4.1. Definition, we may consider F as the Hilbert space $\mathcal{H}$ endowed with the triple product

$$2\{\underline{x}\underline{y}\underline{z}\}=\langle\underline{x},\underline{y}\rangle\underline{z}+\langle\underline{z},\underline{y}\rangle\underline{x}-\langle\underline{x},\bar{\underline{z}}\rangle\bar{\underline{y}}$$

where we write simply $^-$ for the $\mathcal{B}$-conjugation $^{-\mathcal{B}}$. Since the conjugation $^-$ is a surjective real-linear isometry of F, it is necessarily $w^*\to w^*$ continuous. Hence $\underline{a}\in\mathrm{comp}_{w^*}F$ iff $\bar{\underline{a}}\in\mathrm{comp}_{w^*}F$. I.e.

$$\mathrm{comp}_{w^*}F=\mathrm{Span}\,\{\underline{a}\in\mathrm{comp}_{w^*}F\colon\ \underline{a}=\bar{\underline{a}}\}.$$

Suppose $\underline{a}=\bar{\underline{a}}\in\mathrm{comp}_{w^*}F$. Then $\mathcal{H}\ominus(\mathbf{C}\underline{a})=[\mathcal{H}\ominus(\mathbf{C}a)]^-$ and so we can choose an orthonormed sequence $\{\underline{x}_1,\underline{x}_2,\ldots\}$ such that $\underline{x}_n=\bar{\underline{x}}_n\perp\underline{a}$ $(n=1,2,\ldots)$. Since the norm of F is equivalent to the norm of $\mathcal{H}$ (cf. 4.1. Remark (ii)), we have $\underline{x}_n\to 0\ w^*$ because the topologies w* and $\sigma(\mathcal{H},\mathcal{H}^*)$ coincide so in $\overline{\mathrm{B}_1}\mathcal{H}$. On the other hand $-1/2\underline{a}=\{\underline{x}_n\underline{a}\underline{x}_n\}\to 0$ whence $\underline{a}=0$.

5. Main results

In accordance with 1.5. Theorem, we shall consider the decomposition $E_a^\perp\oplus\bigoplus_{F\in\mathcal{M}}F$ of E into the l^∞-direct sums of its continuous part and minimal w^*-closed ideals, respectively, and we shall write π_c,π_F $(F\in\mathcal{M})$ for the corresponding factor projections. Note that

$$x=\Big(w^*\!-\sum_{F\in\mathcal{M}}\pi_Fx\Big)+\pi_cx\qquad(x\in E).\tag{1}$$

5.1. Theorem. *If* $0 \neq a \in \mathrm{comp}_\tau E$ then

$$a = w^* - \sum_{i \in I} \alpha_i t_i$$

for some orthogonal family $\{t_i : i \in I\}$ *of* τ*-compact atoms of* E *and constants* $0 < \alpha_i \leqq \|a\|$ $(i \in I)$.

Proof. Let $a \in \mathrm{comp}_\tau E$. Remark that if we have $a = w^* - \sum_{i \in I} \alpha_i t_i$ for an orthogonal family of atoms and suitable positive constants then, by 1.2., necessarily

$$\alpha_i = \|\alpha_i t_i\| = \|\{t_i t_i a\}\| \leqq \|t_i\|^2 \|a\| = \|a\|,$$

and from 2.2. Lemma it follows also

$$t_i = \alpha_i^{-2} \{a t_i\, a\} \in \mathrm{comp}_\tau E \quad (i \in I).$$

Observe furthermore that $a \in \mathrm{comp}_{w^*} E$ and hence

$$\pi_c a \in \mathrm{comp}_{w^*} E_a^\perp, \quad \pi_F a \in \mathrm{comp}_{w^*} F \quad (F \in \mathcal{M}).$$

