Acta Sci. Math. 46 (1983), 377—379

Two remarks on pointwise periodic topological mappings

I. JOÓ and L. L. STACHÓ

In his investigations [2] concerning the fixed points of biholomorphic automorphisms of the closed unit ball in $C(\Omega)$ spaces, the second author proved, that a pointwise periodic automorphism $T: \Omega \to \Omega$ of a topological F-space Ω is necessarily periodic. (I.e., if for every $x \in \Omega$ there exists a natural number $n = n(x) > 1$ for which $T^n x = x$, then there exists $n_0 > 1$ such that $T^{n_0} x = x$ for every $x \in \Omega$.) His proof made essential use of the abstract properties of the function space $C(\Omega)$ and a lemma stating that the linear operator $\hat{T}: f \mapsto f \circ T$ on $C(\Omega)$ is periodic whenever it is pointwise periodic.

In this note we present a simple elementary generalization of the mentioned theorem about Ω-automorphisms. This may have interest even in itself since so far we have very lacunary information about the structure of automorphisms in abstract topological spaces. Furthermore, we also investigate some extensions of the lemma concerning $\hat{T}$.

1. Quasi F-spaces

Definition. Let Ω be a topological space. We say that Ω is a *quasi F-space* if for every pair of sequences $x_1, x_2, \ldots; y_1, y_2, \ldots$ in Ω such that $\{x_n: n \in N\} \cap \{y_n: n \in N\} = \emptyset$ there exists an infinite index set $I \subset N$ with $\{x_n: n \in I\}^- \cap \{y_n: n \in I\}^- = \emptyset$ (here $^-$ stands for the closure operation in Ω).

Remark. If Ω is a totally regular F-space (for the definition see [1]) then, by a theorem of Henricksen (see [1]), every countable subset is C^*-imbedded in Ω. Hence totally regular F-spaces are all quasi F-spaces. On the other hand, the real line equipped with the topology where the family τ of open sets is given by $\tau = \{G \setminus S$: where G is open in the usual sense, S is countable$\}$ is obviously a quasi F-space but not an F-space.

Received January 3, 1983.

The definition of the quasi-F property can be stated equivalently in the following slightly sharper form.

Lemma. *If Ω is a quasi F-space then for every family of pairwise disjoint sequences $[x_n^{(k)}: n\in N]$ $(k=1, 2, \ldots)$ there exists an infinite index set $I\subset N$ such that $\{x_n^{(k)}: n\in I\}^-\cap\{y_n^{(l)}: n\in I\}^-=\emptyset$ whenever $k\neq l$.*

Proof. By hypothesis, given any J infinite $\subset N$ and $k\neq l$, we may fix $H_{k,l}(J)$ infinite $\subset J$ such that $\{x_n^{(k)}: n\in H_{k,l}(J)\}^-\cap\{x_n^{(l)}: n\in H_{k,l}(J)\}^-=\emptyset$. Now we can define $I_1\supset I_2\supset I_3\supset\ldots$ recursively by $I_1=\mathbf{N}$, $I_{n+1}=H_{n+1,1}\,H_{n+1,2}\ldots H_{n+1,n}(I_n)$.

Clearly we have $\{x_n^{(k)}: n\in J_N\}^-\cap\{x_n^{(l)}: n\in J_N\}^-=\emptyset$ whenever $0<k<l\leqq M$ Therefore the choice

$$I=\left\{\min\{k\in I_M:\ k\geqq M\}:\ M\in N\right\}$$

suits the requirements of the lemma.

Theorem. *Let Ω be a countably compact quasi F-space and let T denote a pointwise periodic continuous mapping of Ω onto itself. Then T is necessarily periodic.*

Proof. Suppose T is not periodic. Then there exists a sequence $x_1, x_2, \ldots\in\Omega$ such that the sequence $p_k=\min\{n>0: T^n x_k=x_k\}$ strictly monotonically tends to ∞ (as $k\to\infty$). Observe that $T^n x_k\neq T^m x_l$ if $k\neq l$ and $0\leqq n<p_k$, $0\leqq m<p_l$. Hence, applying the lemma to the sequence $[x_n^{(k)}: n\in N]$ with

$$x_n^{(k)}=\begin{cases}T^k x_n & \text{if}\quad 0\leqq k<p_n\\ x_n & \text{otherwise}\end{cases}$$

we can find I infinite $\subset N$ such that $\{T^k x_n: n\in I\}^-\cap\{T^l x_n: n\in I\}^-=\emptyset$ for all $k\neq l$. By the countably compactness of Ω there exists an accumulation point $x\in\Omega$ of the sequence $\{x_n: n\in I\}$. But then we have $T^k x\neq T^l x$ whenever $k\neq l$, contradicting the pointwise periodicity of T.

Corollary. *T is a topological automorphism of Ω.*

2. Baire group homomorphisms

In [2] it is shown that a pointwise periodic bounded linear operator on a Banach space is necessarily periodic. The proof of this fact is straightforward if we make full use of the vector structure of the underlying space. However, one can raise the question, what the deeper role of the algebraic considerations here is. The answer is contained in the following substantially sharper result whose proof is, however, also very short.

Theorem. *Let G be a connected topological group endowed with a Baire topology and let U be a pointwise periodic group homomorphism of G into itself. Then U is necessarily periodic.*

Proof. Set $G_n=\{x\in G\colon U^n x=x\}$ $(n=1, 2, \ldots)$. Since U^n is also a continuous group endomorphism of G, G_n is a closed subgroup of G for each n. From the pointwise periodicity of U we obtain $G=\bigcup_{n>0} G_n$. Thus, by the Baire category theorem there exists $n_0>0$ such that the interior of G is not empty. Since G is a subgroup of G, this means that G_{n_0} is also open in G. Therefore, by the connectedness of G, we have $G_{n_0}=G$. That is, $U^{n_0}x=x$ for all $x\in G$.

References

[1] L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand (Princeton, 1960).

[2] L. L. Stachó, On the existence of fixed points for holomorphic automorphisms, *Ann. Mat. Pura Appl., (IV)*, **128** (1980), 207—225.

(I. J.)
EÖTVÖS LORÁND UNIVERSITY
II. CHAIR FOR ANALYSIS
MÚZEUM KRT. 6—8
1088 BUDAPEST, HUNGARY

(L. L. S.)
JÓZSEF ATTILA UNIVERSITY
BOLYAI INSTITUTE
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY

83-2953 — Szegedi Nyomda — F. v.: Dobó József igazgató