Thus taking into account the factorization (1) and that $\mathrm{at}\, E = \lambda \cdot \mathrm{at}\, E$ $(|\lambda| = 1)$, we may complete the proof by showing that $a = w^* - \sum_{i \in J} \alpha_i' t_i'$ for some orthogonal family $\{t_i' : i \in J\} \subset \mathrm{at}\, F$ and constants $\alpha_i' \in \mathbf{C}$ $(i \in I)$ whenever $a \in \mathrm{comp}_{w^*} F$ where $F = E_a^\perp$ or $F \in \mathcal{M}$. (In particular $a = 0$ if $F = E_a^\perp$). If $F \in \mathcal{M}$ is isometrically isomorphic to some finite dimensional Cartan factor then the existence of such an atomic decomposition is well known [13]. In any other case we may assume without loss of generality that F is a weak operator closed subtriple of $\mathcal{L}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ (cf. 1.5. Theorem, 4.3. Proposition, 4.1. Definition). Thus let F be a *wop*-closed subtriple of $\mathcal{L}(\mathcal{H})$ and $a \in \mathrm{comp}_{w^*} F$. Applying polar decomposition, we can write

$$a = u|a|$$

where $|a| := (a^* a)^{1/2}$ and $u \in \mathcal{L}(\mathcal{H})$ is a suitable partial isometry of $\mathcal{H}$ such that

$$\mathrm{ran}\, u = \overline{\mathrm{ran}}\, a, \quad \mathrm{ran}\, u^* = \overline{\mathrm{ran}}\, a^* = \overline{\mathrm{ran}}\, |a|.$$

Observe that

$$u|a|^{2n+1} = \delta_a^n a \in F \qquad (n = 0, 1, 2, \ldots).$$

Hence by choosing a sequence of odd polynomials

$$\varphi_1 \leqq \varphi_2 \ldots \nearrow 1 \quad \text{on} \quad (0, \|a\|]$$

we see that

$$u = wop - \lim_{n \to \infty} \varphi_n(\delta_a) a \in F.$$

Consider now the *wop*-closed subtriple

$$U := \{x \in F : \mathrm{ran}\, x \subset \mathcal{H}_1, \ \mathrm{ran}\, x^* \subset \mathcal{H}_0\},$$

where

$$\mathscr{H}_1 := \operatorname{ran} u, \quad \mathscr{H}_0 := \operatorname{ran} u^*.$$

Clearly the mapping

$$\Psi : U \ni x \mapsto u^* x | \mathscr{H}_0$$

is an isometry from U into $\mathscr{L}(\mathscr{H}_0)$ such that

$$\Psi\{xyz\} = \{(\Psi x)(\Psi y)(\Psi z)\} \quad (x, y, z \in U).$$

Hence its range $\tilde{U}(:=\Psi U)$ is a *wop*-closed subtriple of $\mathscr{L}(\mathscr{H}_0)$. Moreover $1(:=id_{\mathscr{H}_0}) = \Psi u \in \tilde{U}$ and therefore

$$\tilde{x}^* = \{1\tilde{x}1\} \in \tilde{U}, \quad \tilde{x}\tilde{y} + \tilde{y}\tilde{x} = 2\{\tilde{x}1\tilde{y}\} \in U \quad (\tilde{x}, \tilde{y} \in \tilde{U}).$$

Let T be a maximal commutative subfamily of normal operators in $\tilde{U}$ such that

$$|a| = \Psi a \in T.$$

The existence of T is immediate from the Zorn lemma. It also easily follows from the previous remarks that T is a *wop*-closed commutative C*-subalgebra (i.e. von Neumann subalgebra) of $\mathscr{L}(\mathscr{H}_0)$. Since $a \in \operatorname{comp}_{w^*} F$, by 2.1. Lemma also $a \in \operatorname{comp}_{w^*} U$ whence $|a| = \Psi a \in \operatorname{comp}_{w^*} \tilde{U}$ and so

$$0 \neq |a| \in \operatorname{comp}_{w^*} T.$$

Now it follows from 3.2. Corollary that

$$a = \Psi^{-1}|a| = w^*\text{-}\sum_{t \in \mathrm{at}_+ T} \alpha(t) \Psi^{-1} t.$$

Let

$$t \in \mathrm{at}_+ T \quad \text{and} \quad v := \Psi^{-1} t \quad \text{i.e.} \quad v = utp_0$$

where p_0 denotes the orthogonal projection of $\mathscr{H}$ onto $\mathscr{H}_0$. To complete the proof we show that $v \in \mathrm{at}\, F$. Suppose indirectly that there exists $g \in F$ such that $\delta_v g = g \notin \mathbf{C}v$. Necessarily

$$\operatorname{ran} g \subset \operatorname{ran} v \subset \mathscr{H}_1, \quad \operatorname{ran} g^* \subset \operatorname{ran} v^* \subset \mathscr{H}_0,$$

i.e. $g \in U$. Thus we may define $f := \Psi g$. Since $t = \Psi v$ and Ψ is an isomorphism,

$$\delta_t f = f \notin \mathbf{C}t.$$

However, $t \in \mathrm{at}\, T$ and so $f \notin T$. Since t is a positive minimal partial isometry, t is a projection and we have $\delta_t h = h$ iff $\operatorname{ran} h$, $\operatorname{ran} h^* \subset \operatorname{ran} t$ or equivalently iff $h = th = ht$ $(h \in \mathscr{L}(\mathscr{H}_0))$. It follows that also $\delta_t f^* = f^*$ and therefore one of the self-adjoint operators $2\operatorname{Re} f(:= f + f^*)$, $2\operatorname{Im} f(:= i(f^* - f))$ does not belong to T. I.e. there exists

$$h \in \tilde{U} \setminus T, \quad h = h^* = \delta_t h.$$

However, if $z\in T$ then for some $\zeta\in\mathbf{C}$,

$$\zeta t=\delta_t z=\frac{1}{2}tt^*z+\frac{1}{2}zt^*t=tz=zt$$

whence

$$hz=(ht)z=h(tz)=\zeta ht=\zeta h=\zeta th=(zt)h=z(th)=zh.$$

This contradicts the maximality of T.

Corollary. *If* $a\in\operatorname{comp}_\tau E$ *then* $\pi_c a=0$ *and* $\pi_F a=0$ *for infinite dimensional spin factors* $F\in\mathcal{M}$.

Corollary. *If* $a\in\operatorname{comp}_\tau E$ *and* $a=w^*-\sum_{i\in I}\alpha_i t_i$ *with some orthogonal family* $\{t_i\colon i\in I\}\subset\operatorname{at}E$ *then we have*

$$w^*-\sum_{i\in I}\beta_i t_i\in\operatorname{comp}_\tau E$$

whenever $\sup_{i\in I}|\beta_i|<\infty$ *and for every* $\varepsilon>0$ *there exists* $\delta>0$ *with*

$$\{i\in I\colon |\beta_i|\geqq\varepsilon\}\subset\{i\in I\colon |\alpha_i|\geqq\delta\}.$$

Proof. Let $\mathcal{S}:=\{J\subset I\colon \inf_{i\in J}|\alpha_i|>0\}$. Then

$$z_J:=w^*-\sum_{i\in J}t_i,\qquad b_J:=w^*-\sum_{i\in J}\alpha_i^{-2}t_i$$

are well-defined elements of E for $J\in\mathcal{S}$ (since the coefficients are bounded; cf. [6]). By 2.2. Lemma,

$$z_J=\{ab_J a\}\in\operatorname{comp}_\tau E\quad (J\in\mathcal{S})$$

and

$$\operatorname{comp}_\tau E\supset\operatorname{Span}\{z_J\colon J\in\mathcal{S}\}=\Big\{w^*-\sum_{i\in I}\beta_i t_i\colon \inf_{i\in J}|\beta_i|>0\ \ (J\in\mathcal{S})\Big\}$$

since in general we have (for $w^*-\operatorname{Span}\{t_i\colon i\in I\}\simeq w^*-\operatorname{Span}\operatorname{at}T\simeq l^\infty(I)$)

(2) $$\Big\|w^*-\sum_{i\in I}\gamma_i t_i\Big\|=\sup_{i\in I}|\gamma_i|.$$

5.2. The above spectral theorem yields the following improvement of 4.2. Proposition.

Theorem. *Let* F *be a Cartan factor of type* $\leqq 3$. *Then* $a\in\operatorname{comp}_\tau F$ *iff for some sequence* $\alpha_n\downarrow 0$ *and an orthogonal family of atoms* $\{t_1,t_2,\dots\}\subset\operatorname{comp}_\tau F$ *we have* $a=\sum_{n=1}^{\infty}\alpha_n t_n$ *(the sum coverging in norm). In particular*

$$\operatorname{comp}_\tau F=\operatorname{Span}[(\operatorname{at}F)\cap\operatorname{comp}_\tau F].$$

Proof. Since $\mathrm{comp}_\tau F$ is norm-closed and at $F \subset c_0(\mathscr{H})$, the sufficiency is clear (namely, $\dim \mathrm{ran}\, t \leqq 2$ if $t \in \mathrm{at}\, F$; [8]).

Necessity: Let $a \in \mathrm{comp}_\tau F$, $a = w^* - \sum_{i \in I} \alpha_i t_i$ as in 5.1. Theorem. From 4.2. Proposition we know that $a \in c_0(\mathscr{H})$ on the underlying Hilbert space $\mathscr{H}$. But by 1.5. Example, if for some infinite subsequence $\{i_1, i_2, \ldots\}$ of indices we have $\inf\{\alpha_{i_n}: n = 1, 2, \ldots\} > 0$ then $w^* - \sum_{i \in I} \alpha_i t_i$ cannot be a compact operator since the t_i-s are pairwise orthogonal partial isometries. Now the relation $w^* - \sum_{i \in I} \alpha_i t_i = \sum_{i \in I} \alpha_i t_i$ follows from 5.1. (2).

5.3. Classical compact operators can be characterized structurally as norm limits of orthogonal sequences of finite rank operators tending to 0 (in subtriples of $\mathscr{L}(\mathscr{H})$). An analogous class with atoms in the abstract setting instead of finite rank operators admit a description in terms of w-compactness.

Theorem. *We have* $a \in \mathrm{comp}_w E$ *iff* $a = \sum_{n=1}^{\infty} \alpha_n t_n$ *for some orthogonal sequence of atoms in* E *and constants* $\alpha_n \downarrow 0$.

Proof. Necessity: Let $\alpha \in \mathrm{comp}_w E$. By 5.1. Theorem we have

$$a = w^* - \sum_{i \in I} \alpha_i t_i$$

with a suitable orthogonal family of atoms and positive constants, respectively. Consider any infinite index sequence $\{i_1, i_2, \ldots\} \subset I$ such that $\alpha_{i_n} \to \alpha$ for some $\alpha \geqq 0$. We have only to prove that $\alpha = 0$. Set

$$U := \Big\{ \sum_{n=1}^{\infty} \beta_n t_{i_n} : (\beta_1, \beta_2, \ldots) \text{ converges in } \mathbf{C} \Big\}.$$

Now U is a closed subspace of E (cf. 5.1. (2)). The linear functional

$$\Lambda_0 : \sum_{n=1}^{\infty} \beta_n t_n \mapsto \lim_{n \to \infty} \beta_n$$

is well-defined on U and it has norm 1. Thus Λ_0 admits a continuous Hahn—Banach extension Λ to E. Observe that

$$u_n := w^* - \sum_{k \geqq n} t_{i_k} \to 0 \quad w^* \quad (n \to \infty).$$

Hence it follows that $\alpha = \lim_{k \to \infty} \alpha_k = \Lambda_0 (w^* - \sum_{k \geqq n} \alpha_{i_k} t_{i_k}) = \Lambda\{u_n a u_n\}$ $(n = 1, 2, \ldots)$ and so, since $a \in \mathrm{comp}_w E$, indeed

$$\alpha = \lim_{n \to \infty} \Lambda\{u_n a u_n\} = 0.$$

Sufficiency: Since $\text{comp}_w E$ is norm-closed (2.2. Lemma), by 5.1. (2) it suffices to see that $\text{at}\, E \subset \text{comp}_w E$. Let $t \in \text{at}\, E$ be arbitrarily chosen. Since $E = E_a^\perp \oplus \bigoplus_{F \in \mathcal{M}} F$, it follows that $t \in \text{at}\, F$ for some minimal w^*-closed ideal $F \in \mathcal{M}$. Since atoms in Cartan factors are finite rank operators and since the w^*-topology on the unit ball of a Cartan factor coincides with the weak operator topology, it suffices to prove that

$$(1) \qquad \Lambda\{x_i(\underline{e} \otimes \underline{f}^*)x_i\} \to 0$$

whenever

$$\overline{\mathrm{B}_1}\mathcal{L}(\mathcal{H}) \ni x_i \to 0 \ \ wop; \quad \Lambda \in \mathcal{L}(\mathcal{H})^*; \quad \underline{e}, \underline{f} \in \mathcal{H}$$

with a Hilbert space $\mathcal{H}$. Since $\{x_i(\underline{e} \otimes f^*)x_i\} \in \mathrm{c}_0(\mathcal{H})$ in any case, we may write $\Lambda \in \mathrm{c}_0(\mathcal{H})^*$ in (1) instead of the relation $\Lambda \in \mathcal{L}(\mathcal{H})^*$ without loss of generality. However, as it is well-known, $\mathrm{c}_0(\mathcal{H})$ is a predual for $\mathcal{L}(\mathcal{H})$ and therefore (1) is equivalent to saying that $\{x_i(\underline{e} \otimes \underline{f}^*)x_i\} \to 0 \ w^*$ whenever $\overline{\mathrm{B}_1}\mathcal{L}(\mathcal{H}) \ni x_i \to 0 \ w^*$ $(\underline{e}, \underline{f} \in \mathcal{H})$. This latter statement is already established in 4.2. Proposition.

Corollary. *We have* $\text{comp}_w \mathcal{L}(\mathcal{H}) = c_0(\mathcal{H})$, *or more generally* $\text{comp}_w F_k = \text{comp}_{w^*} F_k = c_0(\mathcal{H}) \cap F_k$ *for Cartan factors of type* $k = 1, 2, 3$ *in* $\mathcal{L}(\mathcal{H})$.

Corollary. *We have* $\text{comp}_w E = \text{Span} \bigcup_{F \in \mathcal{M} \setminus \mathcal{S}} \text{at}\, F$ *where* $\mathcal{S}$ *denotes the family of infinite dimensional spin factors of* E.

Example. Let $\mathcal{H} = \bigoplus_{k=1}^\infty \mathcal{H}_k$, $E := \bigoplus_{k=1}^\infty \mathcal{L}(\mathcal{H}_k) \subset \mathcal{L}(\mathcal{H})$ with a Hilbert space $\mathcal{H}$. Then

$$\text{comp}_{w^*} E = \bigoplus_{k=1}^\infty \mathrm{c}_0(\mathcal{H}_k),$$

$$\text{comp}_w E = \Big\{a \in \bigoplus_{k=1}^\infty \mathrm{c}_0(\mathcal{H}_k) \colon \lim_{k \to \infty} \|a|\mathcal{H}_k\| = 0\Big\}.$$

It is easy to see that if $t := \underline{e} \otimes \underline{f}^* \neq 0$ then $t \notin \text{comp}_n \mathcal{L}(\mathcal{H})$. Therefore, by 5.1. Theorem,

$$\text{comp}_n E = \{a \in E \colon a|\mathcal{H}_k = 0 \ \text{ whenever } \ \dim \mathcal{H}_k = \infty\}.$$

In particular

(i) $\text{comp}_{w^*} l^\infty = l^\infty, \quad \text{comp}_w l^\infty = \text{comp}_n l^\infty = \mathrm{c}_0;$

(ii) $\text{comp}_{w^*} \mathcal{L}(l^2) = \text{comp}_w \mathcal{L}(l^2) = \mathrm{c}_0(l^2), \quad \text{comp}_n \mathcal{L}(l^2) = 0.$

5.4. Finally we proceed to the problem: When is $\text{comp}_\tau E$ an ideal in E? The definitive answer seems to require intensive use of the structure theory of the two finite dimensional exceptional Cartan factors. On the other hand, it is not hard to show with operator theoretical technique that if the topology τ has a highly "symmetric" behaviour then $\mathrm{comp}_\tau E$ is an ideal in E.

Lemma. *The surjective linear isometries of a Cartan factor F_k of type $k \leqq 3$ act transitively on* at F_k.

Proof. Let us use the notations of 4.1. Definition along with the description of at F_k in [8, Cor. 8.40].

$k=1$. We have $F_1 = \mathscr{L}(\mathscr{H}, \mathscr{H}_1)$,

$$\mathrm{at}\, F_1 = \{\underline{f} \otimes \underline{e}^*\colon \underline{e} \in \mathscr{H},\ \underline{f} \in \mathscr{H}_1,\ \|\underline{e}\| = \|\underline{f}\| = 1\}.$$

Given any unit vectors $\underline{e}, \underline{e}' \in \mathscr{H}$, $\underline{f}, \underline{f}' \in \mathscr{H}_1$ there exist unitary operators $u \in \mathscr{L}(\mathscr{H})$, $u_1 \in \mathscr{L}(\mathscr{H}_1)$ such that $u\underline{e} = \underline{e}'$, $u\underline{f} = \underline{f}'$. Now, with the surjective linear isometry $\Psi\colon F_1 \ni x \mapsto u_1 x u^*$, we have $\Psi \underline{e} \otimes \underline{f}^* = \underline{e}' \otimes \underline{f}'^*$.

$k=2, 3$. We have $F_2 = \{x \in \mathscr{L}(\mathscr{H}\colon x = \bar{x}^*\}$, $F_3 = \{x \in \mathscr{L}(\mathscr{H})\colon x = -\bar{x}^*\}$,

$$\mathrm{at}\, F_2 = \{\underline{e} \otimes \bar{\underline{e}}^*\colon \underline{e} \in \mathscr{H},\ \|\underline{e}\| = 1\},$$

$$\mathrm{at}\, F_3 = \{\underline{f} \otimes \bar{\underline{e}}^* - \underline{e} \otimes \bar{\underline{f}}^*\colon \underline{e}, \underline{f} \in \mathscr{H},\ \underline{e} \perp \underline{f},\ \|\underline{e}\| = \|\underline{f}\| = 1\}.$$

Given any unit vectors $\underline{e}, \underline{e}', \underline{f}, \underline{f}' \in \mathscr{H}$ such that $\underline{e} \perp \underline{f}$, $\underline{e}' \perp \underline{f}'$ there exists a unitary operator $u \in \mathscr{L}(\mathscr{H})$ with $u\underline{e} = \underline{e}'$, $u\underline{f} = \underline{f}'$. Now the mapping

$$\Psi\colon \mathscr{L}(\mathscr{H}) \ni x \mapsto u x \bar{u}^*$$

is a surjective linear isometry of $\mathscr{L}(\mathscr{H})$ and

$$\Psi F_k = F_k \quad (k = 2, 3),$$

$$\Psi \underline{e} \otimes \bar{\underline{e}}^* = \underline{e}' \otimes \bar{\underline{e}}'^*, \quad \Psi(\underline{f} \otimes \bar{\underline{e}}^* - \underline{e} \otimes \bar{\underline{f}}^*) = \underline{f}' \otimes \bar{\underline{e}}'^* - \underline{e}' \otimes \bar{\underline{f}}'^*.$$

Proposition. *If any surjective linear isometry of E is $\tau \to \tau$ continuous then $\mathrm{comp}_\tau E$ is an ideal in E.*

Proof. We know that surjective linear isometries of E are triple product automorphisms [8] and they are $w^* \to w^*$ continous [6]. Thus if Ψ is a surjective linear isometry of E, $a \in \mathrm{comp}_\tau E$ and $\overline{\mathrm{B}_1 E} \ni x_i \to x\ w^*$ then $\{x_i (\Psi a) x_i\} = \Psi\{(\Psi^{-1} x_i) \cdot a(\Psi^{-1} x_i)\} \to \Psi\{(\Psi^{-1} x) a (\Psi^{-1} x)\} = \{x (\Psi a) x\}\ \tau$, i.e. $\Psi\, \mathrm{comp}_\tau E = \mathrm{comp}_\tau E$ whenever Ψ is $\tau \to \tau$ continous. Since $E = E_a^\perp \otimes \bigoplus_{F \in \mathscr{M}} F$, from the previous lemma it follows by 5.1., 5.2. Theorems that

$$\mathrm{comp}_\tau E = \mathrm{Span} \bigoplus_{\mathscr{N} \in \underline{\mathscr{M}}_\tau} \ \mathrm{Span}\, \mathrm{at}\, F \quad (F \in \mathscr{N})$$

where

$$\underline{\mathscr{M}}_\tau := \big\{\{F \in \mathscr{M}\colon \|\pi_F a\| \geqq 1\}\colon a \in \mathrm{comp}_\tau E\big\},$$

if the surjective linear isometries of E are $\tau\to\tau$ continuous. The right hand side here is obviously an ideal in E.

Corollary. *If any surjective linear isometry of E is $\tau\to\tau$ continuous then there is a family $\underline{\mathcal{M}}_\tau$ of subsets of $\mathcal{M}$ such that we have $a\in\mathrm{comp}_\tau E$ iff $\pi_F a\in\mathrm{comp}_{w^*} F$ $(F\in\mathcal{M})$, $\pi_c\overline{a}=0$ and for all $\varepsilon>0$, there exists $\mathcal{N}\in\underline{\mathcal{M}}_\tau$ such that $\{F\in\mathcal{M}\colon \|\pi_F a\|\geqq\geqq\varepsilon\}\subset\mathcal{N}$.*

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